

RIESZ MEANS OF FOURIER SERIES AND INTEGRALS: STRONG SUMMABILITY AT THE CRITICAL INDEX

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ABSTRACT. We consider spherical Riesz means of multiple Fourier series and some generalizations. While almost everywhere convergence of Riesz means at the critical index $(d-1)/2$ may fail for functions in the Hardy space $h^1(\mathbb{T}^d)$, we prove sharp positive results for strong summability almost everywhere. For functions in $L^p(\mathbb{T}^d)$, $1 < p < 2$, we consider Riesz means at the critical index $d(1/p - 1/2) - 1/2$ and prove an almost sharp theorem on strong summability. The results follow via transference from corresponding results for Fourier integrals. We include an endpoint bound on maximal operators associated with generalized Riesz means on Hardy spaces $H^p(\mathbb{R}^d)$ for $0 < p < 1$.

1. INTRODUCTION

We consider multiple Fourier series of functions on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For $\ell \in \mathbb{Z}^d$ let $e_\ell(x) = e^{2\pi i \langle x, \ell \rangle}$ and define the Fourier coefficients of $f \in L^1(\mathbb{T}^d)$ by $\langle f, e_\ell \rangle = \int_{\mathbb{T}^d} f(y) e^{-2\pi i \langle y, \ell \rangle} dy$. We shall examine the pointwise behavior of (generalized) Riesz means of the Fourier series. Fix a homogeneous distance function ρ , continuous on \mathbb{R}^d , positive and C^∞ on $\mathbb{R}^d \setminus \{0\}$, and satisfying, for some $b > 0$, $\rho(t^b \xi) = t \rho(\xi)$ for all $\xi \in \mathbb{R}^d$. For $f \in L^1(\mathbb{T}^d)$ define the Riesz means of index λ with respect to ρ , by

$$(1.1) \quad \mathcal{R}_t^\lambda f = \sum_{\substack{\ell \in \mathbb{Z}^d \\ \rho(\ell/t) \leq 1}} (1 - \rho(\ell/t))^\lambda \langle f, e_\ell \rangle e_\ell.$$

The classical Riesz means are recovered for $\rho(\xi) = |\xi|$, and when in addition $\lambda = 1$ we obtain the Fejér means. The Bochner-Riesz means are covered with $b = 1/2$ by taking $\rho(\xi) = |\xi|^2$.

It is well known via classical results for Fourier integrals ([34], [40], [31]) and transference ([25], [20], [1]) that for $\lambda > \frac{d-1}{2}$ and $f \in L^1(\mathbb{T}^d)$ we have $\lim_{t \rightarrow \infty} \mathcal{R}_t^\lambda f = f$, both with respect to the L^1 norm and also almost everywhere. For the critical index $\lambda = \frac{d-1}{2}$, it is known that the Riesz means are of weak type $(1, 1)$ and one has convergence in measure ([8], [10]) but Stein [35] showed early that a.e. convergence may fail (see also [40]). Indeed, extending ideas of Bochner, he proved the existence of an $L^1(\mathbb{T}^d)$ function for which the Bochner-Riesz means at index $\frac{d-1}{2}$ diverge almost everywhere, as $t \rightarrow \infty$. Stein's theorem can be seen as an analogue of the theorem by Kolmogorov [23] on the failure of a.e. convergence for Fourier series in $L^1(\mathbb{T})$, see [48, ch. VIII-4]. Later, Stein [37] proved a stronger result showing that even for some functions in the subspace $h^1(\mathbb{T}^d)$ (the local Hardy space) the Bochner-Riesz means at the critical index may diverge almost everywhere. It is then natural to ask what happens if we replace almost everywhere convergence with the weaker notion of strong convergence a.e. (also known as strong summability a.e.) which goes back to Hardy and Littlewood [18].

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Definition. Let $0 < q < \infty$. Given a measurable function $g : (0, \infty) \rightarrow \mathbb{C}$ we say that $g(t)$ converges q -strongly to a , as $t \rightarrow \infty$, if

$$\lim_{T \rightarrow \infty} \left(\frac{1}{T} \int_0^T |g(t) - a|^q dt \right)^{1/q} = 0.$$

If $g(t)$ refers to the partial sum of a series then one also says that the series is strongly H_q summable. Clearly if $\lim_{t \rightarrow \infty} g(t) = a$ then $g(t)$ converges q -strongly to a for all $q < \infty$. Vice versa if $g(t)$ converges q -strongly to a for some $q > 0$ then $g(t)$ is *almost convergent* to a as $t \rightarrow \infty$. That is, there is a (density one) subset $E \subset [0, \infty)$ satisfying

$$(1.2) \quad \lim_{T \rightarrow \infty} \frac{|E \cap [0, T]|}{T} = 1 \quad \text{and} \quad \lim_{\substack{t \rightarrow \infty \\ t \in E}} g(t) = a.$$

See [48, ch.XIII, (7.2)] and also Corollary 6.3 below.

For the classical case of a Fourier series of an $L^1(\mathbb{T})$ function, Zygmund [47] proved that the partial sum $\sum_{|l| \leq t} \langle f, e_l \rangle e_l(x)$ converges q -strongly to $f(x)$ as $t \rightarrow \infty$ a.e. for all $q < \infty$, extending an earlier result by Marcinkiewicz [28] for $q = 2$. Zygmund used complex methods, but in more recent papers one can find alternative approaches with stronger results and some weaker extensions to rectangular partial sums of multiple Fourier series; see, e.g., [30] and [46] and references therein. See also [24] for an overview of recent developments on topics related to the convergence of Fourier series.

Regarding spherical partial sums of multiple Fourier series, q -strong convergence results have been available for $L^p(\mathbb{T}^d)$ functions for the Bochner-Riesz means of index $\lambda > \lambda(p)$ when $p \leq 2$, $q = 2$, where $\lambda(p) = d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ is the critical index (*cf.* [34], [42]). The question of strong convergence a.e. for the Bochner-Riesz means at the critical index $\lambda(1) = \frac{d-1}{2}$, for either $f \in L^1(\mathbb{T}^d)$ or $f \in h^1(\mathbb{T}^d)$ had been left open and was posed by S. Lu in the survey article [27]. We answer this question in the affirmative for $f \in h^1(\mathbb{T}^d)$ for generalized Riesz means with *any* distance function ρ under consideration.

Theorem 1.1. *Let $q < \infty$ and $\lambda(1) = \frac{d-1}{2}$. Then, for all $f \in h^1(\mathbb{T}^d)$ the following statements hold.*

(i) *There is a constant C such that for all $\alpha > 0$,*

$$\text{meas} \left(\left\{ x : \sup_{T > 0} \left(\frac{1}{T} \int_0^T |\mathcal{R}_t^{\lambda(1)} f(x)|^q dt \right)^{1/q} > \alpha \right\} \right) \leq C \alpha^{-1} \|f\|_{h^1}.$$

(ii)

$$\lim_{T \rightarrow \infty} \left(\frac{1}{T} \int_0^T |\mathcal{R}_t^{\lambda(1)} f(x) - f(x)|^q dt \right)^{1/q} = 0 \quad \text{for almost every } x \in \mathbb{T}^d.$$

We remark that for the classical Riesz means (or generalized Riesz means assuming finite type conditions on the cosphere $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$), Theorem 1.1 for the range $q \leq 2$ could have been extracted from [32], although that result is not explicitly stated there. The full range $q < \infty$ obtained here seems to be new. Regarding the question posed for $f \in L^1(\mathbb{T}^d)$, in Section 6, we derive some weaker results including q -strong convergence up to passing to a subsequence.

We now address the question of strong convergence of Riesz means for $L^p(\mathbb{T}^d)$ functions at the critical index $\lambda = \lambda(p)$. In this case, q -strong convergence results may fail for large q . Our next result identifies nearly sharp range of q for which $\mathcal{R}_t^{\lambda(p)} f(x)$ converges q -strongly to $f(x)$ almost everywhere for any $f \in L^p(\mathbb{T}^d)$. We denote by $p' = \frac{p}{p-1}$ the exponent dual to p .

Theorem 1.2. *Let $1 < p < 2$, $q < p'$ and $\lambda(p) = d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. Then, for all $f \in L^p(\mathbb{T}^d)$ the following statements hold.*

(i) *There is a constant C such that for all $\alpha > 0$,*

$$\text{meas}\left(\left\{x \in \mathbb{T}^d : \sup_{T>0} \left(\frac{1}{T} \int_0^T |\mathcal{R}_t^{\lambda(p)} f(x)|^q dt\right)^{1/q} > \alpha\right\}\right) \leq C\alpha^{-p} \|f\|_{L^p(\mathbb{T}^d)}.$$

(ii)

$$\lim_{T \rightarrow \infty} \left(\frac{1}{T} \int_0^T |\mathcal{R}_t^{\lambda(p)} f(x) - f(x)|^q dt\right)^{1/q} = 0 \text{ for almost every } x \in \mathbb{T}^d.$$

(iii) *For suitable $f \in L^p(\mathbb{T}^d)$ statements (i), (ii) fail when $q > p'$.*

Part (ii) in both theorems follow by a standard argument from the respective part (i), using the fact that pointwise (in fact uniform) convergence holds for Schwartz functions. We note that Theorem 1.1 is sharp in view of the above mentioned example by Stein. Moreover, part (iii) of Theorem 1.2 shows that the result is essentially sharp for all $p \in (1, 2)$, but the case $q = p'$ remains open.

We state a special case of Theorem 1.2 for $\lambda(p) = 0$, i.e., for the case of generalized spherical partial sums of Fourier series as a corollary.

Corollary 1.3. *Let $d \geq 2$, $q < \frac{2d}{d-1}$ and $f \in L^{\frac{2d}{d+1}}(\mathbb{T}^d)$. Then*

$$\lim_{T \rightarrow \infty} \left(\frac{1}{T} \int_0^T \left| \sum_{\rho(\ell/t) \leq 1} \langle f, e_\ell \rangle e_\ell(x) - f(x) \right|^q dt\right)^{1/q} = 0 \text{ for almost every } x \in \mathbb{T}^d.$$

In particular, for almost every $x \in \mathbb{T}^d$, the partial sums $\sum_{\rho(\ell/t) \leq 1} \langle f, e_\ell \rangle e_\ell(x)$ are almost convergent to $f(x)$ as $t \rightarrow \infty$, in the sense of (1.2).

We remark that there are analogues of above results for generalized Riesz means of Fourier integral in \mathbb{R}^d :

$$(1.3) \quad R_t^\lambda f(x) = \int_{\rho(\xi/t) \leq 1} (1 - \rho(\xi/t))^\lambda \widehat{f}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi.$$

See §2. Indeed, we derive Theorems 1.1 and 1.2 from corresponding theorems for Fourier integrals in \mathbb{R}^d using transference arguments. Our proof uses somewhat technical arguments on atomic decomposition and Calderón-Zygmund theory. Unlike the proofs of the L^p boundedness of Bochner-Riesz means (such as, say, in [38], [5] and the references therein), our proof does not rely on Fourier restriction theory thanks to the averaging over the dilation parameter t . In particular, the curvature of the cosphere $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$ does not play a role in the argument (*cf.* [10], [11]), which allows us to work with generalized Riesz means with respect to any smooth homogeneous distance function.

This paper. In §2 we formulate Theorems 2.1 and 2.2 on strong convergence for Riesz means of critical index in \mathbb{R}^d and reduce their proof to the main weak type inequality stated in Theorem 2.3. Some preliminary estimates are contained in §3. The proof of the main Theorem 2.3 is given in §4. In §5 we use transference arguments to prove the positive results in Theorems 1.1 and 1.2. In §6 we discuss a weaker result for L^1 functions. In §7 we show the essential sharpness of our L^p results, namely that Theorems 1.2 and 2.2 require the condition $q \leq p'$ (the failure of the maximal theorems for h^1 already follows from Stein's example [37]). In §8 we include the proof of an extension of a theorem by Stein, Taibleson and Weiss ([39]), namely an $H^p \rightarrow L^{p,\infty}$ estimate for the maximal function $\sup_{t>0} |R_t^{\lambda(p)} f(x)|$ associated with generalized Riesz means in Hardy spaces H^p with $p < 1$. Finally, we discuss some open problems in §9.

2. THE MAIN WEAK TYPE ESTIMATE

We first state results on \mathbb{R}^d which are analogous to Theorems 1.1 and 1.2. Let ρ be as in the introduction and we recall the definition of Riesz means R_t^λ for Fourier integrals from (1.3).

Theorem 2.1. *Let $q < \infty$ and $\lambda(1) = \frac{d-1}{2}$. Then, for all $f \in H^1(\mathbb{R}^d)$, for all $\alpha > 0$,*

$$\text{meas}\left(\left\{x \in \mathbb{R}^d : \sup_{T>0} \left(\frac{1}{T} \int_0^T |R_t^{\lambda(1)} f(x)|^q dt\right)^{1/q} > \alpha\right\}\right) \leq C\alpha^{-1} \|f\|_{H^1(\mathbb{R}^d)}.$$

Theorem 2.2. *Let $1 < p < 2$, $q < p'$ and $\lambda(p) = d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. Then, for all $f \in L^p(\mathbb{R}^d)$, for all $\alpha > 0$,*

$$\text{meas}\left(\left\{x \in \mathbb{R}^d : \sup_{T>0} \left(\frac{1}{T} \int_0^T |R_t^{\lambda(p)} f(x)|^q dt\right)^{1/q} > \alpha\right\}\right) \leq C\alpha^{-p} \|f\|_{L^p(\mathbb{R}^d)}.$$

As a consequence of these estimates we obtain

$$\lim_{T \rightarrow 0} \left(\frac{1}{T} \int_0^T |R_t^{\lambda(p)} f(x) - f(x)|^q dt\right)^{1/q} = 0$$

for almost every $x \in \mathbb{R}^d$, for every $f \in L^p(\mathbb{R}^d)$ when $1 < p < 2$ and $f \in h^1(\mathbb{R}^d)$ or $H^1(\mathbb{R}^d)$ when $p = 1$.

2.1. *Contribution near the origin.* Let $v_0 \in C^\infty(\mathbb{R})$ so that $v_0(\rho) = 1$ for $\rho \leq 4/5$ and $v_0(\rho) = 0$ for $\rho \geq 9/10$. It is then standard that the maximal function $\sup_{t>0} |\mathcal{F}^{-1}[v_0(\rho(\cdot/t))(1-\rho(\cdot/t))_+^\lambda \widehat{f}]|$ defines an operator of weak type $(1, 1)$ and bounded on L^p for all $p > 1$. A small complication occurs if ρ is not sufficiently smooth at the origin. Define, for $N > 0$, the functions u, u_N with domain $(0, \infty)$ by $u(\tau) = v_0(\tau)(1-\tau)^\lambda$ and $u_N(s) = u(s^{1/N})$. It is then straightforward to check that for all M

$$\int_0^\infty s^M |u_N^{(M+1)}(s)| ds < \infty$$

and we have the subordination formula ([45])

$$(2.1) \quad u(\rho(\xi)) = u_N(\rho^N(\xi)) = \frac{(-1)^{M+1}}{M!} \int_0^\infty \left(1 - \frac{(\rho(\xi))^N}{s}\right)_+^M s^M u_N^{(M+1)}(s) ds$$

which is proved by integration by parts. Given any $m > 0$ one has $|\mathcal{F}^{-1}[(1-\rho^N)_+^M](x)| \lesssim_m (1+|x|)^{-m}$ provided M and N are large enough. This is used to show that $\sup_{t>0} |\mathcal{F}^{-1}[u \circ \rho(\cdot/t) \widehat{f}]|$ is dominated by a constant times the Hardy-Littlewood maximal function of f (see also Lemma 8.2).

We can now replace the operator R_t^λ in the first part of Theorems 2.1 and 2.2 by S_t^λ defined by

$$(2.2) \quad \widehat{S_t^\lambda f}(\xi) = (1 - v_0(\rho(\xi/t)))(1 - \rho(\xi/t))_+^\lambda \widehat{f}(\xi).$$

2.2. *Further decompositions.* We first recall standard dyadic decompositions on the frequency side. Let $\eta \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ such that η is nonnegative,

$$(2.3) \quad \eta(\xi) = 1 \text{ on } \{\xi : \rho(\xi/t) \in [1/4, 4], 1/2 \leq t \leq 2\}.$$

Define $\mathcal{L}_k f$ by $\widehat{\mathcal{L}_k f}(\xi) = \eta(2^{-k}\xi) \widehat{f}(\xi)$.

We use the nontangential version of the Peetre maximal operators

$$\mathfrak{M}_k f(x) = \sup_{|h| \leq 2^{-k+10}d} |\mathcal{L}_k f(x+h)|$$

and the associated square function

$$(2.4) \quad \mathfrak{S}f(x) = \left(\sum_{k \in \mathbb{Z}} |\mathfrak{M}_k f(x)|^2 \right)^{1/2}.$$

Then

$$(2.5a) \quad \|\mathfrak{S}f\|_{L^1} \leq C\|f\|_{H^1},$$

and

$$(2.5b) \quad \|\mathfrak{S}f\|_{L^p} \leq C_p\|f\|_{L^p}, \quad 1 < p < \infty.$$

see (Peetre [29]).

The inequalities in Theorems 2.1 and 2.2 follow now from

$$\left\| \sup_{T>0} \left(\frac{1}{T} \int_0^T |S_t^{\lambda(p)} f|^q dt \right)^{1/q} \right\|_{L^{p,\infty}} \lesssim \|\mathfrak{S}f\|_p$$

for $1 \leq p < 2$, $q < p'$. Here $L^{p,\infty}$ is the weak type Lorentz space and the expression $\|g\|_{L^{p,\infty}} = \sup_{\alpha>0} \alpha (\text{meas}(\{x : |g(x)| > \alpha\}))^{1/p}$ is the standard quasi-norm on $L^{p,\infty}$. We may, by Hölder's inequality, assume that $2 \leq q < p'$. We can then use

$$(2.6) \quad \sup_{T>0} \left(\frac{1}{T} \int_0^T |S_t^{\lambda(p)} f(x)|^q dt \right)^{1/q} \leq 2^{1/q} \left(\sum_{k \in \mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} |S_t^{\lambda(p)} f(x)|^q dt \right)^{1/q}.$$

We now use the standard idea to decompose the multiplier $(1 - v_0 \circ \rho)(1 - \rho)_+^\lambda$ into pieces supported where $\rho(\xi) \in [1 - 2^{-j}, 1 - 2^{-j-2}]$. Generalizing slightly we assume that we are given C^∞ functions φ_j supported in $[1 - 2^{-j}, 1 - 2^{-j-2}]$ and satisfying

$$\|\partial^n \varphi_j\|_\infty \leq C_n 2^{jn}.$$

for $n = 0, 1, 2, \dots$. Let $I := [1, 2]$. For $t \in I$, $k \in \mathbb{Z}$ define

$$(2.7) \quad \widehat{T_j^k f}(\xi, t) = \varphi_j(\rho(2^{-k}t^{-1}\xi)) \widehat{f}(\xi)$$

We may decompose $S_{2^k t}^\lambda f = \sum_{j \geq 1} 2^{-j\lambda} T_j^k f(x, t)$, with T_j^k of the form in (2.7). The asserted estimates for $S_t^{\lambda(p)}$ follow now from weak type bounds for the expression on the right hand side of (2.6). By (2.3) we have $\eta(2^{-k}\xi) = 1$ whenever $\rho(2^{-k}\xi/t) \in \text{supp}(\varphi_j)$ for any $t \in I$. Thus after changing variables the desired estimate can be recast as

$$\left\| \left(\sum_{k \in \mathbb{Z}} \int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_j^k \mathcal{L}_k f(\cdot, t) \right|^q dt \right)^{1/q} \right\|_{L^{p,\infty}} \lesssim \|\mathfrak{S}f\|_p.$$

Since $\ell^2 \subset \ell^q$ for $q \geq 2$ this follows from the following stronger statement, our main estimate.

Theorem 2.3. *For $1 \leq p < 2$, $\lambda(p) = d(1/p - 1/2) - 1/2$, $q < p'$,*

$$\left\| \left(\sum_{k \in \mathbb{Z}} \left(\int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_j^k \mathcal{L}_k f(\cdot, t) \right|^q dt \right)^{2/q} \right)^{1/2} \right\|_{L^{p,\infty}(\mathbb{R}^d)} \lesssim \|\mathfrak{S}f\|_{L^p(\mathbb{R}^d)}.$$

The theorem will be proved in §4. Some preparatory material is contained in §3.

3. PRELIMINARY ESTIMATES

We gather elementary estimates for the operators T_j^k defined in (2.7).

Lemma 3.1. *For $2 \leq q \leq \infty$,*

$$\left\| \left(\int_1^2 |T_j^k f(\cdot, t)|^q dt \right)^{1/q} \right\|_2 \lesssim 2^{-j/q} \|f\|_2.$$

Proof. Use the convexity inequality, $\|\gamma\|_q \leq \|\gamma\|_2^{2/q} \|\gamma\|_\infty^{1-2/q}$, for $\gamma \in L^q([1, 2])$, and for $\gamma \in C^1$ we have $\|\gamma\|_\infty \lesssim \|\gamma\|_2^{1/2} (\|\gamma\|_2 + \|\gamma'\|_2)^{1/2}$ and hence

$$(3.1) \quad \left(\int_1^2 |\gamma(t)|^q dt \right)^{1/q} \lesssim \left(\int_1^2 |\gamma(t)|^2 dt \right)^{1/2} + \left(\int_1^2 |\gamma(t)|^2 dt \right)^{\frac{1}{2}(\frac{1}{2} + \frac{1}{q})} \left(\int_1^2 |\gamma'(t)|^2 dt \right)^{\frac{1}{2}(\frac{1}{2} - \frac{1}{q})}.$$

We obtain after some standard estimations

$$\left\| \left(\int_1^2 |T_j^k f(\cdot, t)|^2 dt \right)^{1/2} \right\|_2 + 2^{-j} \left\| \left(\int_1^2 \left| \frac{d}{dt} T_j^k f(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2 \lesssim 2^{-j/2} \|f\|_2$$

and then the assertion of the lemma follows from (3.1) applied to $\gamma(t) = T_j^k f(x, t)$, followed by Hölder's inequality in x . \square

To prove the L^1 estimate we rely on a spherical decomposition introduced in [12]. We use a C^∞ partition of unity $\chi_{j,\nu}$, homogeneous of degree 0, so that the restriction of the support of χ_ν to $\{\xi : |\xi| = 1\}$ is supported in a set of diameter $2^{-j/2}$, for fixed j each unit vector is contained in $O(1)$ of the supports of the $\chi_{j,\nu}$ and the indices ν belong to an index set \mathcal{Z}_j with $\#\mathcal{Z}_j = O(2^{j(d-1)/2})$. We may choose this index set so that for every ν , there is a unit vector $\xi_{j,\nu} \in \text{supp}(\chi_{j,\nu})$ so that $\text{dist}(\xi_{j,\nu}, \xi_{j,\nu'}) \geq c2^{-j/2}$ for $\nu \neq \nu'$. We assume that the $\chi_{j,\nu}$ satisfy the natural differential estimates, i.e. $\partial_\xi^\beta \chi_{j,\nu}(\xi) = O(2^{\frac{j}{2}(\beta_1 + \dots + \beta_d)})$. Define $T_{j,\nu}^k$ by

$$(3.2) \quad \widehat{T_{j,\nu}^k f}(\xi, t) = \chi_{j,\nu}(\xi) \varphi_j(\rho(2^{-k}t^{-1}\xi)) \widehat{f}.$$

Let $K_j = \mathcal{F}^{-1}[\varphi_j(\rho(\cdot))]$, and $K_{j,\nu} = \mathcal{F}^{-1}[\varphi_j(\rho(\cdot))\chi_{j,\nu}]$. Let $\Phi_0 \in C_c^\infty(\mathbb{R}^d)$ supported in $\{x : |x| \leq 1\}$ so that $\Phi_0(x) = 1$ for $|x| \leq 1/2$ and, for $n \geq 1$, let $\Phi_n(x) = \Phi_0(2^{-n}x) - \Phi_0(2^{1-n}x)$. Define, for $n = 0, 1, 2, \dots$,

$$\begin{aligned} K_j^n(x) &= K_j(x) \Phi_n(2^{-j}x) \\ K_{j,\nu}^n(x) &= K_{j,\nu}(x) \Phi_n(2^{-j}x) \end{aligned}$$

and

$$\begin{aligned} T_j^{n,k} f(x, t) &= (2^k t)^d K_j^n(2^k t \cdot) * f, \\ T_{j,\nu}^{n,k} f(x, t) &= (2^k t)^d K_{j,\nu}^n(2^k t \cdot) * f. \end{aligned}$$

Then

$$(3.3) \quad T_j^k f = \sum_{\nu \in \mathcal{Z}_j} T_{j,\nu}^k f = \sum_{n=0}^{\infty} T_j^{n,k} f = \sum_{n=0}^{\infty} \sum_{\nu \in \mathcal{Z}_j} T_{j,\nu}^{n,k} f.$$

Lemma 3.2. *Let $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$. Then*

$$|\widehat{K_j^n}(\xi)| \leq C_{M_0, M_1} 2^{-nM_0} (1 + 2^j \text{dist}(\xi, \Sigma_\rho))^{-M_1}.$$

Sketch of Proof. Let $\Psi(x) = \Phi_0(x/2) - \Phi_0(x)$. Then, for $n \geq 1$, we may use that $\widehat{\Psi}$ has vanishing moments and write

$$(3.4) \quad \begin{aligned} \widehat{K}_j^n(\xi) &= \int \varphi_j(\rho(\xi - y)) 2^{(j+n)d} \widehat{\Psi}(2^{j+n}y) dy \\ &= \int_0^1 \frac{(1-s)^{N-1}}{(N-1)!} \int \langle y, \nabla \rangle^N [\varphi_j \circ \rho](\xi - sy) 2^{(j+n)d} \widehat{\Psi}(2^{j+n}y) dy ds \end{aligned}$$

by Taylor's formula. The estimate is now straightforward. When $n = 0$ we just use the first line in (3.4) with Ψ replaced by Φ_0 . \square

For each ν choose $\xi_{j,\nu}$ such that $\rho(\xi_{j,\nu}) = 1$ and $\xi_{j,\nu} \in \text{supp}(\chi_{j,\nu})$. Take $e_{j,\nu} = \frac{\nabla \rho(\xi_{j,\nu})}{|\nabla \rho(\xi_{j,\nu})|}$ and let $P_{j,\nu}$ be the orthogonal projection to $e_{j,\nu}^\perp$, i.e.

$$(3.5) \quad P_{j,\nu} h = h - \langle h, e_{j,\nu} \rangle e_{j,\nu}.$$

Lemma 3.3. *For every $M \geq 0$,*

$$(3.6) \quad \sup_{t \in I} |t^d K_{j,\nu}(tx)| \leq C(M) \frac{2^{-j(d+1)/2}}{(1 + 2^{-j}|x|)^M (1 + 2^{-j/2}|P_{j,\nu}(x)|)^M}.$$

Proof. This is standard (and follows after integration by parts), see, e.g., [12], [11], or [31]. \square

Lemma 3.4. (i) *For $k \in \mathbb{Z}$,*

$$\left\| \sup_{t \in I} |T_j^{n,k} f(\cdot, t)| \right\|_1 \leq C_N 2^j 2^{\frac{d-1}{2}} 2^{-nN} \|f\|_1.$$

(ii) *For $1 < p \leq 2$, $q \leq p'$ and $k \in \mathbb{Z}$,*

$$\left\| \left(\int_I |T_j^{n,k} f(\cdot, t)|^q dt \right)^{1/q} \right\|_p \leq C_N 2^{j(d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2})} 2^{-nN} \|f\|_p.$$

(iii) *For $2 \leq q \leq \infty$,*

$$\left\| \left(\int_1^2 |T_j^{n,k} f(\cdot, t)|^q dt \right)^{1/q} \right\|_2 \lesssim 2^{-j/q} 2^{-nN} \|f\|_2.$$

Proof. Lemma 3.3 easily implies $\left\| \sup_{t \in I} |T_{j,\nu}^{n,k} f(\cdot, t)| \right\|_1 \leq C_N 2^{-nN} \|f\|_1$ and part (i) follows after summing in ν . Using Lemma 3.2 we see that the proof of Lemma 3.1 also gives

$$\left\| \left(\int_1^2 |T_j^{n,k} f(\cdot, t)|^2 dt \right)^{1/2} \right\|_2 \lesssim_N 2^{-nN} 2^{-j/2} \|f\|_2.$$

Part (ii) now follows by complex interpolation.

Part (iii) for $q = 2$ is just the previous displayed inequality. For $q > 2$ it follows by the argument in Lemma 3.1 (cf. (3.1)) applied to $T_j^{n,k}$ in place of T_j^k , in conjunction with Lemma 3.2. \square

4. PROOF OF THEOREM 2.3

The proof combines ideas that were used in the proof of weak type inequalities for Bochner-Riesz means and other radial multipliers, and elsewhere ([15], [8], [9], [10], [32]). It combines atomic decompositions with Calderón-Zygmund estimates using L^r -bounds for $r > p$ in the complement of suitable exceptional sets together with analytic interpolation arguments inspired by [9].

In this section we fix a Schwartz function f whose Fourier transform has compact support in $\mathbb{R}^d \setminus \{0\}$. Observe that then $\mathcal{L}_k f = 0$ for all but a finite number of indices k (depending on f).

This assumption together with the Schwartz bounds can be used to justify the a priori finiteness of various expressions showing up in the arguments below, but they do not enter quantitatively in the estimates.

We need to prove the inequality

$$(4.1) \quad \text{meas}\left\{x \in \mathbb{R}^d : \left(\sum_k \left[\int_I \left|\sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_j^k \mathcal{L}_k f(x, t)\right|^q dt\right]^{2/q}\right)^{1/2} > \alpha\right\} \lesssim \alpha^{-p} \|\mathfrak{S}f\|_p^p,$$

for arbitrary but fixed $\alpha > 0$. The implicit constant does not depend on α and the choice of f .

4.1. *Preliminaries on atomic decompositions.* Let \mathcal{R}_k be the set of dyadic cubes of side length 2^{-k} so that each $R \in \mathcal{R}_k$ is of the form $\prod_{i=1}^d [n_i 2^{-k}, (n_i + 1) 2^{-k}]$ for some $n \in \mathbb{Z}^d$. For $\mu \in \mathbb{Z}$ let

$$\Omega_\mu = \{x : |\mathfrak{S}f(x)| > 2^\mu\}$$

and let \mathcal{R}_k^μ be the set of dyadic cubes of length 2^{-k} with the property that

$$|R \cap \Omega_\mu| \geq |R|/2 \text{ and } |R \cap \Omega_{\mu+1}| < |R|/2.$$

Clearly if $\mathfrak{S}f \in L^p$ then every dyadic cube in \mathcal{R}_k belongs to exactly one of the sets \mathcal{R}_k^μ . We then have ([7])

$$(4.2) \quad \sum_{k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_k^\mu} \int_R |\mathcal{L}_k f|^2 dx \lesssim 2^{2\mu} \text{meas}(\Omega_\mu).$$

For completeness we give the argument. Observe that

$$|\mathcal{L}_k f(x)| \leq \mathfrak{M}_k f(z), \quad \text{for } x, z \in R, R \in \mathcal{R}_k^\mu.$$

Let

$$\tilde{\Omega}_\mu = \{x : M_{HL} \mathbb{1}_{\Omega_\mu} > 10^{-d}\}$$

where M_{HL} denotes the Hardy-Littlewood maximal operator. Then

$$\text{meas}(\tilde{\Omega}_\mu) \lesssim \text{meas}(\Omega_\mu)$$

and we have $\cup_k \cup_{R \in \mathcal{R}_k^\mu} R \subset \tilde{\Omega}_\mu$. Now

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_k^\mu} \|\mathbb{1}_R \mathcal{L}_k f\|_2^2 &\leq \sum_{k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_k^\mu} 2 \int_{R \setminus \Omega_{\mu+1}} |\mathfrak{M}_k f(x)|^2 dx \\ &\leq 2 \int_{\tilde{\Omega}_\mu \setminus \Omega_{\mu+1}} \sum_{k \in \mathbb{Z}} |\mathfrak{M}_k f(x)|^2 dx \leq 2^{2\mu+1} \text{meas}(\tilde{\Omega}_\mu) \leq C 2^{2\mu} \text{meas}(\Omega_\mu) \end{aligned}$$

which yields (4.2).

Next we work with a Whitney decomposition of the open set $\tilde{\Omega}_\mu$, which is a disjoint union of dyadic cubes W , such that

$$\text{diam}(W) \leq \text{dist}(W, \tilde{\Omega}_\mu^c) \leq 4 \text{diam}(W).$$

See [36, ch. VI.1]. We denote by \mathfrak{W}^μ the collection of these Whitney cubes. Each $R \in \mathcal{R}_k^\mu$ is contained in a unique $W(R) \in \mathfrak{W}^\mu$. For each W define

$$(4.3) \quad \mathcal{R}_k^\mu(W) = \{R \in \mathcal{R}_k^\mu : R \subset W\}$$

and

$$\gamma_{W, \mu} = \left(\frac{1}{|W|} \sum_k \sum_{R \in \mathcal{R}_k^\mu(W)} \int_R |\mathcal{L}_k f(y)|^2 dy \right)^{1/2}.$$

Define

$$(4.4) \quad U(x) = \sum_{\mu} \sum_{W \in \mathfrak{W}^{\mu}} \gamma_{W,\mu}^p \mathbb{1}_W(x).$$

Observe that

$$(4.5) \quad \begin{aligned} \|U\|_1 &= \sum_{\mu} \sum_{W \in \mathfrak{W}^{\mu}} |W| \gamma_{W,\mu}^p = \sum_{\mu} \sum_{W \in \mathfrak{W}^{\mu}} |W|^{1-p/2} (|W|^{1/2} \gamma_{\mu,W})^p \\ &\leq \sum_{\mu} \left(\sum_{W \in \mathfrak{W}^{\mu}} |W| \right)^{1-p/2} \left(\sum_{W \in \mathfrak{W}^{\mu}} |W| \gamma_{\mu,W}^2 \right)^{p/2} \\ &\leq \sum_{\mu} |\tilde{\Omega}_{\mu}|^{1-p/2} \left(\sum_k \sum_{W \in \mathfrak{W}^{\mu}} \sum_{R \in \mathcal{R}_k^{\mu}(W)} \|\mathbb{1}_R \mathcal{L}_k f\|_2^2 \right)^{p/2} \\ &\lesssim \sum_{\mu} |\Omega_{\mu}|^{1-p/2} (2^{2\mu} |\Omega_{\mu}|)^{p/2} \lesssim \sum_{\mu} 2^{\mu p} |\Omega_{\mu}|, \end{aligned}$$

by (4.2), and thus

$$(4.6) \quad \sum_{\mu} \sum_{W \in \mathfrak{W}^{\mu}} |W| \gamma_{W,\mu}^p = \|U\|_1 \lesssim \|\mathfrak{S}f\|_p^p.$$

For $\alpha > 0$ let

$$(4.7) \quad \mathcal{O}_{\alpha} = \{x : M_{HL}U > \alpha^p\}$$

and

$$(4.8) \quad \tilde{\mathcal{O}}_{\alpha} = \{x : M_{HL}\mathbb{1}_{\mathcal{O}_{\alpha}}(x) > (10d)^{-d}\}$$

so that $\mathcal{O}_{\alpha} \subset \tilde{\mathcal{O}}_{\alpha}$ and

$$(4.9) \quad \text{meas}(\tilde{\mathcal{O}}_{\alpha}) \lesssim \text{meas}(\mathcal{O}_{\alpha}) \lesssim \alpha^{-p} \|\mathfrak{S}f\|_p^p.$$

Let $\Omega_{\alpha} = \{Q\}$ be the collection of Whitney cubes for the set $\tilde{\mathcal{O}}_{\alpha}$ (cf. [36, ch. VI]) so that

$$\text{diam}(Q) \leq \text{dist}(Q, \tilde{\mathcal{O}}_{\alpha}^c) \leq 4 \text{diam}(Q).$$

In analogy to the usual terminology of “good” and “bad” functions in Calderón-Zygmund theory we split, for fixed α , the collection \mathfrak{W}^{μ} into two subcollections $\mathfrak{W}_{\text{good}}^{\mu} \equiv \mathfrak{W}_{\text{good}}^{\mu}(\alpha)$ and $\mathfrak{W}_{\text{bad}}^{\mu} \equiv \mathfrak{W}_{\text{bad}}^{\mu}(\alpha)$ by setting

$$(4.10) \quad \begin{aligned} \mathfrak{W}_{\text{bad}}^{\mu} &= \{W \in \mathfrak{W}^{\mu} : \gamma_{W,\mu} > \alpha\}, \\ \mathfrak{W}_{\text{good}}^{\mu} &= \{W \in \mathfrak{W}^{\mu} : \gamma_{W,\mu} \leq \alpha\}. \end{aligned}$$

We relate the collection $\mathfrak{W}_{\text{bad}}^{\mu}$ with the collection of Whitney cubes Ω_{α} for the set $\tilde{\mathcal{O}}_{\alpha}$.

Lemma 4.1. *Let $W \in \mathfrak{W}_{\text{bad}}^{\mu}$. Then $W \subset \mathcal{O}_{\alpha}$. Moreover, there is a unique cube $Q = Q(W) \in \Omega_{\alpha}$ containing W .*

Proof. For the first statement, assume otherwise that there is $x \in W \cap \mathcal{O}_{\alpha}^c$ for some $W \in \mathfrak{W}_{\text{bad}}^{\mu}$. Then $U(x) \leq \alpha^p$ and therefore $\gamma_{W,\mu}^p \leq \alpha^p$, which is a contradiction.

For the second statement, we first claim that $W^* \subset \tilde{\mathcal{O}}_{\alpha}$, where W^* is the $10d^{1/2}$ -dilate of W (with same center). The claim follows because for all $y \in W^*$ we have $M_{HL}\mathbb{1}_{\mathcal{O}_{\alpha}}(y) \geq |W|/|W^*| = (10\sqrt{d})^{-d}$ by the first statement. Let x_W be the center of W . Then by the claim

$$\text{dist}(x_W, (\tilde{\mathcal{O}}_{\alpha})^c) \geq \text{dist}(x_W, (W^*)^c) = \frac{\text{diam}(W^*)}{2\sqrt{d}} = 5 \text{diam}(W).$$

Let $Q \in \mathfrak{Q}_\alpha$ such that $x_W \in Q$. Then the last displayed inequality implies

$$5 \operatorname{diam}(W) \leq \operatorname{dist}(x_W, (\tilde{\mathcal{O}}_\alpha)^\complement) \leq \operatorname{diam}(Q) + \operatorname{dist}(Q, (\tilde{\mathcal{O}}_\alpha)^\complement) \leq 5 \operatorname{diam}(Q)$$

and hence $\operatorname{diam}(Q) \geq \operatorname{diam}(W)$. Since both W, Q are dyadic cubes containing x_W this implies $W \subset Q$. Uniqueness of Q follows since the cubes in \mathfrak{Q}_α have disjoint interior. \square

In light of Lemma 4.1, we also set, for a dyadic cube $Q \in \mathfrak{Q}_\alpha$,

$$(4.11) \quad \mathfrak{W}^\mu(Q) = \{W \in \mathfrak{W}_{\text{bad}}^\mu : W \subset Q\}.$$

Lemma 4.2. *Let $Q \in \mathfrak{Q}_\alpha$. Then*

$$\sum_{\mu} \sum_{W \in \mathfrak{W}^\mu(Q)} |W| \gamma_{W, \mu}^p \leq 10^d \alpha^p |Q|.$$

Proof. There is $\tilde{x} \in \tilde{\mathcal{O}}_\alpha^\complement \subset \mathcal{O}_\alpha^\complement$ such that $\operatorname{dist}(\tilde{x}, \tilde{\mathcal{O}}_\alpha^\complement) \leq 4 \operatorname{diam}(Q)$. If Q_* denotes the cube centered at \tilde{x} with diameter equal to $10 \operatorname{diam}(Q)$ then $Q \subset Q_*$. Since $\tilde{x} \in \mathcal{O}_\alpha^\complement$ we have $M_{HL}U(\tilde{x}) \leq \alpha^p$. Hence $\int_Q U \leq \int_{Q_*} U \leq \alpha^p |Q_*| = 10^d \alpha^p |Q|$ and the assertion follows. \square

4.2. *Outline of the proof of the weak type inequalities.* For $R \in \mathcal{R}_k$ let

$$(4.12) \quad e_R(x) = \mathbb{1}_R(x) \mathcal{L}_k f(x)$$

and as in (4.10) split $\mathcal{L}_k f = g^k + b^k$ where

$$(4.13) \quad g^k = \sum_{\mu} \sum_{W \in \mathfrak{W}_{\text{good}}^\mu} \sum_{R \in \mathcal{R}_k^\mu(W)} e_R,$$

$$(4.14) \quad b^k = \sum_{\mu} \sum_{W \in \mathfrak{W}_{\text{bad}}^\mu} \sum_{R \in \mathcal{R}_k^\mu(W)} e_R.$$

In view of (4.9) it suffices to show, for $2 \leq q < \infty$,

$$(4.15) \quad \operatorname{meas}\left\{x : \left(\sum_k \left(\int_I \left|\sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_j^k g^k(x, t)\right|^q dt\right)^{2/q}\right)^{1/2} > \alpha/2\right\} \lesssim \alpha^{-p} \|\mathfrak{S}f\|_p^p$$

and

$$(4.16) \quad \operatorname{meas}\left\{x \in \tilde{\mathcal{O}}_\alpha^\complement : \left(\sum_k \left(\int_I \left|\sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_j^k b^k(x, t)\right|^q dt\right)^{2/q}\right)^{1/2} > \alpha/2\right\} \lesssim \alpha^{-p} \|\mathfrak{S}f\|_p^p.$$

Since $\lambda(p) \geq 1/p - 1 > -1/q$ we can use Lemma 3.1 to bound

$$\begin{aligned} & \left\| \left(\sum_k \left(\int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_j^k g^k(x) \right|^q dt \right)^{2/q} \right)^{1/2} \right\|_2 \\ & \lesssim \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \left\| \left(\sum_k \left(\int_I |T_j^k g^k(x)|^q dt \right)^{2/q} \right)^{1/2} \right\|_2 \\ & \lesssim \sum_{j=1}^{\infty} 2^{-j(\lambda(p) + \frac{1}{q})} \left(\sum_k \|g^k\|_2^2 \right)^{1/2} \lesssim \left(\sum_k \|g^k\|_2^2 \right)^{1/2}. \end{aligned}$$

Hence, by Tshebyshev's inequality, the left hand side of (4.15) is bounded by

$$4\alpha^{-2} \left\| \left(\sum_k \left(\int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_j^k g^k(x) \right|^q dt \right)^{2/q} \right)^{1/2} \right\|_2^2 \lesssim \alpha^{-2} \sum_k \|g^k\|_2^2.$$

Now

$$\begin{aligned} \sum_k \|g^k\|_2^2 &= \sum_k \left\| \sum_{\mu} \sum_{W \in \mathfrak{W}_{\text{good}}^{\mu}} \sum_{R \in \mathcal{R}_k^{\mu}(W)} e_R \right\|_2^2 \leq \sum_k \sum_{\mu} \sum_{W \in \mathfrak{W}_{\text{good}}^{\mu}} \sum_{R \in \mathcal{R}_k^{\mu}(W)} \|e_R\|_2^2 \\ &= \sum_{\mu} \sum_{W \in \mathfrak{W}_{\text{good}}^{\mu}} |W| \gamma_{W,\mu}^2 \lesssim \alpha^{2-p} \sum_{\mu} \sum_{W \in \mathfrak{W}^{\mu}} |W| \gamma_{W,\mu}^p \lesssim \alpha^{2-p} \|\mathfrak{S}f\|_p^p \end{aligned}$$

where we have used $\gamma_{W,\mu} \leq \alpha$ for $W \in \mathfrak{W}_{\text{good}}^{\mu}$. (4.15) follows.

We turn to (4.16). We write $L(Q) = m$ if the side length of Q is 2^m . Define, for $m \geq -k$,

$$(4.17) \quad B_m^k = \sum_{\substack{Q \in \Omega_{\alpha} \\ L(Q)=m}} \sum_{\mu} \sum_{W \in \mathfrak{W}^{\mu}(Q)} \sum_{R \in \mathcal{R}_k^{\mu}(W)} e_R$$

so that $b^k = \sum_{m \geq -k} B_m^k$.

Note that for $R \in \mathcal{R}_k^{\mu}(W)$, we have $L(W) \geq -k$. Then

$$b^k = \sum_{m \geq -k} B_m^k = \sum_{m \geq -k} \sum_{\sigma \geq 0} B_{m,\sigma}^k$$

where

$$(4.18) \quad B_{m,\sigma}^k = \sum_{\substack{Q \in \Omega_{\alpha} \\ L(Q)=m}} \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^{\mu}(Q) \\ L(W)=-k+\sigma}} \sum_{R \in \mathcal{R}_k^{\mu}(W)} e_R.$$

We handle the case of the contributions $T_j^k B_{m,\sigma}^k$ with $m \leq j - k$ differently from those with $m > j - k$. Moreover we distinguish the cases where $|j - k - m| \geq \sigma$ and $|j - k - m| < \sigma$. If we use Tshebyshev's inequality and take into account (4.9) we see that in order to establish (4.16) it suffices to show the following three inequalities, assuming $2 \leq q < p'$ (and hence $p < q' \leq 2$):

$$(4.19) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \sum_{\substack{(m,\sigma): m \leq j-k, \\ 0 \leq \sigma \leq j-m-k}} T_j^k B_{m,\sigma}^k(\cdot, t) \right|^q dt \right]^{q'/q} \right)^{1/q'} \right\|_{L^{q'}(\mathbb{R}^d)}^{q'} \lesssim \alpha^{q'-p} \|\mathfrak{S}f\|_p^p,$$

$$(4.20) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \sum_{\substack{(m,\sigma): m > j-k, \\ 0 \leq \sigma \leq m+k-j}} T_j^k B_{m,\sigma}^k(\cdot, t) \right|^{p/q} dt \right]^{p/q} \right)^{1/p} \right\|_{L^p(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_{\alpha})}^p \lesssim \|\mathfrak{S}f\|_p^p,$$

and

$$(4.21) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j|}} T_j^k B_{m,\sigma}^k(\cdot, t) \right|^q dt \right]^{2/q} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_{\alpha})}^p \lesssim \|\mathfrak{S}f\|_p^p.$$

The proofs will be given in Sections 4.4, 4.5, and 4.6. We shall handle the cases $p = 1$, $2 \leq q < \infty$, and $1 < p < 2$, $2 \leq q < p'$, in a unified way but will need an additional analytic families interpolation argument for $1 < p < 2$.

4.3. *Analytic families.* Fix p, α and consider for $0 \leq \text{Re}(z) \leq 1$ the family of functions

$$(4.22) \quad b_{Q,\sigma}^{k,z} = \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^{\mu}(Q) \\ L(W)=-k+\sigma}} \gamma_{W,\mu,z} \sum_{R \in \mathcal{R}_k^{\mu}(W)} e_R,$$

where for $W \in \mathfrak{W}_{\text{bad}}^{\mu}$

$$\gamma_{W,\mu,z} = \gamma_{W,\mu}^{p(1-z/2)-1}$$

and Q belongs to \mathfrak{Q}_α . Observe that $b_{Q,\sigma}^{k,z}$ is supported in Q . Notice that $z \mapsto \gamma_{W,\mu,z}$ is an entire function for $W \in \mathfrak{W}_{\text{bad}}^\mu$. We also set

$$(4.23) \quad B_{m,\sigma}^{k,z} = \sum_{\substack{Q \in \mathfrak{Q}_\alpha \\ L(Q)=m}} b_{Q,\sigma}^{k,z}$$

and, for $0 \leq \text{Re}(z) \leq 1$, define p_z and $\lambda(p_z)$ by

$$(4.24) \quad \frac{1}{p_z} = 1 - z + \frac{z}{2}, \quad \lambda(p_z) = \frac{d(1-z) - 1}{2}.$$

If $1 < p < 2$ then we set $\vartheta = 2 - 2/p$ so that

$$p_\vartheta = p, \quad \lambda(p_\vartheta) = d\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}, \quad B_{m,\sigma}^{k,\vartheta} = B_{m,\sigma}^k.$$

For $\text{Re}(z) = 1$ we have

Lemma 4.3. *For fixed $k, m \geq -k$ let $\mathcal{N}_{k,m} \subset \mathbb{Z}$. Then*

$$\sum_{k \in \mathbb{Z}} \sum_{m \geq -k} \left\| \sum_{\sigma \in \mathcal{N}_{k,m}} B_{m,\sigma}^{k,z} \right\|_2^2 \lesssim \|\mathfrak{S}f\|_p^p, \quad \text{Re}(z) = 1.$$

Proof. The left hand side is equal to

$$\sum_k \sum_{m \geq -k} \left\| \sum_{\substack{Q \in \mathfrak{Q}_\alpha \\ L(Q)=m}} \sum_{\mu} \sum_{\sigma \in \mathcal{N}_{k,m}} \sum_{\substack{W \in \mathfrak{W}^\mu(Q) \\ L(W)=-k+\sigma}} \gamma_{W,\mu}^{p(1-z/2)-1} \sum_{R \in \mathcal{R}_k^\mu(W)} e_R \right\|_2^2.$$

Let for each W , $Q(W)$ be the unique cube in \mathfrak{Q}_α such that $W \subset Q$. We use that for fixed k the supports of the functions e_R , $R \in \mathcal{R}_k$ have disjoint interior and dominate for $\text{Re}(z) = 1$ the last display by

$$\begin{aligned} & \sum_k \sum_{m \geq -k} \sum_{\substack{Q \in \mathfrak{Q}_\alpha \\ L(Q)=m}} \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^\mu(Q): \\ L(W)+k \in \mathcal{N}_{k,m}}} \gamma_{W,\mu}^{(\frac{p}{2}-1)^2} \sum_{R \in \mathcal{R}_k^\mu(W)} \|e_R\|_2^2 \\ & \lesssim \sum_{\mu} \sum_{W \in \mathfrak{W}_{\text{bad}}^\mu} \gamma_{W,\mu}^{p-2} \sum_{\substack{k:L(W)+k \in \mathcal{N}_{k,L(Q(W))} \\ \mathcal{N}_{k,L(Q(W))}}} \sum_{R \in \mathcal{R}_k^\mu(W)} \|e_R\|_2^2 \leq \sum_{\mu} \sum_{W \in \mathfrak{W}^\mu} \gamma_{W,\mu}^p |W| \lesssim \|\mathfrak{S}f\|_p^p. \quad \square \end{aligned}$$

For $\text{Re}(z) = 0$ we have

Lemma 4.4. *There exists a universal constant C dependent only on the dimension such that for every $Q \in \mathfrak{Q}_\alpha$ and every $\mathcal{N} \subset \mathbb{N} \cup \{0\}$*

$$\int \left(\sum_k \left| \sum_{\sigma \in \mathcal{N}} b_{Q,\sigma}^{k,z}(x) \right|^2 \right)^{1/2} dx \leq C \alpha^p |Q|, \quad \text{if } \text{Re}(z) = 0.$$

Proof. For each $W \in \mathfrak{W}^\mu$ let W_* its double. By Minkowski's inequality the left hand side is dominated by

$$\begin{aligned} & \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^\mu \\ W \subset Q}} \gamma_{W,\mu}^{p-1} \int \left(\sum_{\substack{k: \\ k+L(W) \in \mathcal{N}}} \left| \sum_{R \in \mathcal{R}_k^\mu(W)} e_R(x) \right|^2 \right)^{1/2} dx \\ & \lesssim \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^\mu \\ W \subset Q}} \gamma_{W,\mu}^{p-1} \int_{W_*} \left(\sum_k \sum_{R \in \mathcal{R}_k^\mu(W)} |e_R(x)|^2 \right)^{1/2} dx \end{aligned}$$

which by the Cauchy-Schwarz inequality can be estimated by

$$\sum_{\mu} \sum_{\substack{W \in \mathfrak{M}^{\mu} \\ W \subset Q}} \gamma_{W,\mu}^{p-1} \left(\sum_k \sum_{R \in \mathcal{R}_k^{\mu}(W)} \|e_R\|^2 \right)^{1/2} |W_*|^{1/2} \lesssim \sum_{\mu} \sum_{\substack{W \in \mathfrak{M}^{\mu} \\ W \subset Q}} |W_*| \gamma_{W,\mu}^p \lesssim \alpha^p |Q|.$$

Here we have used Lemma 4.2. \square

4.4. *Proof of (4.19).* Let $1 \leq p < 2$ and $2 \leq q < p'$. The asserted inequality follows from

$$(4.25a) \quad \left\| \left(\sum_k \left(\int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p)} T_j^k B_{j-k-s,\sigma}^k(\cdot, t) \right|^q dt \right)^{q'/q} \right)^{1/q'} \right\|_{q'} \\ \lesssim (1+s)^{1-\frac{2}{q}} 2^{-s(d-1)(\frac{1}{p}-\frac{1}{q'})} \alpha^{p(\frac{1}{p}-\frac{1}{q'})} \|\mathfrak{S}f\|_p^{p/q'}$$

and,

$$(4.25b) \quad \left\| \left(\sum_k \left(\int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p)} T_j^k B_{j-k-s,\sigma}^k(\cdot, t) \right|^q dt \right)^{q'/q} \right)^{1/q'} \right\|_{q'} \\ \lesssim (1+j)^{1-\frac{2}{q}} 2^{-j\frac{d-1}{2}(\frac{1}{p}-\frac{1}{q'})} \alpha^{p(\frac{1}{p}-\frac{1}{q'})} \|\mathfrak{S}f\|_p^{p/q'}, \quad \frac{j}{2} \leq s \leq j.$$

If in addition $p > 1$ we use a complex interpolation argument, embedding $B_{m,\sigma}^k$ in an analytic family of functions, see (4.23).

Define r by

$$(4.26) \quad \frac{1}{r} = \left(\frac{1}{p} - \frac{1}{q} \right) / \left(\frac{2}{p} - 1 \right),$$

so that $1 < r \leq 2$ and for $\vartheta = 2 - 2/p$ we have $(1-\vartheta)(1, \frac{1}{r}) + \vartheta(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{p}, \frac{1}{q'})$. Then by complex interpolation (i.e. the three lines lemma and duality) we deduce (4.25a), (4.25b) from

$$(4.27a) \quad \left\| \left(\sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(pz)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2 \lesssim \|\mathfrak{S}f\|_p^{p/2}, \quad \operatorname{Re}(z) = 1,$$

$$(4.27b) \quad \left\| \left(\sum_k \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(pz)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2 \lesssim \|\mathfrak{S}f\|_p^{p/2}, \quad \frac{j}{2} \leq s \leq j, \quad \operatorname{Re}(z) = 1.$$

and

$$(4.28a) \quad \left\| \left(\sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(pz)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^{r'} dt \right)^{1/r'} \right\|_r \\ \lesssim (1+s)^{\frac{2}{r}-1} 2^{-s(d-1)(1-\frac{1}{r})} \alpha^{p(1-\frac{1}{r})} \|\mathfrak{S}f\|_p^{p/r}, \quad \operatorname{Re}(z) = 0,$$

$$(4.28b) \quad \left\| \left(\sum_k \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(pz)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^{r'} dt \right)^{1/r'} \right\|_r \\ \lesssim (1+j)^{\frac{2}{r}-1} 2^{-j\frac{d-1}{2}(1-\frac{1}{r})} \alpha^{p(1-\frac{1}{r})} \|\mathfrak{S}f\|_p^{p/r}, \quad \frac{j}{2} \leq s \leq j, \quad \operatorname{Re}(z) = 0.$$

We note that for the special case $p = 1$ inequalities (4.28a), (4.28b) with $r = q'$ and $z = 0$ imply inequalities (4.25a), (4.25b) with $p = 1$.

The proof of (4.27a), (4.27b) is straightforward, using orthogonality, i.e. the fact that for each k, t, ξ there are at most five j for which $\varphi_j(\rho(2^{-k}t^{-1}\xi)) \neq 0$. Therefore we get for $\operatorname{Re}(z) = 1$ (and $\operatorname{Re}(\lambda(p_z)) = -1/2$)

$$\begin{aligned} & \left\| \left(\sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_z)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2^2 \\ & \lesssim \sum_k \int_I \sum_{j \geq 2s} 2^j \int |\phi_j(\rho(2^{-k}t^{-1}\xi))|^2 \left| \sum_{0 \leq \sigma \leq s} \widehat{B_{j-k-s,\sigma}^{k,z}}(\xi) \right|^2 d\xi dt \\ & \lesssim \sum_{k \in \mathbb{Z}} \sum_{j \geq 2s} \left\| \sum_{0 \leq \sigma \leq s} B_{j-k-s,\sigma}^{k,z} \right\|_2^2 = \sum_k \sum_{m \geq -k+s} \left\| \sum_{0 \leq \sigma \leq s} B_{m,\sigma}^{k,z} \right\|_2^2 \lesssim \|\mathfrak{G}f\|_p^p, \end{aligned}$$

by Lemma 4.3. Similarly, for fixed j

$$\left\| \left(\sum_k \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_z)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2^2 \lesssim \sum_k \left\| \sum_{0 \leq \sigma \leq s} B_{j-k-s,\sigma}^{k,z} \right\|_2^2 \lesssim \|\mathfrak{G}f\|_p^p.$$

This concludes the proof of (4.27a) and (4.27b).

We now come to the main part of the proof, namely the inequalities (4.28a), (4.28b) when $1 \leq p < 2$ and $\operatorname{Re}(z) = 0$. We fix z with $\operatorname{Re}(z) = 0$ and then use another interpolation inequality based on

$$[L^1(\ell^1(L^\infty)), L^2(\ell^2(L^2))]_\theta = L^r(\ell^r(L^{r'})) \text{ for } \theta = 2 - \frac{2}{r}$$

where Calderón's complex interpolation method is applied to vector-valued L^p spaces (see [2, Theorem 5.1.2]). As a consequence we have

$$\|\cdot\|_{L^r(\ell^r(L^{r'}))} \lesssim \|\cdot\|_{L^1(\ell^1(L^\infty))}^{\frac{2}{r}-1} \|\cdot\|_{L^2(\ell^2(L^2))}^{2-\frac{2}{r}}.$$

Assuming $1 \leq p < 2$, (4.28a), (4.28b) follow from

$$(4.29a) \quad \left\| \left(\sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_z)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2 \lesssim 2^{-s\frac{d-1}{2}} \alpha^{p/2} \|\mathfrak{G}f\|_p^{p/2}, \quad \operatorname{Re}(z) = 0,$$

$$(4.29b) \quad \left\| \left(\sum_k \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_z)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2 \lesssim 2^{-j\frac{d-1}{4}} \alpha^{p/2} \|\mathfrak{G}f\|_p^{p/2}, \quad \frac{j}{2} \leq s \leq j, \quad \operatorname{Re}(z) = 0,$$

and

$$(4.30a) \quad \left\| \sum_k \sup_{t \in I} \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_z)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right| \right\|_1 \lesssim (1+s) \|\mathfrak{G}f\|_p^p, \quad \operatorname{Re}(z) = 0,$$

$$(4.30b) \quad \left\| \sum_k \sup_{t \in I} \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_z)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right| \right\|_1 \lesssim (1+j) \|\mathfrak{G}f\|_p^p, \quad \frac{j}{2} \leq s \leq j, \quad \operatorname{Re}(z) = 0.$$

This proof of (4.29a), (4.29b) is inspired by the work of Christ and Sogge [10], [11]. We use the decomposition (3.3) and orthogonality, first in the j -sum and then, for each j , also in the ν

sums, where $\nu \in \mathcal{Z}_j$. We then see that

$$\begin{aligned}
& \left\| \left(\sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_z)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2^2 \\
& \lesssim \sum_k \sum_{j \geq 2s} \sum_{\nu \in \mathcal{Z}_j} 2^{-j(d-1)} \int_I \left\| \sum_{0 \leq \sigma \leq s} T_{j,\nu}^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right\|_2^2 dt \\
(4.31) \quad & = \sum_m \sum_{k \geq 2s-m} \sum_{\nu \in \mathcal{Z}_{k+m}} 2^{-(k+m)(d-1)} \int_I \left\| \sum_{0 \leq \sigma \leq s} T_{k+m,\nu}^k B_{m-s,\sigma}^{k,z}(\cdot, t) \right\|_2^2 dt.
\end{aligned}$$

We use

$$\int_I \|T_{j,\nu}^k g\|_2^2 dt = \int_I \int \int (2^k t)^d h_{j,\nu}(2^k t(x-y)) g(y) \overline{g(x)} dy dx dt$$

where $h_{j,\nu}(x) = \mathcal{F}^{-1}[|\chi_{j,\nu} \varphi_j(\rho(\cdot))|^2](x)$. The kernel $h_{j,\nu}$ satisfies kernel estimates which are analogous to the right hand side of (3.6), i.e.

$$\sup_{t \in I} |t^d h_{j,\nu}(tx)| \lesssim_N \frac{2^{-j \frac{d+1}{2}}}{(1+2^{-j}|x|)^N (1+2^{-j/2}|P_{j,\nu}(x)|)^N}.$$

Using $j = k + m$ we can then estimate, for $t \in I$

$$\begin{aligned}
& 2^{-(k+m)(d-1)} \left\| \sum_{0 \leq \sigma \leq s} T_{k+m,\nu}^k B_{m-s,\sigma}^{k,z}(\cdot, t) \right\|_2^2 \leq C_N \times \\
& \iint 2^{-(k+m) \frac{d-1}{2}} \frac{2^{-md}}{(1+2^{-m}|x-y|)^N} \frac{1}{(1+2^{-m+\frac{k+m}{2}}|P_{k+m,\nu}(x-y)|)^N} |\beta_{m,s}^{k,z}(y)| dy |\beta_{m,s}^{k,z}(x)| dx,
\end{aligned}$$

with

$$(4.32) \quad \beta_{m,s}^{k,z} := \sum_{0 \leq \sigma \leq s} B_{m-s,\sigma}^{k,z}.$$

Consider a maximal set Z^s of $c2^{-s}$ separated unit vectors η_ς , and let P_ς^s be the orthogonal projection to the orthogonal complement of $\nabla \rho(\eta_\varsigma)$. Notice that for each ς there are $\approx 2^{(d-1)(\frac{1}{2}-s)}$ of the vectors ξ_ν with $\nu \in \mathcal{Z}_j$ which are of distance $\leq C2^{-s}$ to η_ς . For those ν we then have $|\frac{\nabla \rho(\xi_\nu)}{|\nabla \rho(\xi_\nu)|} - \frac{\nabla \rho(\eta_\varsigma)}{|\nabla \rho(\eta_\varsigma)|}| = O(2^{-s})$. Consequently, for those ν , and $j = k + m \geq 2s$

$$\begin{aligned}
& \frac{2^{-md}}{(1+2^{-m}|x-y|)^N} \frac{1}{(1+2^{-m+\frac{k+m}{2}}|P_{k+m,\nu}(x-y)|)^N} \\
& \lesssim_N \frac{2^{-md}}{(1+2^{-m}|x-y|)^N} \frac{1}{(1+2^{-m+s}|P_\varsigma^s(x-y)|)^N}
\end{aligned}$$

and there are $O(2^{(d-1)(\frac{k+m}{2}-s)})$ indices $\nu \in \mathcal{Z}_{k+m}$ for which we may use this inequality. Then, setting

$$(4.33) \quad A_{k,m,\varsigma}(x) = \int 2^{-s(d-1)} \frac{2^{-md}}{(1+2^{-m}|x-y|)^N} \frac{1}{(1+2^{-m+s}|P_\varsigma^s(x-y)|)^N} |\beta_{m,s}^{k,z}(y)| dy,$$

we get by the above considerations

$$(4.31) \lesssim \sum_{\varsigma \in Z^s} \sum_m \sum_{k \geq 2s-m} \int A_{k,m,\varsigma}(x) |\beta_{m,s}^{k,z}(x)| dx$$

$$(4.34) \lesssim \sum_{\varsigma \in Z^s} \sum_m \int \left(\sum_{k \geq 2s-m} [A_{k,m,\varsigma}(x)]^2 \right)^{1/2} \left(\sum_k |\beta_{m,s}^{k,z}(x)|^2 \right)^{1/2} dx.$$

We first establish that

$$(4.35) \quad \sup_m \sum_{\varsigma \in Z^s} \left\| \left(\sum_{k \geq 2s-m} |A_{k,m,\varsigma}|^2 \right)^{1/2} \right\|_{\infty} \lesssim \alpha^p 2^{-s(d-1)}.$$

For each dyadic cube Q let y_Q be the center of Q . Using (4.33) we estimate for fixed $x \in \mathbb{R}^d$

$$\left(\sum_{k \geq 2s-m} |A_{k,m,\varsigma}(x)|^2 \right)^{1/2} \lesssim 2^{-s(d-1)} \times$$

$$\sum_{\substack{Q: \\ L(Q)=m-s}} \frac{2^{-md}}{(1+2^{-m}|x-y_Q|)^N (1+2^{-m+s}|P_{\varsigma}^s(x-y_Q)|)^N} \int \left(\sum_k \left| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{k,z}(y) \right|^2 \right)^{1/2} dy$$

and using Lemma 4.4 we bound this expression by

$$2^{-s(d-1)} \sum_{\substack{Q: \\ L(Q)=m-s}} \frac{2^{-md}}{(1+2^{-m}|x-y_Q|)^N (1+2^{-m+s}|P_{\varsigma}^s(x-y_Q)|)^N} \alpha^p |Q|$$

$$\lesssim \alpha^p 2^{-s(d-1)} \int \frac{2^{-md}}{(1+2^{-m}|x-w|)^N (1+2^{-m+s}|P_{\varsigma}^s(x-w)|)^N} dw \lesssim \alpha^p 2^{-2s(d-1)}.$$

We sum over $\varsigma \in Z^s$ and use that $\#Z^s = O(2^{s(d-1)})$ to obtain (4.35).

Combining (4.35) and (4.34) we obtain

$$(4.31) \lesssim 2^{-s(d-1)} \alpha^p \sum_m \sum_{\substack{Q \in \Omega_{\alpha}: \\ L(Q)=m-s}} \left\| \left(\sum_k \left| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{k,z} \right|^2 \right)^{1/2} \right\|_1.$$

Finally, by Lemma 4.4 again

$$\sum_m \sum_{\substack{Q \in \Omega_{\alpha}: \\ L(Q)=m-s}} \left\| \left(\sum_k \left| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{k,z} \right|^2 \right)^{1/2} \right\|_1 \lesssim \sum_{Q \in \Omega_{\alpha}} |Q| \alpha^p \lesssim \alpha^p |\tilde{\mathcal{O}}_{\alpha}| \lesssim \|\mathfrak{S}f\|_p^p,$$

by (4.9). This finishes the proof of (4.29a).

The proof of (4.29b) uses the same idea. We estimate for fixed $j \in [s/2, s]$, $\text{Re}(z) = 0$,

$$(4.36) \quad \left\| \left(\sum_k \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_z)} T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2^2$$

$$\lesssim 2^{-j(d-1)} \sum_k \int_I \left\| T_j^k \beta_{j-k,s}^{k,z}(\cdot, t) \right\|_2^2 \lesssim 2^{-j(d-1)} \sum_{\nu \in \mathcal{Z}_j} \sum_k \int_I \left\| T_{j,\nu}^k \beta_{j-k,s}^{k,z}(\cdot, t) \right\|_2^2 dt$$

$$\lesssim 2^{-j(d-1)} \sum_{\nu \in \mathcal{Z}_j} \sum_k \int \mathcal{A}_{k,j,\nu}(x) |\beta_{j-k,s}^{k,z}(x)| dx$$

where again $\beta_{m,s}^{k,z}$ is as in (4.32) and

$$\mathcal{A}_{k,j,\nu}(x) := \int \frac{2^{kd}2^{-j\frac{d+1}{2}}}{(1+2^{k-j}|x-y|)^N(1+2^{k-\frac{j}{2}}|P_{j,\nu}(x-y)|)^N} |\beta_{j-k,s}^{k,z}(y)| dy.$$

Now $\mathcal{A}_{k,j,\nu}(x) \lesssim$

$$\int \frac{2^{kd}2^{-j\frac{d+1}{2}}}{(1+2^{k-j}|x-w|)^N(1+2^{k-\frac{j}{2}}|P_{j,\nu}(x-w)|)^N} dw \sup_{\substack{Q \in \Omega_\alpha \\ L(Q)=j-k-s}} \frac{1}{|Q|} \int \left| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{k,z}(y) \right| dy$$

which is bounded by $C\alpha^p$. Consequently

$$\begin{aligned} (4.36) &\lesssim 2^{-j(d-1)} \sum_{\nu \in \mathcal{Z}_j} \alpha^p \sum_k \sum_{\substack{Q \in \Omega_\alpha \\ L(Q)=j-k-s}} \left\| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{k,z} \right\|_1 \\ &\lesssim 2^{-j\frac{d-1}{2}} \alpha^p \sum_{Q \in \Omega_\alpha} \left\| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{j-s-L(Q),z} \right\|_1 \\ &\lesssim 2^{-j\frac{d-1}{2}} \alpha^p \sum_{Q \in \Omega_\alpha} \left\| \left(\sum_k \left| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{k,z} \right|^2 \right)^{1/2} \right\|_1 \lesssim 2^{-j\frac{d-1}{2}} \alpha^p \|\mathfrak{S}f\|_p^p, \end{aligned}$$

by Lemma 4.4.

We now turn to the proof of (4.30a), (4.30b), here still $\operatorname{Re}(z) = 0$. We estimate the left hand side of (4.30a) using Lemma 3.4 by

$$\sum_k \sum_{j \geq 2s} 2^{-j\frac{d-1}{2}} \sum_{0 \leq \sigma \leq s} \left\| \sup_{t \in I} |T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t)| \right\|_1 \lesssim \sum_k \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} \|B_{j-k-s,\sigma}^{k,z}\|_1$$

and the right hand side is dominated by

$$\begin{aligned} &\sum_{0 \leq \sigma \leq s} \sum_k \sum_{j \geq 2s} \sum_\mu \sum_{\substack{Q \in \Omega_\alpha: \\ L(Q)=j-k-s}} \sum_{\substack{W \in \mathfrak{W}^\mu(Q) \\ L(W)=-k+\sigma}} \gamma_{W,\mu}^{p-1} \left\| \sum_{R \in \mathcal{R}_k^\mu(W)} e_R \right\|_1 \\ &\lesssim \sum_{0 \leq \sigma \leq s} \sum_k \sum_{j \geq 2s} \sum_\mu \sum_{\substack{Q \in \Omega_\alpha: \\ L(Q)=j-k-s}} \sum_{\substack{W \in \mathfrak{W}^\mu(Q) \\ L(W)=-k+\sigma}} \gamma_{W,\mu}^{p-1} |W|^{1/2} \left(\sum_{R \in \mathcal{R}_k^\mu(W)} \|e_R\|_2^2 \right)^{1/2} \\ &\lesssim \sum_{0 \leq \sigma \leq s} \sum_\mu \sum_{W \in \mathfrak{W}^\mu} \gamma_{W,\mu}^p |W| \lesssim (1+s) \|\mathfrak{S}f\|_p^p \end{aligned}$$

The left hand side of (4.30b) is estimated for fixed $j \in [s, 2s]$ by

$$2^{-j\frac{d-1}{2}} \sum_k \sum_{0 \leq \sigma \leq s} \left\| \sup_{t \in I} |T_j^k B_{j-k-s,\sigma}^{k,z}(\cdot, t)| \right\|_1 \lesssim \sum_{0 \leq \sigma \leq s} \sum_k \|B_{j-k-s,\sigma}^{k,z}\|_1$$

and the subsequent estimation is as for (4.30a). This concludes the proof of (4.19). \square

4.5. *Proof of (4.20).* It suffices to show, assuming $1 \leq p < 2$, $q = p'$ that for some $a(p, q) > 0$ and $s \geq 0$

$$\left\| \left(\sum_k \left[\int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \sum_{0 \leq \sigma \leq s} T_j^k B_{j-k+s,\sigma}^k(\cdot, t) \right|^q dt \right]^{p/q} \right)^{1/p} \right\|_{L^p(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_\alpha)} \lesssim 2^{-a(p,q)s} \|\mathfrak{S}f\|_p.$$

When $p > 1$ we use the analytic family of functions in (4.23). It suffices to prove the inequalities (4.37)

$$\left\| \left(\sum_k \int_I \left| \sum_{j=1}^{\infty} 2^{-j\lambda(pz)} \sum_{0 \leq \sigma \leq s} T_j^k B_{j-k+s,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_{L^2(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_\alpha)} \lesssim \|\mathfrak{G}f\|_p^{p/2}, \quad \operatorname{Re}(z) = 1,$$

and

$$(4.38) \quad \left\| \sum_k \sup_{t \in I} \left| \sum_{j=1}^{\infty} 2^{-j\lambda(pz)} \sum_{0 \leq \sigma \leq s} T_j^k B_{j-k+s,\sigma}^{k,z}(\cdot, t) \right| \right\|_{L^1(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_\alpha)} \lesssim 2^{-\varepsilon s} \|\mathfrak{G}f\|_p^p, \quad \operatorname{Re}(z) = 0,$$

for some $\varepsilon > 0$.

To show (4.37) we replace the $L^2(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_\alpha)$ norm by the $L^2(\mathbb{R}^d)$ norm and argue exactly as in the proof of (4.27a), using Lemma 4.3.

To show (4.38) it suffices to prove, after Minkowski's inequality for the σ -summation (involving $O(1+s)$ terms),

$$(4.39) \quad \left\| \sum_k \sup_{t \in I} \left| \sum_{j=1}^{\infty} 2^{-j\lambda(pz)} T_j^k B_{j-k+s,\sigma}^{k,z}(\cdot, t) \right| \right\|_{L^1(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_\alpha)} \lesssim 2^{-\varepsilon s} \|\mathfrak{G}f\|_p^p, \quad \operatorname{Re}(z) = 0, \quad 0 \leq \sigma \leq s.$$

For the proof observe that, for $t \in I$, $T_j^{n,k} B_{j-k+s,\sigma}^{k,z}(\cdot, t)$ is supported in $\tilde{\mathcal{O}}_\alpha$ when $n \leq s$ and thus does not contribute to the $L^1(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_\alpha)$ norm. We then use the simple L^1 estimate in Lemma 3.4, part (i), for $n > s$ and $\operatorname{Re}(\lambda(pz)) = (d-1)/2$ to estimate the left hand side of (4.39) by a constant times

$$\begin{aligned} & 2^{-sN} \sum_k \sum_j \|B_{j-k+s,\sigma}^{k,z}\|_1 \\ & \lesssim 2^{-sN} \sum_k \sum_j \sum_{\substack{Q \in \mathfrak{Q}_\alpha \\ L(Q)=j-k+s}} \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^\mu(Q) \\ L(W)=-k+\sigma}} \gamma_{W,\mu}^{p-1} \left\| \sum_{R \in \mathcal{R}_k^\mu(W)} e_R \right\|_1. \end{aligned}$$

We interchange the sums and note that each W is contained in a unique cube $Q \in \mathfrak{Q}_\alpha$, and thus because of the disjointness of the cubes in \mathfrak{Q}_α the (j, Q) sums corresponding to a fixed W collap to a single term. Hence we can bound the previous expression by C_N times

$$\begin{aligned} & 2^{-sN} \sum_k \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^\mu \\ L(W)=-k+\sigma}} \gamma_{W,\mu}^{p-1} |W|^{1/2} \left(\sum_{R \in \mathcal{R}_k^\mu(W)} \|e_R\|_2^2 \right)^{1/2} \\ & \lesssim 2^{-sN} \sum_{\mu} \sum_{W \in \mathfrak{W}^\mu} \gamma_{W,\mu}^p |W| \lesssim 2^{-sN} \|\mathfrak{G}f\|_p^p. \end{aligned}$$

This completes the proof of (4.20). \square

4.6. *Proof of (4.21).* The estimate follows from the inequalities

$$(4.40) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{j \geq 1} 2^{-j\lambda(p)} \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j|, \\ \sigma \geq j}} T_j^k B_{m,\sigma}^k(\cdot, t) \right|^q dt \right]^{2/q} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d \setminus \tilde{\mathcal{O}}_\alpha)} \lesssim \|\mathfrak{G}f\|_p$$

and

$$(4.41) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{j \geq 1} 2^{-j\lambda(p)} \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j|, \\ \sigma < j}} T_j^k B_{m,\sigma}^k(\cdot, t) \right|^q dt \right]^{p/q} \right)^{1/p} \right\|_{L^p(\mathbb{R}^d)} \lesssim \|\mathfrak{G}f\|_p.$$

4.6.1. *Proof of (4.40).* We use the decomposition $T_j^k = \sum_{n>0} T_j^{n,k}$ and use Minkowski's inequality for the j and n sums. When $j+n \leq \sigma$ the support of $T_j^{k,n} B_{m,\sigma}^k(\cdot, t)$ is contained in $\tilde{\mathcal{O}}_\alpha$, for all $t \in I$. Thus in (4.40) we only need to consider the terms with $|m+k-j| < \sigma$ and $j \leq \sigma \leq j+n$. Since $\lambda(p) + 1/q > 0$ it suffices to show for fixed $j \geq 1$, that

$$(4.42) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j| \\ j \leq \sigma \leq j+n}} T_j^{n,k} B_{m,\sigma}^k(\cdot, t) \right|^q dt \right]^{2/q} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-n} 2^{-j/q} \|\mathfrak{S}f\|_p.$$

This follows from

$$(4.43) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j| \\ j \leq \sigma \leq j+n}} T_j^{n,k} B_{m,\sigma}^{k,z}(\cdot, t) \right|^q dt \right]^{2/q} \right)^{1/2} \right\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-n} 2^{-j/q} \|\mathfrak{S}f\|_p^{p/2}, \quad \operatorname{Re}(z) = 1,$$

and

$$(4.44) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j| \\ j \leq \sigma \leq j+n}} T_j^{n,k} B_{m,\sigma}^{k,z}(\cdot, t) \right|^q dt \right]^{2/q} \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-n} 2^{-j/q} \|\mathfrak{S}f\|_p^p, \quad \operatorname{Re}(z) = 0.$$

By Lemma 3.4, part (iii), the left hand side of (4.43) is

$$\left(\sum_k \left\| \left(\int_I \left| \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j| \\ j \leq \sigma \leq j+n}} T_j^{n,k} B_{m,\sigma}^{k,z}(\cdot, t) \right|^q dt \right)^{1/q} \right\|_2^2 \right)^{1/2} \lesssim 2^{-n-j/q} \left(\sum_k \left\| \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j| \\ j \leq \sigma \leq j+n}} B_{m,\sigma}^{k,z} \right\|_2^2 \right)^{1/2}.$$

Recall that

$$\operatorname{supp}(B_{m,\sigma}^{k,z}) \subset \bigcup_{\substack{Q \in \Omega_\alpha \\ L(Q)=m}} Q.$$

Therefore, for $\operatorname{Re}(z) = 1$ we have

$$\left(\sum_k \left\| \sum_{\substack{m,\sigma: \\ \sigma > |m+k-j| \\ j \leq \sigma \leq j+n}} B_{m,\sigma}^{k,z} \right\|_2^2 \right)^{1/2} = \left(\sum_k \sum_{m \geq -k} \left\| \sum_{\substack{\sigma: \\ \sigma > |m+k-j| \\ j \leq \sigma \leq j+n}} B_{m,\sigma}^{k,z} \right\|_2^2 \right)^{1/2} \lesssim \|\mathfrak{S}f\|_p^p,$$

by Lemma 4.3. Hence (4.43) follows.

We now turn to the proof of (4.44), where $\operatorname{Re}(z) = 0$. For $W \in \mathfrak{W}_{\text{bad}}^\mu$ let $Q(W)$ be the unique cube in Ω_α containing W . We can split

$$B_{m,\sigma}^{k,z} = \sum_{\mu \in \mathbb{Z}} \sum_{\substack{W \in \mathfrak{W}_{\text{bad}}^\mu \\ L(W) = -k + \sigma}} \tilde{B}_{m,\sigma,\mu,W}^{k,z}$$

where

$$\tilde{B}_{m,\sigma,\mu,W}^{k,z} = \begin{cases} \gamma_{W,\mu,z} \sum_{R \in \mathcal{R}_k^\mu(W)} e_R, & \text{if } L(Q(W)) = m \text{ and } L(W) = -k + \sigma, \\ 0 & \text{if either } L(Q(W)) \neq m \text{ or } L(W) \neq -k + \sigma. \end{cases}$$

Observe that for $j \leq \sigma$, $L(W) = -k + \sigma$, the function $T_j^{n,k} \tilde{B}_{m,\sigma,\mu,W}^{k,z}$ is supported in a 2^{n+3} -dilate of W (with respect to its center). Hence, by the Minkowski and Cauchy-Schwarz inequalities

we estimate for fixed j, n

$$\begin{aligned} & \left\| \left(\sum_k \left[\int_I \left| \sum_{\substack{m, \sigma: \\ \sigma > |m+k-j| \\ \sigma \leq j+n}} T_j^{n,k} B_{m,\sigma}^{k,z}(\cdot, t) \right|^q dt \right]^{2/q} \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \\ & \lesssim \sum_{\mu} \sum_{W \in \mathfrak{W}_{\text{bad}}^{\mu}} 2^{nd/2} |W|^{1/2} \left\| \left(\sum_{\substack{k: L(Q(W))+k-j \\ < L(W)+k \leq j+n}} \left[\int_I \left| T_j^{n,k} \tilde{B}_{L(Q(W)), L(W)+k, \mu, W}^{k,z}(\cdot, t) \right|^q dt \right]^{2/q} \right)^{1/2} \right\|_2 \end{aligned}$$

which by an application of Lemma 3.4 is bounded by

$$\begin{aligned} & C_N 2^{-n(N-d/2)} 2^{-j/q} \sum_{\mu} \sum_{W \in \mathfrak{W}_{\text{bad}}^{\mu}} |W|^{1/2} \left(\sum_{\substack{k: \\ |L(Q(W))+k-j| < L(W)+k \\ L(W)+k \leq j+n}} \left\| \gamma_{W, \mu, z} \sum_{R \in \mathcal{R}_k^{\mu}(W)} e_R \right\|_2^2 \right)^{1/2} \\ & \lesssim 2^{-n-j/q} \sum_{\mu} \sum_{W \in \mathfrak{W}^{\mu}} |W|^{1/2} \gamma_{W, \mu}^{p-1} \left(\sum_k \sum_{R \in \mathcal{R}_k^{\mu}(W)} \|e_R\|_2^2 \right)^{1/2} \\ & \lesssim 2^{-n-j/q} \sum_{\mu} \sum_{W \in \mathfrak{W}^{\mu}} |W| \gamma_{W, \mu}^p \lesssim 2^{-n-j/q} \|\mathfrak{S}f\|_p^p. \end{aligned}$$

4.6.2. *Proof of (4.41).* By Minkowski's inequality (4.41) follows if we can prove for fixed $\sigma > 0$,

$$(4.45) \quad \left\| \left(\sum_k \left[\int_I \left| \sum_{j > \sigma} 2^{-j\lambda(p)} \sum_{\substack{m: \\ \sigma > |m+k-j|}} T_j^k B_{m,\sigma}^k(\cdot, t) \right|^q dt \right]^{p/q} \right)^{1/p} \right\|_{L^p(\mathbb{R}^d)} \lesssim (1 + \sigma)^{1/p} 2^{-\sigma d(\frac{1}{q} - \frac{1}{p'})} \|\mathfrak{S}f\|_p.$$

When $p > 1$ we use complex interpolation to deduce this from

$$(4.46) \quad \left\| \left(\sum_k \int_I \left| \sum_{j > \sigma} 2^{-j\lambda(pz)} \sum_{\substack{m: \\ \sigma > |m+k-j|}} T_j^k B_{m,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_{L^2(\mathbb{R}^d)} \lesssim (1 + \sigma)^{1/2} \|\mathfrak{S}f\|_p^{p/2}, \quad \text{Re}(z) = 1,$$

and, with $\frac{1}{q_0} = (\frac{1}{p} - \frac{1}{q'}) / (\frac{2}{p} - 1)$,

$$(4.47) \quad \left\| \sum_k \left(\int_I \left| \sum_{j > \sigma} 2^{-j\lambda(pz)} \sum_{\substack{m: \\ \sigma > |m+k-j|}} T_j^k B_{m,\sigma}^{k,z}(\cdot, t) \right|^{q_0} dt \right)^{1/q_0} \right\|_{L^1(\mathbb{R}^d)} \lesssim (1 + \sigma) 2^{-\sigma d/q_0} \|\mathfrak{S}f\|_p^p, \quad \text{Re}(z) = 0.$$

Note that $1/q_0 = 1 - 1/r$ where r is as in (4.26), and we have $(1 - \vartheta)(1, \frac{1}{q_0}) + \vartheta(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{p}, \frac{1}{q})$ for $\vartheta = 2/p'$.

We first consider the inequality for $\text{Re}(z) = 1$. We can use the orthogonality of the functions $\varphi_j(\rho(\cdot/t))$ to estimate

$$\begin{aligned} & \left\| \left(\sum_k \int_I \left| \sum_{j > \sigma} 2^{-j\lambda(pz)} \sum_{\substack{m: \\ \sigma > |m+k-j|}} T_j^k B_{m,\sigma}^{k,z}(\cdot, t) \right|^2 dt \right)^{1/2} \right\|_2 \\ & \lesssim \left(\sum_k \int_I \sum_{j > \sigma} 2^j \left\| \sum_{\substack{m: \\ \sigma > |m+k-j|}} T_j^k B_{m,\sigma}^{k,z}(\cdot, t) \right\|_2^2 dt \right)^{1/2} \lesssim \left(\sum_k \sum_{j > \sigma} \left\| \sum_{\substack{m: \\ \sigma > |m+k-j|}} B_{m,\sigma}^{k,z} \right\|_2^2 \right)^{1/2}. \end{aligned}$$

We use the disjointness of the cubes in Ω_α and then interchange the m, j summations. Using that for fixed m, k there are $O(1 + \sigma)$ terms in the j summation, we bound the last expression by

$$\begin{aligned} \left(\sum_k \sum_{j>\sigma} \sum_{\substack{m \geq -k: \\ \sigma > |m+k-j|}} \|B_{m,\sigma}^{k,z}\|_2^2 \right)^{1/2} &\lesssim (1 + \sigma)^{1/2} \left(\sum_k \sum_{m \geq -k} \|B_{m,\sigma}^{k,z}\|_2^2 \right)^{1/2} \\ &\lesssim (1 + \sigma)^{1/2} \|\mathfrak{S}f\|_p^{p/2} \end{aligned}$$

where in the last line we have applied Lemma 4.3 to conclude (4.46).

We now turn to (4.47). We split $T_j^k = \sum_{n=0}^\infty T_j^{n,k}$, set

$$b_{W,\mu,z}^k = \gamma_{W,\mu,z} \sum_{R \in \mathcal{R}_k^\mu(W)} e_R$$

and estimate the left hand side of (4.47) by

$$(4.48) \quad \sum_k \sum_{n \geq 0} \sum_{j > \sigma} 2^{-j \frac{d-1}{2}} \sum_{\substack{m: \\ \sigma > |m+k-j|}} \sum_{\substack{Q \in \Omega_\alpha \\ L(Q)=m}} \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^\mu(Q) \\ L(W)=-k+\sigma}} \left\| \left(\int_I |T_j^{n,k} b_{W,\mu,z}^k(\cdot, t)|^{q_0} dt \right)^{1/q_0} \right\|_1.$$

We bound for fixed W , with $L(W) = -k + \sigma$,

$$\begin{aligned} &\left\| \left(\int_I |T_j^{n,k} b_{W,\mu,z}^k(\cdot, t)|^{q_0} dt \right)^{1/q_0} \right\|_1 \\ &\lesssim 2^{(-k+j+n)d/q_0} \left\| \left(\int_I |T_j^{n,k} b_{W,\mu,z}^k(\cdot, t)|^{q_0} dt \right)^{1/q_0} \right\|_{q'_0} \\ &\lesssim 2^{(-k+j+n)d/q_0} 2^{j(d(\frac{1}{q'_0} - \frac{1}{2}) - \frac{1}{2})} 2^{-nN} \|b_{W,\mu,z}^k\|_{q'_0}, \end{aligned}$$

by Lemma 3.4, part (ii). Hence after summing in n

$$(4.48) \lesssim \sum_k \sum_{j > \sigma} \sum_{\substack{m: \\ \sigma > |m+k-j|}} \sum_{\substack{Q \in \Omega_\alpha \\ L(Q)=m}} \sum_{\mu} \sum_{\substack{W \in \mathfrak{W}^\mu(Q) \\ L(W)=-k+\sigma}} 2^{-kd/q_0} \|b_{W,\mu,z}^k\|_{q'_0}.$$

Observe that for $L(W) + k = \sigma$,

$$\begin{aligned} 2^{-kd/q_0} \|b_{W,\mu,z}^k\|_{q'_0} &\leq 2^{-kd/q_0} |W|^{1/q'_0 - 1/2} \|b_{W,\mu,z}^k\|_2 \\ &\lesssim 2^{-\sigma d/q_0} |W|^{1/2} \gamma_{W,\mu}^{p-1} \left(\sum_{R \in \mathcal{R}_k^\mu(W)} \|e_R\|_2^2 \right)^{1/2} \leq 2^{-\sigma d/q_0} |W| \gamma_{W,\mu}^p. \end{aligned}$$

We interchange summations and use that, for fixed $W \in \mathfrak{W}_{\text{bad}}^\mu$,

$$\#\{j \geq \sigma : |L(Q(W)) + \sigma - L(W) - j| < \sigma\} = O(1 + \sigma).$$

We then obtain

$$(4.48) \lesssim 2^{-\sigma d/q_0} (1 + \sigma) \sum_{\mu} \sum_{W \in \mathfrak{W}^\mu} |W| \gamma_{W,\mu}^p \lesssim 2^{-\sigma d/q_0} (1 + \sigma) \|\mathfrak{S}f\|_p^p.$$

This completes the proof of (4.41), and then (4.21) and finally the proof of Theorem 2.3.

5. PROOFS OF THEOREMS 1.1 AND 1.2

In this section we use the Theorems proved in \mathbb{R}^d and transference argument to establish the corresponding versions for periodic functions. Such transference arguments go back to De Leeuw [25]. See also [20] for transference of maximal operators and [26], [14] inequalities in Hardy spaces on \mathbb{T}^d . In our presentation we rely on the method in [14].

5.1. *The $h^1(\mathbb{T}^d) \rightarrow L^{1,\infty}(\mathbb{T}^d)$ bound.* We identify functions f on \mathbb{T}^d with functions on \mathbb{R}^d satisfying $f(x+n) = f(x)$ for all $n \in \mathbb{Z}^d$. Let $Q^0 = [-\frac{1}{2}, \frac{1}{2}]^d$.

Let

$$h_\lambda(s) = (1 - v_0(s))(1 - s)_+^\lambda$$

and $\mathcal{S}_t^\lambda f = \sum_{\ell \in \mathbb{Z}^d} h(\rho(\ell/t)) \langle f, e_\ell \rangle e_\ell$. Let $\lambda(1) = \frac{d-1}{2}$. After a reduction analogous to the one in §2.1 we need to prove the bound

$$\left\| \left(\sum_{k>0} \int_I |\mathcal{S}_{2^{kt}}^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{T}^d)} \lesssim \|f\|_{h^1(\mathbb{T}^d)}.$$

By normalizing we may assume that $\|f\|_{h^1(\mathbb{T}^d)} = 1$.

By the atomic decomposition for periodic functions ([17], [14]) we may assume that

$$f = f_0 + \sum_{Q \in \mathcal{Q}} c_Q a_Q$$

where $f_0 \in L^2$, $\|f_0\|_2 \lesssim 1$, where \mathcal{Q} is a collection of cubes of sidelength at most $1/4$ which intersect the fundamental cube Q^0 and where a_Q is periodic and supported in $Q + \mathbb{Z}^d$, satisfying $\|a_Q\|_{L^2(Q^0)} \leq |Q|^{-1/2}$ and

$$(5.1) \quad \int_Q a_Q(x) P(x) dx = 0$$

for all polynomials of degree at most $2d$. Moreover

$$(5.2) \quad \|f\|_{h^1} \approx \|f_0\|_2 + \sum_{Q \in \mathcal{Q}} |c_Q| \approx 1.$$

The contribution acting on f_0 is taken care of by standard L^2 estimates.

Now let $\gamma = (\gamma_1, \dots, \gamma_d) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}^d =: \Gamma$ and let $Q^\gamma = \gamma + Q^0$. We can then split the family of cubes \mathcal{Q} into 3^d disjoint families \mathcal{Q}_γ so that each cube $Q \in \mathcal{Q}_\gamma$ has the property that its double is contained in the cube Q^γ . By periodicity, and the monotone convergence theorem, it suffices to prove for each finite subset \mathcal{N} of \mathbb{N} , and for each $\gamma \in \Gamma$,

$$(5.3) \quad \sup_{\alpha>0} \alpha \text{ meas} \left(\left\{ x \in Q^\gamma : \left(\sum_{k \in \mathcal{N}} \int_I |\mathcal{S}_{2^{kt}}^{\lambda(1)} [\sum_{Q \in \mathcal{Q}_\gamma} c_Q a_Q]|^q dt \right)^{1/q} > \alpha \right\} \right) \lesssim 1.$$

It suffices to show for every finite subset \mathcal{F}^γ of \mathcal{Q}_γ

$$(5.4) \quad \sup_{\alpha>0} \alpha \text{ meas} \left(\left\{ x \in Q_\gamma : \left(\sum_{k \in \mathcal{N}} \int_I |\mathcal{S}_{2^{kt}}^{\lambda(1)} [\sum_{Q \in \mathcal{F}^\gamma} c_Q a_Q]|^q dt \right)^{1/q} > \alpha \right\} \right) \lesssim \sum_{Q \in \mathcal{F}^\gamma} |c_Q|$$

where the implicit constant is independent of \mathcal{F}^γ . To see the reduction we split $\mathcal{Q}_\gamma = \cup_{n=0}^\infty \mathcal{F}^{\gamma,n}$ where $\mathcal{F}^{\gamma,n}$ is finite and $\sum_{Q \in \mathcal{F}^{\gamma,n}} |c_Q| \leq 2^{-n}$. By using the result of Stein and N. Weiss on adding $L^{1,\infty}$ functions [41, Lemma 2.3] the left hand side in (5.3) is bounded by $C \sum_{n=0}^\infty (1+n)2^{-n} \lesssim 1$, as claimed.

In order to prove (5.4) we can renormalize again, replacing c_Q with $c_Q(\sum_{Q' \in \mathcal{F}^\gamma} |c_{Q'}|)^{-1}$ and α with $\alpha(\sum_{Q' \in \mathcal{F}^\gamma} |c_{Q'}|)^{-1}$. It therefore remains to prove for every finite subset \mathcal{F}^γ of \mathcal{Q}^γ , and for $\sum_{Q \in \mathcal{F}^\gamma} |c_Q| = 1$, that

$$(5.5) \quad \sup_{\alpha > 0} \alpha \operatorname{meas} \left(\left\{ x \in Q^\gamma : \left(\sum_{k \in \mathcal{N}} \int_I |\mathcal{S}_{2^{kt}}^{\lambda(1)} [\sum_{Q \in \mathcal{F}^\gamma} c_Q a_Q]|^q dt \right)^{1/q} > \alpha \right\} \right) \lesssim 1.$$

where the implicit constant is independent of \mathcal{F}^γ .

Now fix $\alpha > 0$. Let $\phi \in C^\infty$ supported in $\{x : |x| \leq 1\}$ such that $\int \phi(x) dx = 1$ and let $\phi_\varepsilon = \varepsilon^{-d} \phi(\varepsilon^{-1} \cdot)$. Choose ε_Q be small, less than one tenth of the sidelength of Q so that in addition $\|\phi_{\varepsilon(Q)} * a_Q - a_Q\|_2 < \alpha^{1/2}$. Let $\tilde{a}_Q = \phi_{\varepsilon(Q)} * a_Q$. Then by Tshebyshev's inequality and standard L^2 estimates (such as in §3)

$$\begin{aligned} & \operatorname{meas} \left(\left\{ x \in Q^\gamma : \left(\sum_{k \in \mathcal{N}} \int_I |\mathcal{S}_{2^{kt}}^{\lambda(1)} [\sum_{Q \in \mathcal{F}^\gamma} c_Q (a_Q - \tilde{a}_Q)]|^q dt \right)^{1/q} > \alpha \right\} \right) \\ & \lesssim \alpha^{-2} \left\| \left(\sum_{k \in \mathcal{N}} \int_I |\mathcal{S}_{2^{kt}}^{\lambda(1)} [\sum_{Q \in \mathcal{F}^\gamma} c_Q (a_Q - \tilde{a}_Q)]|^q dt \right)^{1/q} \right\|_2^2 \\ & \lesssim \alpha^{-2} \left(\sum_{Q \in \mathcal{F}^\gamma} |c_Q| \|a_Q - \tilde{a}_Q\|_2 \right)^2 \lesssim \alpha^{-1} \left(\sum_{Q \in \mathcal{F}^\gamma} |c_Q| \right)^2 \lesssim \alpha^{-1}; \end{aligned}$$

here we have used the normalization $\sum_Q |c_Q| \leq 1$.

It suffices to show that

$$(5.6) \quad \operatorname{meas} \left(\left\{ x \in Q^\gamma : \left(\sum_{k \in \mathcal{N}} \int_I |\mathcal{S}_{2^{kt}}^{\lambda(1)} [\sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q]|^q dt \right)^{1/q} > \alpha \right\} \right) \lesssim \alpha^{-1}.$$

We shall now follow the argument in [14] and set

$$(5.7) \quad \Psi(x) = \prod_{i=1}^d (1 - x_i^2/4)_+, \quad \Psi_N^\gamma(x) = \Psi(N^{-1}(x - \gamma)).$$

As in [14] we use the following formula, valid for g in the Schwartz space of \mathbb{T}^d , and for $x \in \mathbb{R}^d$,

$$(5.8) \quad \Psi_N^\gamma(x) \mathcal{S}_{2^{kt}}^\lambda g(x) - \mathcal{S}_{2^{kt}}^\lambda [\Psi_N^\gamma g](x) = \sum_{\ell \in \mathbb{Z}^d} \langle g, e_\ell \rangle e_\ell(x) \int \left[h_\lambda(\rho(\frac{\ell}{2^{kt}})) - h_\lambda(\rho(\frac{\ell + N^{-1}\xi}{2^{kt}})) \right] \widehat{\Psi}(\xi) e^{2\pi i \langle x - \gamma, N^{-1}\xi \rangle} d\xi.$$

As the Fourier coefficients $\langle g, e_\ell \rangle$ decay rapidly, $\widehat{\Psi} \in L^1$ and h_λ is Hölder continuous for $\lambda > 0$ this implies

$$(5.9) \quad \lim_{N \rightarrow \infty} \sup_{t \in I} \sup_{x \in \mathbb{R}^d} |\Psi_N^\gamma(x) \mathcal{S}_{2^{kt}}^\lambda g(x) - \mathcal{S}_{2^{kt}}^\lambda [\Psi_N^\gamma g](x)| = 0,$$

for $k \in \mathcal{N}$.

Next we observe that $\Psi_N^\gamma(x) \geq (3/4)^d$ for all $x \in m + Q^\gamma$, when $-N \leq m_i \leq N$ for $i = 1, \dots, d$. Using periodicity we see that the left hand side of (5.6) is equal to

$$\begin{aligned} & (2N+1)^{-d} \sum_{\substack{-N \leq m_i \leq N \\ i=1, \dots, d}} \operatorname{meas} \left(\left\{ x \in m + Q^\gamma : \left(\sum_{k \in \mathcal{N}} \int_I |\mathcal{S}_{2^{kt}}^{\lambda(1)} [\sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q](x)|^q dt \right)^{1/q} > \alpha \right\} \right) \\ & \leq (2N+1)^{-d} \operatorname{meas} \left(\left\{ x \in \mathbb{R}^d : \left(\sum_{k \in \mathcal{N}} \int_I |\Psi_N^\gamma(x) \mathcal{S}_{2^{kt}}^{\lambda(1)} [\sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q](x)|^q dt \right)^{1/q} > (3/4)^d \alpha \right\} \right). \end{aligned}$$

Consider the periodic C^∞ function $g = \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q$ and apply (5.9). Hence there is $N_0 = N_0(g, \alpha, \mathcal{N})$ such that for every $x \in \mathbb{R}^d$ and $N > N_0$,

$$\left(\sum_{k \in \mathcal{N}} \int_I \left| \Psi_N^\gamma(x) \mathcal{S}_{2^{k_t}}^{\lambda(1)} \left[\sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right](x) - \mathcal{S}_{2^{k_t}}^{\lambda(1)} \left[\Psi_N^\gamma \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right](x) \right|^q dt \right)^{1/q} < (3/4)^d \alpha / 2.$$

Assuming $N > N_0$ in what follows we see that it suffices to bound

$$(5.10) \quad (2N + 1)^{-d} \text{meas} \left(\left\{ x \in \mathbb{R}^d : \left(\sum_{k \in \mathcal{N}} \int_I \left| \mathcal{S}_{2^{k_t}}^{\lambda(1)} \left[\Psi_N^\gamma \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right](x) \right|^q dt \right)^{1/q} > (3/4)^d \alpha / 2 \right\} \right).$$

Define for $Q \in \mathcal{F}^\gamma$, $m \in \mathbb{Z}^d$,

$$a_{Q,m}(y) = \mathbb{1}_{m+Q^\gamma}(y) \Psi_N^\gamma(y) \tilde{a}_Q(y).$$

Then the support of $a_{Q,m}$ is in the interior of $m + Q^\gamma$ and Ψ_N^γ coincides on the support of $a_{Q,m}$ with a bounded polynomial of degree $2d$. Hence $a_{Q,m}$ is an L^2 function supported on the double of Q , such that $\int a_{Q,m}(y) dy = 0$ and such that $\|a_{Q,m}\|_2 \lesssim |Q|^{-1/2}$. Moreover $a_{Q,m}$ is nontrivial only when $|m_i| \leq 2N$ for $i = 1, \dots, d$. This implies

$$\left\| \Psi_N^\gamma \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right\|_{H^1(\mathbb{R}^d)} \leq \sum_{\substack{-2N \leq m_i \leq 2N \\ i=1, \dots, d}} \sum_{Q \in \mathcal{F}^\gamma} |c_Q| \|a_{Q,m}\|_{H^1(\mathbb{R}^d)} \lesssim (4N + 1)^d.$$

We now apply Theorem 2.1 to see that the left hand side of (5.10) is bounded by

$$C \alpha^{-1} (2N + 1)^{-d} \left\| \Psi_N^\gamma \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right\|_{H^1(\mathbb{R}^d)} \lesssim (2N + 1)^{-d} (4N + 1)^d \alpha^{-1} \lesssim \alpha^{-1}$$

which finishes the proof of the theorem. \square

5.2. *The $L^p \rightarrow L^{p,\infty}$ bound.* The proof is similar (but more straightforward), therefore we will be brief. Now $\lambda(p)$ can be negative, but we have $\lambda > -1/q$. The limiting relation (5.9) is now replaced by

$$(5.11) \quad \lim_{N \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \left(\int_I \left| \Psi_N^\gamma(x) \mathcal{S}_{2^{k_t}}^{\lambda(p)} g(x) - \mathcal{S}_{2^{k_t}}^{\lambda(p)} [\Psi_N^\gamma g](x) \right|^q dt \right)^{1/q} = 0, \quad k \in \mathcal{N}.$$

Here, we consider $g \in \mathcal{S}(\mathbb{T}^d)$. We sketch a proof of (5.11), based on (5.8).

We start by observing that

$$(5.12) \quad \int_I |h_\lambda(\rho(\zeta/t))|^q dt \leq C,$$

uniformly in $\zeta \in \mathbb{R}^d$. To see this, note that $\rho(\zeta/t) = \rho(\zeta)t^{-1/b}$ and we may assume that $\rho(\zeta) \sim 1$ due to the support of h_λ . Therefore, (5.12) follows by a change of variable. From this observation, we may reduce (5.11) to

$$(5.13) \quad \lim_{N \rightarrow \infty} \left(\int_I \left| h_\lambda(\rho(\frac{\ell}{2^{k_t}})) - h_\lambda(\rho(\frac{\ell + N^{-1}\xi}{2^{k_t}})) \right|^q dt \right)^{1/q} = 0$$

for fixed l, k, ξ using (5.8), Minkowski's inequality and the dominated convergence theorem.

For (5.13), we argue as follows. Let $h \in L^q(J)$ for a compact subinterval J of $(0, \infty)$. Then for any $a > 0$

$$\lim_{\delta \rightarrow 0} \left(\int_J |h(as) - h((a + \delta)s)|^q ds \right)^{1/q} = 0$$

and the limit is uniform if a is taken from a compact subset of $(0, \infty)$. This is easily seen for smooth h and follows for general $h \in L^q(J)$ by an approximation argument. Changing variables $s = t^{-1/b}$ we obtain that for any compact subinterval $I \subset (0, \infty)$

$$(5.14) \quad \lim_{\delta \rightarrow 0} \left(\int_I |h(at^{-1/b}) - h((a + \delta)t^{-1/b})|^q dt \right)^{1/q} = 0.$$

Then (5.13) follows from (5.14) with $h = h_\lambda$, $\delta = \rho((\ell + N^{-1}\xi)/2^k) - \rho(\ell/2^k)$ and $a = \rho(\ell/2^k)$ using the homogeneity and continuity of ρ .

Finally, using (5.11) we get, for sufficiently large N ,

$$\begin{aligned} & \text{meas} \left(\left\{ x \in Q^0 : \left(\sum_{k \in \mathcal{N}} \int_I |S_{2^k t}^{\lambda(p)} g|^q dt \right)^{1/q} > \alpha \right\} \right) \\ & \lesssim (2N + 1)^{-d} \text{meas} \left(\left\{ x \in \mathbb{R}^d : \left(\sum_{k \in \mathcal{N}} \int_I |S_{2^k t}^{\lambda(p)} [\Psi_N^0 g](x)|^q dt \right)^{1/q} > (3/4)^d \alpha / 2 \right\} \right) \end{aligned}$$

and by Theorem 2.2 we bound the right hand side by

$$C(2N + 1)^{-d} \alpha^{-p} \|\Psi_N^0 g\|_{L^p(\mathbb{R}^d)}^p \lesssim \alpha^{-p} \|g\|_{L^p(\mathbb{T}^d)}^p.$$

Remark. It is also possible to build a proof of Theorem 1.2 from Theorem 2.2 using modifications of a duality argument by deLeeuw [25], see also [40] and [20].

6. AN L^1 RESULT

We currently do not have an analogue of Theorem 1.1 for general functions in $L^1(\mathbb{T}^d)$. We formulate a weaker result which is essentially a consequence of Theorem 1.1.

Theorem 6.1. (i) Let $f \in L^1(\mathbb{T}^d)$. Then for all $q < \infty$, and $\lambda(1) = \frac{d-1}{2}$,

$$\lim_{T \rightarrow \infty} \left\| \left(\frac{1}{T} \int_0^T |\mathcal{R}_t^{\lambda(1)} f - f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{T}^d)} = 0.$$

(ii) The analogous statement holds on $L^1(\mathbb{R}^d)$ with $R_t^{\lambda(1)} f$ in place of $\mathcal{R}_t^{\lambda(1)} f$.

Proof. Since the convergence holds for Schwartz function one can by a standard approximation argument reduce the proof of (ii) to the inequality

$$(6.1) \quad \sup_{T > 0} \left\| \left(\frac{1}{T} \int_0^T |R_t^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}.$$

Similarly the proof of (i) is reduced to a corresponding inequality on \mathbb{T}^d , with the supremum in T extended over $T \geq 1$. The weak type (1, 1) inequality in the \mathbb{T}^d case follows from the \mathbb{R}^d case by the transference arguments of §5. Therefore, it suffices to show (6.1).

By the maximal estimate in §2.1 it remains to prove

$$(6.2) \quad \left\| \left(\frac{1}{T} \int_0^T |S_t^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}$$

where $S_t^{\lambda(1)}$ is as in (2.2). We may assume $q \geq 2$. Now

$$\left(\frac{1}{T} \int_0^T |S_t^{\lambda(1)} f(x)|^q dt \right)^{1/q} \leq \sum_{l=0}^{\infty} 2^{-l/q} \left(\frac{1}{T 2^{-l}} \int_{T 2^{-l}}^{T 2^{-l+1}} |S_t^{\lambda(1)} f(x)|^q dt \right)^{1/q}$$

and we claim the inequality

$$(6.3) \quad \sup_{A>0} \left\| \left(\frac{1}{A} \int_A^{2A} |S_t^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}} \leq C_q \|f\|_1.$$

Assuming that (6.3) is verified we can deduce that the left hand side of (6.2) is bounded by $C_q \tilde{C} \sum_{l>0} (1+l) 2^{-l/q} \|f\|_1 \lesssim_q \|f\|_1$, by the theorem of Stein and N. Weiss [41, Lemma 2.3] on summing $L^{1,\infty}$ functions.

Let η be as (2.3). Then our main result, Theorem 2.3, yields for all $A > 0$

$$\left\| \left(\frac{1}{A} \int_A^{2A} |S_t^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{R}^d)} \leq C_q \|\mathcal{F}^{-1}[\eta(A^{-1}\cdot)] * f\|_{H^1(\mathbb{R}^d)}.$$

Since η is C^∞ and compactly supported away from the origin we have

$$\|\mathcal{F}^{-1}[\eta(A^{-1}\cdot)] * f\|_{H^1(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}$$

uniformly in A . This yields (6.3) and concludes the proof of (6.2). \square

As an immediate consequence of Theorem 6.1 we get

Corollary 6.2. *Let $f \in L^1(\mathbb{T}^d)$. There is a subsequence $T_j \rightarrow \infty$ such that*

$$(6.4) \quad \lim_{j \rightarrow \infty} \left(\frac{1}{T_j} \int_0^{T_j} |\mathcal{R}_t^{\lambda(1)} f(x) - f(x)|^q dt \right)^{1/q} = 0 \text{ a.e.}$$

Arguing as in [48, ch. XIII.7] or [46, §4] we get

Corollary 6.3. *Let $f \in L^1(\mathbb{T}^d)$. For almost every $x \in \mathbb{T}^d$ there is a measurable set $E = E(f, x)$ of upper density one, i.e. satisfying*

$$(6.5) \quad \limsup_{T \rightarrow \infty} \frac{|E \cap [0, T]|}{T} = 1$$

such that

$$\lim_{\substack{t \rightarrow \infty \\ t \in E}} \mathcal{R}_t^{\lambda(1)} f(x) = f(x).$$

For convenience of the reader we give a proof.

Proof. Fix x such that (6.4) in Corollary 6.2 holds and let $g(t) = |\mathcal{R}_t^{\lambda(1)} f(x) - f(x)|$. We may assume that T_j is increasing in j . For $m = 1, 2, \dots$ let $E_m = \{t : g(t) \leq 1/m\}$. By Tshebyshev's inequality we have

$$\frac{|E_m^c \cap [0, T_j]|}{T_j} \leq m^q \frac{1}{T_j} \int_0^{T_j} g(t)^q dt$$

which by assumption tends to 0 as $j \rightarrow \infty$. Hence $\lim_{j \rightarrow \infty} T_j^{-1} |E_m \cap [0, T_j]| = 1$. Thus we may choose a strictly increasing sequence j_m of positive integers such that $T_j^{-1} |E_m \cap [0, T_j]| > 1 - m^{-1}$ for $j \geq j_m$. Let $E = [0, T_{j_1}] \cup \bigcup_{m=1}^{\infty} (E_m \cap [T_{j_m}, T_{j_{m+1}}])$. Since the sets E_m are decreasing we have

$$|E \cap [0, T_{j_{m+1}}]| \geq |E_m \cap [0, T_{j_{m+1}}]| \geq (1 - m^{-1}) T_{j_{m+1}}$$

and hence $\limsup_{T \rightarrow \infty} T^{-1} |E \cap [0, T]| = 1$. Now $E \cap [T_{j_m}, \infty] \subset E_m$ and thus $g(t) \leq m^{-1}$ on this set. It follows that $g(t) \rightarrow 0$ as $t \rightarrow \infty$ within E . \square

It would be desirable to replace the lim sup in (6.5) by the lim inf. The proof of the corollary shows that this would require the existence a.e. of the limit in (6.4) for *all* sequences $T_j \rightarrow \infty$. However this we can currently only prove for functions in h^1 .

7. SHARPNESS

In this section we show that Theorems 1.2 and 2.2 fail for $q > p'$. We shall first reduce the argument for Fourier series to the one for Fourier integrals by a familiar transplantation method and then modify an argument that was used by Tao to obtain necessary conditions for the Bochner-Riesz maximal operator, see [43, sect.5], and also the work by Carbery and Soria [6] where a related argument appears in the context of localization results for Fourier series. Note that the almost everywhere convergence assertion in part (ii) of Theorem 1.2 also fails for $q > p'$, by Stein-Nikishin theory ([35]).

7.1. *Fourier series.* We have for $f \in L^p(\mathbb{T}^d)$

$$(7.1) \quad \left\| \sup_{T>0} \left(\frac{1}{T} \int_0^T |\mathcal{R}_t^\lambda f|^q dt \right)^{1/q} \right\|_{L^{p,\infty}(\mathbb{T}^d)} \geq \sup_{T>0} \left\| \left(\frac{1}{T} \int_0^T |\mathcal{R}_t^\lambda f|^q dt \right)^{1/q} \right\|_{L^{p,\infty}(\mathbb{T}^d)}$$

and our necessary condition will follow from Proposition 7.2 below and the following result.

Lemma 7.1. *Let $1 \leq p \leq 2$. Suppose that for some $C > 0$*

$$(7.2) \quad \sup_{\|f\|_{L^p(\mathbb{T}^d)}=1} \sup_{T>0} \left\| \left(\frac{1}{T} \int_0^T |\mathcal{R}_t^\lambda f|^q dt \right)^{1/q} \right\|_{L^{p,\infty}(\mathbb{T}^d)} \leq C.$$

Then also

$$(7.3) \quad \sup_{\|f\|_{L^p(\mathbb{R}^d)}=1} \sup_{T>0} \left\| \left(\frac{1}{T} \int_0^T |R_t^\lambda f|^q dt \right)^{1/q} \right\|_{L^{p,\infty}(\mathbb{R}^d)} \leq C.$$

Proof. By scaling, density of C_c^∞ functions in L^p and the monotone convergence theorem it suffices to show for all $f \in C_c^\infty(\mathbb{R}^d)$, all compact sets K , all $\delta \in (0, 1)$, all $\varepsilon \in (0, 1)$ and all $\alpha > 0$

$$\text{meas} \left(\left\{ x \in K : \left(\int_\delta^1 |R_t^\lambda f(x)|^q dt \right)^{1/q} > \alpha \right\} \right) \leq C^p (1 - \varepsilon)^{-p} \alpha^p \|f\|_p^p.$$

Fix such f , α , δ , ε and K . For large $L \in \mathbb{N}$ define

$$V_{L,t}^\lambda f(x) = \sum_{\ell \in \mathbb{Z}^d} L^{-d} \widehat{f}(L^{-1}\ell) (1 - \rho(t^{-1}L^{-1}\ell)_+^\lambda) e^{2\pi i L^{-1}\langle x, \ell \rangle}.$$

Then $V_{L,t}^\lambda f(x)$ is a Riemann sum for the integral representing $R_t^\lambda f(x)$. Hence we have

$$\lim_{L \rightarrow \infty} V_{L,t}^\lambda f(x) = R_t^\lambda f(x)$$

with the limit uniform in $t \in [\delta, 1]$, $x \in K$. We may therefore choose L such that

$$\text{supp}(f(L \cdot)) \subset \{x : |x| < 1/4\} \text{ and } K \subset LQ^0$$

with $Q^0 = [-1, 2, 1/2]^d$, and

$$\sup_{\delta \leq t \leq 1} \sup_{x \in K} |R_t^\lambda f(x) - V_{L,t}^\lambda f(x)| < \alpha \varepsilon.$$

It remains to show

$$(7.4) \quad \text{meas} \left(\left\{ x \in K : \left(\int_\delta^1 |V_{L,t}^\lambda f(x)|^q dt \right)^{1/q} > \alpha(1 - \varepsilon) \right\} \right) \leq C^p (1 - \varepsilon)^{-p} \alpha^p \|f\|_p^p.$$

Observe that for $w \in Q^0$

$$\left(\int_\delta^1 |V_{L,t}^\lambda f(Lw)|^q dt \right)^{1/q} = \left(\frac{1}{L} \int_{\delta L}^L \left| \sum_{\ell \in \mathbb{Z}^d} L^{-d} \widehat{f}(L^{-1}\ell) (1 - \rho(\ell/s)_+^\lambda) e^{2\pi i \langle w, \ell \rangle} \right|^q ds \right)^{1/q}.$$

Let $f_L^{\text{per}}(w) = \sum_{\kappa \in \mathbb{Z}^d} f(L(w + \kappa))$. Then by the Poisson summation formula the Fourier coefficients of the periodic function f_L^{per} are given by $\langle f_L^{\text{per}}, e_\ell \rangle = L^{-d} \widehat{f}(L^{-1}\ell)$. Hence the expression on the right hand side of the last display is equal to $(L^{-1} \int_{\delta L}^L |\mathcal{R}_t^\lambda f_L^{\text{per}}|^q dt)^{1/q}$. Replacing K by the larger set LQ_0 and then changing variables $x = Lw$ we see that the expression on the left hand side of (7.4) is dominated by

$$\begin{aligned} & L^d \text{meas} \left(\left\{ w \in Q^0 : \left(\frac{1}{L} \int_{\delta L}^L |\mathcal{R}_t^\lambda f_L^{\text{per}}(w)|^q dt \right)^{1/q} > \alpha(1 - \varepsilon) \right\} \right) \\ & \leq L^d C^p \alpha^{-p} (1 - \varepsilon)^{-p} \int_{Q_0} |f_L^{\text{per}}(w)|^p dw \end{aligned}$$

where the last inequality follows by assumption (7.2). Since the support of $f(L \cdot)$ is contained in Q_0 one immediately gets

$$\|f_L^{\text{per}}\|_{L^p(Q^0)}^p = \|f(L \cdot)\|_{L^p(\mathbb{R}^d)}^p = L^{-d} \|f\|_{L^p(\mathbb{R}^d)}^p.$$

This shows (7.4) and concludes the proof. \square

7.2. Fourier integrals. Using the \mathbb{R}^d analogue of (7.1) we reduce the sharpness claim in Theorem 2.2 to the following proposition.

Proposition 7.2. *Let $1 < p \leq 2$, and $\lambda > -1/2$. Assume that there is a constant $C > 0$ such that*

$$(7.5) \quad \sup_{T>1} \left\| \left(\frac{1}{T} \int_0^T |R_t^\lambda f|^q dt \right)^{1/q} \right\|_{L^{p,\infty}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}$$

for all Schwartz functions f . Then

$$\lambda \geq \lambda(p) + \frac{1}{2} \left(\frac{1}{p'} - \frac{1}{q} \right).$$

In particular, if (7.5) holds for $\lambda = \lambda(p)$, then $q \leq p'$.

Proof. We note that the inequality with a given ρ is equivalent to the inequality with $\rho \circ A$ where A is any rotation.

Let $\xi^\circ \in \Sigma_\rho$ such that $|\xi^\circ|$ is maximal. Then the Gaussian curvature does not vanish at ξ^0 . Choose small neighborhoods U_1, U_0 of ξ^0 in Σ_ρ such that $\overline{U}_1 \subset U_0$, the Gauss map is injective in a neighborhood of \overline{U}_0 and the curvature is bounded below on U_0 . Let γ be homogeneous of degree zero, $\gamma(\xi) \neq 0$ for $\xi \in U_1$ with γ supported on the closure of the cone generated by U_0 . Let $n(\xi_0) = \frac{\nabla \rho(\xi^\circ)}{|\nabla \rho(\xi^\circ)|}$ the outer normal at ξ_0 , let $\Gamma_\varepsilon = \{x \in \mathbb{R}^d : \left| \frac{x}{|x|} - n(\xi_0) \right| \leq 2\varepsilon\}$, with ε so small that Γ_ε is contained in the cone generated by the normal vectors $\nabla \rho(\xi)$ with $\xi \in U_1$. Let, for $R \gg 1$, $\Gamma_{\varepsilon,R} = \{x \in \Gamma_\varepsilon : |x| \geq R\}$. By the choice of ε there is, for each $x \in \Gamma_\varepsilon$, a unique $\Xi(x) \in \Sigma_\rho$, so that $\gamma(\Xi(x)) \neq 0$ and so that x is normal to Σ_ρ at $\Xi(x)$. Clearly $x \mapsto \Xi(x)$ is homogeneous of degree zero on Γ , smooth away from the origin. By a rotation we may assume

$$(7.6) \quad n(\xi^\circ) = (0, \dots, 0, 1).$$

By §2.1 inequality (7.5) also implies the similar inequality where $R_t^\lambda f$ is replaced with $S_t^\lambda f$ and S_t^λ is as in (2.2). Let $h_\lambda(s) = (1 - v_0(s))(1 - s)_+^\lambda$ and

$$K_{\lambda,t}(x) = t^d \mathcal{F}^{-1}[\gamma h_\lambda \circ \rho](tx).$$

Observe that $K_{\lambda,t} * f = S_t^\lambda f_\gamma$ with $\widehat{f}_\gamma = \gamma \widehat{f}$. By the Hörmander multiplier theorem γ is a Fourier multiplier of L^p and we see that (7.5) implies that

$$(7.7) \quad \sup_{T>0} \left\| \left(\frac{1}{T} \int_0^T |K_{\lambda,t} * f|^q dt \right)^{1/q} \right\|_{L^{p,\infty}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}.$$

We now derive an asymptotic expansion for $K_{\lambda,1}(x)$ when $x \in \Gamma_{\varepsilon,R}$. Recall that ρ is homogeneous of degree $1/b$, i.e. $\rho(t^b \xi) = t\rho(\xi)$. We use generalized polar coordinates $\xi = \rho^b \xi(\omega)$ where $\omega \rightarrow \xi(\omega)$ is a parametrization of Σ_ρ in a neighborhood of U_0 . Then

$$\begin{aligned} d\xi &= b\rho^{db-1} d\rho \langle \xi(\omega), n(\xi(\omega)) \rangle \left(\det \left(\frac{\partial \xi}{\partial \omega} \right)^\top \frac{\partial \xi}{\partial \omega} \right)^{1/2} d\omega \\ &= \rho^{db-1} d\rho |\nabla \rho(\xi')|^{-1} d\sigma(\xi'), \quad \xi' = \xi(\omega). \end{aligned}$$

Here we have used Euler's homogeneity relation $b\langle \xi, \nabla \rho(\xi) \rangle = \rho(\xi)$ for vectors on Σ_ρ . Then

$$(7.8) \quad K_{\lambda,1}(x) = \int_0^\infty h_\lambda(\rho) \rho^{bd-1} \int_{\Sigma_\rho} \gamma(\xi') e^{2\pi i \rho \langle \xi', x \rangle} \frac{d\sigma(\xi')}{|\nabla \rho(\xi')|} d\rho.$$

We use the method of stationary phase and get for $x \in \Gamma_{\varepsilon,R}$

$$(7.9) \quad K_{\lambda,1}(x) = I(x) + \sum_{j=1}^N II_j(x) + III(x)$$

where

$$I(x) = c \int_0^\infty h_\lambda(\rho) \rho^{bd-1-\frac{d-1}{2}} e^{2\pi i \rho \langle \Xi(x), x \rangle} d\rho \frac{\gamma(\Xi(x)) |\nabla \rho(\Xi(x))|^{-1}}{(\langle \Xi(x), x \rangle)^{\frac{d-1}{2}} |\text{curv}(\Xi(x))|^{1/2}}$$

where $\text{curv}(\Xi(x))$ is the Gaussian curvature at $\Xi(x)$ and $c \neq 0$, and

$$II_j(x) = c_j \int_0^\infty h_\lambda(\rho) \rho^{bd-1-\frac{d-1}{2}-j} e^{2\pi i \rho \langle \Xi(x), x \rangle} d\rho \frac{\gamma_j(\Xi(x))}{(\langle \Xi(x), x \rangle)^{\frac{d-1}{2}+j} |\text{curv}(\Xi(x))|^{1/2}}$$

where γ_j is smooth. For the remainder term we get

$$|III(x)| \lesssim_N \|h\|_1 |x|^{-N}, \quad x \in \Gamma_{\varepsilon,R}.$$

In the resulting ρ integrals we use asymptotics for the one-dimensional Fourier transform of h_λ , cf. [13, §2.8], and see that for $x \in \Gamma_{\varepsilon,R}$,

$$\int_0^\infty h_\lambda(\rho) \rho^{bd-\frac{d+1}{2}} e^{2\pi i \rho \langle \Xi(x), x \rangle} d\rho = C_\lambda \langle \Xi(x), x \rangle^{-\lambda-1} e^{2\pi i \langle \Xi(x), x \rangle} + O(\langle \Xi(x), x \rangle^{-\lambda-2}),$$

with similar asymptotics for the ρ -integrals in the terms II_j .

Now set for $x \in \Gamma_\varepsilon$, $H(x) = \langle \Xi(x), x \rangle$ and use Euler's homogeneity relation to write

$$H(x) = |x| \left\langle \Xi(x), \frac{\nabla \rho(\Xi(x))}{|\nabla \rho(\Xi(x))|} \right\rangle = |x| \frac{\rho(\Xi(x))}{b |\nabla \rho(\Xi(x))|} = \frac{|x|}{b |\nabla \rho(\Xi(x))|}.$$

If ε is small we then have for $t|x| \gg R$,

$$K_{\lambda,t}(x) = A_\lambda(x, t) + B_\lambda(x, t), \quad |x'| \leq \varepsilon^2 |x_d|$$

where

$$(7.10) \quad \begin{aligned} A_\lambda(x, t) &= C(\lambda) t^{d-\frac{d+1}{2}-\lambda} G(x) e^{2\pi i t H(x)} \\ \text{where } G(x) &= H(x)^{-\frac{d+1}{2}-\lambda} \frac{\gamma(\Xi(x)) |\nabla \rho(\Xi(x))|^{-1}}{|\text{curv}(\Xi(x))|^{1/2}} \end{aligned}$$

and

$$B_\lambda(x, t) \lesssim t^{d-\frac{d+3}{2}-\lambda} H(x)^{-\frac{d+3}{2}-\lambda}.$$

Recall (7.6) and split $y = (y', y_d)$. We now let

$$P(T, \varepsilon) = \{y : |y'| \leq T^{-1}\varepsilon, |y_d| \leq T^{-1/2}\varepsilon\}$$

and define

$$f_T(y) = \mathbb{1}_{P(T, \varepsilon)}(y)e^{2\pi i \varepsilon T y_d}.$$

Then

$$(7.11) \quad \|f_T\|_p \lesssim T^{\frac{1}{p}(\frac{1}{2}-d)}.$$

We examine the integrals $K_{\lambda, t} * f_T(x)$ for $|x| \approx 1$ and $R \ll t \approx \varepsilon T$. We may obtain a lower bound for the absolute value of this integral if we can choose t for given x such that

$$(7.12) \quad 2\pi(\varepsilon T y_d + tH(x-y) - tH(x)) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \text{ for all } y \in \text{supp}(f_T).$$

As the Gauss map is invertible near ξ° we observe that H is smooth and homogeneous of degree 1. We have $\nabla H(x) = \xi^\circ + O(\varepsilon)$ and thus $\partial_{x_d} H(x) \geq c > 0$. Now

$$(7.13) \quad \begin{aligned} & \varepsilon T y_d + tH(x-y) - tH(x) = \\ & -t \sum_{i=1}^{d-1} y_i \partial_{x_i} H(x) + y_d(\varepsilon T - t\partial_{x_d} H(x)) + t \sum_{i,j=1}^d y_i y_j \int_0^1 (1-s) \partial_{x_i x_j}^2 H(x-sy) ds. \end{aligned}$$

The first and the third term on the right hand side are $O(\varepsilon)$ when $y \in \text{supp}(f_T)$. We choose t in the interval

$$(7.14) \quad I_{x, T} = \left[\frac{\varepsilon T}{\partial_{x_d} H(x)} - \varepsilon T^{1/2}, \frac{\varepsilon T}{\partial_{x_d} H(x)} + \varepsilon T^{1/2} \right].$$

We assume that ε is chosen so small that $I_{x, T} \subset [0, T]$. If $t \in I_{x, T}$ the second term on the right hand side of (7.13) will be $O(\varepsilon)$ as well so that (7.12) is satisfied.

We now split

$$K_{\lambda, t} * f_T(x) = \mathcal{J}_1(x, t) + \mathcal{J}_2(x, t) + \mathcal{J}_3(x, t)$$

with

$$\begin{aligned} \mathcal{J}_1(x, t) &= C(\lambda) G(x) e^{2\pi i t H(x)} t^{\frac{d-1}{2}-\lambda} \int e^{2\pi i (T\varepsilon y_d + tH(x-y) - tH(x))} \mathbb{1}_{P(T, \varepsilon)}(y) dy, \\ \mathcal{J}_2(x, t) &= C(\lambda) t^{\frac{d-1}{2}-\lambda} \int e^{2\pi i (T\varepsilon y_d + tH(x-y))} (G(x-y) - G(x)) \mathbb{1}_{P(T, \varepsilon)}(y) dy, \\ \mathcal{J}_3(x, t) &= \int B_\lambda(x-y, t) f_T(y) dy. \end{aligned}$$

We estimate these terms for

$$(7.15) \quad x \in \Omega := \{x : |x'| \leq \varepsilon^2 |x_d|, 1/2 \leq |x_d| \leq 1\}, \quad t \in I_{x, T}.$$

Then by (7.12) the real part of the integrand in the definition of $\mathcal{J}_1(x, t)$ is bounded below by $2^{-1/2} \mathbb{1}_{P(T, \varepsilon)}(y)$ and therefore, for $x \in \Omega$,

$$\begin{aligned} |\mathcal{J}_1(x, t)| &\geq C G(x) t^{\frac{d-1}{2}-\lambda} \int \mathbb{1}_{P(T, \varepsilon)}(y) dy \\ &\geq c t^{\frac{d-1}{2}-\lambda} T^{\frac{1}{2}-d}. \end{aligned}$$

Moreover,

$$\begin{aligned} |\mathcal{J}_2(x, t)| &\lesssim t^{\frac{d-1}{2}-\lambda} \varepsilon T^{-1/2} T^{\frac{1}{2}-d} \\ |\mathcal{J}_3(x, t)| &\lesssim t^{\frac{d-3}{2}-\lambda} T^{\frac{1}{2}-d}. \end{aligned}$$

Hence for small ε and $t|x| \gg R$, $t \in I_{x,T}$ the term $|\mathcal{J}_1(x,t)|$ is significantly larger than the terms $|\mathcal{J}_2(x,t)|$ and $|\mathcal{J}_3(x,t)|$. Consequently, by $|I_{x,T}| \geq \varepsilon T^{1/2}$, and assuming (7.15) we get

$$\begin{aligned} \left(\frac{1}{T} \int_0^T |K_{\lambda,t} * f_T(x)|^q dt \right)^{1/q} &\geq \left(\frac{1}{T} \int_{I_{x,T}} |K_{\lambda,t} * f_T(x)|^q dt \right)^{1/q} \\ &\geq c\varepsilon^{1/q} T^{-1/2q} (\varepsilon T)^{\frac{d-1}{2}-\lambda} T^{1/2-d} = c\varepsilon T^{-\frac{d}{2}-\lambda-\frac{1}{2q}} \end{aligned}$$

and thus

$$\left\| \left(\frac{1}{T} \int_0^T |K_{\lambda,t} * f_T|^q dt \right)^{1/q} \right\|_{L^{p,\infty}} \gtrsim_\varepsilon T^{-\frac{d}{2}-\lambda-\frac{1}{2q}} T^{\frac{d}{p}-\frac{1}{2p}} \|f_T\|_p$$

which for $T \rightarrow \infty$ implies $\lambda \geq \lambda(p) + \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{q})$. \square

8. MAXIMAL FUNCTIONS ON $H^p(\mathbb{R}^d)$ FOR $p < 1$

We now consider the maximal operator associated with the generalized Riesz means when they act on functions or distributions in the Hardy space $H^p(\mathbb{R}^d)$ for $p < 1$. The following result generalizes one by Stein, Taibleson and Weiss [39] for the standard Bochner-Riesz means. Other generalizations for specific rough ρ were considered in [19] and the references therein.

Let R_t^λ as in (1.3).

Theorem 8.1. *For $0 < p < 1$, $\lambda(p) = d(1/p - 1/2) - 1/2$ we have for all $f \in H^p(\mathbb{R}^d)$*

$$\left\| \sup_{t>0} |R_t^{\lambda(p)} f| \right\|_{L^{p,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{H^p(\mathbb{R}^d)}.$$

We use the same reductions as in §2. Write, for $t > 0$

$$R_t^\lambda f(x) = \mathcal{F}^{-1}[u(\rho(\cdot/t))\widehat{f}](x) + \sum_{j=1}^{\infty} 2^{-j\lambda} T_j f(x, t)$$

where u is as in §2.1 and $\widehat{T_j f}(\xi, t) = \varphi_j(\rho(\xi/t))\widehat{f}(\xi)$ with ϕ_j as in §2.2. This is similar to (2.7) (except that now t ranges over $(0, \infty)$). The functions u , φ_j depend on λ but satisfy uniform estimates as λ is taken over a compact subset of \mathbb{R} . Let

$$\mathcal{M}_0 f(x) = \sup_{t>0} |\mathcal{F}^{-1}[u(\rho(\cdot/t))\widehat{f}](x)|$$

and for $j \geq 1$,

$$\mathcal{M}_j f(x) = \sup_{t>0} |T_j f(x, t)|.$$

We then have

$$(8.1) \quad \sup_{t>0} |\mathcal{R}_t^{\lambda(p)} f(x)| \leq \mathcal{M}_0 f(x) + \sum_{j \geq 1} 2^{-j\lambda(p)} \mathcal{M}_j f(x)$$

and we shall derive a weak type inequality on H^p for the right hand side in (8.1). The ingredients are $H^p \rightarrow L^p$ bounds for the maximal operators \mathcal{M}_0 and \mathcal{M}_j .

Let M be a nonnegative integer. We recall that a function a supported on a ball B is a (p, M) atom if $\|a\|_\infty \leq \text{vol}(B)^{-1/p}$ and $\int a(x)P(x)dx = 0$ for all polynomials of degree at most M . By the atomic decomposition it suffices to check the $H^p \rightarrow L^p$ bounds on (p, M) atoms for every non-negative integer $M > d(p^{-1} - 1) - 1$. The bound for $\mathcal{M}_0 a$ is straightforward:

Lemma 8.2. *Let $M + 1 > d(p^{-1} - 1)$ and let a be a (p, M) -atom. For $0 < p \leq 1$ we have*

$$\|\mathcal{M}_0 a\|_p \lesssim 1$$

Proof. This follows by a variant of the argument in §2.1. Define

$$\mathfrak{M}_{N_1, N_2} f = \sup_{\tau > 0} |\mathcal{F}^{-1}[(1 - \rho(\cdot/\tau)^{N_1})_+^{N_2} \widehat{f}]|.$$

Let N_1, N_2 be large so that \mathfrak{M}_{N_1, N_2} maps H^p to L^p . By the subordination formula (2.1) we have

$$(8.2) \quad \sup_{t > 0} |\mathcal{F}^{-1}[u(\rho(\cdot/t))\widehat{a}](x)| \lesssim |\mathfrak{M}_{N_1, N_2} a(x)| \frac{1}{N_2!} \int_0^\infty s^{N_2} |u_{N_1}^{(N_2+1)}(s)| ds$$

and the integral is finite. Hence we get the desired L^p bound for $\mathcal{M}_0 a$. \square

8.1. *The main $H^p \rightarrow L^p$ bound.*

Proposition 8.3. *Let $0 < p \leq 1$, $j \geq 1$, $\nu \in \mathcal{Z}_j$. Let $M + 1 > d(p^{-1} - 1)$ and let a be a (p, M) -atom. Then*

$$\|\mathcal{M}_j a\|_p \lesssim 2^{j(d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2})}.$$

We further decompose $T_j f(x, t) = \sum_{\nu \in \mathcal{Z}_j} T_{j, \nu} f(x, t)$ where we use the homogeneous partition of unity as in (3.2). Let for $\nu \in \mathcal{Z}_j$,

$$\mathcal{M}_{j, \nu} f(x) = \sup_{t > 0} |T_{j, \nu} f(x, t)|.$$

Then $\mathcal{M}_j f(x) \leq \sum_{\nu \in \mathcal{Z}_j} \mathcal{M}_{j, \nu} f(x)$. Since $\#\mathcal{Z}_j = O(2^{j(d-1)/2})$ we can use the triangle inequality in L^p , $p \leq 1$, to see that the proposition follows from

$$(8.3) \quad \|\mathcal{M}_{j, \nu} a\|_p \lesssim 2^{j \frac{d+1}{2} (\frac{1}{p} - 1)}.$$

We proceed with the proof of (8.3).

By translation and scaling, we may assume that a is supported in the ball B of radius 1 centered at the origin, $\|a\|_\infty \leq 1$ and $\int a(x) P(x) dx = 0$ for all polynomials of degree $\leq M$. By a rotation we may also assume that $\nabla \rho(\xi_{j, \nu})$ is parallel to $(1, 0, \dots, 0)$ and thus writing $x = (x_1, x')$ we have

$$(8.4) \quad |\partial^\alpha K_{j, \nu}(x)| \leq C_{N_1, N_2, \alpha} \frac{2^{-j(d+1)/2}}{(1 + 2^{-j}|x_1|)^{N_1} (1 + 2^{-j/2}|x'|)^{N_2}}$$

for all multiindices $\alpha \in \mathbb{N}_0^d$ and all $N_1, N_2 \geq 0$. See e.g. [11] or [31]. Let

$$\mathcal{D} = \{(x_1, x') \in \mathbb{R}^d : |x_1| \leq 5 \cdot 2^j, |x'| \leq 5 \cdot 2^{j/2}\}.$$

In the following subsections, we estimate the L^p -quasi-norm of $\mathcal{M}_{j, \nu} a(x)$ over \mathcal{D} and \mathcal{D}^c , respectively, using the cancellation condition for the atom when $x \in \mathcal{D}^c$.

8.1.1. *Estimation over \mathcal{D} .* Let

$$\mathcal{D}_0 = \{(x_1, x') \in \mathbb{R}^d : |x_1| \leq 5, |x'| \leq 5\}$$

$$\mathcal{D}_1 = \{(x_1, x') \in \mathbb{R}^d : |x_1| \leq 5 \cdot 2^{j/2}, |x'| \leq 5\}$$

and

$$(8.5) \quad E = \{(x_1, x') \in \mathbb{R}^d : |x'| \geq 2^{-j/2}|x_1|\}.$$

We derive the following pointwise estimates

$$(8.6) \quad \mathcal{M}_{j,\nu}a(x) \lesssim \begin{cases} 1 & \text{if } x \in \mathcal{D}_0, \\ 2^{\epsilon j/2}|x_1|^{-(1+\epsilon)} & \text{if } x \in \mathcal{D}_1 \setminus \mathcal{D}_0, \\ 2^{-j/2}|x'|^{-d} & \text{if } x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E, \\ 2^{j\frac{d-1}{2}}|x_1|^{-d} & \text{if } x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E^{\mathbb{C}}. \end{cases}$$

If we use this for $0 < \epsilon < \frac{1}{p} - 1$ then straightforward integrations give the desired bound

$$(8.7) \quad \|\mathcal{M}_{j,\nu}a\|_{L^p(\mathcal{D})} \lesssim 2^{j\frac{d+1}{2}(\frac{1}{p}-1)}.$$

To verify (8.6) first observe the pointwise bound $\mathcal{M}_{j,\nu}a(x) \leq \sup_{t>0} \|t^d K_{j,\nu}(t \cdot)\|_1 \|a\|_\infty \lesssim 1$. This gives (8.6) for $x \in \mathcal{D}_0$. Secondly for any $x \in \mathcal{D}_1 \setminus \mathcal{D}_0$ and $y \in B_1(0)$, we have $|x_1 - y_1| \gtrsim |x_1|$. Using (8.4) with $N_1 = 1 + \epsilon$ and $N_2 = d - 1 - \epsilon$, we have

$$\begin{aligned} |t^d K_{j,\nu}(t \cdot) * a(x)| &\lesssim t^d 2^{-j(d+1)/2} (2^{-j}t|x_1|)^{-(1+\epsilon)} \int_{|y'|\leq 1} (2^{-j/2}t|x' - y'|)^{-(d-1-\epsilon)} dy' \\ &\lesssim 2^{\epsilon j/2}|x_1|^{-(1+\epsilon)} \end{aligned}$$

for all $x \in \mathcal{D}_1 \setminus \mathcal{D}_0$.

Assume that $x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E$. Then $|x'| \geq 5$ and thus $|x' - y'| \geq c|x'|$ for some $c > 0$ for all $|y'| \leq 1$. Using (8.4) with $N_1 = 0$ and $N_2 = d$, we have

$$|t^d K_{j,\nu}(t \cdot) * a(x)| \lesssim t^d 2^{-j(d+1)/2} (2^{-j/2}t|x'|)^{-d} = 2^{-j/2}|x'|^{-d}.$$

Finally, when $x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E^{\mathbb{C}}$, we have $|x_1 - y_1| \geq c|x_1|$ and necessarily $|x_1| \geq 2^{-j/2}$. If we put $N_1 = d$, $N_2 = 0$ in (8.4) we get

$$|t^d K_{j,\nu}(t \cdot) * a(x)| \lesssim t^d 2^{-j(d+1)/2} (2^{-j}t|x_1|)^{-d} = 2^{j(d-1)/2}|x_1|^{-d}.$$

This concludes the proof of the pointwise estimate (8.6) which implies (8.7).

8.1.2. *Estimation over $\mathcal{D}^{\mathbb{C}}$* . When $x \in \mathcal{D}^{\mathbb{C}}$ we use the cancellation of the atom and Taylor's formula to write

$$\begin{aligned} t^d K_{j,\nu}(t \cdot) * a(x) &= t^d \int \left(K_{j,\nu}(tx - ty) - \sum_{n=1}^M \frac{\langle -ty, \nabla \rangle^n K_{j,\nu}(tx)}{n!} \right) a(y) dy \\ &= \frac{(-1)^{M+1}}{M!} t^{d+M+1} \int_0^1 (1-s)^M \int \langle y, \nabla \rangle^{M+1} K_{j,\nu}(tx - sty) a(y) dy ds. \end{aligned}$$

We now use (8.4) for the derivatives of order $M+1$. Also notice that if E is as in (8.5) we have $|x'| \geq 5 \cdot 2^{j/2}$ for $x \in \mathcal{D}^{\mathbb{C}} \cap E$ and $|x_1| \geq 5 \cdot 2^j$ for $x \in \mathcal{D}^{\mathbb{C}} \cap E^{\mathbb{C}}$. We obtain

$$\mathcal{M}_{j,\nu}a(x) \lesssim \begin{cases} 2^{jM/2}|x'|^{-d-1-M} & \text{if } x \in \mathcal{D}^{\mathbb{C}} \cap E, \\ 2^{j(M+\frac{d+1}{2})}|x_1|^{-d-M-1} & \text{if } x \in \mathcal{D}^{\mathbb{C}} \cap E^{\mathbb{C}}. \end{cases}$$

where for $x \in \mathcal{D}^{\mathbb{C}} \cap E$ we took $N_1 = 0$, $N_2 = d + M + 1$ in (8.4) and for $x \in \mathcal{D}^{\mathbb{C}} \cap E^{\mathbb{C}}$ we took $N_1 = d + M + 1$ and $N_2 = 0$. Hence

$$\|\mathcal{M}_{j,\nu}a\|_{L^p(\mathcal{D}^{\mathbb{C}} \cap E)} \lesssim 2^{jM/2} \left(\int_{|x'| \gtrsim 2^{j/2}} |x'|^{-(d+1+M)p} \int_{|x_1| \lesssim 2^{j/2}|x'|} dx_1 dx' \right)^{1/p}$$

and

$$\|\mathcal{M}_{j,\nu}a\|_{L^p(\mathcal{D}^{\mathbb{C}} \cap E^{\mathbb{C}})} \lesssim 2^{j(M+\frac{d+1}{2})} \left(\int_{|x_1| \gtrsim 2^j} |x_1|^{-(d+1+M)p} \int_{|x'| \lesssim 2^{-j/2}|x_1|} dx' dx_1 \right)^{1/p}.$$

Both integrals are $\lesssim 2^{j\frac{d+1}{2}(\frac{1}{p}-1)}$ provided $p > \frac{d}{d+1+M}$, which is the hypothesis on p and M . This concludes the proof of (8.3). \square

8.2. *Proof of Theorem 8.1, conclusion.* As a crucial ingredient we shall use the generalized triangle inequality for $L^{p,\infty}$, namely

$$(8.8) \quad \left\| \sum_l f_l \right\|_{L^{p,\infty}} \lesssim A_p \left(\sum_l \|f_l\|_{L^{p,\infty}}^p \right)^{1/p}$$

which holds with $A_p = O((1-p)^{-1/p})$ as $p \rightarrow 1-$. See either the paper by Kalton [21] or the paper by Stein-Taibleson-Weiss [39]. By Lemma 8.2 it suffices to prove

$$(8.9) \quad \left\| \sum_{j \geq 1} 2^{-j\lambda(p)} \mathcal{M}_j f \right\|_{L^{p,\infty}} \lesssim \|f\|_{H^p}$$

and by (8.8) and the atomic decomposition we may assume that f is a (p, M) -atom a , with $M+1 > d(p^{-1}-1)$. By dilation and translation invariance we may assume that a is function supported in $\{x : |x| \leq 1\}$ such that $\|a\|_\infty \leq 1$ and such that $\int a(x)P(x)dx = 0$ for all polynomials of degree $\leq M$. Because of this normalization we notice that (up to a harmless constant) the function a is also a (p_1, M) and a (p_0, M) atom where $p_1 < p < p_0 < 1$ and we pick p_1 is sufficiently close to p such that $M+1 > d(p_1^{-1}-1)$.

We need to verify for all $\alpha > 0$

$$(8.10) \quad \text{meas}(\{x : \sum_{j \geq 1} 2^{-j\lambda(p)} \mathcal{M}_j a > \alpha\}) \lesssim \alpha^{-p}.$$

By Proposition 8.3 we have for every $j \geq 1$

$$(8.11) \quad \left\| 2^{-j\lambda(p)} \mathcal{M}_j a \right\|_{p_i} \lesssim 2^{j(\lambda(p_i)-\lambda(p))}.$$

We employ a variant of an interpolation argument in [4] to estimate

$$\text{meas}(\{x : \sum_{j \geq 1} 2^{-j\lambda(p)} \mathcal{M}_j a > \alpha\}) \leq I + II$$

where I is the measure of the set on which $\sum_{2^j \leq \alpha^{-p/d}} 2^{-j\lambda(p)} \mathcal{M}_j a > \alpha/2$ and II is the measure of the set on which $\sum_{2^j > \alpha^{-p/d}} 2^{-j\lambda(p)} \mathcal{M}_j a > \alpha/2$. By Tshebyshev's inequality

$$I \leq (2/\alpha)^{p_1} \left\| \sum_{\substack{j \in \mathbb{N}: \\ 2^j \leq \alpha^{-p/d}} 2^{-j\lambda(p)} \mathcal{M}_j a \right\|_{p_1}^{p_1},$$

$$II \leq (2/\alpha)^{p_0} \left\| \sum_{\substack{j \in \mathbb{N}: \\ 2^j > \alpha^{-p/d}} 2^{-j\lambda(p)} \mathcal{M}_j a \right\|_{p_0}^{p_0}.$$

Apply (8.11) to obtain

$$\begin{aligned} I + II &\lesssim \alpha^{-p_1} \sum_{2^j \leq \alpha^{-p/d}} 2^{j(\lambda(p_1)-\lambda(p))p_1} + \alpha^{-p_0} \sum_{2^j > \alpha^{-p/d}} 2^{j(\lambda(p_0)-\lambda(p))p_0} \\ &= \alpha^{-p_1} \sum_{2^{jd} \leq \alpha^{-p}} 2^{jd(1-\frac{p_1}{p})} + \alpha^{-p_0} \sum_{2^{jd} > \alpha^{-p}} 2^{jd(1-\frac{p_0}{p})} \lesssim \alpha^{-p}. \end{aligned}$$

This yields (8.10) and concludes the proof. \square

Remark. Versions of the Fan-Wu transference argument in §5.1 for maximal functions and h^p for $p < 1$ can be used to prove a theorem for Riesz means of Fourier series analogous to Theorem 8.1.

9. OPEN PROBLEMS

9.1. *Spaces near L^1 .* For $f \in L^1(\mathbb{T}^d)$ it remains open whether the Riesz means $\mathcal{R}_t^{\lambda(p)}f(x)$ converge q -strongly a.e. for any $q < \infty$. In particular can one upgrade in Corollary 6.3 the conclusion of upper density one of $E(f, x)$ to density one?

It may also be interesting to investigate strong convergence a.e. for spaces intermediate between L^1 and $L \log L$.

9.2. *The case $q = p'$.* For $f \in L^p(\mathbb{T}^d)$, $1 < p < 2$, prove or disprove that $\mathcal{R}_t^{\lambda(p)}f(x)$ converges q -strongly a.e. when $q = p'$. For $f \in h^1(\mathbb{T}^d)$, is there a version of Rodin's theorem [30] in one dimension, that applies to Riesz means at the critical index $\lambda(1) = \frac{d-1}{2}$ where the L^q -average norm in t -variable is replaced by a *BMO*-average?

9.3. *Problems involving nonisotropic dilations.* One can ask the same questions for quasi-radial Riesz means when the isotropic dilation group is replaced by a nonisotropic dilation group t^P where P is a matrix with positive eigenvalues and ρ satisfies $\rho(t^P\xi) = t\rho(\xi)$. It turns out that the results depend on the geometry of the surface in relation to the eigenvectors of P . In the case that $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$ has nonvanishing curvature everywhere one has almost everywhere convergence for $\lambda > \frac{d-1}{2}$, but there are other examples where a.e. convergence fails for $\lambda < d/2$, see [22] for details. Even in the case of nonvanishing curvature we have currently no endpoint results for strong convergence of $\mathcal{R}_t^\lambda f$, for the critical $\lambda = \lambda(p)$ when the dilations are nonisotropic.

9.4. *Almost everywhere convergence.* For $1 < p < 2$ the problem of a.e. convergence, and the critical q for strong summability for $\lambda > \lambda(p)$ is wide open. Optimal results for the maximal operators are currently known only for the subspace L_{rad}^p of radial L^p functions, see [16]. For general L^p functions results that improve on Stein's classical theorem for a.e. convergence of Riesz means of index $> (d-1)(1/p - 1/2)$ are currently only known in two dimensions, see Tao's paper [44].

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