TWO PROBLEMS ASSOCIATED WITH CONVEX FINITE TYPE DOMAINS

Alexander Iosevich Eric Sawyer Andreas Seeger

Abstract. We use scaling properties of convex surfaces of finite line type to derive new estimates for two problems arising in harmonic analysis. For Riesz means associated to such surfaces we obtain sharp L^p estimates for $p > 4$, generalizing the Carleson-Sjölin theorem. Moreover we obtain estimates for the remainder term in the lattice point problem associated to convex bodies; these estimates are sharp in some instances involving sufficiently flat boundaries.

1. Introduction

Let Ω be a convex domain in $\mathbb{R}^d$ with smooth boundary. We assume that $\partial\Omega$ is of *finite line type*, that is, at each point each tangent line has finite order of contact.

We discuss two problems in this paper. Both problems have in common that progress can be made using some approximate scaling properties of $\partial\Omega$. We derive an extension of the Carleson-Sjölin theorem concerning L^p convergence results for Riesz means defined by a distance function associated to Ω; we assume that $1 \le p \le 4/3$. We also give asymptotics for the number of integer lattice points inside large dilates of Ω; the bounds for the error terms are sharp in some cases where there exist points with all lines tangent to the boundary having high order of contact with $\partial\Omega$.

1.1. Riesz means. We assume that the origin belongs to the interior of Ω. Let $\rho : \mathbb{R}^d \to [0,\infty)$ be homogeneous of degree 1 be the Minkowski functional associated to Ω; *i.e.* ρ is homogeneous of degree one, so that $\rho(\xi) = 1$ if $\xi \in \partial\Omega$. The boundary $\Sigma_\rho := \partial\Omega$ is then the unit sphere for the generalized distance function ρ. The Bochner-Riesz operator associated to ρ is defined by

$$\widehat{S_{\lambda,\rho} f}(\xi) = (1-\rho(\xi))_+^\lambda \widehat{f}(\xi); \tag{1.1}$$

here our definition of the Fourier transform is $\widehat{f}(\xi) = \int f(y) e^{-i\langle y,\xi\rangle} dy$. It is well known that if $1 \le p < \infty$ the L^p boundedness of the Bochner-Riesz operator implies L^p convergence of the Riesz means $\mathcal{F}^{-1}[(1-\rho/t)_+^\lambda \widehat{f}]$ to the limit f if $f \in L^p$ and $t \to \infty$.

A necessary condition for L^p boundedness is

$$\lambda > \lambda(p) = d|1/p - 1/2| - 1/2 \tag{1.2}$$

Indeed in view of the compact support of the multiplier it is necessary for L^p boundedness that the inverse Fourier transform of $(1-\rho)_+^\lambda$ belongs to L^p. Using standard asymptotic expansions one can show (working near points on Σ_ρ where the curvature does not vanish) that (1.2) is necessary for $\mathcal{F}^{-1}[(1-\rho)_+^\lambda] \in L^p$.

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It is known [9], [29] that the validity of an L^2 restriction theorem for the Fourier transform implies the L^p boundedness of the Bochner-Riesz operator. Since Σ_ρ is of finite type, say $\le n$, it follows from [3] that the Fourier transform of $\widehat{d\sigma}(\xi)$ of a smooth density carried by Σ_ρ is $O(|\xi|^{-\mu})$ for some μ with $\mu \ge (d-1)/n$. Using the appropriate versions of the Stein-Tomas restriction theorem ([10]) one can show that L^p boundedness holds for $1 \le p \le 2(\mu+1)/(\mu+2)$ and $\lambda > \lambda(p)$ (*cf.* [29]). Note that $2(\mu+1)/(\mu+2) = (2n+2d-2)/(2n+d-1)$ for the example $x_d = \sum_{i=1}^{d-1} x_i^n$ with even n, so that the range obtained in this way is small for large n.

Theorem 1.1. *Suppose that $d \ge 2$, $1 \le p \le 4/3$, $\lambda > d(1/p - 1/2) - 1/2$ and that Σ_ρ is of finite line type. Then $S_{\lambda,\rho}$ is bounded on $L^p(\mathbb{R}^d)$.*

It is conjectured that L^p boundedness holds for the same range of exponents as for the sphere. The conjecture for the sphere is that L^p boundedness should hold for $\lambda > \lambda(p)$ for $p < 2d/(d+1)$. This is currently known only in two dimensions, see Carleson and Sjölin [4]. Sjölin [28] extended this result to arbitrary planar domains with smooth boundary, for some variants concerning convex domains in the plane with nonsmooth boundary see also the more recent paper by Ziesler and the third author [27]. For partial results in higher dimensions, in the case that the Gauß curvature of Σ_ρ does not vanish, we refer to Bourgain [1] and for background to [29]. Our proof of Theorem 1.1 uses a variant of Córdoba's geometrical proof [6] of the Carleson-Sjölin theorem and rescaling.

1.2 Multitype and an estimate for the Fourier transform of surface carried measure.

A precise estimate of the Fourier transforms of surface carried measure is due to Bruna, Nagel and Wainger [3]. Let $\Sigma = \partial\Omega$ and $H_P(\Sigma)$ the affine tangent plane at $P \in \Sigma$, and let

$$B(P,\delta) = \{y \in \Sigma : \operatorname{dist}(y, H_P(\Sigma)) < \delta\}. \tag{1.3}$$

Then

$$|\widehat{d\sigma}(\xi)| \le C\big[|B(P_+, |\xi|^{-1})| + |B(P_-, |\xi|^{-1})|\big] \tag{1.4}$$

where $P_\pm$ are the points on Σ for which ξ is a normal vector and $|B|$ denotes the surface measure of B. For many problems it is important to know not just the size of the balls but also the distribution function of $x \mapsto |B(x,\delta)|$ and how it relates to the notions of multitype and type. We review the definition of multitype which is implicit in Schulz [26], see also [17].

Consider a smooth real valued function Φ defined in a neighborhood of the origin in a $d-1$-dimensional Euclidean vector space $\mathbb{E}^{d-1}$ so that $\Phi(0) = \nabla\Phi(0) = 0$. We say that a vector v in $\mathbb{E}^{d-1}$ has contact of order at least $n+1$ if

$$\Phi(sv) = O(s^{n+1}) \quad \text{if} \quad s \to 0.$$

The sets

$$S^n = \{v \in \mathbb{E}^n : \ v \text{ has contact of order at least } n+1\} \tag{1.5}$$

are linear subspaces of $\mathbb{E}^{d-1}$ and there are even integers $m_1, \dots, m_k$ so that $m_1 < \cdots < m_k$, $1 \le k \le d-1$ and $m_0 := m_1 - 1 \ge 1$ and

$$0 = S^{m_k} \subsetneq \cdots \subsetneq S^{m_0} := \mathbb{E}^{d-1}; \tag{1.6}$$

moreover the sequence is maximal, in the sense that $S^n = S^{m_k}$ if $m_{k-1} < n \le m_k$. Define

$$a_i = m_j \quad \text{if } d-1-\dim S^{m_{j-1}} < i \le d-1-\dim S^{m_j},\ j = 1,\dots,k. \tag{1.7}$$

The $d-1$-tuple $\mathfrak{a} = (a_1,\dots,a_{d-1})$ is then called the multitype of Φ at 0.

To illustrate the above definitions consider a convex body whose boundary passes through the origin and nearby is given by the equation $x_d = \sum_{i=1}^{d-1} |x_i|^{a_i}$ where the a_i are even integers, with $a_i \le a_{i+1}$, $1 \le i \le d-2$. In this case the multitype is $(a_1,\dots,a_{d-1})$ and the subspaces S^m above are $S^m = \mathrm{span}(\{e_i : a_i > m\})$ (and $S^m = \{0\}$ if $m \ge a_{d-1}$.)

We now fix $P \in \Sigma$, choose a unit normal n_P and parametrize Σ near P as a graph over its tangent plane at P. Thus the parametrization is given by

$$v = \Gamma(v) \mapsto P + v + \Phi(v) n_P \tag{1.8}$$

for $v \in T_P\Sigma$, and Φ is a convex function vanishing of second order at the origin. We perform the above construction for $\Phi(v)$ defined on $\mathbb{E}^{d-1} = T_P\Sigma$ and obtain a flag of subspaces

$$0 = S_P^{m_k} \subsetneq \cdots \subsetneq S_P^{m_0} = T_P\Sigma. \tag{1.9}$$

Let W_j be the orthogonal complement of $S_P^{m_j}$ in $S_P^{m_{j-1}}$, $j = 1,\dots,k$, then

$$T_P\Sigma = W_1 \oplus \cdots \oplus W_k \tag{1.10}$$

We denote by Π_j^P the orthonormal projection on $T_P\Sigma$ to W_j. We also have a similar decomposition and projections Π_j^P to W_j^* on $T_P^*\Sigma$, here we let W_j^* the space of linear functionals on W_j extended by 0 on the orthogonal complement of W_j. We can extend these projections to linear maps on $T_P^*\mathbb{R}^d \simeq (\mathbb{R}^d)^*$ by defining $\Pi_j^P n_P = 0$.

On $T_P^*\Sigma$ we define a nonisotropic distance function ρ_* by

$$\rho_*(\eta) = \sum_{j=1}^{k} |\Pi_j^P \eta|^{\frac{m_j}{m_j - 1}}; \tag{1.11}$$

here $|\cdot|$ denotes the Euclidean distance in W_j. If $\xi \in T_P^*\mathbb{R}^d$ is taken from a suitable conic neighborhood of n_P and Π^P denotes the projection to $T_P^*\Sigma$ we define

$$\Theta_P(\xi) = \rho_*\Big(\frac{\Pi^P \xi}{\langle \xi, n_P\rangle}\Big). \tag{1.12}$$

Finally we set for $l \le d-2$

$$\nu_l(P) = \sum_{i=l}^{d-1} a_i^{-1} = \sum_{j=1}^{k} \frac{\dim S_P^{m_{j-1}} - \dim S_P^{m_j}}{m_j} \tag{1.13}$$

and write $\nu(P) \equiv \nu_1(P)$. An alternative description of $\nu(P)$ (see [16]) is

$$\nu(P) = \sup\{q : \mathrm{dist}(\cdot, H_P\Sigma) \in L^q(\Sigma)\}; \tag{1.14}$$

in fact for $q = \nu(P)$ the function $\mathrm{dist}(\cdot, H_P\Sigma)^{-1}$ belongs to the space $L^{q,\infty}(\Sigma)$.

Our result for the Fourier transform of surface carried measure is

Proposition 1.2. *Let $P \in \partial\Omega$. Then there is a neighborhood U of P and a conic neighborhood V of $\{\pm n_P\}$ in $\mathbb{R}^d$ so that for all $\chi \in C_0^\infty(U)$ and all $\xi \in V$ with $|\xi| \geq 1$ we have*

$$|\widehat{\chi d\sigma}(\xi)| \lesssim \|\chi\|_{C^N} \min\{|\xi|^{-\nu}, |\xi|^{-\frac{1}{2}-\nu_2}[\Theta_P(\xi)]^{\nu-\nu_2-\frac{1}{2}}\};$$

here $\|\chi\|_{C^N} = \max_{\alpha \leq N} \|\chi^{(\alpha)}\|_{L^\infty(U)}$ and N is sufficiently large.

In this statement $N > d + m_k$ will suffice. Note the proposition is an improvement over previous results only in the case where all the principal curvatures vanish (and thus $a_1 > 2$).

1.3. A lattice point estimate.

Let

$$\mathcal{N}_\Omega(t) = \operatorname{card}(t\Omega \cap \mathbb{Z}^d). \tag{1.15}$$

It is well known (and elementary) that $\mathcal{N}_\Omega(t)$ is asymptotic to $t^d \operatorname{vol}(\Omega)$ as $t \to \infty$ and that the error term

$$E_\Omega(t) = \mathcal{N}(t) - t^d \operatorname{vol}(\Omega) \tag{1.16}$$

as $O(t^{d-1})$. Moreover if $\partial\Omega$ has suitable curvature properties then the error term improves; in particular if the Fourier transform of the surface measure on the boundary satisfies $\widehat{d\sigma}(\xi) = O(|\xi|^{-\alpha})$ then the classical method (see *e.g.* [11], [13, Theorem 7.7.16] and [24]) yields $E_\Omega(t) = O(t^{d-1-\frac{\alpha}{d-\alpha}})$. This estimate however is not sharp, and several authors beginning with van der Corput have obtained improvements for the case of nonvanishing Gauß curvature; see the monographs by Krätzel [18] and Huxley [14], and in particular the papers by Krätzel and Nowak [20] and recent improvements by W. Müller [22] for results on general convex bodies with nonvanishing curvature in higher dimensions. In [24, I], [25] Randol obtained better estimates for the case of convex domains in the plane with finite type boundary; these are sharp for $\Omega = \{x : x_1^k + x_2^k \leq 1\}$ where $k \geq 4$ is even. See also Nowak [23] for more refined results. Generalizations to domains of the form $\Omega = \{x : x_1^k + \ldots + x_d^k \leq 1\}$ are in [24, II], [19].

Here we give a version for general convex bodies with finite type boundary in higher dimensions. Let $\nu(P) = \nu_1(P)$ and $\nu_2(P)$ as in (1.13) above.

Theorem 1.3. *Let*

$$\nu = \min_{P \in \partial\Omega} \nu(P), \qquad \mu = \frac{1}{2} + \min_{P \in \partial\Omega} \nu_2(P).$$

Then there is a constant C depending on Ω so that

$$|E_\Omega(t)| \leq C_\Omega(1 + t^{d-1-\nu} + t^{d-1-\frac{\mu}{d-\mu}}). \tag{1.17}$$

Specifically, if Γ is the set of all points $P \in \partial\Omega$ at which all principal curvatures vanish then

$$E_\Omega(t) = \sum_{P \in \Gamma} t^{d-1-\nu(P)} G_P(t) + O(t^{d-1-\frac{\mu}{d-\mu}}) \tag{1.18}$$

where $G_P(t)$ is bounded as $t \to \infty$. If the normal line determined by n_P coincides with $\mathbb{R}e_i$ for some $i \in \{1, \ldots, d\}$ then $\limsup_{t\to\infty} |G_P(t)| > 0$.

We note that the number $\mu/(d-\mu)$ is greater then $(2d-1)^{-1}$ since $\mu > 1/2$. In particular if the Gauß curvature only vanishes at one point at the surface and if $\nu < \mu/(d-\mu)$ then there

is $A \in SO(d)$ so that $\limsup_{t\to\infty} t^{\nu-d+1}|E_{A\Omega}(t)|$ is positive (for other model cases compare [19], [23]). Note that the sum in (1.18) over $P \in \Gamma$, since Γ is a discrete subset of $\partial\Omega$ (as noted in [16], *cf.* the proof of Lemma 2.2. below). We remark that it is well known that for almost all rotations $A \in SO(d)$ the error terms $E_{A\Omega}(t)$ improve, see [5], [31], [32], [23], [15] and [2].

We shall derive the estimate for the Fourier transform in Proposition 1.2 in the next section. §3 contains the application to the lattice point problem. In §4 and §5 we prove results on Bochner-Riesz multipliers; here we first consider the case of one nonvanishing principal curvature and then in §5 the case of convex domains.

Notation: Given two quantities A, B we write $A \lesssim B$ if there is an absolute positive constant C so that $A \le CB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

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2. An estimate for Fourier transforms of surface carried measures

We begin by reviewing some facts about classes of convex functions in [3], [26], [16], [17].

Let $B_T \subset \mathbb{R}^n$ denote the open ball of radius T centered at 0; it is always assumed that $T \le 1$.

Fix a flag $\mathfrak{V}$ of subspaces $0 = \mathbb{V}_k \subsetneq \cdots \subsetneq \mathbb{V}_0$ of $\mathbb{E}^{d-1}$, with $\mathbb{V}_0 = \mathbb{E}^{d-1}$, and let $m = (m_1, \dots, m_k)$ be a k-tuple of even positive integers with $m_1 < \cdots < m_k$. For $0 < b \le M$, $N \in \mathbb{Z}^+$, $N > m_k$, let $\mathfrak{S}_T^{d-1}(b, M, \mathfrak{V}, m, N)$ be the class of all $C_N(\overline{B_T})$ functions g with the property that

$$\begin{aligned}
&g(0) = \nabla g(0) = 0\\
&\frac{d^2}{(dt)^2} g(x+t\theta)\big|_{t=0} \ge 0 \text{ for all } \theta \in S^{d-2}, x \in B_T\\
&\max_{2\le j\le m_l} \Big|\big(\frac{d}{dt}\big)^j g(x+t\theta)\big|_{t=0}\Big| \ge b \text{ for all } \theta \in S^{d-2} \cap \mathbb{V}_{l-1}, x \in B_T\\
&\max_{|\alpha|\le N} \Big|\big(\frac{\partial}{\partial x}\big)^\alpha g(x)\Big| \le M \text{ for all } x \in B_T.
\end{aligned} \tag{2.1}$$

Here S^{d-2} denotes the unit sphere in $\mathbb{E}^d$. We also define $\mathfrak{a}(\mathfrak{V}, m) = (a_1(\mathfrak{V}, m), \dots, a_l(\mathfrak{V}, m))$ by

$$a_i(\mathfrak{V}, m) = m_j(\mathfrak{V}, m) \quad \text{if } d-1-\dim \mathbb{V}_{j-1} < i \le d-1-\dim \mathbb{V}_j, \tag{2.2}$$

in analogy to (1.7).

Now if $P \in \Sigma$ (with $\Sigma = \partial\Omega$ as in the introduction) and $\mathbb{E}^{d-1} = T_P\Sigma$ then let $\mathbb{V}_j = S_P^{m_j} \subset T_P\Sigma$ as in (1.5). Let Φ be as in (1.8). Then there is $T > 0$ and a neigborhood $\mathcal{U}$ of 0 so that for all $w \in \mathcal{U}$ the functions $y \mapsto \Phi(w+y) - \Psi(w) - \langle y, \nabla_w \Phi(w)\rangle$ belong to $\mathfrak{S}_T^n(b, M, \mathfrak{V}, m, N)$; moreover there are positive constants c_0, C_0, C_1 so that

$$\mathcal{B}(w, \delta) = \{y : |\Phi(y) - \Phi(w) - \langle \nabla_w \Phi(w), y - w\rangle| \le \delta\} \tag{2.3}$$

belongs to B_T if $\delta \le c_0 T^{m_k}$ and satisfies

$$\operatorname{meas}(\mathcal{B}(w, \delta)) \le C\delta^\nu; \tag{2.4}$$

see Proposition 2.1 in [17].

Lemma 2.1. *Suppose that $\Phi \in \mathfrak{S}_T^{d-1}(b, M, \mathfrak{V}, m, N)$ and suppose that $\mathfrak{a} = (a_1, \dots, a_{d-1})$ is the multitype at the origin. Let $\Psi^w(y) = \Phi(y) - \Phi(w) - \langle \nabla_w \Phi(w), y - w\rangle$ and let $\mathfrak{a}(w) = (a_1(w), \dots, a_{d-1}(w))$ be the multitype of Ψ^w at the origin. Then there is a neighborhood $\mathcal{U}$ of the origin so that $a_i(w) \le a_i$ for $i = 1, \dots, d-1$ and all $w \in \mathcal{U}$.*

Proof. Let S^{m_i} be as in (1.5) and let $\ell > \dim S^{m_i}$. Recall that $S^n = S^{m_{j-1}}$ for $m_j < n \le m_{j-1}$. Using continuity and compactness arguments together with the definition of the spaces S^{m_i} we see that there is a neighborhood $\mathcal{U} \subset \widetilde{\mathcal{U}}$ of the origin so that for every $w \in \widetilde{\mathcal{U}}$, every $y \in \mathcal{U}$ and every ℓ-tuple of orthonormal vectors $\{u_1, \dots, u_\ell\}$

$$\sum_{i=1}^{\ell} \sum_{s \le m_j} \left| (\langle u_i, \nabla_y \rangle)^s \Psi^w(y) \right| \ge b_0 > 0. \tag{2.5}$$

The result of the Lemma follows quickly from the definition of the multitype. $\square$

We now let Σ denote the graph of Φ. On $T_0\Sigma = \mathbb{R}^{d-1}$ we define a nonisotropic distance function ρ by

$$\rho(y) = \sum_{j=1}^{k} |\Pi_j y|^{m_j}; \tag{2.6}$$

note that that the unit ball for ρ^* in (1.11) is the polar set for the unit ball for ρ.

The following Lemma gives an improvement of estimates in [16] and [17]. A rescaling argument is used as in those papers; the present improvement is obtained using a more careful argument for the rescaled pieces.

Lemma 2.2. *Let Φ be a convex smooth function defined in a neighborhood of the origin in $\mathbb{R}^{d-1}$, so that $\Phi(0) = \nabla\Phi(0) = 0$. Let $\mathfrak{V}$ be the flag of subspaces $\{S^{m_j}\}$ defined as in (1.5). Let $\mathfrak{a}$ be the multitype of Φ near 0, $\mathcal{B}(w, \delta)$ as in (2.3) and ρ as in (2.6). Let $\nu = \sum_{i=1}^{d-1} a_i^{-1}$, $\nu_2 = \sum_{i=2}^{d-1} a_i^{-1}$.*

Then there is a neighborhood $\mathcal{U}$ of the origin and $\delta_0 > 0$ so that for all $0 < \delta \le \delta_0$ and all $w \in \mathcal{U}$

$$\mathrm{meas}(\mathcal{B}(w, \delta)) \le C\delta^\alpha [\rho(w)]^{\nu - \alpha}, \quad \nu \le \alpha \le \frac{1}{2} + \nu_2.$$

Proof. We may assume that $a_1 > 2$ since otherwise the theorem follows already from the estimate (2.4). Let $\{u_1, \dots, u_{d-1}\}$ an orthonormal basis of $\mathbb{R}^{d-1}$ so that

$$S^{m_j} = \mathrm{span}\{u_i,\, d - 1 - \dim S^{m_j} < i \le d - 1\} \tag{2.7}$$

for $j = 0, \dots, k-1$. By performing a rotation we may assume that the u_i are the standard coordinate vectors.

Define dilations A_t by

$$A_t x = (t^{\frac{1}{a_1}} x, \dots, t^{\frac{1}{a_{d-1}}} x). \tag{2.8}$$

According to [26], [16] we may split

$$\Phi(x) = Q(x) + R(x)$$

where Q is a convex polynomial satisfying

$$Q(A_t x) = tQ(x) \tag{2.9}$$

and

$$0 < |Q(x)| \le C_1|x||\nabla Q(x)| \le C_2|x|^2 \sum_{i,j} \Big|\frac{\partial^2 Q}{\partial x_i \partial x_j}(x)\Big|. \tag{2.10}$$

and the remainder term R satisfies

$$\Big|s^{-1}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\big(R(A_s x)\big)\Big| \lesssim s^{1/m} \tag{2.11}$$

for $|x| \le T$ and all multiindices $\alpha = (\alpha_1, \dots, \alpha_{d-1})$ with $|\alpha| \le N$. Since Q is positive away from the origin and homogeneous with respect to dilations (A_t) we have that

$$Q(y) \approx \rho(y)$$

where ρ is as in (2.6); in fact $\rho(y) \approx \sum_{i=1}^{d-1} |\langle y, u_i\rangle|^{a_i}$.

Set $\Phi_\ell(y) = 2^\ell \Phi(A_{2^{-\ell}} y)$ and note that $\Phi_\ell(y) = Q(y) + R_\ell(y)$ where R_ℓ and its derivatives tend to zero uniformly on compact sets, as $\ell \to \infty$.

Denote by $\mathfrak{a}(w) = (a_1(w), \dots, a_{d-1}(w))$ the multitype of Q at w. Then $\mathfrak{a}(0) = \mathfrak{a}$ and by Lemma 2.1 there is $M > 0$ so that $a_i(w) \le a_i$ for $0 \le \rho(w) \le 2^{-M+2}$ and, by (2.10/11), $a_1(w) = 2$ for $0 < \rho(w) \le 2^{-M+2}$; note that nothing is said about the position of the spaces $S^m(w)$. Now for any point w there is an open ball $U(w)$ of radius $T(w)/4$ and a flag $\mathfrak{V}(w)$ consisting of $l(w)$ nested subspaces and an $l(w)$-tuple $m(w)$ so that for $x \in U(w)$ the functions

$$h \mapsto Q^x(h) = Q(x+h) - Q(x) - \langle \nabla Q(x), h\rangle$$

belong to a class $\mathfrak{S}^{d-1}_{T(w)}(b(w), M(w), \mathfrak{V}(w), m(w), N)$ so that $a_i(\mathfrak{V}(w), m(w)) \ge a_i$ and $a_1(\mathfrak{V}(w), m(w)) = 2$.

By the metric property of the nonisotropic balls $\mathcal{B}(w, \delta)$ there are constants $C_2 \gg C_1 \gg 1$ and $\delta_1 \ll 1$ so that

$$\mathcal{B}(y, \delta) \subset \{x : C_1^{-1}\rho(y) \le \rho(x) \le C_1\rho(y)\} \quad \text{if } \rho(y) \ge C_2\delta; \tag{2.12}$$

we may assume that $C_1 \ge 2^{2M+4}$.

We shall now show that there are constants $c_0 > 0$, $C_0 > 1$ so that for $2^{-\ell} \le c_0$

$$|\mathcal{B}(y, \delta)| \le \delta^\alpha 2^{\ell(\alpha-\nu)} \text{ if } 2^{-l-M} \le \rho(y) \le 2^{-l-M+1},\ \delta \le C_0^{-1}2^{-M-\ell},\ 0 \le \alpha \le \nu_2 + \frac{1}{2}. \tag{2.13}$$

Let

$$W = \{y : C_1^{-2}2^{-M-2} \le \rho(y) \le C_1^{-1}2^{-M+2}\} \tag{2.14}$$

which because of $C_1 \ge 2^{2M+4}$ is contained in the open ball of radius 2^{-M} centered at the origin. We may cover the compact annulus W by finitely many open balls U_i with center $w_i \in W$ and radius $T(w_i)/4$ so that $Q^x \in \mathfrak{S}^{d-1}_{T(w_i)}(b(w_i), M(w_i), \mathfrak{V}(w_i), m(w_i), N)$ provided that $|x - w_i| \le T(w_i)/2$.

Since Φ_ℓ converges to Q in the C^N-topology uniformly on compact sets. There is a positive constant c_0 so that for $2^{-\ell} \le c_0$ the functions

$$h \mapsto \Phi_\ell(x+h) - \Phi_\ell(x) - \langle \nabla \Phi_\ell(x), h\rangle \tag{2.15}$$

belong to $\mathfrak{S}^{d-1}_{T(w_i)}(\frac{b(w_i)}{2}, 2M(w_i), \mathfrak{V}(w_i), m(w_i), N)$ if $|x-w_i| \le T(w_i)/2$. By the finite type property there is a $\delta_0 > 0$ so that for $\gamma \le \delta_0$ and $x \in U_i$ the caps

$$W_\ell(x,\gamma) \subset \{z : |\Phi_\ell(z) - \Phi_\ell(x) - \langle \nabla \Phi_\ell(x), z-x\rangle| \le \gamma\}$$

are contained in the double of U_i; moreover we have

$$|W_\ell(x,\gamma)| \le C\gamma^{\frac12+\nu_2}, \quad \gamma \le \delta_0, \tag{2.16}$$

by the analogue of (2.4) with exponent $1/2+\nu_2$; here C is independent of ℓ.

Now in order to show that (2.13) holds we assume that $C_1^{-1}2^{-l-M} \le \rho(y) \le C_1^{-1}2^{-l-M+1}$ and observe that the image of $\mathcal{B}(y,\delta)$ under the linear transformation A_{2^ℓ} is $W_\ell(A_{2^\ell}y, 2^\ell\delta)$ which is contained in W, in fact in a U_i if $2^\ell\delta \le \delta_0$. Since $\det A_{2^\ell} = 2^{\ell\nu}$ we have $|\mathcal{B}(y,\delta)| \lesssim 2^{-\ell\nu}|W_\ell(A_{2^\ell}y, 2^\ell\delta)|$ and (2.13) follows.

Finally if $\delta \le C_0^{-1}2^{-M-\ell}$ we use $|\mathcal{B}(y,\delta)| = O(\delta^\nu)$ instead and observe that in this range $\delta^\alpha 2^{\ell(\alpha-\nu)} \lesssim \delta^\nu$, provided that $\alpha \ge \nu$. This together with (2.13) proves the asserted statement. □

Lemma 2.3. *Let Φ, $\mathfrak{V}$, $\mathfrak{a}$, $\mathcal{U}$ be as in Lemma 2.2, $N > d + a_{d-1}$. For $\xi \in \mathbb{R}^d$ define*

$$F(\xi) = \int \chi(y) e^{-\imath(\langle \xi', y\rangle + \xi_d \Phi(y))} dy.$$

There is a neighborhood $\widetilde{\mathcal{U}} \subset \mathcal{U}$ of the origin and a conic neighborhood $\mathcal{V}$ of e_d so that for $\xi \in \mathcal{V}$

$$|F(\xi)| \le C\|\chi\|_{C^N}|\xi|^{-\alpha}\Big(\sum_j \Big[\frac{\Pi_j \xi}{|\xi_d|}\Big]^{\frac{m_j}{m_j-1}}\Big)^{\nu-\alpha}, \quad \nu \le \alpha \le \frac12 + \nu_2, \tag{2.17}$$

Proof. We may assume that (2.7) holds and that the u_i's form the standard basis in $\mathbb{R}^{d-1}$. Observe that then

$$\sum_j |\Pi_j \eta|^{\frac{m_j}{m_j-1}} \approx \sum_{i=1}^{d-1} |\xi_i|^{a_i'}$$

with $a_i' = a_i/(a_i-1)$.

Assume that $s/2 \le \rho(x) \le 2s$ and s is small. Then $|A_{1/s}x| \approx 1$ and $|Q_{x_i}(A_{1/s}x)| \le C$. But $Q_{x_i}(A_{1/s}x) = s^{-1+1/a_i}Q_{x_i}(x)$ so that $|Q_{x_i}(x)| \lesssim s^{1-\frac{1}{a_i}}$. Similarly by (2.11) the remainder term R_{x_i} satisfies the same estimate so that

$$|\Phi_{x_i}(x)| \lesssim \Big(\sum_{k=1}^{d-1} |x_k|^{a_k}\Big)^{1-\frac{1}{a_i}}$$

for small x and therefore

$$\sum_{i=1}^{d-1} |\Phi_{x_i}(x)|^{a_i'} \lesssim \sum_{k=1}^{d-1} |x_k|^{a_k}.$$

Now let $x(\xi)$ be the unique point at which ξ is normal to the graph of Φ. By the Bruna-Nagel-Wainger estimate for the Fourier transform (1.4) and Lemma 2.2 we have that

$$|F(\xi)| \lesssim |\xi|^{-\alpha}\rho(x(\xi))^{\nu-\alpha}$$

and since $x(\xi)$ is determined by $\xi_i/\xi_d = \pm\Phi_{x_i}(x(\xi))$ for $i = 1, \dots, d-1$, the estimate (2.17) follows. □

3. Lattice point estimates

In this section we prove Theorem 1.3. We use a variant of the classical proof (see Randol [24] for the two-dimensional case). Choose $\zeta \in C_0^\infty(\mathbb{R}^d)$ so that ζ is nonnegative, $\zeta(x) = 0$ if $|x| \geq 1$ and $\int \zeta(x)dx = 1$. Define $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(\varepsilon^{-1}x)$. We work with the ε-regularization $\chi_\Omega * \zeta_\varepsilon$ of the characteristic function of Ω and define

$$\mathcal{N}_\varepsilon(t) = \sum_{k\in\mathbb{Z}^d} \chi_{t\Omega} * \zeta_{\varepsilon t}(k).$$

By the Poisson summation formula

$$\begin{aligned}\mathcal{N}_\varepsilon(t) &= \sum_{k\in\mathbb{Z}^d} t^d \widehat{\chi_\Omega}(2\pi tk)\widehat{\zeta}(2\pi\varepsilon tk)\\ &= t^d \operatorname{vol}(\Omega) + \mathcal{R}_\varepsilon(t)\end{aligned} \tag{3.1}$$

where

$$\mathcal{R}_\varepsilon(t) = \sum_{k\neq 0} t^d \widehat{\chi_\Omega}(2\pi tk)\widehat{\zeta}(2\pi\varepsilon tk).$$

By the divergence theorem

$$\widehat{\chi_\Omega}(\xi) = \int_\Omega e^{-\imath\langle x,\xi\rangle}dx = \imath\sum_{i=1}^d \frac{\xi_i}{|\xi|^2}F_i(\xi) \tag{3.2}$$

where

$$F_i(\xi) = \int_\Sigma n_i(y)e^{-\imath\langle y,\xi\rangle}d\sigma(y) \tag{3.3}$$

and n_i denotes the i^{th} component of the outer normal vector n_P.

Let Γ be the set of points $P \in \Sigma$ at which all principal curvatures vanish. As noticed in [16] it follows from (2.10/11) that the set Γ is discrete, thus finite by compactness. For every $P \in \Gamma$ we choose a narrow conic symmetric neighborhood V_P of the normals $\{\pm n_P\}$, a small neighborhood U_P of P in Σ and a C_0^∞ function χ_P whose restriction to Σ vanishes off $\mathcal{U}$ and so that χ_P equals one in a neighborhood of P. We may arrange these neighborhoods so that the sets $\overline{V}_P \cap \{\xi : |\xi| \geq 1\}$, $P \in \Gamma$ are pairwise disjoint and that the normals to all points in a neighborhood of $\overline{U}_P$ are contained in V_P (thus the $\overline{U}_P$'s are disjoint too).

Define

$$F_{i,P}(\xi) = \int_\Sigma \chi_P(y)n_i(y)e^{-\imath\langle y,\xi\rangle}d\sigma(y)$$

If the cones V_P are chosen sufficiently narrow, we have

$$F_{i,P}(\xi) \lesssim \begin{cases} \min\{|\xi|^{-\nu(P)},\, \xi^{-(\frac{1}{2}+\nu_2(P))}\}\Theta_P(\frac{\Pi^P\xi}{\langle n_P,\xi\rangle}) & \text{if } \xi \in V_P \\ C_N|\xi|^{-N} & \text{if } \xi \notin V_P. \end{cases} \tag{3.4}$$

The estimate for $\xi \in V_P$ follows from Proposition 1.2, and the estimate for $\xi \notin V_P$ follows by a simple integration by parts; namely if $t \mapsto \gamma(t)$ parametrizes Σ near P then $|\langle\gamma'(t),\xi\rangle| \approx |\xi|$ for $\gamma(t) \in U_P$ and $\xi \notin V_P$.

Moreover by the Bruna-Nagel-Wainger estimate we have

$$|F_i(\xi) - \sum_{P\in\Gamma} F_{i,P}(\xi)| \lesssim |\xi|^{-\mu}, \qquad \mu = \frac{1}{2} + \inf_{P\in\Sigma} \nu_2(P) \tag{3.5}$$

here we used the definition of Γ and the fact that χ_P equals one near P.

We now estimate the remainder term $R_\varepsilon(t)$ where $\varepsilon \ll 1/t$ will be suitably chosen. Let dist_∞ denote the distance taken with respect to the ℓ^∞ metric in $\mathbb{R}^d$, or $\mathbb{Z}^d$. For $P \in \Gamma$ let

$$\begin{aligned}
\mathcal{A}_P &= \{k \in V_P \cap \mathbb{Z}^d : k \neq 0, \mathrm{dist}_\infty(k, \mathbb{R}n_P) \leq 3/4\}\\
\mathcal{B}_P &= \{k \in V_P \cap \mathbb{Z}^d : k \neq 0, \mathrm{dist}_\infty(k, \mathbb{R}n_P) > 3/4\}\\
\mathcal{C} &= \{k \in \mathbb{Z}^d : k \neq 0, k \notin \cup_{P\in\Gamma} V_P\}.
\end{aligned}$$

Let

$$\begin{aligned}
\mathfrak{A}^i_P(t) &= \sum_{k\in\mathcal{A}_P} t^d \widehat{\zeta}(2\pi\varepsilon tk)\frac{2\pi k_i}{|2\pi k|^2} F_{i,P}(2\pi tk)\\
\mathfrak{B}^i_P(t) &= \sum_{k\in\mathcal{B}_P} t^d \widehat{\zeta}(2\pi\varepsilon tk)\frac{2\pi k_i}{|2\pi k|^2} F_{i,P}(2\pi tk)\\
\mathfrak{C}^i_P(t) &= \sum_{k\in\mathcal{C}} t^d \widehat{\zeta}(2\pi\varepsilon tk)\frac{2\pi k_i}{|2\pi k|^2} F_{i,P}(2\pi tk)\\
\mathfrak{D}^i(t) &= \sum_{k\neq 0} t^d \widehat{\zeta}(2\pi\varepsilon tk)\frac{2\pi k_i}{|2\pi k|^2} (F_i(2\pi tk) - \sum_{P\in\Gamma} F_{i,P}(2\pi tk))
\end{aligned}$$

then

$$\mathcal{R}_\varepsilon(t) = \sum_{i=1}^d (\mathfrak{D}^i(t) + \sum_{P\in\Gamma} (\mathfrak{A}^i_P(t) + \mathfrak{B}^i_P(t) + \mathfrak{C}^i_P(t))). \tag{3.6}$$

When evaluating $\mathfrak{A}^i_P$ we essentially sum over integers in a tubular neighborhood of a line and by the estimate (2.4) we certainly get

$$|\mathfrak{A}^i_P(t)| \lesssim \sum_{k\in\mathcal{A}_P} t^d |tk|^{-1-\nu} \lesssim t^{d-1-\nu}. \tag{3.7}$$

Next for the estimation of $\mathfrak{D}^i_P$ we use the rapid decay estimate in (3.4) to obtain

$$|\mathfrak{D}^i_P(t)| \lesssim \sum_{k\neq 0} t^d |tk|^{-N} \lesssim t^{d-N} \tag{3.8}$$

and for $\mathfrak{C}^i_P$ we use (3.5) which yields

$$|\mathfrak{C}^i_P(t)| \lesssim C_N \sum_{k\neq 0} t^d (1+|\varepsilon tk|)^{-N}(1+|tk|)^{-\mu-1} \lesssim \varepsilon^{\mu+1-d} \tag{3.9}$$

Finally

$$|\mathfrak{B}^i_P(t)| \lesssim \sum_{\substack{k\neq 0\\ k\in V_P}} t^d |tk|^{-\frac{3}{2}-\nu_2(P)} \Theta_P\big(\tfrac{\Pi^{P_k}}{\langle k, n_P\rangle}\big)(1+|\varepsilon tk|)^{-N}$$

and we claim that for $\lambda \geq 1$

$$\sum_{\substack{|k|\approx\lambda \\ k\in V_P}} t^d |tk|^{-\frac{3}{2}-\nu_2(P)} \Theta_P\big(\tfrac{\Pi^P k}{\langle k, n_P\rangle}\big)(1+|\varepsilon t k|)^{-N} \lesssim \lambda^{d-\frac{3}{2}-\nu_2(P)} \min\{1, (\lambda\varepsilon t)^{-N}\} \tag{3.10}$$

which implies

$$|\mathfrak{B}^i_P(t)| \lesssim \varepsilon^{\frac{3}{2}+\nu_2(P)-d} \lesssim \varepsilon^{-(d-1-\mu)}. \tag{3.11}$$

We verify (3.10). Let $\mathfrak{a} = \mathfrak{a}(P)$ be the multitype at P. In view of $\operatorname{dist}(k, \mathbb{R}n_P) \geq 3/4$ it is straightforward to check that

$$\Theta_P\big(\tfrac{\Pi^P k}{\langle k, n_P\rangle}\big) \approx \Theta_P\big(\tfrac{\Pi^P \xi}{\langle \xi, n_P\rangle}\big) \quad \text{if } |\xi - k|_\infty \leq 1/2,\ k \in \mathcal{B}_P.$$

Thus we may replace the sum in (3.10) by an integral. After performing a suitable rotation in this integral we have to show that

$$\int_{|\xi_d|\approx\lambda} \int_{|\xi'|\leq\lambda} |\xi|^{-3/2-\nu_2(P)} \Big(\sum_{i=1}^{d-1} \frac{|\xi_i|^{a_i'}}{|\xi|^{a_i'}}\Big)^{\nu-\nu_2(P)-\frac{1}{2}} d\xi' d\xi_d \lesssim \lambda^{d-\frac{3}{2}-\nu_2(P)}. \tag{3.12}$$

Now $(\sum_{i=1}^{d-1}(|\xi_i|/|\xi|)^{a_i'})^{\nu-\nu_2-\frac{1}{2}} \lesssim (|\xi_1|/|\lambda|)^{a_1'(1/a_1-1/2)}$ with $a_1'(1/a_1 - 1/2) > -1$, and therefore the integral in (3.12) is bounded by

$$\lambda^{d-1-3/2-\nu_2(P)} \int_{|\xi_1|\leq\lambda} (|\xi_1|/|\lambda|)^{a_1'(1/a_1-1/2)} d\xi_1 \lesssim \lambda^{d-3/2-\nu_2(P)}.$$

This shows (3.10).

To finish the proof we note that

$$N_\varepsilon(t(1-C\varepsilon)) \leq \mathcal{N}_\Omega(t) \leq N_\varepsilon(t(1+C\varepsilon))$$

where C is a constant depending only on the geometry of Ω. Thus, by taking into account the leading term in (3.1) we see that

$$E_\Omega(t) \lesssim (t^{d-1-\nu} + t^d\varepsilon + \varepsilon^{-(d-1-\mu)})$$

and the desired estimate follows if we choose $\varepsilon = t^{-d/(d-\mu)}$. This completes the proof of (1.17).

Lower bounds. To show (1.18) we work with our choice $\varepsilon = \varepsilon(t) = t^{-d/(d-\mu)}$. For (1.18) we simply set

$$G_P(t) = \sum_{i=1}^{d} t^{\nu(P)-d-1} \mathfrak{A}^i_P(t)$$

which we already showed to be bounded above. However we have to verify the claim that $\limsup_{t\in\infty} |G_P(t)| > 0$ in the case where $n_P = \pm e_i$.

We now assume that $n_P = e_i$ (the case $n_P = -e_i$ is handled in the same way). Then define

$$G_P(t) = t^{\nu(P)+1-d} \sum_{\kappa\in\mathbb{Z}\setminus\{0\}} (2\pi)^{-1} t^d \widehat{\zeta}(2\pi t^{-\mu/(d-\mu)}\kappa e_i) \operatorname{sign}(\kappa) |\kappa|^{-1} F_{i,P}(2\pi t \kappa e_i).$$

We split this sum into parts $G_P(t) = I(t) + II(t)$ where

$$I_P(t) = (2\pi)^{-1} t^{\nu(P)+1} \sum_{\kappa\in\mathbb{Z}\setminus\{0\}} \operatorname{sign}(\kappa)|\kappa|^{-1} F_{i,P}(2\pi t\kappa e_i)$$
$$II_P(t) = (2\pi)^{-1} t^{\nu(P)+1} \sum_{\kappa\in\mathbb{Z}\setminus\{0\}} (1-\widehat{\zeta}(2\pi t^{-\mu/(d-\mu)}\kappa e_i))\operatorname{sign}(\kappa)|\kappa|^{-1} F_{i,P}(2\pi t\kappa e_i).$$

For the estimation of II we note that $|(1-\widehat{\zeta}(2\pi t^{-\mu/(d-\mu)}\kappa e_i))| \lesssim \min\{1, t^{-\mu/(d-\mu)\kappa}\}$ with and since $F_{i,P}(2\pi t\kappa e_i) = O((t\kappa)^{-\nu}$ we get the estimate

$$|II(t)| \lesssim t^{-\frac{\mu}{d-\mu}}.$$

To examine $I(t)$ we parametrize by our assumption on $n_P = e_i$

$$F_{i,P}(2\pi t\kappa e_i) = e^{-\imath\kappa\langle P,e_i\rangle}\int_{y'\in\mathbb{R}^{d-1}} \chi_0(y')(1+|\nabla\Phi(y')|^2)^{1/2} e^{\imath\kappa\Phi(y')}dy'$$

where $\Phi \equiv \Phi^P$ is convex, vanishes of second order at the origin of $\mathbb{R}^{d-1}$ and has multitype $\mathfrak{a}(P)$ there; χ_0 is smooth, compactly supported and equal to one in a neighborhood of the origin. By the convexity $\langle P, n_P\rangle = \langle P, e_i\rangle \neq 0$. To examine the integral we may use an asymptotic expansion derived in Schulz [26] (stated there for $\kappa\to\infty$, but the statement for $\kappa\to-\infty$ follows similarly). We obtain

$$F_{i,P}(2\pi t\kappa e_i) = e^{-2\pi\imath t\kappa\langle P,e_i\rangle}\kappa^{-\nu}c_0(P)e^{\frac{\pi i}{2\nu}\operatorname{sign}(\kappa)} + O(\kappa^{-\nu-\eta})$$

where $c_0(P) > 0$ and η is the reciprocal of the least common multiple of $a_1,\dots,a_n$. Thus

$$I(t) = c_0(P)\pi^{-1}\sum_{\kappa>0}|\kappa|^{-\nu-1}\sin\big(2\pi\kappa t\langle P,e_i\rangle - \pi/(2\nu)\big) + O(\kappa^{-\nu-1-\eta}).$$

The sum defines a periodic function which is not identically zero, by the uniqueness theorem for Fourier series. Combining this with the estimation for the error term $II(t)$ we see that $\limsup_{t\to\infty}|G_P(t)| > 0$.

Remark. For almost all rotations the estimates for the error term improves. There is $r > 2$ so that

$$|E_{A\Omega}(t)| \le \mathcal{C}(A)t^{d-1-\frac{d-1}{d+1}}\log^{1/r}(2+t)$$

(indeed $\mathcal{C}$ is in $L^q(SO(d))$ for $q < r$. As in [2] this is proved using a result on the maximal function

$$M(\theta) = \sup_{r>0} r^{(d+1)/2}|\widehat{\chi_\Omega}(r\theta)|$$

which was shown by Svensson [30] to be in $L^{q_0}(S^{d-1})$ for some $q_0 > 2$ (under our assumption of finite line type, see also Randol [25] for a similar result with additional real analyticity assumption). Indeed, let $\mathcal{R}_{\varepsilon,A}(t) = \sum_{k\neq0}\chi_\Omega(2\pi tAk)\widehat{\zeta}(2\pi\varepsilon tk)$ and

$$\mathcal{M}_j(A) = \sup_{2^j\le t\le 2^{j+1}}|\mathcal{R}_{\varepsilon_j,A}(t)|, \qquad \text{with } \varepsilon_j = 2^{-2jd/(d+1)}$$

then for $q \le q_0$

$$\begin{aligned}\|\mathcal{M}_j\|_{L^q(SO(d))} &\le 2^{jd}\sum_{k\neq 0}(1+|\varepsilon_j 2^j|k|)^{-N}(2^j|k|)^{-(d+1)/2}\Big(\int |M(A\tfrac{k}{|k|})|^q dA\Big)^{1/q}\\ &\lesssim 2^{j(d-1-\frac{d-1}{d+1})}\|M\|_{L^q(S^{d-1})}\end{aligned}$$

by the (standard) choice of ε_j. But

$$|E_{A\Omega}(t)|t^{-(d-1-\frac{d-1}{d+1})}\log^{-1/r}(2+t) \lesssim 1+\Big(\sum_{j>0}|\mathcal{M}_j(A)2^{-j(d-1-\frac{d-1}{d+1})}(1+j)^{-1/r}|^q\Big)^{1/q}$$

which is in $L^q(SO(d))$ for $r < q_0$.

We remark that the methods in W. Müller's paper [22] could be used to improve the above bound to $|E_{A\Omega}(t)| \le \mathcal{C}(A)t^{d-1-\frac{d-1}{d+1}-\beta}$ where $\beta = \beta(\Omega) > 0$ and $\mathcal{C}$ is finite almost everywhere.

4. Bochner-Riesz multipliers - the case of one nonvanishing principal curvature

In this section we shall prove a general theorem concerning multipliers of Bochner-Riesz type associated to surfaces with at least one nonvanishing principal curvature. Then, in the subsequent section, we shall deduce Theorem 1.1 by rescaling arguments.

In what follows M_p will be the space of Fourier multipliers on $L^p(\mathbb{R}^d)$; $\|m\|_{M_p}$ is the operator norm of the operator T_m defined by $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$.

We split variables in $\mathbb{R}^d$ as $\xi = (\widetilde{\xi}, \xi_d)$ and in the statement of the Proposition we further split $\widetilde{\xi} = (\xi_1, \xi') \in \mathbb{R}\times\mathbb{R}^{d-2}$. The proof of the following result uses the ideas from the two-dimensional case, see [9], [6].

Proposition 4.1. *Let $\varepsilon > 0$, $N \ge d+1+2/\varepsilon$ and let $g \in C^N(\mathbb{R}^{d-1})$. Suppose that there is a cube U centered at the origin and $a > 0$ so that*

$$\frac{\partial^2 g}{\partial \xi_1^2}(\xi_1, \xi') \ge a$$

in U. Let χ be supported in U and let ϕ be a smooth function supported in $(1/2, 2)$. Let $0 < \delta \ll 1$ and

$$m_\delta(\xi) = \chi(\xi)\phi(\delta^{-1}(\xi_d - g(\xi_1, \xi'))).$$

Then

$$\|m_\delta\|_{M_4} \le C_\varepsilon \delta^{-\frac{d-2}{4}-\varepsilon},$$

where C_ε depends only on a, ε, U, the $C^N(U)$ norms of the functions g, χ and the C^{d+1} norm of ϕ.

Proof. We may assume that U is the unit cube, and that the support of χ has small diameter. We decompose $m_\delta = \sum_k m_{\delta,k}$ where $k = (k_2, \dots, k_{d-1})$ ranges over $(d-2)$-tuples of integers $k_i \le C\delta^{-1/2}$ and

$$m_{\delta,k}(\xi) = m_\delta(\xi)\prod_{i=2}^{d-1}\psi(\delta^{-1/2}\xi_i - k_i)$$

for suitable $\psi \in C_0^\infty$ satisfying $\sum_{n=-\infty}^{\infty} \psi(s-n) = 1$, so that supp $\psi \subset [-1,1]$. Let $\widetilde{\psi} \in C_0^\infty([-2,2])$ so that $\widetilde{\psi}$ is equal to 1 on the support of ψ.

Denote by T_k the convolution operator with Fourier multiplier $m_{\delta,k}$ and by R_k the convolution operator with Fourier multiplier $\widetilde{\psi}(\delta^{-1/2}\xi' - k)$. Note that $\|R_k\|_{L^p \to L^p} \le C$, $1 \le p \le \infty$. Then for $2 \le p \le \infty$

$$\Big\| \sum_k R_k g_k \Big\|_p \lesssim \Big(\sum_k \|g_k\|_p^{p'} \Big)^{1/p'}$$

which follows for $p = \infty$ from Minkowski's inequality and for $p = 2$ by orthogonality; for $2 < p < \infty$ one uses interpolation. Since $T_k = R_k T_k R_k$ it follows that

$$\Big\| \sum_k T_k \Big\|_{L^4 \to L^4} \le C\delta^{-(d-2)/4} \sup_k \|T_k\|_{L^4 \to L^4}$$

and therefore it suffices to show that

$$\|T_k\|_{L^4 \to L^4} \lesssim \delta^{-\varepsilon}. \tag{4.1}$$

The estimate (4.1) is proved using arguments in Córdoba [6] which we will sketch. For $\nu \in \mathbb{Z}$ we define operators $T_{k,\nu}$ and S_ν by $\widehat{S_\nu f}(\xi) = \widetilde{\psi}(\delta^{-1/2}\xi_1 - \nu)$ and $\widehat{T_{k,\nu} f}(\xi) = \psi(\delta^{-1/2}\xi_1 - \nu)\widehat{T_k f}(\xi)$. Then $T_k = \sum_\nu T_{k,\nu} S_\nu f$ where the sum is extended over integers ν with $|\nu| \ll \delta^{-1/2}$ since we assume that the support of χ is small.

Now

$$\begin{aligned} \Big\| \sum_\nu T_{k,\nu} S_\nu f \Big\|_4^2 &= \Big\| \sum_{\nu,\nu'} (T_{k,\nu} S_\nu f)(T_{k,\nu'} S_{\nu'} f) \Big\|_2 \\ &\le \sum_{\ell : 2^\ell \delta^{1/2} \ll 1} \Big\| \sum_{\substack{(\nu,\nu'): \\ |\nu - \nu'| \approx 2^\ell}} (T_{k,\nu} S_\nu f)(T_{k,\nu'} S_{\nu'} f) \Big\|_2 \end{aligned} \tag{4.2}$$

It can be checked that the family of functions $(T_{k,\nu} S_\nu f)(T_{k,\nu'} S_{\nu'} f)$ has an orthogonality property which implies that

$$\Big\| \sum_{\substack{(\nu,\nu') \\ |\nu - \nu'| \approx 2^\ell}} (T_{k,\nu} S_\nu f)(T_{k,\nu'} S_{\nu'} f) \Big\|_2 \lesssim \Big\| \Big(\sum_\nu |T_{k,\nu} S_\nu f|^2 \Big)^{1/2} \Big\|_4^2. \tag{4.3}$$

The proof of (4.3) is based on an idea of C. Fefferman [9]; in higher dimensions one uses the following

Lemma 4.2. *Suppose that $a' \in \mathbb{R}^{d-2}$, $|a'| \ll 1$, and the vectors $\widetilde{\xi}$, $\widetilde{\eta}$, $\widetilde{\zeta}$, $\widetilde{\omega}$ satisfy*

(i) $\xi + \eta - \widetilde{\zeta} - \widetilde{\omega} = 0$,

(ii) $\xi_1 > \zeta_1 > 0$, $\eta_1 < \omega_1 < 0$,

(iii) $|\widetilde{\xi}|, |\widetilde{\eta}|, |\widetilde{\zeta}|, |\widetilde{\omega}| \in [2^{\ell-1}\delta^{1/2}, 2^{\ell+1}\delta^{1/2}]$,

(iv) ξ', η', ζ' *and* ω' *belong to the cube of sidelength* $4\delta^{1/2}$ *centered at* a'.

Then

$$g(\widetilde{\xi}) + g(\widetilde{\eta}) - g(\widetilde{\zeta}) - g(\widetilde{\omega}) \ge c2^\ell \delta^{1/2} \big(|\xi_1 - \zeta_1| + |\eta_1 - \omega_1| \big) \tag{4.4}$$

In (4.4), c depends only on the lower bound of $g_{\xi_1\xi_1}$ and the C^4 norm of g in supp χ.

Sketch of Proof. A Taylor expansion about the origin yields

$$g(\widetilde{\xi}) + g(\widetilde{\eta}) - g(\widetilde{\zeta}) - g(\widetilde{\omega}) = I + II + III + IV$$

where

$$\begin{aligned} I &= \frac{1}{2} g_{\xi_1\xi_1}(0)(\xi_1^2 + \eta_1^2 - \zeta_1^2 - \omega_1^2) \\ II &= \frac{1}{2}\big(\xi_1 \langle g_{\xi_1\xi'}(0), \xi'\rangle + \eta_1 \langle g_{\xi_1\xi'}(0), \eta'\rangle - \zeta_1 \langle g_{\xi_1\xi'}(0), \zeta'\rangle - \omega_1 \langle g_{\xi_1\xi'}(0), \omega'\rangle\big) \\ III &= \frac{1}{2}\big(\langle \xi', g_{\xi'\xi'}(0)\xi'\rangle + \langle \eta', g_{\xi'\xi'}(0)\eta'\rangle - \langle \zeta', g_{\xi'\xi'}(0)\zeta'\rangle - \langle \omega', g_{\xi'\xi'}(0)\omega'\rangle\big) \\ IV &= r(\widetilde{\xi}) + r(\widetilde{\eta}) - r(\widetilde{\zeta}) - r(\widetilde{\omega}) \end{aligned}$$

where r vanishes of third order at the origin. (4.4) is proved by verifying

$$\begin{aligned} I &\approx 2^\ell \delta^{1/2}(|\xi_1 - \zeta_1| + |\eta_1 - \omega_1|) \\ II &\le C 2^\ell \delta \\ III &\le C\delta \\ IV &\le C 2^{2\ell}\delta(|\xi_1 - \zeta_1| + |\eta_1 - \omega_1|). \end{aligned}$$

The straightforward calculation is omitted; we note that formula (6.30) in [21] turns out to be useful in order to carry it out. □

Proof of Proposition 2.1, cont. By (4.3) it remains to show that

$$\Big\| \Big(\sum_\nu |T_{k,\nu} S_\nu f|^2 \Big)^{1/2} \Big\|_4 \lesssim \delta^{-\varepsilon} \|f\|_4. \tag{4.5}$$

Let $\Gamma_k(t) = (-\nabla_{\widetilde{\xi}} g(t, \delta^{1/2}k), 1)$ which gives a one parameter family of vectors normal to Σ_ρ.

For $\sigma \ge 2$ let $\mathcal{R}_{k,\sigma}$ be the set of all cylinders whose base is a $d-2$ dimensional ball of radius s and whose height is σs (any $s > 0$), so that the axis is parallel to $\Gamma_k(t)$ for some $|t| \le 1$.

Define the maximal function

$$M_{k,\sigma} f(x) = \sup_{\substack{x \in R \\ R \in \mathcal{R}_{k,\sigma}}} \frac{1}{|R|} \int_R |f(y)| dy.$$

Then arguing as in [6] and using standard estimates for the kernel of $T_{k,\nu}$ we see that

$$\int \sum_\nu |T_{k,\nu} S_\nu f(x)|^2 w(x) dx \lesssim \int \sum_\nu |S_\nu f(x)|^2 M_{k,\delta^{-1/2}} w(x) dx.$$

The L^p norm of $(\sum_\nu |S_\nu f|^2)^{1/2}$ is bounded by the L^p norm of f, for $p \ge 2$ (see [6]) and therefore we can finish our proof by using duality and showing that

$$\|M_{k,\sigma} f\|_2 \le C_\varepsilon \sigma^\varepsilon \|f\|_2 \tag{4.6}$$

uniformly in k.

If we knew that for every ξ the function $t \mapsto \langle \xi, \Gamma_k(t)\rangle$ changed sign at most M times then it would follow from a result by Córdoba [7] that (4.6) holds with σ^ε replaced by $C_1 M[\log\sigma]^{C_2}$. This hypothesis may not be satisfied, but we can get around this point by a simple approximation. Namely devide $[-1,1]$ into $\sigma^{\varepsilon/2}$ intervals $[a_j, b_j]$ of lengths $\sigma^{-\varepsilon/2}$. Let $P_{k,j}(t)$ be the vector valued Taylor polynomial of degree $[2/\varepsilon]$ of $\nabla_{\widetilde{\xi}} g(\cdot, \delta^{1/2}k)$ expanded about a_j, and let $\Gamma_{k,j}(t) = (-P_{k,j}(1), 1)$. Then $|\Gamma_k(t) - \Gamma_{k,j}(t)| \le C\sigma^{-1}$ for $t \in [a_j, b_j]$.

Let $\mathcal{R}_{k,\sigma,j}$ be the set of all cylinders whose base is a $d-2$-dimensional ball of radius s whose height is σs, so that the axis is parallel to $\Gamma_{k,j}(t)$ for some $|t| \le 1$. If $M_{k,\sigma,j}$ denotes the associated maximal operator then it is immediate that $M_{k,\sigma} f \le \sum_j M_{k,\sigma,j} f$ where the sum contains only $O(\sigma^{\varepsilon/2})$ terms. Córdoba's result yields the L^2 bound $C_\varepsilon[\log\sigma]^{C_2}$ for each $M_{k,\sigma,j}$. This finishes the proof of (4.6). $\square$

5. Proof of Theorem 1.1

The L^1 version of the theorem is well known, and therefore by an interpolation argument one has to show the boundedness on $L^{4/3}(\mathbb{R}^d)$, or, equivalently, on $L^4(\mathbb{R}^d)$.

We split $(1-\rho(\xi))_+^\lambda = h_0(\rho(\xi)) + h_1(\rho(\xi))$ where h_0 is supported in $\{t : t \le 1-\epsilon_0\}$ for suitable small ϵ_0 and h_1 is supported in $\{t : t > 1 - 2\epsilon_0\}$. Then $h_0(\rho(\xi))$ is a Fourier multiplier in M_1; the mild singularity at the origin can be handled e.g. by an averaging argument in [8, p. 248], replacing ρ by ρ^N for large N.

Let $\xi^0 \in \Sigma_\rho$. It suffices to show that there exists a neighborhood V of ξ^0 (in $\mathbb{R}^d$) so that $h_1(\rho(\xi))\widetilde{\chi}$ is a multiplier on $\mathbb{R}^d$ for $\lambda > (d-2)/4$ if $\widetilde{\chi} \in C^\infty$ and supported in V. The multiplier norm is invariant under rotations and we may assume that Σ_ρ can be parametrized as a graph $\xi_d = G(\widetilde{\xi})$, $\widetilde{\xi} \in \mathbb{R}^{d-1}$ near ξ^0, so that $\rho(\xi) < 1$ if $\xi_d > G(\widetilde{\xi})$. We write

$$\chi(\xi) h_1(\rho(\xi)) = \chi(\xi) H(\xi) (\xi_d - G(\widetilde{\xi}))_+^\lambda \quad \text{where} \quad H(\xi) = \Big(\frac{1-\rho(\xi)}{\xi_d - G(\widetilde{\xi})}\Big)^\lambda.$$

A Taylor expansion of ρ about $\xi_d = G(\widetilde{\xi})$ shows that H is smooth on supp χ; therefore by the algebra property of M_p it suffices to show that $\widetilde{\chi}(\xi)(\xi_d - G(\xi_1, \xi'))_+^\lambda$ belongs to M_4 if supp $\widetilde{\chi}$ is sufficiently close to ξ^0.

Let $\mathfrak{a} = (a_1, \dots, a_{d-1})$ be the multitype of Σ_ρ at ξ^0, in the sense of §1.2. By an affine transformation we may assume that $\xi^0 = 0$, $G(0) = \nabla G(0) = 0$ and that $G = Q + R$ where Q and R are as in the proof of Lemma 2.2: The function Q is mixed homogeneous of degree $(a_1, \dots, a_{d-1})$, i.e. if $A_s(\widetilde{\xi}) = (s^{\frac{1}{a_1}}\xi_1, \dots, s^{\frac{1}{a_{d-1}}}\xi_{d-1})$ then Q satisfies $Q(A_s(\widetilde{\xi})) = sQ(\widetilde{\xi})$. The remainder term R satisfies $\Big|s^{-1}\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\big(R(A_s\widetilde{\xi})\big)\Big| \le C_{M,N} s^{1/m}$ for small x and s and all multiindices $\alpha = (\alpha_1, \dots, \alpha_{d-1})$ with $|\alpha| \le N$. In particular $|R(\widetilde{\xi})| \le Q(\widetilde{\xi})/10$ if $Q(\widetilde{\xi}) \le 2^{-r_0+2}$ for suitably large r_0.

Next we set $R_r(\widetilde{\xi}) = 2^r R(A_{2^{-r}}\widetilde{\xi})$, so that $G_r = Q + R_r$ tends to G in the C^∞ topology, as $r \to \infty$. Since the Hessian of Q has rank 1 where $1/4 < Q(\widetilde{\xi}) \le 4$ (see (2.10)) the same is true for $G_r = Q + R_r$ if r is large; we may assume that the matrix norm of $(Q+R_r)''$ is bounded below uniformly in r if $r \ge r_0$.

Let ϕ_1 be supported in $(1/2,2)$ such that $\sum_{k\geq 0}\phi_1(2^k s)=1$ for $0<s\leq 1$. Then we have to show a bound for the M_4 norm of

$$\kappa_j(\xi)=\widetilde{\chi}(\xi)\phi_1(2^j(\xi_d-G(\xi_1,\xi')))(\xi_d-G(\xi_1,\xi'))_+^\lambda.$$

Here we may assume that $\widetilde{\chi}(\xi)=0$ when $Q(\widetilde{\xi})\geq 2^{-r_0}$.

We now perform a further decomposition in terms of $G(\widetilde{\xi})$. Let $\eta\in C_0^\infty(\mathbb{R})$ so that $\eta(s)=1$ if $|s|\leq 1/2$ and $\eta(s)=0$ if $|s|\geq 1$; also let $\eta_0=\eta$ and for integer $r>0$ let $\eta_r(s)=\eta(2^{-r}s)-\eta(2^{-r+1}s)$. Let

$$\kappa_{j,n}(\xi)=\kappa_j(\xi)\eta_n(2^jG(\widetilde{\xi}))$$

so that $\kappa_{j,n}$ is supported where $|\xi_d-G(\widetilde{\xi})|\approx 2^{-j}$ and $G(\widetilde{\xi})\approx 2^{n-j}$ if $n\geq 0$ and $G(\widetilde{\xi})\lesssim 2^{-j}$ if $n=0$. Using the assumption on the support of the cutoff function $\widetilde{\chi}$ we see that $\kappa_{j,n}=0$ for $j\leq n+r_0$.

For the pieces $\kappa_{j,n}$ we employ a scaling argument (for a similar argument in two dimensions see [12]). For the scaling we use the dilations $\xi\mapsto(A_{2^{n-j}}(\widetilde{\xi}),2^{n-j}\xi_d)$. Define for $n>0$

$$\widetilde{\kappa}_{j,n}(\widetilde{\xi},\xi_d)=\phi_1(2^n(\xi_d-G_{j-n}(\xi_1,\xi')))(\xi_d-G_{j-n}(\xi_1,\xi'))_+^\lambda\eta_1(G_{j-n}(\widetilde{\xi}));$$

for $n=0$ we use the same formula but with η_1 replaced by $\eta=\eta_0$. Then

$$\kappa_{j,n}(A_{2^{n-j}}\widetilde{\xi},2^{n-j}\xi_d)=2^{(n-j)\lambda}\widetilde{\chi}(A_{2^{n-j}}\widetilde{\xi},2^{n-j}\xi_d)\widetilde{\kappa}_{j,n}(\widetilde{\xi},\xi_d)$$

so that

$$\|\kappa_{j,n}\|_{M_p}\lesssim 2^{(n-j)\lambda}\|\widetilde{\kappa}_{j,n}\|_{M_p}$$

It is now easy to see that the C^4 norm of $\widetilde{\kappa}_{j,0}$ is $\lesssim 2^{-j\lambda}$ and $\widetilde{\kappa}_{j,0}$ is supported in a fixed ball with diameter independent of j.

Therefore

$$\|\widetilde{\kappa}_{j,0}\|_{M_p}\lesssim 2^{-j\lambda},\qquad 1\leq p\leq\infty.$$

Note that for $j-n\geq r_0$ the multipliers $\widetilde{\kappa}_{j,n}$ are supported where $1/4<Q(\widetilde{\xi})<4$, and by construction the matrix norm of G''_{j-n} is in this region bounded above and below, for $j-n\geq r_0$. We may apply Proposition 4.1 (with $\delta=2^{-n}$), to see that for $0<n\leq j-r_0$

$$\|\widetilde{\kappa}_{j,n}\|_{M_4}\lesssim 2^{(n-j)\lambda}2^{-n(\lambda-\frac{d-2}{4})}$$

and the assertion of Theorem 1.1 follows by summing over $0<n\leq j-r_0$, $j>0$. $\square$

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A. Iosevich, Mathematics Department, University of Missouri, Columbia, MO 65211, USA

E-mail address: iosevich@wolff.math.missouri.edu

E. Sawyer, Department of Mathematics and Statistics, 1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada

E-mail address: sawyer@mcmaster.ca

A. Seeger, Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA

E-mail address: seeger@math.wisc.edu