

# SHARP $L^p$ -BOUNDS FOR THE WAVE EQUATION ON GROUPS OF HEISENBERG TYPE

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ABSTRACT. Consider the wave equation associated with the Kohn Laplacian on groups of Heisenberg type. We construct parametrices using oscillatory integral representations and use them to prove sharp  $L^p$  and Hardy space regularity results.

## INTRODUCTION

Given a second order differential operator  $L$  on a suitable manifold we consider the Cauchy problem for the associated wave equation

$$(1) \quad (\partial_\tau^2 + L)u = 0, \quad u|_{\tau=0} = f, \quad \partial_\tau u|_{\tau=0} = g.$$

This paper is a contribution to the problem of  $L^p$  bounds of the solutions at fixed time  $\tau$ , in terms of  $L^p$  Sobolev norms of the initial data  $f$  and  $g$ . This problem is well understood if  $L$  is the standard Laplacian  $-\Delta$  (i.e. defined as a positive operator) in  $\mathbb{R}^d$  (Miyachi [18], Peral [28]), or the Laplace-Beltrami operator on a compact manifold ([30]) of dimension  $d$ . In this case (1) is a strictly hyperbolic problem and reduces to estimates for Fourier integral operators associated to a local canonical graph. The known sharp regularity results in this case say that if  $\gamma(p) = (d-1)|1/p - 1/2|$  and the initial data  $f$  and  $g$  belong to the  $L^p$ -Sobolev spaces  $L^p_{\gamma(p)}$  and  $L^p_{\gamma(p)-1}$ , resp., then the solution  $u(\cdot, \tau)$  at fixed time  $\tau$  (say  $\tau = \pm 1$ ) belongs to  $L^p$ .

In the absence of strict hyperbolicity, the classical Fourier integral operator techniques do not seem available anymore and it is not even clear how to efficiently construct parametrices for the solutions; consequently the  $L^p$  regularity problem is largely open. However some considerable progress has been made for the specific case of an invariant operator on the Heisenberg group  $\mathbb{H}_m$  which is often considered as a model case for more general situations. Recall that coordinates on  $\mathbb{H}_m$  are given by  $(z, u)$ , with  $z = x + iy \in \mathbb{C}^m$ ,  $u \in \mathbb{R}$ , and the group law is given by  $(z, u) \cdot (z', u') = (z + z', u + u' - \frac{1}{2} \operatorname{Im}(z \cdot \overline{z}'))$ . A basis of left invariant vector fields is given by  $X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial u}$ ,  $Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial u}$ ,

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and we consider the Kohn Laplacian

$$L = - \sum_{j=1}^m (X_j^2 + Y_j^2).$$

This operator is perhaps the simplest example of a nonelliptic sum of squares operator in the sense of Hörmander [10]. In view of the Heisenberg group structure it is natural to analyze the corresponding wave group using tools from noncommutative Fourier analysis. The operator  $L$  is essentially self-adjoint on  $C_0^\infty(G)$  (this follows from the methods used in [27]) and the solution of (1) can be expressed using the spectral theorem in terms of functional calculus; it is given by

$$u(\cdot, \tau) = \cos(\tau\sqrt{L})f + \frac{\sin(\tau\sqrt{L})}{\sqrt{L}}g.$$

We are then aiming to prove estimates of the form

$$(2) \quad \|u(\cdot, \tau)\|_p \lesssim \|(I + \tau^2 L)^{\frac{\gamma}{2}} f\|_p + \|\tau(I + \tau^2 L)^{\frac{\gamma}{2}-1} g\|_p.$$

involving versions of  $L^p$ -Sobolev spaces defined by the subelliptic operator  $L$ . Alternatively, one can consider equivalent uniform  $L^p \rightarrow L^p$  bounds for operators  $a(\tau\sqrt{L})e^{\pm i\tau\sqrt{L}}$  where  $a$  is a standard (constant coefficient) symbol of order  $-\gamma$ . Note that it suffices to prove those bounds for times  $\tau = \pm 1$ , after a scaling using the automorphic dilations  $(z, u) \mapsto (rz, r^2u)$ ,  $r > 0$ .

A first study about the solutions to (1) has been undertaken by Nachman [26] who showed that the wave operator on  $\mathbb{H}_m$  has a fundamental solution whose singularities lie on the cone  $\Gamma$  formed by the characteristics through the origin. He showed that the singularity set  $\Gamma$  has a far more complicated structure for  $\mathbb{H}_m$  than the corresponding cone in the Euclidean case. The fundamental solution is given by a series involving Laguerre polynomials and Nachman was able to examine the asymptotic behavior as one approaches a generic singular point on  $\Gamma$ . However his method does not seem to yield uniform estimates in a neighborhood of the singular set which are crucial for obtaining  $L^p$ -Sobolev estimates for solutions to (1).

In [25] the first author and E. Stein were able to derive nearly sharp  $L^1$  estimates (and by interpolation also  $L^p$  estimates, leaving open the interesting endpoint bounds). Their approach relied on explicit calculations using Gelfand transforms for the algebra of radial  $L^1$  functions on the Heisenberg group, and the geometry of the singular support remained hidden in this approach. Later, Greiner, Holcman and Kannai [7] used contour integrals and an explicit formula for the heat kernel on the Heisenberg group to derive an integral formula for the fundamental solution of the wave equation on  $\mathbb{H}^m$  which exhibits the singularities of the wave kernel. We shall follow a somewhat different approach, which allows us to link the geometrical picture to a decomposition of the joint spectrum of  $L$  and the operator  $U$  of differentiation in the central direction (see also Strichartz [33]); this linkage is crucial to prove optimal  $L^p$  regularity estimates.

In order to derive parametrices we will use a subordination argument based on stationary phase calculations to write the wave operator as an integral involving Schrödinger operators for which explicit formulas are available ([6], [12]). This will yield some type of oscillatory integral representation of the kernels, as in the theory of Fourier integral operators which will be amenable to proving  $L^p$  estimates. Unlike in the classical theory of Fourier integral operators ([11]) our phase functions are not smooth everywhere and have substantial singularities; this leads to considerable complications. Finally, an important point in our proof is the identification of a suitable Hardy space for the problem, so that  $L^p$  bounds can be proved by interpolation of  $L^2$  and Hardy space estimates. We then obtain the following sharp  $L^p$  regularity result which is a direct analogue of the result by Peral [28] and Miyachi [18] on the wave equation in the Euclidean setting.

**Theorem.** *Let  $d = 2m + 1$ ,  $1 < p < \infty$ , and  $\gamma \geq (d - 1)|1/p - 1/2|$ . Then the operators  $(I + \tau^2 L)^{-\gamma/2} \exp(\pm i\tau\sqrt{L})$  extend to bounded operators on  $L^p(\mathbb{H}^m)$ . The solutions  $u$  to the initial value problem (1) satisfy the Sobolev type inequalities (2).*

Throughout the paper we shall in fact consider the more general situation of *groups of Heisenberg type*, introduced by Kaplan [13]. These include groups with center of dimension  $> 1$ . The extension of the above result for the wave operator to groups of Heisenberg type and further results will be formulated in the next section.

## 1. THE RESULTS FOR GROUPS OF HEISENBERG TYPE

1.1. *Groups of Heisenberg type.* Let  $d_1, d_2$  be positive integers, with  $d_1$  even, and consider a Lie algebra  $\mathfrak{g}$  of Heisenberg type, where  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , with  $\dim \mathfrak{g}_1 = d_1$  and  $\dim \mathfrak{g}_2 = d_2$ , and

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_2 \subset \mathfrak{z}(\mathfrak{g}) ,$$

$\mathfrak{z}(\mathfrak{g})$  being the center of  $\mathfrak{g}$ . Now  $\mathfrak{g}$  is endowed with an inner product  $\langle \cdot, \cdot \rangle$  such that  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are orthogonal subspaces and, and if we define for  $\mu \in \mathfrak{g}_2^* \setminus \{0\}$  the symplectic form  $\omega_\mu$  on  $\mathfrak{g}_1$  by

$$(3) \quad \omega_\mu(V, W) := \mu([V, W]) ,$$

then there is a unique skew-symmetric linear endomorphism  $J_\mu$  of  $\mathfrak{g}_1$  such that

$$(4) \quad \omega_\mu(V, W) = \langle J_\mu(V), W \rangle$$

(here, we also used the natural identification of  $\mathfrak{g}_2^*$  with  $\mathfrak{g}_2$  via the inner product). Then on a Lie algebra of Heisenberg type

$$(5) \quad J_\mu^2 = -|\mu|^2 I$$

for every  $\mu \in \mathfrak{g}_2^*$ . As the corresponding connected, simply connected Lie group  $G$  we then choose the linear manifold  $\mathfrak{g}$ , endowed with the Baker-Campbell-Hausdorff product

$$(V_1, U_1) \cdot (V_2, U_2) := (V_1 + V_2, U_1 + U_2 + \frac{1}{2}[V_1, V_2]).$$

As usual, we identify  $X \in \mathfrak{g}$  with the corresponding left-invariant vector field on  $G$  given by the Lie-derivative

$$Xf(g) := \frac{d}{dt}f(g \exp(tX))|_{t=0},$$

where  $\exp : \mathfrak{g} \rightarrow G$  denotes the exponential mapping, which agrees with the identity mapping in our case.

Let us next fix an orthonormal basis  $X_1, \dots, X_{d_1}$  of  $\mathfrak{g}_1$ , as well as an orthonormal basis  $U_1, \dots, U_{d_2}$  of  $\mathfrak{g}_2$ . We may then identify  $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$  and  $G$  with  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$  by means of the basis  $X_1, \dots, X_{d_1}, U_1, \dots, U_{d_2}$  of  $\mathfrak{g}$ . Then our inner product on  $\mathfrak{g}$  will agree with the canonical Euclidean product  $v \cdot w = \sum_{j=1}^{d_1+d_2} v_j w_j$  on  $\mathbb{R}^{d_1+d_2}$ , and  $J_\mu$  will be identified with a skew-symmetric  $d_1 \times d_1$  matrix. We shall also identify the dual spaces of  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  with  $\mathbb{R}^{d_1}$  and  $\mathbb{R}^{d_2}$ , respectively, by means of this inner product. Moreover, the Lebesgue measure  $dx du$  on  $\mathbb{R}^{d_1+d_2}$  is a bi-invariant Haar measure on  $G$ . By

$$(6) \quad d := d_1 + d_2$$

we denote the topological dimension of  $G$ . The group law on  $G$  is then given by

$$(7) \quad (x, u) \cdot (x', u') = (x + x', u + u' + \frac{1}{2}\langle \vec{J}x, x' \rangle)$$

where  $\langle \vec{J}x, x' \rangle$  denotes the vector in  $\mathbb{R}^{d_2}$  with components  $\langle J_{U_i}x, x' \rangle$ .

Let

$$(8) \quad L := - \sum_{j=1}^{d_1} X_j^2$$

denote the sub-Laplacian corresponding to the basis  $X_1, \dots, X_{d_1}$  of  $\mathfrak{g}_1$ .

In the special case  $d_2 = 1$  we may assume that  $J_\mu = \mu J, \mu \in \mathbb{R}$ , where

$$(9) \quad J := \begin{pmatrix} 0 & I_{d_1/2} \\ -I_{d_1/2} & 0 \end{pmatrix}$$

and  $I_{d_1/2}$  is the identity matrix on  $\mathbb{R}^{d_1/2}$ . In this case  $G$  is the *Heisenberg group*  $\mathbb{H}_{d_1/2}$ , discussed in the introduction.

Finally, some dilation structures and the corresponding metrics will play an important role in our proofs; we shall work with both isotropic and nonisotropic dilations. First, the natural dilations on the Heisenberg type groups are the automorphic dilations

$$(10) \quad \delta_r(x, u) := (rx, r^2u), \quad r > 0,$$

on  $G$ . We work with the *Koranyi norm*

$$\|(x, u)\|_{\text{Ko}} := (|x|^4 + |4u|^2)^{1/4}$$

which is a homogeneous norm with respect to the dilations  $\delta_r$ . Moreover, if we denote the corresponding balls by

$$Q_r(x, u) := \{(y, v) \in G : \|(y, v)^{-1} \cdot (x, u)\|_{\text{Ko}} < r\}, \quad (x, u) \in G, \quad r > 0,$$

then the volume  $|Q_r(x, u)|$  is given by

$$|Q_r(x, u)| = |Q_1(0, 0)| r^{d_1 + 2d_2}.$$

Recall that  $d_1 + 2d_2 = d + d_2$  is the *homogeneous dimension* of  $G$ .

We will also have to work with a variant of the ‘Euclidean’ balls, i.e. ‘isotropic balls’ skewed by the Heisenberg translation, denoted by  $Q_{r,E}(x, u)$ .

$$(11) \quad \begin{aligned} Q_{r,E}(x, u) &:= \{(y, v) \in G : |(y, v)^{-1}(x, u)|_E < r\}, \\ &= \{(y, v) \in G : |x - y| + |u - v + \frac{1}{2}\langle \vec{J}x, y \rangle| < r\}; \end{aligned}$$

here

$$|(x, u)|_E := |x| + |u|$$

is comparable with the standard Euclidean norm  $(|x|^2 + |u|^2)^{1/2}$ . Observe that the balls  $Q_r(x, u)$  and  $Q_{r,E}(x, u)$  are the left-translates by  $(x, u)$  of the corresponding balls centered at the origin.

1.2. *The main results.* We consider symbols  $a$  of class  $S^{-\gamma}$ , i.e. satisfying the estimates

$$(12) \quad \left| \frac{d^j}{(ds)^j} a(s) \right| \leq c_j (1 + |s|)^{-\gamma-j}$$

for all  $j = 0, 1, 2, \dots$ . Our main boundedness result is

**Theorem 1.1.** *Let  $1 < p < \infty$ ,  $\gamma(p) := (d-1)|1/p - 1/2|$  and  $a \in S^{-\gamma(p)}$ . Then for  $-\infty < \tau < \infty$ , the operators  $a(\tau\sqrt{L})e^{i\tau\sqrt{L}}$  extend to bounded operators on  $L^p(G)$ .*

*The solutions  $u$  to the initial value problem (1) satisfy the Sobolev type inequalities (2), for  $\gamma \geq \gamma(p)$ .*

Our proof also gives sharp  $L^1$  estimates for operators with symbols supported in dyadic intervals.

**Theorem 1.2.** *Let  $\chi \in C_c^\infty$  supported in  $(1/2, 2)$  and let  $\lambda \geq 1$ . Then the operators  $\chi(\lambda^{-1}\tau\sqrt{L})e^{\pm i\tau\sqrt{L}}$  extend to bounded operators on  $L^1(G)$ , with operator norms  $O(\lambda^{\frac{d-1}{2}})$ .*

In view of the invariance under automorphic dilations it suffices to prove these results for  $\tau = \pm 1$ , and by symmetry considerations, we only need to consider  $\tau = 1$ .

An interesting question posed in [25] concerns the validity of an appropriate result in the limiting case  $p = 1$  (such as a Hardy space bound).

Here the situation is more complicated than in the Euclidean case because of the interplay of isotropic and nonisotropic dilations. The usual Hardy spaces  $H^1(G)$  are defined using the nonisotropic automorphic dilations (10) together with the Koranyi balls. This geometry is not appropriate for our problem; instead the estimates for our kernels require a Hardy space that is defined using isotropic dilations (just as in the Euclidean case) and yet is compatible with the Heisenberg group structure. On the other hand we shall use a dyadic decomposition of the spectrum of  $L$  which corresponds to a Littlewood-Paley decomposition using nonisotropic dilations.

This space  $h_{\text{iso}}^1(G)$  is a variant of the isotropic local or (nonhomogeneous) Hardy space in the Euclidean setting. To define it we first introduce the appropriate notion of atoms. For  $0 < r \leq 1$  we define a  $(P, r)$  atom as a function  $b$  supported in the isotropic Heisenberg ball  $Q_{r,E}(P)$  with radius  $r$  centered at  $P$  (cf. (11)), such that  $\|b\|_2 \leq r^{-d/2}$ , and  $\int b = 0$  if  $r \leq 1/2$ . A function  $f$  belongs to  $h_{\text{iso}}^1(G)$  if  $f = \sum c_\nu b_\nu$  where  $b_\nu$  is a  $(P_\nu, r_\nu)$  atom for some point  $P_\nu$  and some radius  $r_\nu \leq 1$ , and the sequence  $\{c_\nu\}$  is absolutely convergent. The norm on  $h_{\text{iso}}^1(G)$  is given by

$$\inf \sum_{\nu} |c_\nu|$$

where the infimum is taken over representations of  $f$  as a sum  $f = \sum_{\nu} c_\nu b_\nu$  where the  $b_\nu$  are atoms. It is easy to see that  $h_{\text{iso}}^1(G)$  is a closed subspace of  $L^1(G)$ . The spaces  $L^p(G)$ ,  $1 < p < 2$ , are complex interpolation spaces for the couple  $(h_{\text{iso}}^1(G), L^2(G))$  (see §10) and by an analytic interpolation argument Theorem 1.1 can be deduced from an  $L^2$  estimate and the following  $h_{\text{iso}}^1 \rightarrow L^1$  result.

**Theorem 1.3.** *Let  $a \in S^{-\frac{d-1}{2}}$ . Then the operators  $a(\sqrt{L})e^{\pm i\sqrt{L}}$  map the isotropic Hardy space  $h_{\text{iso}}^1(G)$  boundedly to  $L^1(G)$ .*

The norm in the Hardy space  $h_{\text{iso}}^1(G)$  is not invariant under the automorphic dilations (10). It is not currently known whether there is a suitable Hardy space result which can be used for interpolation and works for all  $a(\tau\sqrt{L})e^{i\tau\sqrt{L}}$  with bounds uniform in  $\tau$ .

**1.3. Spectral multipliers.** If  $m$  is a bounded spectral multiplier, then clearly the operator  $m(L)$  is bounded on  $L^2(G)$ . An important question is then under which additional conditions on the spectral multiplier  $m$  the operator  $m(L)$  extends from  $L^2 \cap L^p(M)$  to an  $L^p$ -bounded operator, for a given  $p \neq 2$ .

Fix a non-trivial cut-off function  $\chi \in C_0^\infty(\mathbb{R})$  supported in the interval  $[1, 2]$ ; it is convenient to assume that  $\sum_{k \in \mathbb{Z}} \chi(2^k s) = 1$  for all  $s > 0$ . Let  $L_\alpha^2(\mathbb{R})$  denote the classical Sobolev-space of order  $\alpha$ . Hulanicki and Stein (see Theorem 6.25 in [5]), proved analogs of the classical Mikhlin-Hörmander

multiplier theorem on stratified groups, namely the inequality

$$(13) \quad \|m(L)\|_{L^p \rightarrow L^p} \leq C_{p,\alpha} \sup_{t>0} \|\chi m(t)\|_{L_\alpha^2},$$

for sufficiently large  $\alpha$ . By the work of M. Christ [2], and also Mauceri-Meda [16], the inequality (13) holds true for  $\alpha > (d+d_2)/2$ , in fact they established a more general result for all stratified groups. Observe that, in comparison to the classical case  $G = \mathbb{R}^d$ , the homogeneous dimension  $d + d_2$  takes over the role of the Euclidean dimension  $d$ . However, for the special case of the Heisenberg groups it was shown by E.M. Stein and the first author [24] that (13) holds for the larger range  $\alpha > d/2$ . This result, as well as an extension to Heisenberg type groups has been proved independently by Hebisch [9], and Martini [15] showed that Hebisch's argument can be used to prove a similar result on Métivier groups. Here we use our estimate on the wave equation to prove, only for Heisenberg type groups, a result that covers a larger class of multipliers:

**Theorem 1.4.** *Let  $G$  be a group of Heisenberg type, with topological dimension  $d$ . Let  $m \in L^\infty(\mathbb{R})$ , let  $\chi \in C_0^\infty$  be as above, let*

$$\mathfrak{A}_R := \sup_{t>0} \int_{|s|\geq R} |\mathcal{F}_\mathbb{R}^{-1}[\chi m(t)](s)| s^{\frac{d-1}{2}} ds$$

and assume

$$(14) \quad \|m\|_\infty + \int_2^\infty \mathfrak{A}_R \frac{dR}{R} < \infty.$$

Then the operator  $m(\sqrt{L})$  is of weak type  $(1,1)$  and bounded on  $L^p(G)$ ,  $1 < p < \infty$ .

*Remarks.* (i) Let  $H^1(G)$  be the standard Hardy space defined using the automorphic dilations (10). Our proof shows that under condition (14),  $m(\sqrt{L})$  maps  $H^1(G)$  to  $L^1(G)$ .

(ii) Note that by an application of the Cauchy-Schwarz inequality and Plancherel's theorem that the condition

$$\sup_{t>0} \|\chi m(t)\|_{L_\beta^2} < \infty, \text{ for some } \beta > d/2$$

implies  $\mathfrak{A}_R \lesssim_\gamma R^{\frac{d}{2}-\beta}$  for  $R \geq 2$  and thus Theorem 1.4 covers and extends the above mentioned multiplier results in [24], [9].

(iii) More refined results for fixed  $p > 1$  could be deduced by interpolation, but such results would likely not be sharp.

## 2. SOME NOTATION

**2.1. Smooth cutoff functions.** We denote by  $\zeta_0$  an even  $C^\infty$  function supported in  $(-1,1)$  and assume that  $\zeta_0(s) = 1$  for  $|s| \leq 9/16$ . Let  $\zeta_1(s) = \zeta_0(s/2) - \zeta_0(s)$  so that  $\zeta_1$  is supported in  $(-2, -1/2) \cup (1/2, 2)$ . If we set

$\zeta_j(s) = \zeta_1(2^{1-j}s)$  then  $\zeta_j$  is supported in  $(-2^j, -2^{j-2}) \cup (2^{j-2}, 2^j)$  and we have  $1 = \sum_{j=0}^{\infty} \zeta_j(s)$  for all  $s \in \mathbb{R}$ .

Let  $\eta_0$  be a  $C^\infty$  function supported in  $(-\frac{5\pi}{8}, \frac{5\pi}{8})$  which has the property that  $\eta_0(s) = 1$  for  $|s| \leq \frac{3\pi}{8}$  and satisfies  $\sum_{k \in \mathbb{Z}} \eta_0(t - k\pi) = 1$  for all  $t \in \mathbb{R}$ . For  $l = 1, 2, \dots$  let  $\eta_l(s) = \eta(2^{l-1}s) - \eta_0(2^l s)$  so that  $\eta_0(s) = \sum_{l=1}^{\infty} \eta_l(s)$  for  $s \neq 0$ .

**2.2. Inequalities.** We use the notation  $A \lesssim B$  to indicate  $A \leq CB$  for some constant  $C$ . We sometimes use the notation  $A \lesssim_\kappa B$  to emphasize that the implicit constant depends on the parameter  $\kappa$ . We use  $A \approx B$  if  $A \lesssim B$  and  $B \lesssim A$ .

**2.3. Other notation.** We use the definition

$$\widehat{f}(\xi) \equiv \mathcal{F}f(\xi) = \int f(y) e^{-2\pi i \langle \xi, y \rangle} dy$$

for the Fourier transform in Euclidean space  $\mathbb{R}^d$ .

The convolution on  $G$  is given by

$$f * g(x, u) = \int f(y, v) g(x - y, u - v + \frac{1}{2} \langle \vec{J}x, y \rangle) dy dv.$$

### 3. BACKGROUND ON GROUPS OF HEISENBERG TYPE AND THE SCHRÖDINGER GROUP

For more on the material reviewed here see, e.g., [4], [20] and [22].

**3.1. The Fourier transform on a group of Heisenberg type.** Let us first briefly recall some facts about the unitary representation theory of a Heisenberg type group  $G$ . In many contexts, it is useful to establish analogues of the Bargmann-Fock representations of the Heisenberg group for such groups [14] (compare also [29],[3]). For our purposes, it will be more convenient to work with Schrödinger type representations. It is well-known that these can be reduced to the case of the Heisenberg group  $\mathbb{H}_{d_1/2}$  whose product is given by  $(z, t) \cdot (z', t') = (z + z', t + t' + \frac{1}{2}\omega(z, z'))$ , where  $\omega$  denotes the *canonical symplectic form*  $\omega(z, w) := \langle Jz, w \rangle$ , with  $J$  is as in (9). For the convenience of the reader, we shall outline this reduction to the Heisenberg group.

Let us split coordinates  $z = (x, y) \in \mathbb{R}^{d_1/2} \times \mathbb{R}^{d_1/2}$  in  $\mathbb{R}^{d_1}$ , and consider the associated natural basis of left-invariant vector fields of the Lie algebra of  $\mathbb{H}_{d_1/2}$ ,

$$\tilde{X}_j := \partial_{x_j} - \frac{1}{2}y_j \partial_t, \quad \tilde{Y}_j := \partial_{y_j} + \frac{1}{2}x_j \partial_t, \quad j = 1, \dots, \frac{d_1}{2}, \quad \text{and } T := \partial_t.$$

For  $\tau \in \mathbb{R} \setminus \{0\}$ , the *Schrödinger representation*  $\rho_\tau$  of  $\mathbb{H}_{d_1/2}$  acts on the Hilbert space  $L^2(\mathbb{R}^{d_1/2})$  as follows:

$$[\rho_\tau(x, y, t)h](\xi) := e^{2\pi i \tau(t + y \cdot \xi + \frac{1}{2}y \cdot x)} h(\xi + x), \quad h \in L^2(\mathbb{R}^{d_1/2}).$$



This is an irreducible, unitary representation, and every irreducible unitary representation of  $\mathbb{H}_{d_1/2}$  which acts non-trivially on the center is in fact unitarily equivalent to exactly one of these, by the Stone-von Neumann theorem (a good reference for these and related results is for instance [4]; see also [20]).

Next, if  $\pi$  is any unitary representation, say, of a Heisenberg type group  $G$ , we denote by

$$\pi(f) := \int_G f(g)\pi(g) dg, \quad f \in L^1(G),$$

the associated representation of the group algebra  $L^1(G)$ . For  $f \in L^1(G)$  and  $\mu \in \mathfrak{g}_2^* = \mathbb{R}^{d_2}$ , it will also be useful to define the partial Fourier transform  $f^\mu$  of  $f$  along the center by

$$(15) \quad f^\mu(x) \equiv \mathcal{F}_2 f(x, \mu) := \int_{\mathbb{R}^{d_2}} f(x, u) e^{-2\pi i \mu \cdot u} du \quad (x \in \mathbb{R}^{d_1}).$$

Going back to the Heisenberg group (where  $\mathfrak{g}_2^* = \mathbb{R}$ ), if  $f \in \mathcal{S}(\mathbb{H}_{d_1/2})$ , then it is well-known and easily seen that

$$\rho_\tau(f) = \int_{\mathbb{R}^{d_1}} f^{-\tau}(z) \rho_\tau(z, 0) dz$$

defines a trace class operator on  $L^2(\mathbb{R}^{d_1/2})$ , and its trace is given by

$$(16) \quad \text{tr}(\rho_\tau(f)) = |\tau|^{-d_1/2} \int_{\mathbb{R}} f(0, 0, t) e^{2\pi i \tau t} dt = |\tau|^{-d_1/2} f^{-\tau}(0, 0),$$

for every  $\tau \in \mathbb{R} \setminus 0$ .

From these facts, one derives the Plancherel formula for our Heisenberg type group  $G$ . Given  $\mu \in \mathfrak{g}_2^* = \mathbb{R}^{d_2}$ ,  $\mu \neq 0$ , consider the matrix  $J_\mu$  as in (4). By (5) we have  $J_\mu^2 = -I$  if  $|\mu| = 1$ , and  $J_\mu$  has only eigenvalues  $\pm i$ . Since it is orthogonal there exists an orthonormal basis

$$X_{\mu,1}, \dots, X_{\mu, \frac{d_1}{2}}, Y_{\mu,1}, \dots, Y_{\mu, \frac{d_1}{2}}$$

of  $\mathfrak{g}_1 = \mathbb{R}^{d_1}$  which is symplectic with respect to the form  $\omega_\mu$ , i.e.,  $\omega_\mu$  is represented by the standard symplectic matrix  $J$  in (9) with respect to this basis.

This means that, for every  $\mu \in \mathbb{R}^{d_2} \setminus \{0\}$ , there is an orthogonal matrix  $R_\mu = R_{\frac{\mu}{|\mu|}} \in O(d_1, \mathbb{R})$  such that

$$(17) \quad J_\mu = |\mu| R_\mu J^t R_\mu.$$

Condition (17) is in fact equivalent to  $G$  being of Heisenberg type.

Now consider the subalgebra  $L_{\text{rad}}^1(G)$  of  $L^1(G)$ , consisting of all ‘radial’ functions  $f(x, u)$  in the sense that they depend only on  $|x|$  and  $u$ . As for Heisenberg groups ([4],[20]), this algebra is commutative for arbitrary Heisenberg type groups ([29]), i.e.,

$$(18) \quad f * g = g * f \quad \text{for every } f, g \in L_{\text{rad}}^1(G).$$

This can indeed be reduced to the corresponding result on Heisenberg groups by applying the partial Fourier transform in the central variables.

The following lemma is easy to check and establishes a useful link between representations of  $G$  and those of  $\mathbb{H}_{d_1/2}$ .

**Lemma 3.1.** *The mapping  $\alpha_\mu : G \rightarrow \mathbb{H}_{d_1/2}$ , given by*

$$\alpha_\mu(z, u) := ({}^tR_\mu z, \frac{\mu \cdot u}{|\mu|}), \quad (z, u) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},$$

*is an epimorphism of Lie groups. In particular,  $G/\ker \alpha_\mu$  is isomorphic to  $\mathbb{H}_{d_1/2}$ , where  $\ker \alpha_\mu = \mu^\perp$  is the orthogonal complement of  $\mu$  in the center  $\mathbb{R}^{d_2}$  of  $G$ .*

Given  $\mu \in \mathbb{R}^{d_2} \setminus \{0\}$ , we can now define an irreducible unitary representation  $\pi_\mu$  of  $G$  on  $L^2(\mathbb{R}^{d_1})$  by putting

$$\pi_\mu := \rho_{|\mu|} \circ \alpha_\mu.$$

Observe that then  $\pi_\mu(0, u) = e^{2\pi i \mu \cdot u} I$ . In fact, any irreducible representation of  $G$  with central character  $e^{2\pi i \mu \cdot u}$  factors through the kernel of  $\alpha_\mu$  and hence, by the Stone-von Neumann theorem, must be equivalent to  $\pi_\mu$ .

One then computes that, for  $f \in \mathcal{S}(G)$ ,

$$\pi_\mu(f) = \int_{\mathbb{R}^{d_1}} f^{-\mu}(R_\mu z) \rho_{|\mu|}(z, 0) dz,$$

so that the trace formula (16) yields the analogous trace formula

$$\mathrm{tr} \pi_\mu(f) = |\mu|^{-\frac{d_1}{2}} f^{-\mu}(0)$$

on  $G$ . The Fourier inversion formula in  $\mathbb{R}^{d_2}$  then leads to

$$f(0, 0) = \int_{\mu \in \mathbb{R}^{d_2} \setminus \{0\}} \mathrm{tr} \pi_\mu(f) |\mu|^{\frac{d_1}{2}} d\mu.$$

When applied to  $\delta_{g^{-1}} * f$ , we arrive at the Fourier inversion formula

$$(19) \quad f(g) = \int_{\mu \in \mathbb{R}^{d_2} \setminus \{0\}} \mathrm{tr} (\pi_\mu(g)^* \pi_\mu(f)) |\mu|^{\frac{d_1}{2}} d\mu, \quad g \in G.$$

Applying this to  $f^* * f$  at  $g = 0$ , where  $f^*(g) := \overline{f(g^{-1})}$ , we obtain the Plancherel formula

$$(20) \quad \|f\|_2^2 = \int_{\mu \in \mathbb{R}^{d_2} \setminus \{0\}} \|\pi_\mu(f)\|_{HS}^2 |\mu|^{\frac{d_1}{2}} d\mu,$$

where  $\|T\|_{HS} = (\mathrm{tr} (T^* T))^{1/2}$  denotes the Hilbert-Schmidt norm.

3.2. *The Sub-Laplacian and the group Fourier transform.* Let us next consider the group Fourier transform of our sub-Laplacian  $L$  on  $G$ .

We first observe that  $d\alpha_\mu(X) = {}^tR_\mu X$  for every  $X \in \mathfrak{g}_1 = \mathbb{R}^{d_1}$ , if we view, for the time being, elements of the Lie algebra as tangential vectors at the identity element. Moreover, by (17), we see that

$${}^tR_\mu X_{\mu,1}, \dots, {}^tR_\mu X_{\mu,d_1/2}, {}^tR_\mu Y_{\mu,1}, \dots, {}^tR_\mu Y_{\mu,d_1/2}$$

forms a symplectic basis with respect to the canonical symplectic form  $\omega$  on  $\mathbb{R}^{d_1}$ . We may thus assume without loss of generality that this basis agrees with our basis  $\tilde{X}_1, \dots, \tilde{X}_{d_1/2}, \tilde{Y}_1, \dots, \tilde{Y}_{d_1/2}$  of  $\mathbb{R}^{d_1}$ , so that

$$d\alpha_\mu(X_{\mu,j}) = \tilde{X}_j, \quad d\alpha_\mu(Y_{\mu,j}) = \tilde{Y}_j, \quad j = 1, \dots, d_1/2.$$

By our construction of the representation  $\pi_\mu$ , we thus obtain for the derived representation  $d\pi_\mu$  of  $\mathfrak{g}$  that

$$(21) \quad d\pi_\mu(X_{\mu,j}) = d\rho_{|\mu|}(\tilde{X}_j), \quad d\pi_\mu(Y_{\mu,j}) = d\rho_{|\mu|}(\tilde{Y}_j), \quad j = 1, \dots, d_1/2.$$

Let us define the sub-Laplacians  $L_\mu$  on  $G$  and  $\tilde{L}$  on  $\mathbb{H}_{d_1/2}$  by

$$L_\mu := - \sum_{j=1}^{d_1/2} (X_{\mu,j}^2 + Y_{\mu,j}^2), \quad \tilde{L} := - \sum_{j=1}^{d_1/2} (\tilde{X}_j^2 + \tilde{Y}_j^2),$$

where from now on we consider elements of the Lie algebra again as left-invariant differential operators. Then, by (21),

$$d\pi_\mu(L_\mu) = d\rho_{|\mu|}(\tilde{L}).$$

Moreover, since the basis  $X_{\mu,1}, \dots, X_{\mu,d_1/2}, Y_{\mu,1}, \dots, Y_{\mu,d_1/2}$  and our original basis  $X_1, \dots, X_{d_1}$  of  $\mathfrak{g}_1$  are both orthonormal bases, it is easy to verify that the distributions  $L\delta_0$  and  $L_\mu\delta_0$  agree. Since  $Af = f * (A\delta_0)$  for every left-invariant differential operator  $A$ , we thus have  $L = L_\mu$ , hence

$$(22) \quad d\pi_\mu(L) = d\rho_{|\mu|}(\tilde{L}).$$

But, it follows immediately from our definition of Schrödinger representation  $\rho_\tau$  that  $d\rho_\tau(\tilde{X}_j) = \partial_{\xi_j}$  and  $d\rho_\tau(\tilde{Y}_j) = 2\pi i \tau \xi_j$ , so that  $d\rho_{|\mu|}(\tilde{L}) = -\Delta_\xi + (2\pi|\mu|)^2|\xi|^2$  is a rescaled Hermite operator (*cf.* also [4]), and an orthonormal basis of  $L^2(\mathbb{R}^{d_1/2})$  is given by the tensor products

$$h_\alpha^{|\mu|} := h_{\alpha_1}^{|\mu|} \otimes \dots \otimes h_{\alpha_{d_1/2}}^{|\mu|}, \quad \alpha \in \mathbb{N}^{d_1/2},$$

where  $h_k^\mu(x) := (2\pi|\mu|)^{1/4} h_k((2\pi|\mu|)^{1/2}x)$ , and

$$h_k(x) = c_k (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}$$

denotes the  $L^2$ -normalized Hermite function of order  $k$  on  $\mathbb{R}$ . Consequently,

$$(23) \quad d\pi_\mu(L)h_\alpha^{|\mu|} = 2\pi|\mu| \left( \frac{d_1}{2} + 2|\alpha| \right) h_\alpha^{|\mu|}, \quad \alpha \in \mathbb{N}^{d_1/2}.$$

It is also easy to see that

$$(24) \quad d\pi_\mu(U_j) = 2\pi i\mu_j I, \quad j = 1, \dots, d_2.$$

Now, the operators  $L, -iU_1, \dots, -iU_{d_2}$  form a set of pairwise strongly commuting self-adjoint operators, with joint core  $\mathcal{S}(G)$ , so that they admit a joint spectral resolution, and we can thus give meaning to expressions like  $\varphi(L, -iU_1, \dots, -iU_{d_2})$  for each continuous function  $\varphi$  defined on the corresponding joint spectrum. For simplicity of notation we write

$$U := (-iU_1, \dots, -iU_{d_2}).$$

If  $\varphi$  is bounded, then  $\varphi(L, U)$  is a bounded, left invariant operator on  $L^2(G)$ , so that it is a convolution operator

$$\varphi(L, U)f = f * K_\varphi, \quad f \in \mathcal{S}(G),$$

with a convolution kernel  $K_\varphi \in \mathcal{S}'(G)$  which will also be denoted by  $\varphi(L, U)\delta$ . Moreover, if  $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^{d_2})$ , then  $\varphi(L, U)\delta \in \mathcal{S}(G)$  (cf. [23]). Since functional calculus is compatible with unitary representation theory, we obtain in this case from (23), (24) that

$$(25) \quad \pi_\mu(\varphi(L, U)\delta)h_\alpha^{|\mu|} = \varphi\left(2\pi|\mu|\left(\frac{d_1}{2} + 2|\alpha|\right), 2\pi\mu\right)h_\alpha^{|\mu|}$$

(this identity in combination with the Fourier inversion formula could in fact be taken as the definition of  $\varphi(L, U)\delta$ ). In particular, the Plancherel theorem implies then that the operator norm on  $L^2(G)$  is given by

$$(26) \quad \|\varphi(L, U)\| = \sup\{|\varphi(|\mu|\left(\frac{d_1}{2} + 2q\right), \mu)| : \mu \in \mathbb{R}^{d_2}, q \in \mathbb{N}\}.$$

Finally, observe that

$$(27) \quad K_\phi^\mu = \varphi(L^\mu, 2\pi\mu)\delta;$$

this follows for instance by applying the unitary representation induced from the character  $e^{2\pi i\mu \cdot u}$  on the center of  $G$  to  $K_\varphi$ .

We shall in fact only work with functions of  $L$  and  $|U|$ , defined by

$$\pi_\mu(\varphi(L, |U|)\delta)h_\alpha^{|\mu|} = \varphi\left(2\pi|\mu|\left(\frac{d_1}{2} + 2|\alpha|\right), 2\pi|\mu|\right)h_\alpha^{|\mu|}.$$

Observe that if  $\varphi$  depends only on the second variable, then  $\varphi(|U|)$  is just the radial convolution operator acting only in the central variables, given by

$$(28) \quad \mathcal{F}_{\mathbb{R}^{d_2}}[\varphi(|U|)f](x, \mu) = \varphi(2\pi|\mu|)\mathcal{F}_{\mathbb{R}^{d_2}}f(x, \mu) \quad \text{for all } \mu \in (\mathbb{R}^{d_2})^*.$$

3.3. *Partial Fourier transforms and twisted convolution.* For  $\mu \in \mathfrak{g}_2^*$ , one defines the  $\mu$ -twisted convolution of two suitable functions (or distributions)  $\varphi$  and  $\psi$  on  $\mathfrak{g}_1 = \mathbb{R}^{d_1}$  by

$$(\varphi *_\mu \psi)(x) := \int_{\mathbb{R}^{d_1}} \varphi(x-y)\psi(y)e^{-i\pi\omega_\mu(x,y)} dy$$

where  $\omega_\mu$  is as in (3). Then, with  $f^\mu$  as in (15),

$$(f * g)^\mu = f^\mu *_\mu g^\mu,$$

where  $f * g$  denotes the convolution product of the two functions  $f, g \in L^1(G)$ . Accordingly, the vector fields  $X_j$  are transformed into the  $\mu$ -twisted first order differential operators  $X_j^\mu$  such that  $(X_j f)^\mu = X_j^\mu f^\mu$ , and the sub-Laplacian is transformed into the  $\mu$ -twisted Laplacian  $L^\mu$ , i.e.,

$$(L f)^\mu = L^\mu f^\mu = - \sum_{j=1}^{d_1} (X_j^\mu)^2,$$

say for  $f \in \mathcal{S}(G)$ . A computation shows that explicitly

$$(29) \quad X_j^\mu = \partial_{x_j} + i\pi\omega_\mu(\cdot, X_j).$$

3.4. *The Schrödinger group*  $\{e^{itL^\mu}\}$ . It will be important for us that the Schrödinger operators  $e^{itL^\mu}$ ,  $t \in \mathbb{R}$ , generated by  $L^\mu$  can be computed explicitly.

**Proposition 3.2.** (i) For  $f \in \mathcal{S}(G)$ ,

$$(30) \quad e^{itL^\mu} f = f *_\mu \gamma_t^\mu, \quad t \geq 0,$$

where  $\gamma_t^\mu \in \mathcal{S}'(\mathbb{R}^{d_1})$  is a tempered distribution.

(ii) For all  $t$  such that  $\sin(2\pi t|\mu|) \neq 0$  the distribution  $\gamma_t^\mu$  is given by

$$(31) \quad \gamma_t^\mu(x) = 2^{-d_1/2} \left( \frac{|\mu|}{\sin(2\pi t|\mu|)} \right)^{d_1/2} e^{-i\frac{\pi}{2}|\mu| \cot(2\pi t|\mu|)|x|^2}.$$

(iii) For all  $t$  such that  $\cos(2\pi t|\mu|) \neq 0$  the Fourier transform of  $\gamma_t^\mu$  is given by

$$(32) \quad \widehat{\gamma}_t^\mu(\xi) = \frac{1}{(\cos(2\pi t|\mu|))^{d_1/2}} e^{i\frac{2\pi}{|\mu|} \tan(2\pi t|\mu|)|\xi|^2}.$$

Indeed, for  $\mu \neq 0$ , let us consider the symplectic vector space  $V := \mathfrak{g}_1$ , endowed with the symplectic form  $\sigma := \frac{1}{|\mu|}\omega_\mu$ . Notice first that, because of (5), the volume form  $\sigma^{\wedge(d_1/2)}$ , i.e., the  $d_1/2$ -fold exterior product of  $\sigma$  with itself, can be identified with Lebesgue measure on  $\mathbb{R}^{d_1}$ . As in [19], we then associate to the pair  $(V, \sigma)$  the Heisenberg group  $\mathbb{H}_V$ , with underlying manifold  $V \times \mathbb{R}$  and endowed with the product

$$(v, u)(v', u') := (v + v', u + u' + \frac{1}{2}\sigma(v, v')).$$

It is then common to denote for  $\tau \in \mathbb{R}$  the  $\tau$ -twisted convolution by  $\times_\tau$  in place of  $*_\tau$  (compare §5 in [19]). The  $\mu$ -twisted convolution associated to

the group  $G$  will then agree with the  $|\mu|$ -twisted convolution  $\times_{|\mu|}$  defined on the symplectic vector space  $(V, \sigma)$ . Moreover, if we identify the  $X_j \in V$  also with left-invariant vector fields on  $\mathbb{H}_V$ , then (29) shows that

$$X_j^\mu = \partial_{x_j} + i\pi|\mu|\sigma(\cdot, X_j)$$

agrees with the corresponding  $|\mu|$ -twisted differential operators  $\tilde{X}_j^{|\mu|}$  defined in [19].

Accordingly, our  $\mu$ -twisted Laplacian  $L^\mu$  will agree with the  $|\mu|$ -twisted Laplacian

$$\tilde{L}_S^{|\mu|} = \tilde{\mathcal{L}}_{-I}^\mu = \sum_{j=1}^{d_1} (\tilde{X}_j^{|\mu|})^2$$

associated to the symmetric matrix  $A := -I$  in [19]. Here,

$$S = -A \frac{1}{|\mu|} J_\mu = \frac{1}{|\mu|} J_\mu.$$

Consequently,

$$e^{itL^\mu} = e^{it\tilde{L}_S^{|\mu|}}.$$

From Theorem 5.5 in [19] we therefore obtain that for  $f \in L^2(V)$

$$\exp\left(\frac{it}{|\mu|} \tilde{L}_S^{|\mu|}\right) f = f \times_{|\mu|} \Gamma_{t, iS}^{|\mu|}, \quad t \geq 0,$$

where,  $\Gamma_{t, iS}^{|\mu|}$  is a tempered distribution whose Fourier transform is given by

$$\widehat{\Gamma_{t, iS}^{|\mu|}}(\xi) = \frac{1}{\sqrt{\det(\cos 2\pi itS)}} e^{-\frac{2\pi}{|\mu|} \sigma(\xi, \tan(2\pi itS)\xi)}$$

whenever  $\det(\cos(2\pi itS)) \neq 0$ . Since  $S^2 = -I$  because of (5), one sees that

$$\sin(2\pi itS) = i \sin(2\pi t)S, \quad \cos(2\pi itS) = \cos(2\pi t)I.$$

Note also that  $\sigma(\xi, \eta) = \langle S\xi, \eta \rangle$ . We thus see that (30) and (32) hold true, and the formula (31) follows by Fourier inversion (*cf.* Lemma 1.1 in [21]).

#### 4. AN APPROXIMATE SUBORDINATION FORMULA

We shall use Proposition 3.2 and the following subordination formula to obtain manageable expressions for the wave operators.

**Proposition 4.1.** *Let  $\chi_1 \in C^\infty$  so that  $\chi_1(s) = 1$  for  $s \in [1/4, 4]$ . Let  $g$  be a  $C^\infty$  function supported in  $(1/2, 2)$ . Then there are  $C^\infty$  functions  $a_\lambda$  and  $\rho_\lambda$ , depending linearly on  $g$ , with  $a_\lambda$  supported in  $[1/16, 4]$ , and  $\rho_\lambda$  supported in  $[1/4, 4]$ , so that for all  $K = 2, 3, \dots$ ,  $N_1, N_2 \geq 0$ , and all  $\lambda \geq 1$*

$$(33) \quad \sup_s \left| \partial_s^{N_1} \partial_\lambda^{N_2} a_\lambda(s) \right| \leq c(K) \lambda^{-N_2} \sum_{\nu=0}^K \|g^{(\nu)}\|_\infty, \quad N_1 + N_2 < \frac{K-1}{2},$$

$$(34) \quad \sup_s |\partial_s^{N_1} \partial_\lambda^{N_2} \rho_\lambda(s)| \leq c(K, N_2) \lambda^{N_1+1-K} \sum_{\nu=0}^K \|g^{(\nu)}\|_\infty, \quad N_1 \leq K-2.$$

and the formula

$$(35) \quad g(\lambda^{-1}\sqrt{L})e^{i\sqrt{L}} = \chi_1(\lambda^{-2})L\sqrt{\lambda} \int e^{i\frac{\lambda}{4s}} a_\lambda(s) e^{isL/\lambda} ds + \rho_\lambda(\lambda^{-2}L)$$

holds. For any  $N \in \mathbb{N}$ , the functions  $\lambda^N \rho_\lambda$  are uniformly bounded in the topology of the Schwartz-space, and the operators  $\rho_\lambda(\lambda^{-2}L)$  are bounded on  $L^p(G)$ ,  $1 \leq p \leq \infty$ , with operator norm  $O(\lambda^{-N})$ .

For explicit formulas of  $a_\lambda$  and  $\rho_\lambda$  see Lemma 4.3 below. The proposition is essentially an application of the method of stationary phase where we keep track on how  $a_\lambda$ ,  $\rho_\lambda$  depend on  $g$ . We shall need an auxiliary lemma.

**Lemma 4.2.** *Let  $K \in \mathbb{N}$  and  $g \in C^K(\mathbb{R})$ . Let  $\zeta_1 \in C^\infty(\mathbb{R})$  be supported in  $(1/2, 2) \cup (-2, -1/2)$  and  $\Lambda \geq 1$  and  $\ell \geq 0$ . Then, for all nonnegative integers  $M$ ,*

$$(36) \quad \left| \int y^{2M} g(y) \zeta_1(\Lambda^{1/2} 2^{-\ell} y) e^{i\Lambda y^2} dy \right| \leq C_{M,K} 2^{-2\ell K} (2^\ell \Lambda^{-1/2})^{1+2M} \sum_{j=0}^K (2^\ell \Lambda^{-1/2})^j \|g^{(j)}\|_\infty.$$

Moreover, for  $0 \leq m < \frac{K-1}{2}$ ,

$$(37) \quad \left| \left( \frac{d}{d\Lambda} \right)^m \int g(y) e^{i\Lambda y^2} dy \right| \leq C_K \Lambda^{-m-\frac{1}{2}} \sum_{j=0}^K \Lambda^{-j/2} \|g^{(j)}\|_\infty.$$

*Proof.* By induction on  $K$  we prove the following assertion labeled

( $\mathcal{A}_K$ ): If  $g \in C^K$  then

$$(38) \quad \int y^{2M} g(y) \zeta_1(\Lambda^{1/2} 2^{-\ell} y) e^{i\Lambda y^2} dy = \Lambda^{-K} \sum_{j=0}^K \int g^{(j)}(y) \zeta_{j,K,M,\Lambda}(y) e^{i\Lambda y^2} dy$$

where  $\zeta_{j,K,M,\Lambda}$  is supported on  $\{y : |y| \in [2^{\ell-1}\Lambda^{-1/2}, 2^{\ell+1}\Lambda^{-1/2}]\}$  and, for  $0 \leq j \leq K$ , satisfies the differential inequalities

$$(39) \quad |\zeta_{j,K,M,\Lambda}^{(n)}(y)| \leq C(j, K, M, n) (2^{-\ell} \Lambda^{1/2})^{n-2M} 2^{-\ell(2K-j)} \Lambda^{K-j/2}.$$

Clearly this assertion implies (36).

We set  $\zeta_{0,0,M,\Lambda}(y) = y^{2M} \zeta_1(\Lambda^{1/2} 2^{-\ell} y)$  and the claim ( $\mathcal{A}_K$ ) is immediate for  $K = 0$ . It remains to show that the implication ( $\mathcal{A}_K$ )  $\implies$  ( $\mathcal{A}_{K+1}$ ), holds for all  $K \geq 0$ .

Assume  $(\mathcal{A}_K)$  for some  $K \geq 0$  and let  $g \in C^{K+1}$ . We let  $0 \leq j \leq K$  and examine the  $j$ th term in the sum in (38). Integration by parts yields

$$\begin{aligned} & \int g^{(j)}(y) \zeta_{j,K,M,\Lambda}(y) e^{i\Lambda y^2} dy \\ &= i \int \left[ \frac{g^{(j+1)}(y)}{2y\Lambda} \zeta_{j,K,M,\Lambda}(y) + g^{(j)}(y) \frac{d}{dy} \left( \frac{\zeta_{j,K,M,\Lambda}(y)}{2y\Lambda} \right) \right] e^{i\Lambda y^2} dy. \end{aligned}$$

The sum  $\Lambda^{-K} \sum_{j=0}^K \int g^{(j)}(y) \zeta_{j,K,M,\Lambda}(y) e^{i\Lambda y^2} dy$  can now be rewritten as

$$\Lambda^{-K-1} \sum_{\nu=0}^{K+1} \int g^{(\nu)}(y) \zeta_{\nu,K+1,M,\Lambda}(y) e^{i\Lambda y^2} dy$$

where

$$\begin{aligned} \zeta_{0,K+1,M,\Lambda}(y) &= i \frac{d}{dy} \left( \frac{\zeta_{0,K,M,\Lambda}(y)}{2y} \right), \\ \zeta_{\nu,K+1,M,\Lambda}(y) &= i \frac{d}{dy} \left( \frac{\zeta_{\nu,K,M,\Lambda}(y)}{2y} \right) + i \frac{\zeta_{\nu-1,K,M,\Lambda}(y)}{2y}, \quad 1 \leq \nu \leq K, \\ \zeta_{K+1,K+1,M,\Lambda}(y) &= i \frac{\zeta_{K,K,M,\Lambda}(y)}{2y}. \end{aligned}$$

On the support of the cutoff functions we have  $|y| \geq 2^{\ell-1} \Lambda^{-1/2}$  and the asserted differential inequalities for the functions  $\zeta_{\nu,K+1,M,\Lambda}$  can be verified using the Leibniz rule. This finishes the proof of the implication  $(\mathcal{A}_K) \implies (\mathcal{A}_{K+1})$  and thus the proof of (36).

We now prove (37). Let  $\zeta_0$  be an even  $C^\infty$  function supported in  $(-1, 1)$  and assume that  $\zeta_0(s) = 1$  for  $|s| \leq 1/2$ . Let  $\zeta_1(s) = \zeta_0(s/2) - \zeta_0(s)$  so that  $\zeta_1$  is supported in  $[-2, -1/2] \cup [1/2, 2]$ , as in the statement of (36). We split the left hand side of (37) as  $\sum_{\ell=0}^\infty I_{\ell,m}$  where

$$I_{\ell,m} = \int (iy^2)^m g(y) \zeta_1(\Lambda^{1/2} 2^{-\ell} y) e^{i\Lambda y^2} dy, \quad \text{for } \ell > 0$$

and  $I_{0,m}$  is defined similarly with  $\zeta_0(\Lambda^{1/2} y)$  in place of  $\zeta_1(\Lambda^{1/2} 2^{-\ell} y)$ . Clearly  $|I_{0,m}| \lesssim \Lambda^{-m-1/2} \|g\|_\infty$  and by (36)

$$I_{\ell,m} \lesssim_{m,K} \sum_{j=0}^K 2^{-\ell(2K-2m-j-1)} \Lambda^{-\frac{1+2m+j}{2}} \|g^{(j)}\|_\infty.$$

Since  $j \leq K$  we can sum in  $\ell$  if  $m < \frac{K-1}{2}$  and the assertion (37) follows.  $\square$

**Lemma 4.3.** *Let  $K \in \mathbb{N}$  and let  $g \in C^K(\mathbb{R})$  be supported in  $(1/2, 2)$ , and let  $\chi_1 \in C_c^\infty(\mathbb{R})$  so that  $\chi_1(x) = 1$  on  $(1/4, 4)$ . Also let  $\varsigma$  be a  $C_0^\infty(\mathbb{R})$  function supported in  $[1/9, 3]$  with the property that  $\varsigma(s) = 1$  on  $[1/8, 2]$ . Then*

$$(40) \quad g(\sqrt{x}) e^{i\lambda\sqrt{x}} = \chi_1(x) \left[ \sqrt{\lambda} \int e^{i\frac{\lambda}{4s}} a_\lambda(s) e^{i\lambda s x} ds + \tilde{\rho}_\lambda(x) \right]$$



where  $a_\lambda$  is supported in  $[\frac{1}{16}, 4]$ , and

$$(41) \quad a_\lambda(s) = \pi^{-1} \sqrt{\lambda} \varsigma(s) \int (y + \frac{1}{2s}) g(y + \frac{1}{2s}) e^{-i\lambda s y^2} dy$$

and

$$(42) \quad \mathcal{F}[\tilde{\rho}_\lambda](\xi) = (1 - \varsigma(\frac{2\pi\xi}{\lambda})) \mathcal{F}[g(\sqrt{\cdot}) e^{i\lambda\sqrt{\cdot}}](\xi).$$

Let  $\rho_\lambda = \chi_1 \tilde{\rho}_\lambda$ . Then the estimates (33) and (34) hold for all  $\lambda \geq 1$ .

*Proof.* Let  $\Psi_\lambda$  be the Fourier transform of  $x \mapsto g(\sqrt{x}) e^{i\lambda\sqrt{x}}$ , i.e.

$$(43) \quad \Psi_\lambda(\xi) = \int g(\sqrt{x}) e^{i\lambda\sqrt{x}} e^{-2\pi i \xi x} dx = \int 2s g(s) e^{i(\lambda s - 2\pi \xi s^2)} ds$$

Observe that  $g(\sqrt{x}) = 0$  for  $x \notin (1/4, 4)$ , thus  $g(\sqrt{x}) = \chi_1(x) g(\sqrt{x})$ . By the Fourier inversion formula we have

$$g(\sqrt{x}) e^{i\lambda\sqrt{x}} = \chi_1(x) (v_\lambda(x) + \rho_\lambda(x))$$

where

$$(44) \quad \begin{aligned} v_\lambda(x) &= \int \varsigma(\frac{2\pi\xi}{\lambda}) \Psi_\lambda(\xi) e^{2\pi i x \xi} d\xi \\ \tilde{\rho}_\lambda(x) &= \int (1 - \varsigma(\frac{2\pi\xi}{\lambda})) \Psi_\lambda(\xi) e^{2\pi i x \xi} d\xi \end{aligned}$$

so that  $\tilde{\rho}_\lambda$  is as in (42).

We first consider  $\tilde{\rho}_\lambda$  and verify that the inequalities (34) hold. On the support of  $1 - \varsigma(2\pi\xi/\lambda)$  we have either  $|2\pi\xi| \leq \lambda/8$  or  $|2\pi\xi| \geq 2\lambda$ . Clearly, on the support of  $g$  we have  $|\partial_s(\lambda s - 2\pi\xi s^2)| \geq \lambda/2$  if  $|2\pi\xi| \leq \lambda/8$  and  $|\partial_s(\lambda s - 2\pi\xi s^2)| \geq |2\pi\xi|/2$  if  $|2\pi\xi| \geq 2\lambda$ . Integration by parts in (43) yields

$$|\partial_\xi^{M_1} \partial_\lambda^{M_2} [(1 - \varsigma(2\pi\xi/\lambda)) \Psi_\lambda(\xi)]| \leq C_{M_1, M_2, K} \|g\|_{C^K} (1 + |\xi| + |\lambda|)^{-K}.$$

Thus, if  $N_1 \leq K - 2$ ,

$$\begin{aligned} \left| \left( \frac{d}{dx} \right)^{N_1} \tilde{\rho}_\lambda(x) \right| &= \left| \int (2\pi\xi)^{N_1} (1 - \varsigma(2\pi\xi/\lambda)) \Psi_\lambda(\xi) e^{2\pi i x \xi} d\xi \right| \\ &\leq C_{N_1, K} \|g\|_{C^K} \int \frac{(1 + |\xi|)^{N_1}}{(1 + |\xi| + |\lambda|)^K} d\xi \leq C'_{N_1, K} \|g\|_{C^K} \lambda^{-K + N_1 + 1}. \end{aligned}$$

This yields (34) for  $N_2 = 0$ , and the same argument applies to the  $\lambda$ -derivatives.

It remains to represent the function  $\lambda^{-1/2} v_\lambda$  as the integral in (40). Let

$$(45) \quad \tilde{g}(s) = 2s g(s).$$

By a change of variable we may write

$$(46) \quad \Psi_\lambda(\xi) = e^{\frac{i\lambda^2}{8\pi\xi}} \int \tilde{g}(y + \frac{\lambda}{4\pi\xi}) e^{-2\pi i \xi y^2} dy.$$

We compute from (44), (46),

$$v_\lambda(x) = \lambda \int \varsigma(s) e^{i\frac{\lambda}{4s} + i\lambda s x} \lambda^{-1/2} a_\lambda(s) ds$$

where

$$a_\lambda(s) = (2\pi)^{-1} \sqrt{\lambda} \varsigma(s) \int \tilde{g}(y + \frac{1}{2s}) e^{-i\lambda s y^2} dy,$$

i.e.  $a_\lambda$  is as in (41). In order to show the estimate (33) observe

$$2\pi \partial_\lambda^{N_2} (\lambda^{-1/2} a_\lambda(s)) = \varsigma(s) \int \tilde{g}(y + \frac{1}{2s}) (-i s y^2)^{N_2} e^{-i\lambda s y^2} dy$$

and then by the Leibniz rule  $\partial_s^{N_1} \partial_\lambda^{N_2} [\lambda^{-1/2} a_\lambda(s)]$  is a linear combination of terms of the form

$$(47) \quad \left(\frac{d}{ds}\right)^{N_3} [\varsigma(s) s^{N_2}] \int y^{2N_2} (\lambda y^2)^{N_5} \left(\frac{d}{ds}\right)^{N_4} [\tilde{g}(y + \frac{1}{2s})] e^{i\lambda s y^2} dy$$

where and  $N_3 + N_4 + N_5 = N_1$ . By estimate (37) in Lemma 4.2 we see that the term (47) is bounded (uniformly in  $s \in [1/9, 3]$ ) by a constant times

$$\lambda^{-N_2 - \frac{1}{2}} \left\| \left(\frac{d}{ds}\right)^{N_4} [\tilde{g}(\cdot + \frac{1}{2s})] \right\|_{C^{K-N_4}}$$

provided that  $N_2 + N_5 < (K - N_4 - 1)/2$ . This condition is satisfied if  $N_1 + N_2 < (K - 1)/2$  and under this condition we get

$$\sup_s |\partial_s^{N_1} \partial_\lambda^{N_2} [\lambda^{-1/2} a_\lambda(s)]| \lesssim \lambda^{-N_2 - \frac{1}{2}} \|g\|_{C^K}.$$

Now (33) is a straightforward consequence.  $\square$

*Proof of Proposition 4.1.* The identity (35) is an immediate consequence of the spectral resolution  $L = \int_{\mathbb{R}^+} x dE_x$ , Lemma 4.3 (applied with  $x/\lambda$  in place of  $x$ ) and Fubini's theorem. Note that in view of the symbol estimates (34) any Schwartz norm of  $\rho_\lambda(\lambda^{-2} \cdot)$  is  $O(\lambda^{-N})$  for every  $N \in \mathbb{N}$ . The statement on the operator norms of  $\rho_\lambda(\lambda^{-2} L)$  follows then from the known multiplier theorems (such as the original one by Hulanicki and Stein, see [12], [5]).  $\square$

Thus in order to get manageable formulas for our wave operators it will be important to get explicit formulas for the Schrödinger group  $e^{isL}$ ,  $s \in \mathbb{R}$ .

## 5. BASIC DECOMPOSITIONS OF THE WAVE OPERATOR AND STATEMENTS OF REFINED RESULTS

We consider operators  $a(\sqrt{L})e^{i\sqrt{L}}$  where  $a \in S^{(d-1)/2}$  (satisfying (12) with  $\gamma = \frac{d-1}{2}$ ). We split off the part of the symbol supported near 0. Let  $\chi_0 \in C_c^\infty(\mathbb{R})$  be supported in  $[-1, 1]$ ; then we observe that the operator  $\chi_0(\sqrt{L}) \exp(i\sqrt{L})$  extends to a bounded operator on  $L^p(G)$ , for  $1 \leq p \leq \infty$ . To see this we decompose  $\chi_0(\sqrt{\tau})e^{i\sqrt{\tau}} = \chi_0(\sqrt{\tau}) + \sum_{n=0}^\infty \alpha_n(\tau)$ ,  $\tau > 0$ , where

$$\alpha_n(\tau) = \chi_0(\sqrt{\tau})(e^{i\sqrt{\tau}} - 1)(\zeta_0(2^{n-1}\tau) - \zeta_0(2^n\tau))$$

where  $\zeta_0$  is as in §2.1. Clearly  $\chi_0(\sqrt{\cdot}) \in C_0^\infty$ . Thus by Hulanicki's theorem [12] the convolution kernel of  $\chi_0(\sqrt{L})$  is a Schwartz function and hence  $\chi_0(\sqrt{L})$  is bounded on  $L^1(G)$ . Moreover the functions  $2^{n/2}\alpha_n(2^{-n}\cdot)$  belong to a bounded set of Schwartz functions supported in  $[-2, 2]$ . By dilation invariance and again Hulanicki's theorem the convolution kernels of

$2^{n/2}\alpha_n(2^{-n}L)$  are Schwartz functions and form a bounded subset of the Schwartz space  $\mathcal{S}(G)$ . Thus, by rescaling, the operator  $\alpha_n(L)$  is bounded on  $L^1(G)$  with operator norm  $O(2^{-n/2})$ . We may sum in  $n$  and obtain the desired bounds for  $\chi_0(\sqrt{\tau})e^{i\sqrt{\tau}}$ .

The above also implies that for any  $\lambda$  the operator  $\chi(\lambda^{-1}\sqrt{L})\exp(i\sqrt{L})$  is bounded on  $L^1$  (with a polynomial and nonoptimal growth in  $\lambda$ ). Thus, in what follows it suffices to consider symbols  $a \in S^{-(d-1)/2}$  with the property that  $a(s) = 0$  in a neighborhood of 0. Then

$$(48) \quad a(\sqrt{L})e^{i\sqrt{L}} = \sum_{j>C} 2^{-j\frac{d-1}{2}} g_j(\sqrt{2^{-2j}L})e^{i\sqrt{L}},$$

where the  $g_j$  form a family of smooth functions supported in  $(1/2, 2)$  and bounded in the  $C_0^\infty$  topology. In many calculations below when  $j$  is fixed we shall also use the parameter  $\lambda$  for  $2^j$ .

Let  $\chi_1$  be a smooth function such that

$$(49a) \quad \text{supp}(\chi_1) \subset (2^{-10}, 2^{10}),$$

$$(49b) \quad \chi_1(s) = 1 \text{ for } s \in (2^{-9}, 2^9).$$

By Proposition 4.1 and Lemma 4.3 we may thus write

$$(50) \quad a(\sqrt{L})e^{i\sqrt{L}} = m_{\text{negl}}(L) + \sum_{j>100} 2^{-j\frac{d-1}{2}} \chi_1(2^{-2j}L)m_{2^j}(L),$$

where the ‘‘negligible’’ operator  $m_{\text{negl}}(L)$  is a convolution with a Schwartz kernel,

$$(51) \quad m_\lambda(\rho) = \sqrt{\lambda} \int e^{i\lambda/(4\tau)} a_\lambda(\tau) e^{i\tau\rho/\lambda} d\tau, \quad \text{with } \lambda = 2^j,$$

and the  $a_\lambda$  form a family of smooth functions supported in  $(1/16, 4)$ , bounded in the  $C_0^\infty$  topology.

We shall use the formulas (31), which give explicit expressions for the partial Fourier transform in the central variables of the Schwartz kernel of  $e^{itL}$ . In undoing this partial Fourier transform, it will be useful to recall from §3 that if  $\rho_1$  denotes the spectral parameter for  $L$  then the joint spectrum of the operators  $L$  and  $|U|$  is contained in the closure of

$$(52) \quad \{(\rho_1, \rho_2) : \rho_2 \geq 0, \rho_1 = (\frac{d_1}{2} + 2q)\rho_2 \text{ for some nonnegative integer } q\}.$$

As the phase in (31) exhibits periodic singularities it natural to introduce an equally spaced decomposition in the central Fourier variable (i.e., in the spectrum of the operator  $|U|$ ). Let  $\eta_0$  be a  $C^\infty$  function such that

$$(53a) \quad \text{supp}(\eta_0) \subset (-\frac{5\pi}{8}, \frac{5\pi}{8}),$$

$$(53b) \quad \eta_0(s) = 1 \text{ for } s \in (-\frac{3\pi}{8}, \frac{3\pi}{8}),$$

$$(53c) \quad \sum_{k \in \mathbb{Z}} \eta_0(t - k\pi) = 1 \text{ for } t \in \mathbb{R}.$$

We decompose

$$(54) \quad \chi_1(\lambda^{-2}L)m_\lambda(L) = \sum_{k=0}^{\infty} \chi_1(\lambda^{-2}L)T_\lambda^k,$$

where

$$(55) \quad T_\lambda^k = \lambda^{1/2} \int e^{i\lambda/(4\tau)} a_\lambda(\tau) \eta_0\left(\frac{\tau}{\lambda}|U| - k\pi\right) e^{i\tau L/\lambda} d\tau.$$

The description (52) of the joint spectrum of  $L$  and  $|U|$  gives a restriction on the summation in  $k$ . Namely the operator  $\eta_0(\frac{\tau}{\lambda}|U| - k\pi)\chi_1(\lambda^{-2}L)$  is identically zero unless there exist positive  $\rho_1$  and  $\rho_2$  with  $\rho_1 \geq \rho_2 d_1/2$  such that  $\frac{\lambda^2}{5} < \rho_1 < 5\lambda^2$  and  $(k\pi - \frac{5\pi}{8})\frac{\lambda}{\tau} < \rho_2 < (k\pi + \frac{5\pi}{8})\frac{\lambda}{\tau}$  for some  $\tau \in (\frac{1}{16}, 4)$ . A necessary condition for these two conditions to hold simultaneously is of course  $\frac{d_1}{2}(k\pi - \frac{5\pi}{8})\frac{\lambda}{4} \leq 5\lambda^2$  and since  $d_1 \geq 2$  and  $\lambda \geq 1$  we see that the sum in (54) extends only over  $k$  with

$$(56) \quad 0 \leq k < 8\lambda.$$

We now derive formulas for the convolution kernels of  $T_\lambda^k$ , which we denote by  $K_\lambda^k$ . The identity (31) first gives formulas for the partial Fourier transforms  $\mathcal{F}_{\mathbb{R}^{d_2}} K_\lambda^k$ . Applying the Fourier inversion formula we get

$$(57) \quad K_\lambda^k(x, u) = \lambda^{1/2} \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}} e^{i\frac{\lambda}{4\tau}} a_\lambda(\tau) \eta_0(2\pi|\mu|\frac{\tau}{\lambda} - k\pi) \times \\ \left( \frac{|\mu|}{2 \sin(2\pi|\mu|\tau/\lambda)} \right)^{d_1/2} e^{-i|x|^2 \frac{\pi}{2} |\mu| \cot(2\pi|\mu|\tau/\lambda)} d\tau e^{2\pi i \langle u, \mu \rangle} d\mu.$$

We note that the term  $|\mu| \cot(2\pi t|\mu|)$  in (57) is singular for  $2t|\mu| \in \mathbb{Z} \setminus \{0\}$  and therefore we shall treat the operator  $T_\lambda^0$  separately from  $T_\lambda^k$  for  $k > 0$ . We shall see that  $T_\lambda^0$ , and the operators  $\sum_j \chi(2^{-2j}L)T_{2^j}^0$  can be handled using known results about Fourier integral operator, while the operators  $T_{2^j}^k$  need a more careful treatment due to the singularities of the phase function. We shall see that the decomposition into the operators  $T_{2^j}^k$  encodes useful information on the singularities of the wave kernels.

In §7, §8 we shall prove the following  $L^1$  estimates

**Theorem 5.1.** (i) For  $\lambda \geq 2^{10}$

$$(58) \quad \|T_\lambda^0\|_{L^1 \rightarrow L^1} \lesssim \lambda^{(d-1)/2}.$$

(ii) For  $\lambda \geq 2^{10}$ ,  $k = 1, 2, \dots$ ,

$$(59) \quad \|T_\lambda^k\|_{L^1 \rightarrow L^1} \lesssim k^{-\frac{d_1+1}{2}} \lambda^{(d-1)/2}.$$

Note that  $d_1 \geq 2$  and thus the estimates (59) can be summed in  $k$ . Hence Theorem 1.2 is an immediate consequence of Theorem 5.1.

*Dyadic decompositions.* For the Hardy space bounds we shall need to combine the dyadic pieces in  $j$  and also refine the dyadic decomposition in (50).

Define

$$(60) \quad V_j = 2^{-j(d-1)/2} \chi_1(2^{-2j}L) T_{2^j}^0$$

$$(61) \quad W_j = 2^{-j(d-1)/2} \chi_1(2^{-2j}L) (m_{2^j}(L) - T_{2^j}^0)$$

In section §6 we shall use standard estimates on Fourier integral operators to prove

**Theorem 5.2.** *The operator  $\mathcal{V} = \sum_{j>100} V_j$  extends to a bounded operator from  $h_{\text{iso}}^1$  to  $L^1$ .*

We further decompose the pieces  $W_j$  in (61) and let

$$(62) \quad \begin{aligned} W_{j,0} &= \zeta_0(2^{-j}|U|) W_j \\ W_{j,n} &= \zeta_1(2^{-j-n}|U|) W_j; \end{aligned}$$

here again  $\zeta_0, \zeta_1$  as in §2.1, i.e.  $\zeta_0$  supported in  $(-1, 1)$ ,  $\zeta_1$  supported in  $\pm(1/2, 2)$  so that  $\zeta_0 + \sum_j \zeta_1(2^{1-j}\cdot) \equiv 1$ .

By the description (52) of the joint spectrum of  $L$  and  $|U|$  and the support property (49a) we also have

$$\chi_1(2^{-2j}L) \zeta_1(2^{-j-n}|U|) = 0 \text{ when } 2^{2j+10} \leq 2^{j+n-1},$$

i.e when  $j \leq n - 11$  and thus

$$(63) \quad W_{j,n} = 0 \text{ when } n \geq j + 11.$$

Observe from (52), as in the discussion following (55) that, for  $k = 1, 2, \dots$ ,

$$\zeta_0(2^{-j}\rho_2) \eta_0\left(\frac{\tau}{2^j}\rho_2 - k\pi\right) = 0 \text{ for } \tau \in \left(\frac{1}{16}, 4\right), \rho_2 \geq 0, \text{ if } 2^j \leq (k - \frac{5}{8})\pi 2^j/4,$$

and

$$\begin{aligned} \zeta_1(2^{j-n}\rho_2) \eta_0\left(\frac{\tau}{2^j}\rho_2 - k\pi\right) &= 0 \text{ for } \tau \in \left(\frac{1}{16}, 4\right), \rho_2 \geq 0, \\ &\text{if } 2^{j+n+1} \leq 2^j(k - \frac{5}{8})\pi/4 \text{ or } 16 \cdot 2^j(k + \frac{5}{8})\pi \leq 2^{j+n-1}. \end{aligned}$$

Thus we have for  $k = 1, 2, \dots$ ,

$$\begin{aligned} \zeta_0(2^{-j}|U|) T_{2^j}^k &= 0 \text{ when } k \geq 2, \\ \zeta_1(2^{-j-n}|U|) T_{2^j}^k &= 0 \text{ when } k \notin [2^{n-8}, 2^{n+2}]. \end{aligned}$$

Let

$$(64) \quad \mathcal{J}_n = \begin{cases} \{1\}, & n = 0, \\ \{k : 2^{n-8} \leq k \leq 2^{n+2}\}, & n \geq 1. \end{cases}$$

Then by (54) we have  $m_{2^j}(L) - T_{2^j}^0 = \sum_{k=1}^{\infty} T_{2^j}^k$  and therefore we get

$$(65a) \quad W_{j,0} = 2^{-j(d-1)/2} \chi_1(2^{-2j}L) \zeta_0(2^{-j}|U|) \sum_{k \in \mathcal{J}_0} T_{2^j}^k,$$

$$(65b) \quad W_{j,n} = 2^{-j(d-1)/2} \chi_1(2^{-2j}L) \zeta_1(2^{-j-n}|U|) \sum_{k \in \mathcal{J}_n} T_{2^j}^k.$$

Observe that Theorem 5.1 implies

$$(66) \quad \|W_{j,n}\|_{L^1 \rightarrow L^1} \lesssim 2^{-n(d_1-1)/2}$$

uniformly in  $j$ .

Define for  $n = 0, 1, 2, \dots$

$$(67) \quad \mathcal{W}_n = \sum_{j > 100} W_{j,n}$$

Theorem 1.3 will then be a consequence of Theorem 5.2 and

**Theorem 5.3.** *The operators  $\mathcal{V}$  and  $\mathcal{W}_n$  are bounded from  $h_{\text{iso}}^1$  to  $L^1$ ; moreover*

$$(68) \quad \|\mathcal{W}_n\|_{h_{\text{iso}}^1 \rightarrow L^1} \lesssim (1+n)2^{-n(d_1-1)/2}$$

The proofs will be given in §6 and §9.

## 6. FOURIER INTEGRAL ESTIMATES

In this section we shall reduce the proof of the estimates for  $T_\lambda^0$  and  $\mathcal{V}$  in Theorems 5.1 and 5.3 to standard bounds for Fourier integral operators in [30] or [1].

We will prove a preliminary lemma that allows us to add or suppress  $\chi_1(\lambda^{-2}L)$  from the definition of  $T_\lambda^0$ .

**Lemma 6.1.** *For  $\lambda > 2^{10}$  we have*

$$\|T_\lambda^0 - \chi_1(\lambda^{-2}L)T_\lambda^0\|_{L^1 \rightarrow L^1} \lesssim C_N \lambda^{-N}$$

for any  $N$ .

*Proof.* The operator  $T_\lambda^0 - \chi_1(\lambda^{-2}L)T_\lambda^0$  can be written as  $b_\lambda(|L|, |U|)$  where

$$b_\lambda(\rho_1, \rho_2) = \lambda^{1/2} (1 - \chi_1(\lambda^{-2}\rho_1)) \lambda^{1/2} \int a_\lambda(\tau) e^{i\varphi(\tau, \rho_1, \lambda)} \eta_0(\tau \rho_2 / \lambda) d\tau$$

with

$$\varphi(\tau, \rho_1, \lambda) = \frac{\lambda}{4\tau} + \frac{\tau \rho_1}{\lambda}.$$

Only the values of  $\rho_1 \leq \lambda^2 2^{-9}$  and  $\rho_1 \geq 2^9 \lambda^2$  are relevant. Now

$$\frac{\partial \varphi}{\partial \tau} = -\frac{\lambda}{4\tau^2} + \frac{\rho_1}{\lambda}$$

and  $(\partial/\partial \tau)^n \varphi = c_n \lambda \tau^{-n-1}$  for  $n \geq 2$ . Note that for  $\rho_1 \geq 2^9 \lambda^2$  we have  $|\varphi'_\tau| \geq \rho_1/\lambda - (16^2/4)\lambda \geq \rho_1 \lambda^{-1} (1 - 2^{-9} 2^6) \geq \rho_1/(2\lambda)$ . Similarly for  $\rho_1 \leq$

$2^{-9}\lambda^2$  we have  $|\varphi'_r| \geq \lambda/16 - 16 \cdot 2^{-9}\lambda \geq 2^{-5}\lambda$ . Use integrations by parts to conclude that

$$\left| \frac{\partial^{n_1+n_2}[b_\lambda(\lambda^2 \cdot, \lambda \cdot)]}{(\partial \rho_1)^{n_1}(\partial \rho_2)^{n_2}}(\rho_1, \rho_2) \right| \leq C_{n_1, n_2, N} \lambda^{-N}$$

and in view of the compact support of  $b_\lambda(\lambda^2 \rho_1, \lambda \rho_2)$  the assertion can be deduced from a result in [23] (or alternatively from Hulanicki's result [12] and a Fourier expansion in  $\rho_2$ ).  $\square$

The convolution kernel for  $T_\lambda^0$ . It is given by

$$K_\lambda^0(x, u) = \lambda^{1/2} \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}} e^{i\frac{\lambda}{4s} a_\lambda(s)} \eta_0(2\pi|\mu|\frac{s}{\lambda}) \times \\ \left( \frac{|\mu|}{2 \sin(2\pi|\mu|s/\lambda)} \right)^{d_1/2} e^{-i|x|^2 \frac{\pi}{2} |\mu| \cot(2\pi|\mu|s/\lambda)} ds e^{2\pi i \langle u, \mu \rangle} d\mu.$$

We introduce frequency variables  $\theta = (\omega, \sigma)$  on the cone

$$(69) \quad \Gamma_\delta = \{\theta = (\omega, \sigma) \in \mathbb{R}^{d_2} \times \mathbb{R} : |\omega| \leq (\pi - \delta)\sigma, \sigma > 0\},$$

Set

$$\omega = \frac{\pi\mu}{2}, \quad \sigma = \frac{\lambda}{4s}.$$

Note that  $\sigma \approx \lambda$  for  $s \in \text{supp}(a_\lambda)$ . We note that we will consider the case  $\delta = \pi/4$  in view of the support of  $\eta_0$  but any choice of  $\delta \in (0, \pi/4)$  is permissible with some constants below depending on  $\delta$ .

If we set

$$(70) \quad g(\tau) := \tau \cot \tau,$$

the above integral becomes

$$(71) \quad K_\lambda^0(x, u) = \iint e^{i\Psi(x, u, \omega, \sigma)} \beta_\lambda(\omega, \sigma) d\omega d\sigma$$

with

$$\Psi(x, u, \omega, \sigma) = \sigma(1 - |x|^2 g(|\omega|/\sigma)) + \langle 4u, \omega \rangle$$

and

$$\beta_\lambda(\omega, \sigma) = 4^{-1} \left( \frac{2}{\pi} \right)^{\frac{d_1}{2} + d_2} \lambda^{3/2} \sigma^{\frac{d_1}{2} - 2} a_\lambda\left(\frac{\lambda}{4\sigma}\right) \eta_0\left(\frac{|\omega|}{|\sigma|}\right) \left( \frac{|\omega|}{\sigma} \right) \left( \frac{|\omega|}{\sigma} \right)^{d_1/2}.$$

The  $\beta_\lambda$  are symbols of order  $\frac{d_1-1}{2}$  uniformly in  $\lambda$ , and supported in  $\Gamma$ . The same applies to  $\sum_{k>10} \beta_{2^k}$ .

We will need formulas for the derivatives of  $\Psi$  with respect to the frequency variables  $\theta = (\omega, \sigma)$ :

$$(72) \quad \frac{\partial \Psi}{\partial \omega_i} = 4u_i - |x|^2 \frac{\omega_i}{\sigma} \frac{g'(\frac{|\omega|}{\sigma})}{\frac{|\omega|}{\sigma}} \\ \frac{\partial \Psi}{\partial \sigma} = 1 - |x|^2 \left( g\left(\frac{|\omega|}{\sigma}\right) - \frac{|\omega|}{\sigma} g'\left(\frac{|\omega|}{\sigma}\right) \right)$$

Now  $g$  is analytic for  $|\tau| < 2\pi$  and we have

$$(73a) \quad g'(\tau) = \frac{\sin(2\tau) - 2\tau}{2\sin^2 \tau}$$

$$(73b) \quad g''(\tau) = \frac{2(\tau \cos \tau - \sin \tau)}{\sin^3 \tau}$$

Observe that

$$g'(\tau) < 0 \text{ and } g''(\tau) < 0 \text{ for } 0 < \tau < \pi.$$

Moreover as  $\tau \rightarrow 0$ ,

$$g(\tau) = 1 - \tau^2/3 + O(\tau^4)$$

and hence  $g'(0) = 0$  and  $g''(0) = -2/3$ . The even expression

$$g(\tau) - \tau g'(\tau) = 1 + \int_0^\tau (-s g''(s)) ds$$

will frequently occur; from the above we get

$$(74) \quad \begin{aligned} g(\tau) - \tau g'(\tau) &\geq 1, \text{ for } 0 \leq |\tau| < \pi, \\ |g(\tau) - \tau g'(\tau)| &\leq 10, \text{ for } 0 \leq |\tau| < 3\pi/4. \end{aligned}$$

**Lemma 6.2.** *We have*

$$(75) \quad |K_\lambda^0(x, u)| \lesssim \lambda^{\frac{d_1+2d_2+1}{2}-N} (|x|^2 + |u|)^{-N}, \quad |x|^2 + 4|u| > 2.$$

and

$$(76) \quad |K_\lambda^0(x, u)| \lesssim \lambda^{\frac{d_1+2d_2+1}{2}-N} (1 + |u|)^{-N}, \quad |x|^2 \leq 1/20.$$

*Proof.* If  $|x| \geq \sqrt{2}$  we may integrate by parts with respect to  $\sigma$  (using (74)), and obtain

$$|K_\lambda^0(x, u)| \lesssim_N \lambda^{\frac{d_1+2d_2+1}{2}-N} |x|^{-N}, \quad |x| \geq \sqrt{2}.$$

If  $|u| \leq 10|x|^2$  this also yields (75). Since  $\max_{|\tau| \leq 3\pi/4} |g'(\tau)| \leq 3\pi/2$  we have  $|\nabla_\omega \Psi| \geq 4|u| - (3\pi/2)|x|^2$  and hence  $|\nabla_\omega \Psi| \geq |u|$  when  $|u| \geq 10|x|^2$ . Thus integration by parts in  $\omega$  yields

$$|K_\lambda^0(x, u)| \lesssim_N \lambda^{\frac{d_1+2d_2+1}{2}-N} |u|^{-N}, \quad |u| \geq 10|x|^2.$$

This proves (75).

Since  $|g'(\tau)| \leq 3\pi$  for  $|\tau| \leq 3\pi/2$  we have  $|\nabla_\omega \Psi| \geq 2|u|$  if  $|x|^2 \leq 2|u|/3\pi$  and  $|\Psi_\sigma| \geq 1/2$  if  $|x|^2 \leq 1/20$ . Integrations by parts imply (76).  $\square$

*Fourier integral operators.* Let  $\rho \ll 10^{-2}$ . Let  $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  so that

$$\chi(x, u, y, v) = 0 \text{ for } \begin{cases} |y| + |v| \geq \rho, \\ |x - y| < 1/20, \\ |x - y|^2 + |u - v| \geq 4. \end{cases}$$

Let

$$b_\lambda(x, y, u, v, \omega, \sigma) = \chi(x, u, y, v) \beta_\lambda(\omega, \sigma),$$



let as before  $g(\tau) = \tau \cot \tau$ , and let

$$(77) \quad \begin{aligned} \Phi(x, u, y, v, \omega, \sigma) &= \Psi(x - y, u - v + \frac{1}{2}\langle \vec{J}x, y \rangle, \omega, \sigma) \\ &= \sigma(1 - |x - y|^2 g(|\omega|/\sigma)) + \sum_{i=1}^{d_2} (4u_i - 4v_i - 2x^\top J_i y) \omega_i. \end{aligned}$$

Let  $\mathfrak{F}_\lambda$  be the Fourier integral operator with Schwartz kernel

$$(78) \quad \mathcal{K}_\lambda(x, u, y, v) = \iint e^{i\Phi(x, u, y, v, \omega, \sigma)} b_\lambda(\omega, \sigma) d\omega d\sigma.$$

Given Lemma 6.2 it suffices to prove the inequalities

$$(79) \quad \|\mathfrak{F}_\lambda\|_{L^1 \rightarrow L^1} \leq \lambda^{\frac{d-1}{2}}.$$

and

$$(80) \quad \left\| \sum_{k>C} 2^{-k(d-1)/2} \mathfrak{F}_{2^k} \right\|_{h^1 \rightarrow L^1} < \infty.$$

To this end we apply results in [30] on Fourier integral operators associated with canonical graphs and now check the required hypotheses.

*Analysis of the phase function  $\Phi$ .* We compute the first derivatives:

$$\begin{aligned} \Phi_{x_j} &= -2\sigma(x_j - y_j)g\left(\frac{|\omega|}{\sigma}\right) - 2 \sum_{i=1}^{d_2} \omega_i e_j^\top J_i y \\ \Phi_{u_i} &= 4\omega_i \\ \Phi_{\omega_i} &= -|x - y|^2 g'\left(\frac{|\omega|}{\sigma}\right) \frac{\omega_i}{|\omega|} + 4u_i - 4v_i - 2x^\top J_i y \\ \Phi_\sigma &= (1 - |x - y|^2 g\left(\frac{|\omega|}{\sigma}\right)) + |x - y|^2 \frac{|\omega|}{\sigma} g'\left(\frac{|\omega|}{\sigma}\right) \end{aligned}$$

For the second derivatives we have, with  $\delta_{jk}$  denoting the Kronecker delta and  $J^\omega = \sum_{i=1}^{d_2} \omega_i J_i$

$$\begin{aligned} \Phi_{x_j y_k} &= 2\sigma g\left(\frac{|\omega|}{\sigma}\right) \delta_{jk} - 2e_j^\top J^\omega e_k, \\ \Phi_{x_j v_l} &= 0, \\ \Phi_{x_j \omega_l} &= -2(x_j - y_j)g'\left(\frac{|\omega|}{\sigma}\right) \frac{\omega_l}{|\omega|} - 2e_j^\top J_l y, \\ \Phi_{x_j \sigma} &= 2(x_j - y_j) \left( -g\left(\frac{|\omega|}{\sigma}\right) + \frac{|\omega|}{\sigma} g'\left(\frac{|\omega|}{\sigma}\right) \right), \end{aligned}$$

and

$$\Phi_{u_i y_k} = 0, \quad \Phi_{u_i v_l} = 0, \quad \Phi_{u_i \omega_l} = 4\delta_{il}, \quad \Phi_{u_i \sigma} = 0.$$

Moreover

$$\begin{aligned} \Phi_{\omega_i y_k} &= 2(x_k - y_k)g'\left(\frac{|\omega|}{\sigma}\right) \frac{\omega_i}{|\omega|} - 2x^\top J_i e_k \\ \Phi_{\omega_i v_l} &= -4\delta_{il} \\ \Phi_{\omega_i \omega_l} &= -|x - y|^2 \left( g'\left(\frac{|\omega|}{\sigma}\right) \frac{\delta_{il} |\omega|^2 - \omega_i \omega_l}{|\omega|^3} + g''\left(\frac{|\omega|}{\sigma}\right) \frac{\omega_i \omega_l}{\sigma |\omega|^2} \right) \\ \Phi_{\omega_i \sigma} &= |x - y|^2 \frac{\omega_i}{\sigma^2} g''\left(\frac{|\omega|}{\sigma}\right) \end{aligned}$$

and

$$\begin{aligned}\Phi_{\sigma y_k} &= 2(x_k - y_k)\left(g\left(\frac{|\omega|}{\sigma}\right) - \frac{|\omega|}{\sigma}g'\left(\frac{|\omega|}{\sigma}\right)\right) \\ \Phi_{\sigma v_l} &= 0 \\ \Phi_{\sigma \omega_l} &= |x - y|^2 \frac{\omega_l}{\sigma^2} g''\left(\frac{|\omega|}{\sigma}\right) \\ \Phi_{\sigma \sigma} &= -|x - y|^2 \frac{|\omega|^2}{\sigma^3} g''\left(\frac{|\omega|}{\sigma}\right)\end{aligned}$$

The required  $L^2$  boundedness properties follow if we can show that associated canonical relation is locally the graph of a canonical transformation; this follows from the invertibility of the matrix

$$(81) \quad \begin{pmatrix} \Phi_{xy} & \Phi_{xv} & \Phi_{x\omega} & \Phi_{x\sigma} \\ \Phi_{uy} & \Phi_{uv} & \Phi_{u\omega} & \Phi_{u\sigma} \\ \Phi_{\omega y} & \Phi_{\omega v} & \Phi_{\omega\omega} & \Phi_{\omega\sigma} \\ \Phi_{\sigma y} & \Phi_{\sigma v} & \Phi_{\sigma\omega} & \Phi_{\sigma\sigma} \end{pmatrix},$$

see [11]. This matrix is given by

$$\begin{pmatrix} 2\sigma g I_{d_1} - 2J^\omega & 0 & (*)_{13} & 2(x - y)(\tau g' - g) \\ 0 & 0 & 4I_{d_2} & 0 \\ (*)_{31} & -4I_{d_2} & (*)_{33} & (*)_{34} \\ 2(x - y)^\top(g - \tau g') & 0 & (*)_{43} & -|x - y|^2 \sigma^{-1} \tau^2 g'' \end{pmatrix},$$

where  $\tau = \frac{|\omega|}{\sigma}$ ,  $g, g', g''$  are evaluated at  $\tau = \frac{|\omega|}{\sigma}$ , and  $x - y$  is considered a  $d_1 \times 1$  matrix,  $(*)_{13}$  is a  $d_1 \times d_2$ -matrix,  $(*)_{31}$  is a  $d_2 \times d_1$ -matrix,  $(*)_{33}$  is a  $d_2 \times d_2$ -matrix,  $(*)_{34}$  is a  $d_2 \times 1$ -matrix, and  $(*)_{43} = (*)_{34}^\top$ .

The determinant  $D$  of the displayed matrix is equal to

$$(82) \quad D = 16^{d_2} \det \begin{pmatrix} 2\sigma g I_{d_1} - 2J^\omega & 2(x - y)(\tau g' - g) \\ 2(x - y)^\top(g - \tau g') & -|x - y|^2 \sigma^{-1} \tau^2 g'' \end{pmatrix}.$$

To compute this we use the formula

$$\begin{pmatrix} I & 0 \\ a^\top & 1 \end{pmatrix} \begin{pmatrix} A & -b \\ b^\top & \gamma \end{pmatrix} \begin{pmatrix} I & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & -Aa - b \\ a^\top A + b^\top & -a^\top Aa - 2a^\top b + \gamma \end{pmatrix}.$$

If  $A$  is invertible we can choose  $a = -A^{-1}b$ . Since  $b^\top S b = 0$  for the skew symmetric matrix  $S = (A^{-1})^\top - A^{-1}$  this choice of  $a$  yields the matrix

$$\begin{pmatrix} A & 0 \\ -b^\top (A^{-1})^\top A + b^\top & -b^\top (A^{-1})^\top b - 2b^\top A^{-1}b + \gamma \end{pmatrix} = \begin{pmatrix} A & 0 \\ * & \gamma + b^\top A^{-1}b \end{pmatrix}$$

and hence

$$(83) \quad \det \begin{pmatrix} A & -b \\ b^\top & \gamma \end{pmatrix} = (\gamma + b^\top A^{-1}b) \det(A).$$

**Lemma 6.3.** *Let  $c, \Lambda \in \mathbb{R}$ ,  $c^2 + \Lambda^2 \neq 0$ . Let  $S$  be a skew symmetric  $d_1 \times d_1$ -matrix satisfying  $S^2 = -\Lambda^2 I$ . Then  $cI + S$  is invertible with*

$$(cI + S)^{-1} = \frac{c}{c^2 + \Lambda^2} I - \frac{1}{c^2 + \Lambda^2} S,$$

and  $\det(cI + S) = (c^2 + \Lambda^2)^{\frac{d_1}{2}}$ .

*Proof.*  $(cI + S)(cI + S)^* = (cI + S)(cI - S) = c^2I - S^2 = (c^2 + \Lambda^2)I$ .  $\square$

In our situation (82) we have  $A = cI + S$ , with

$$\begin{aligned} c &= 2\sigma g\left(\frac{|\omega|}{\sigma}\right), \\ S &= -2J^\omega, \end{aligned}$$

moreover,

$$\begin{aligned} \Lambda &= 2|\omega|, \\ \gamma &= -|x - y|^2 \sigma^{-1} \left(\frac{|\omega|}{\sigma}\right)^2 g''\left(\frac{|\omega|}{\sigma}\right), \\ b &= 2(x - y) \left(g\left(\frac{|\omega|}{\sigma}\right) - \frac{|\omega|}{\sigma} g'\left(\frac{|\omega|}{\sigma}\right)\right). \end{aligned}$$

In particular, if we recall that  $\tau = |\omega|/\sigma$ , we see that

$$\det A = \left((2\sigma g(\tau))^2 + (2|\omega|)^2\right)^{\frac{d_1}{2}} = (2\sigma)^{d_1} \left(\frac{\tau}{\sin \tau}\right)^{d_1}.$$

Moreover,

$$\begin{aligned} &\gamma + b^\top A^{-1} b \\ &= |x - y|^2 \left( -\frac{|\omega|^2}{\sigma^3} g''\left(\frac{|\omega|}{\sigma}\right) + 4 \left(g\left(\frac{|\omega|}{\sigma}\right) - \frac{|\omega|}{\sigma} g'\left(\frac{|\omega|}{\sigma}\right)\right)^2 \frac{2\sigma g\left(\frac{|\omega|}{\sigma}\right)}{4\sigma^2 g\left(\frac{|\omega|}{\sigma}\right)^2 + 4|\omega|^2} \right) \\ &= \frac{|x - y|^2}{\sigma} \left( -\tau^2 g''(\tau) + 2(g(\tau) - \tau g'(\tau))^2 \frac{g(\tau)}{g(\tau)^2 + \tau^2} \right). \end{aligned}$$

From (73a), we get

$$g(\tau) - \tau g'(\tau) = \left(\frac{\tau}{\sin \tau}\right)^2,$$

and in combination with (73b) this implies after a calculation that

$$\gamma + b^\top A^{-1} b = \frac{|x - y|^2}{\sigma} 2 \left(\frac{\tau}{\sin \tau}\right)^2.$$

Thus we see from (83) that the determinant of the matrix (81) is given by

$$(84) \quad D = 2^{d_1+4d_2+1} \sigma^{d_1-1} \left(\frac{|\omega|}{\sin \frac{|\omega|}{\sigma}}\right)^{d_1+2}.$$

This shows that  $D > 0$  for  $\frac{|\omega|}{\sigma} \in [0, \pi)$ , and  $D \sim \sigma^{d_1-1}$  for  $\frac{|\omega|}{\sigma} \in [0, \pi - \delta]$ , for every sufficiently small  $\delta > 0$ . In particular, the matrix (81) is invertible for  $\frac{|\omega|}{\sigma} \in [0, \pi - \delta]$ .

We now write

$$\mathfrak{F}_\lambda f(x) = \int K_\lambda(x, y) f(y) dy$$

where  $K_\lambda$  is given by our oscillatory integral representation (78). In that formula we have  $d_2 + 1$  frequency variables  $d_2 + 1$ , and thus, given any  $\alpha \in \mathbb{R}$

the operator convolution with  $\sum_{k>C} \mathfrak{F}_{2^k} 2^{-k\alpha}$  is a Fourier integral operator of order

$$\frac{d_1 - 1}{2} - \alpha - \frac{d - (d_2 + 1)}{2} = -\alpha.$$

With these observations we can now apply the boundedness result of [30] and deduce that

$$\|\mathfrak{F}_\lambda f\|_1 \lesssim \lambda^{\frac{d-1}{2}} \|f\|_1$$

and

$$\left\| \sum_{k>C} 2^{-k(d-1)/2} \mathcal{F}_{2^k} f_\rho \right\|_1 \lesssim 1$$

for standard  $h_1$  atoms supported in  $B_\rho$ . But atoms associated to balls centered at the origin are also atoms in our Heisenberg Hardy space  $h_{\text{iso}}^1$ . Thus if we also take into account Lemma 6.2 and use translation invariance under Heisenberg translations we get

$$\left\| \sum_{k \geq 0} T_{2^k}^0 f \right\|_1 \lesssim \|f\|_{h_{\text{iso}}^1}.$$

*Remark.* We also have

$$\begin{pmatrix} \Phi_{\omega\omega} & \Phi_{\omega\sigma} \\ \Phi_{\sigma\omega} & \Phi_{\sigma\sigma} \end{pmatrix} = |x - y|^2 \begin{pmatrix} -\left(g'(\frac{|\omega|}{\sigma}) \frac{I_{d_2} |\omega|^2 - \omega\omega^\top}{|\omega|^3} + g''(\frac{|\omega|}{\sigma}) \frac{\omega\omega^\top}{\sigma|\omega|^2}\right) & \frac{\omega}{\sigma^2} g''(\frac{|\omega|}{\sigma}) \\ \frac{\omega^\top}{\sigma^2} g''(\frac{|\omega|}{\sigma}) & -\frac{|\omega|^2}{\sigma^3} g''(\frac{|\omega|}{\sigma}) \end{pmatrix}$$

which has maximal rank  $d_2 + 1 - 1 = d_2$ . Thus the above result can also be deduced from Beals [1], via the equivalence of phase functions theorem.

## 7. THE OPERATORS $T_\lambda^k$

We now consider the operator  $T_k^\lambda$ , for  $k \geq 1$ , as defined in (55). In view of the singularities of  $\cot$  we need a further decomposition in terms of the distance to the singularities. For  $l = 1, 2, \dots$  let  $\eta_l(s) = \eta_0(2^{l-1}s) - \eta_0(2^l s)$  so that

$$\eta_0(s) = \sum_{l=1}^{\infty} \eta_l(s) \text{ for } s \neq 0.$$

Define

$$(85) \quad T_\lambda^{k,l} = \lambda^{1/2} \int e^{i\frac{\lambda}{4\tau}} a_\lambda(\tau) \eta_l\left(\frac{\tau}{\lambda}|U| - k\pi\right) e^{i\tau L/\lambda} d\tau;$$

then

$$(86) \quad T_\lambda^k = \sum_{l=1}^{\infty} T_\lambda^{k,l}.$$

From the formula (57) for the kernels  $K_\lambda^k$  we get a corresponding formula for the kernels  $K_\lambda^{k,l}$ , namely

$$K_\lambda^{k,l}(x, u) = \lambda^{1/2} \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}} e^{i\frac{\lambda}{4\tau}} a_\lambda(\tau) \eta_l(2\pi|\mu|\frac{\tau}{\lambda} - k\pi) \times \\ \left( \frac{|\mu|}{2 \sin(2\pi|\mu|\tau/\lambda)} \right)^{d_1/2} e^{-i|x|^2 \frac{\pi}{2} |\mu| \cot(2\pi|\mu|\tau/\lambda)} d\tau e^{2\pi i \langle u, \mu \rangle} d\mu.$$

Now we use polar coordinates in  $\mathbb{R}^{d_2}$  and the fact that the Fourier transform of the surface carried measure on the unit sphere in  $\mathbb{R}^{d_2}$  is given by

$$(2\pi)^{d_2/2} \mathcal{J}_{d_2}(2\pi|u|), \text{ with } \mathcal{J}_{d_2}(\sigma) := \sigma^{-\frac{d_2-2}{2}} J_{\frac{d_2-2}{2}}(\sigma)$$

(the standard Bessel function formula, *cf.* [32], p.154). Thus

$$K_\lambda^{k,l}(x, u) = \lambda^{1/2} \int_0^\infty \int_{\mathbb{R}} e^{i\frac{\lambda}{4\tau}} a_\lambda(\tau) \eta_l(2\pi\tau\rho/\lambda - k\pi) \times \\ \left( \frac{\rho}{2 \sin(2\pi\tau\rho/\lambda)} \right)^{d_1/2} e^{-i\frac{\pi}{2}|x|^2 \rho \cot(2\pi\tau\rho/\lambda)} d\tau (2\pi)^{d_2/2} \mathcal{J}_{d_2}(2\pi\rho|u|) \rho^{d_2-1} d\rho.$$

In this integral we introduce new variables

$$(87) \quad (s, t) = \left( \frac{1}{4\tau}, \frac{2\pi\tau\rho}{\lambda} \right),$$

so that  $(\tau, \rho) = ((4s)^{-1}, 2\lambda ts/\pi)$  with  $d\tau d\rho = \lambda(2\pi s)^{-1} ds dt$ . Then we obtain for  $k \geq 1$

$$(88) \quad K_\lambda^{k,l}(x, u) = \lambda^{d_2 + \frac{d_1+1}{2}} \times \\ \iint \beta_\lambda(s) \eta_l(t - k\pi) \left( \frac{t}{\sin t} \right)^{d_1/2} t^{d_2-1} e^{i\lambda s \psi(t, |x|)} \mathcal{J}_{d_2}(4s\lambda t|u|) ds dt$$

where

$$(89) \quad \psi(t, r) = 1 - r^2 t \cot t$$

and

$$(90) \quad \beta_\lambda(s) = 2^{\frac{3d_2}{2}-2} \pi^{-\frac{d_1+d_2}{2}} a_\lambda\left(\frac{1}{4s}\right) s^{\frac{d_1}{2}+d_2-2};$$

thus  $\beta_\lambda$  is  $C^\infty$  with bounds uniform in  $\lambda$ , and  $\beta_\lambda$  is also supported in  $[1/16, 4]$ .

In the next two sections we shall prove the  $L^1$  estimates

$$(91) \quad \sum_{k < 8\lambda} \sum_{l=0}^\infty \iint \lambda^{-\frac{d-1}{2}} |K_\lambda^{k,l}(x, u)| dx du = O(1)$$

and Theorem 5.1 and then also Theorem 1.2 will follow by summing the pieces. Moreover we shall give some refined estimates which will be used in the proof of Theorem 5.3.

7.1. *An  $L^\infty$  bound for the kernels.* The expression

$$(92) \quad \mathfrak{C}_{\lambda,k,l} = \lambda^{1+\frac{d_2}{2}} k^{d_2-1} (2^l k)^{\frac{d_1}{2}}$$

will frequently appear in pointwise estimates, namely as upper bounds for the integrand in the integral defining  $\lambda^{-\frac{d-1}{2}} K_\lambda^{k,l}$ . Note that

$$(93) \quad \|\lambda^{-\frac{d-1}{2}} K_\lambda^{k,l}\|_\infty \lesssim 2^{-l} \mathfrak{C}_{\lambda,k,l};$$

the additional factor of  $2^{-l}$  occurs since the integration in  $t$  is over the union of two intervals of length  $\approx 2^{-l}$ .

7.2. *Formulas for the phase functions.* For later reference we gather some formulas for the  $t$ -derivatives of the phase  $\psi(t, r) = 1 - r^2 t \cot t$ :

$$(94a) \quad \psi_t(t, r) = r^2 \left( \frac{t}{\sin^2 t} - \cot t \right)$$

$$(94b) \quad = r^2 \left( \frac{2t - \sin(2t)}{2 \sin^2 t} \right);$$

moreover

$$(95) \quad \psi_{tt}(t, r) = \frac{2r^2}{\sin^3 t} (\sin t - t \cos t) = \frac{2r^2}{\sin^3 t} \int_0^t \tau \sin \tau \, d\tau.$$

Observe that  $\psi_{tt} = 0$  when  $\tan t = t$  and  $t \neq 0$  and thus  $\psi_{tt}(t, r) \approx r^2$  for  $0 \leq t \leq \frac{3\pi}{4}$ , namely, we use  $\frac{2\sqrt{2}}{3\pi}t \leq \sin t \leq t$  to get the crude estimate

$$(96) \quad \pi^{-1} r^2 < \psi_{tt}(t, r) < \pi^3 r^2, \quad 0 < t \leq \frac{3\pi}{4}.$$

It is also straightforward to establish estimates for the higher derivatives:

$$(97) \quad |\partial_t^n \psi(t, r)| \lesssim r^2, \quad |t| \leq 3\pi/4$$

and

$$(98) \quad \partial_t^n \psi(t, r) = O\left(\frac{r^2 |t|}{|\sin t|^{n+1}}\right),$$

for all  $t$ .

7.3. *Asymptotics in the main case*  $|u| \gg (k\lambda)^{-1}$ . We shall see in the next section that there are straightforward  $L^1$  bounds in the region where  $|u| \lesssim (k+1)^{-1} \lambda^{-1}$ . We therefore concentrate on the region

$$\{(x, u) : |u| \geq C(k+1)^{-1} \lambda^{-1}\}$$

where we have to take into account the oscillation of the terms  $\mathcal{J}_{d_2}(4s\lambda t|u|)$ .

The standard asymptotics for Bessel functions imply that for

$$(99) \quad \mathcal{J}_{d_2}(\sigma) = e^{-i|\sigma|} \varpi_1(|\sigma|) + e^{i|\sigma|} \varpi_2(|\sigma|), \quad |\sigma| \geq 2,$$

where  $\varpi_1, \varpi_2 \in S^{-(d_2-1)/2}$  are supported in  $\mathbb{R} \setminus [-1, 1]$ .

Thus we may split, for  $|u| \gg (k+1)^{-1} \lambda^{-1}$ ,

$$(100) \quad \lambda^{-\frac{d-1}{2}} K_\lambda^{k,l}(x, u) = A_\lambda^{k,l}(x, u) + B_\lambda^{k,l}(x, u)$$

where, with  $\mathfrak{C}_{\lambda,k,l}$  defined in (92),

$$(101) \quad A_\lambda^{k,l}(x, u) = \mathfrak{C}_{\lambda,k,l} \iint \eta_{\lambda,k,l}(s, t) e^{i\lambda s(\psi(t, |x|) - 4t|u|)} \varpi_1(4\lambda st|u|) dt ds,$$

and

$$(102) \quad B_\lambda^{k,l}(x, u) = \mathfrak{C}_{\lambda,k,l} \iint \eta_{\lambda,k,l}(s, t) e^{i\lambda s(\psi(t, |x|) + 4t|u|)} \varpi_2(4\lambda st|u|) dt ds;$$

here, as before  $\psi(t, r) = 1 - r^2 t \cot t$  and, with  $\beta_\lambda$  as in (90),

$$(103a) \quad \eta_{\lambda,0}(s, t) = \beta_\lambda(s) \eta_0(t) \left( \frac{t}{\sin t} \right)^{d_1/2} t^{d_2-1},$$

$$(103b) \quad \eta_{\lambda,k,l}(s, t) = \beta_\lambda(s) \eta_l(t - k\pi) \left( \frac{t/k}{2^l \sin t} \right)^{d_1/2} (t/k)^{d_2-1}.$$

Note that  $\|\partial_s^{N_1} \partial_t^{N_2} \eta_{\lambda,k,l}\|_\infty \leq C_{N_1, N_2} 2^{lN_2}$ . Moreover if

$$(104) \quad J_{k,l} := (k\pi - 2^{-l} \frac{5\pi}{4}, k\pi - 2^{-l} \frac{3\pi}{8}] \cup [k\pi + 2^{-l} \frac{3\pi}{8}, k\pi + 2^{-l} \frac{5\pi}{4})$$

then

$$(105) \quad \eta_{\lambda,k,l}(s, t) \neq 0 \implies t \in J_{k,l}.$$

The main contribution in our estimates comes from the kernels  $A_\lambda^{k,l}$  while the kernels  $B_\lambda^{k,l}$  are negligible terms with rather small  $L^1$  norm. The latter will follow from the support properties of  $\eta_{\lambda,k,l}$  and the observation that

$$\partial_t(\psi(t, |x|) + 4t|u|) \neq 0, \quad (x, u) \neq (0, 0);$$

cf. (94b). As a consequence only the kernels  $A_\lambda^{k,l}$  will exhibit the singularities of the kernel away from the origin.

**7.4. The phase functions and the singular support.** We introduce polar coordinates in  $\mathbb{R}^{d_1}$  and  $\mathbb{R}^{d_2}$  (scaled by a factor of 4 in the latter) and set

$$r = |x|, \quad v = 4|u|.$$

We define for all  $v \in \mathbb{R}$ ,

$$(106) \quad \phi(t, r, v) := \psi(t, r) - tv = 1 - r^2 t \cot t - tv.$$

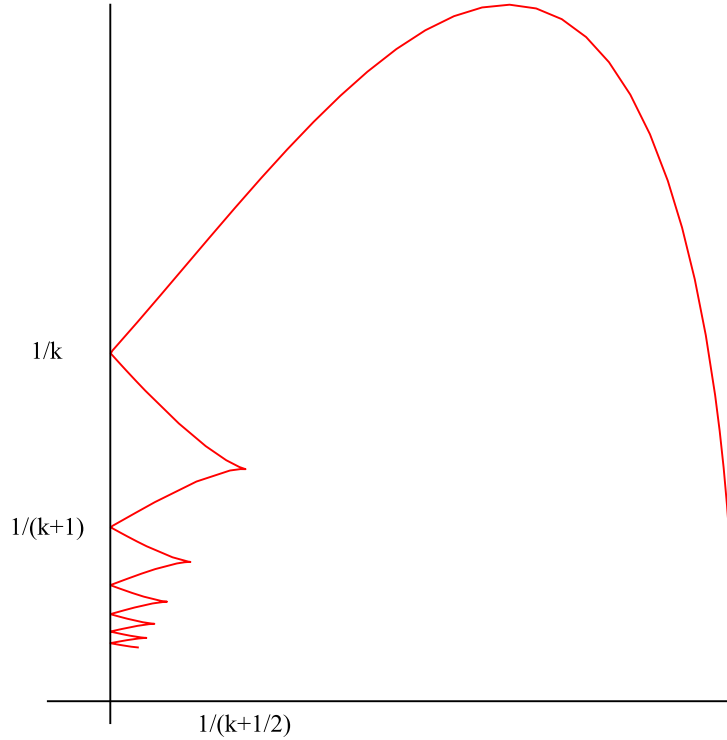
Then from (94b) and (94a)

$$(107) \quad \begin{aligned} \phi_t(t, r, v) &= r^2 \left( \frac{2t - \sin(2t)}{2 \sin^2 t} \right) - v \\ &= \frac{r^2 t}{\sin^2 t} - \frac{1}{t} + \frac{\phi(t, r, v)}{t}. \end{aligned}$$

Moreover  $\phi_{tt} = \psi_{tt}$ , and we will use the formulas (95) and (98) for the derivatives of  $\phi_t$ .

If we set

$$(108) \quad \begin{aligned} r(t) &= \left| \frac{\sin t}{t} \right|, \quad v(t) = \frac{1}{t} - \frac{\sin(2t)}{2t^2} \\ r(0) &= 1, \quad v(0) = 0 \end{aligned}$$

FIGURE 1.  $\{\pi(r(t), v(t)) : t > 0\}$ 

then we have

$$(109a) \quad \phi_t(t, r, v) = \frac{v(t)}{r^2(t)} r^2 - v = -\left(v - v(t) - v(t) \frac{r^2 - r(t)^2}{r(t)^2}\right),$$

$$(109b) \quad \phi(t, r, v) = \frac{r(t)^2 - r^2}{r(t)^2} + t\phi_t(t, r, v).$$

Thus

$$(110) \quad \phi(t, r, v) = \phi_t(t, r, v) = 0 \iff (r, v) = (r(t), v(t)).$$

Only the points  $(r, v)$  for which there exists a  $t$  satisfying (110) may contribute to the singular support  $\Gamma$  of  $e^{i\sqrt{L}}\delta_0$ . One recognizes the result by Nachman [26] who showed for the Heisenberg group that the singular support of the convolution kernel of  $e^{i\sqrt{L}}$  consists of those  $(x, u)$  for which there is a  $t > 0$  with  $(|x|, 4|u|) = (r(t), v(t))$ .

The figure pictures the singular support, including the contribution near  $|u| = 0$  and  $|x|$  near 1. However we have taken care of the corresponding



estimates in §6, and thus we are only interested in the above formulas for  $t > 3\pi/8$ .

For later reference we gather some formulas and estimates for the derivatives of  $r(t)$  and  $v(t)$ . For the vector of first derivatives we get, for  $t \notin \pi\mathbb{Z}$ ,

$$(111) \quad \begin{pmatrix} r'(t) \\ v'(t) \end{pmatrix} = \frac{\sin t - t \cos t}{t^2} \begin{pmatrix} -\text{sign}((\sin t)/t) \\ 2t^{-1} \cos t \end{pmatrix}$$

with  $r'(t) = O(t)$  and  $v'(t) - \frac{2}{3} = O(t)$  as  $t \rightarrow 0$ . Hence, for  $t \notin \pi\mathbb{Z}$ ,

$$(112) \quad \frac{v'(t)}{r'(t)} = -\text{sign}((\sin t)/t) \frac{2 \cos t}{t} = -2r(t) \cot t.$$

Clearly all derivatives of  $t$  and  $v$  extend to functions continuous at  $t = 0$ . Further computation yields for positive  $t \notin \pi\mathbb{Z}$ ,  $\nu \geq 1$ ,

$$(113a) \quad \text{sign}\left(\frac{\sin t}{t}\right) r^{(\nu)}(t) = \sum_{n=1}^{\nu+1} a_{n,\nu} t^{-n} \sin t + \sum_{n=1}^{\nu} b_{n,\nu} t^{-n} \cos t$$

and

$$(113b) \quad v^{(\nu)}(t) = \gamma_\nu t^{-\nu-1} + \sum_{n=1}^{\nu+1} c_{n,\nu} t^{-n-1} \sin 2t + \sum_{n=1}^{\nu} d_{n,\nu} t^{-n-1} \cos 2t;$$

here  $a_{n,\nu} = c_{n,\nu} = 0$  if  $n - \nu$  is even, and  $b_{n,\nu} = d_{n,\nu} = 0$  if  $n - \nu$  is odd; moreover  $\gamma_\nu = (-1)^\nu (\nu - 1)!$ , and  $a_{1,\nu} = (-1)^{\nu/2}$  for  $\nu = 2, 4, \dots$ . For the coefficients in the first derivatives formula we get  $b_{1,1} = 1$ ,  $a_{2,1} = -1$ ,  $d_{1,1} = -1$ , and  $c_{2,1} = 1$ . For the second derivatives, we have the coefficients  $a_{1,2} = -1$ ,  $b_{2,2} = -2$ ,  $a_{3,2} = 2$ ,  $c_{1,2} = 2$ ,  $d_{2,2} = 4$ ,  $c_{3,2} = -3$ . Consequently, for the second derivatives we get the estimates

$$(114) \quad |r''(t)| \lesssim t^{-1} |\sin t| + (1+t)^{-2}, \quad |v''(t)| \lesssim t^{-2} |\sin 2t| + (1+t)^{-3}.$$

Also,  $|r^{(\nu)}(t)| \lesssim_\nu (1+t)^{-1}$ , and  $|v^{(\nu)}(t)| \lesssim_\nu (1+t)^{-2}$  for all  $t > 0$ .

## 8. $L^1$ ESTIMATES

In this section we prove the essential  $L^1$  bounds needed for the proof of Theorem 1.2. We may assume that  $\lambda$  is large.

In what follows we frequently need to perform repeated integrations by parts in the presence of oscillatory terms with nonlinear phase functions and we start with a standard calculus lemma which will be used several times.

8.1. *Two preliminary lemmata.* Let  $\eta \in C_0^\infty(\mathbb{R}^n)$  and let  $\Phi \in C^\infty$  so that  $\nabla\Phi \neq 0$  in the support of  $\eta$ . Then, after repeated integration by parts,

$$(115) \quad \int e^{i\lambda\Phi(y)} \eta(y) dy = (i/\lambda)^N \int e^{i\lambda\Phi(y)} \mathcal{L}^N \eta(y) dy$$

where the operator  $\mathcal{L}$  is defined by

$$(116) \quad \mathcal{L}a = \text{div}\left(\frac{a\nabla\Phi}{|\nabla\Phi|^2}\right).$$

In order to analyze the behavior of  $\mathcal{L}^N$  we shall need a lemma. We use multiindex notation, i.e. for  $\beta = (\beta^1, \dots, \beta^n) \in (\mathbb{N} \cup \{0\})^n$  we write  $\partial^\beta = \partial_{y_1}^{\beta^1} \cdots \partial_{y_n}^{\beta^n}$  and let  $|\beta| = \sum_{i=1}^n \beta^i$  be the order of the multiindex.

**Lemma 8.1.** *Let  $\mathcal{L}$  be as in (116). Then  $\mathcal{L}^N a$  is a linear combination of  $C(N, n)$  terms of the form*

$$\frac{\partial^\alpha a \prod_{\nu=1}^j \partial^{\beta_\nu} \Phi}{|\nabla \Phi|^{4N}}$$

where  $2N \leq j \leq 4N - 1$  and  $\alpha, \beta_1, \dots, \beta_j$  are multiindices in  $(\mathbb{N} \cup \{0\})^n$  with  $1 \leq |\beta_\nu| \leq |\beta_{\nu+1}|$ , satisfying

- (1)  $0 \leq |\alpha| \leq N$ ,
- (2)  $|\beta_\nu| = 1$  for  $\nu = 1, \dots, 2N$ ,
- (3)  $|\alpha| + \sum_{\nu=1}^j |\beta_\nu| = 4N$ ,
- (4)  $\sum_{\nu=1}^j (|\beta_\nu| - 1) = N - |\alpha|$ .

*Proof.* Use induction on  $N$ . We omit the straightforward details.  $\square$

*Remark:* In dimension  $n = 1$  we see that  $\mathcal{L}^N a$  is a linear combination of  $C(N, 1)$  terms of the form

$$\frac{a^{(\alpha)}}{(\Phi')^\alpha} \prod_{\beta \in \mathfrak{J}} \frac{\Phi^{(\beta)}}{(\Phi')^\beta},$$

where  $\mathfrak{J}$  is a set of integers  $\beta \in \{2, \dots, N + 1\}$  with the property that  $\sum_{\beta \in \mathfrak{J}} (\beta - 1) = N - \alpha$ . If  $\mathfrak{J}$  is the empty set then we interpret the product as 1.

In what follows we shall often use the following

**Lemma 8.2.** *Let  $\Lambda > 0$ ,  $\rho > 0$ ,  $n \geq 1$  and  $N > \frac{n+1}{2}$ . Then*

$$\int_{-\infty}^{\infty} \frac{(1 + \Lambda|v|)^{-\frac{n-1}{2}} |v|^{n-1}}{(1 + \Lambda|\rho - v|)^N} dv \lesssim_n \begin{cases} \Lambda^{-\frac{n+1}{2}} \rho^{\frac{n-1}{2}} & \text{if } \Lambda\rho \geq 1, \\ \Lambda^{-n} & \text{if } \Lambda\rho \leq 1. \end{cases}$$

We omit the proof. Lemma 8.2 will usually be applied after using integration by parts with respect to the  $s$ -variable, with the parameters  $n = d_2$  and  $\Lambda = \lambda k$ .

8.2. *Estimates for  $|u| \lesssim (k + 1)^{-1} \lambda^{-1}$ .* We begin by proving an  $L^1$  bound for the part of the kernels  $K_\lambda^{k,l}$  for which the terms  $\mathcal{J}_{d_2}(4s\lambda t|u|)$  have no significant oscillation, i.e. for the region where  $|u| \leq C(\lambda k)^{-1}$  (or  $|u| \lesssim \lambda^{-1}$  if  $k = 0$ ).

**Lemma 8.3.** *Let  $\lambda \geq 1$ ,  $k \geq 1$ ,  $l \geq 1$ . Then*

$$(117) \quad \iint_{|u| \lesssim (\lambda k)^{-1}} |\lambda^{-\frac{d-1}{2}} K_\lambda^{k,l}(x, u)| dx du \lesssim (2^l k)^{-1} \lambda^{1-\frac{d}{2}}.$$

*Proof.* First we integrate the pointwise bound (93) over the region where  $|x| \leq (\lambda k 2^l)^{-1/2}$ ,  $|u| \leq (\lambda k)^{-1}$  and obtain

$$\begin{aligned} & \iint_{\substack{|x| \leq C(\lambda k 2^l)^{-1/2} \\ |u| \leq C(\lambda k)^{-1}}} |\lambda^{-\frac{d_1-1}{2}} K_\lambda^{k,l}(x, u)| dx du \\ & \lesssim 2^{-l} \mathfrak{E}_{\lambda,k,l} (\lambda k 2^l)^{-d_1/2} (\lambda k)^{-d_2} = (2^l k)^{-1} \lambda^{1-\frac{d_1+d_2}{2}}. \end{aligned}$$

If  $|x| \geq C(\lambda k 2^l)^{-1/2}$  then from (94b), (98) we get that  $|\psi_t(t, |x|)| \gtrsim 2^{2l} k |x|^2$  on the support of  $\eta_l(t - k\pi)$ , moreover  $(\partial/\partial t)^{(n)} \psi(t, |x|) = O(|x|^2 k 2^{l(n+1)})$ . The  $n$ th  $t$ -derivative of  $\eta_l(t - k\pi) \mathcal{J}_{d_2}(4s\lambda t|u|)$  is  $O(2^{ln})$ . Thus an integration by parts gives

$$\lambda^{-\frac{d_1-1}{2}} |K_\lambda^{k,l}(x, u)| \leq C_N 2^{-l} \mathfrak{E}_{\lambda,k,l} (\lambda 2^l k |x|^2)^{-N}$$

for  $|x| \geq (\lambda k 2^l)^{-1/2}$  and  $|u| \leq (\lambda k)^{-1}$ . The bound  $O((2^l k)^{-1} \lambda^{1-\frac{d_1}{2}})$  follows by integration by parts.  $\square$

8.3. *Estimates for  $|u| \gg (k+1)^{-1} \lambda^{-1}$ .* We now proceed to give  $L^1$  estimates for the kernels  $A_\lambda^{k,l}$  and  $B_\lambda^{k,l}$  for  $k \geq 1$ , in the region where  $|u| \gg (k\lambda)^{-1}$ .

8.3.1. *An estimate for small  $x$ .* As a first application we prove  $L^1$  estimates for  $|x| \lesssim (2^l \lambda k)^{-1/2}$ ,  $k \geq 1$ .

**Lemma 8.4.** *Let  $C \geq 1$ . Then*

$$(118) \quad \iint_{\substack{(x,u): \\ |x| \leq C(2^l \lambda k)^{-1/2}}} [ |A_\lambda^{k,l}(x, u)| + |B_\lambda^{k,l}(x, u)| ] dx du \lesssim_C (2^l k)^{-1} \lambda^{-\frac{d_1-1}{2}}.$$

*Proof.* Integration by parts with respect to  $s$  yields

$$(119) \quad |A_\lambda^{k,l}(x, u)| + |B_\lambda^{k,l}(x, u)| \lesssim_N \sum_{\pm} \frac{\mathfrak{E}_{\lambda,k,l}}{(1 + \lambda k |u|)^{\frac{d_2-1}{2}}} \int_{|t-k\pi| \lesssim 2^{-l}} (1 + \lambda k |\pm 4u| - |x|^2 \cot t + t^{-1})^{-N} dt.$$

We first integrate in  $u$ . Notice that by Lemma 8.2 we have for fixed  $t$  and fixed  $r \leq (2^l \lambda k)^{-1/2}$

$$\int_0^\infty \frac{(1 + \lambda k v)^{-\frac{d_2-1}{2}} v^{d_2-1}}{(1 + \lambda k |\pm |v| - r^2 \cot t + t^{-1}|)^N} dv \lesssim \lambda^{-\frac{d_2+1}{2}} k^{-d_2}.$$

We integrate in  $x$  over a set of measure  $\lesssim (2^l k \lambda)^{-d_1/2}$  and then in  $t$  (over an interval of length  $\approx 2^{-l}$ ) and (118) follows.  $\square$

8.3.2.  $L^1$ -bounds for  $B_\lambda^{k,l}$ .

**Lemma 8.5.** For  $\lambda \geq 1$ ,  $0 < k \leq 8\lambda$ ,

$$(120) \quad \|B_\lambda^{k,l}\|_1 \lesssim (2^l k)^{-1} \lambda^{-\frac{d_1-1}{2}}.$$

*Proof.* The bound for the region with  $|x| \lesssim (2^l \lambda k)^{-1/2}$  (for which there is no significant oscillation in the  $t$  integral) is proved in Lemma 8.4.

Consider the region where  $|x| \approx 2^m (2^l \lambda k)^{-1/2}$ . We perform  $N_1$  integration by parts in  $t$  followed by  $N_2$  integrations by parts with respect to  $s$ . Denote by  $\mathcal{L}_t$  the operator defined by  $\mathcal{L}_t g = \partial_t \left( \frac{g(t)}{\psi_t(t, |x|) + 4|u|} \right)$ . Then

$$B_\lambda^{k,l}(x, u) = \mathfrak{C}_{\lambda,k,l}(i/\lambda)^{N_1} \times \\ \iint e^{i\lambda s(\psi(t, |x|) + 4t|u|)} \frac{(I - \partial_s^2)^{N_2} [s^{-N_1} \mathcal{L}_t^{N_1} \{\eta_{\lambda,k,l}(s, t) \varpi_2(4\lambda s t |u|)\}]}{(1 + \lambda^2 |\psi(t, |x|) + 4t|u||^2)^{N_2}} dt ds$$

From (94b),

$$|\partial_t(\psi(t, |x|) + 4t|u|)| \gtrsim 2^{2l} k |x|^2 + 4|u| \gtrsim 2^{2m+l} \lambda^{-1}.$$

Moreover, for  $\nu \geq 2$ ,  $\partial_t^\nu \psi = O(2^{2m+l\nu} \lambda^{-1})$  and  $\nu$  differentiations of the amplitude produce factors of  $2^{l\nu}$ . Thus we obtain the bound

$$|B_\lambda^{k,l}(x, u)| \lesssim \frac{\mathfrak{C}_{\lambda,k,l}}{(1 + 4\lambda k |u|)^{\frac{d_2-1}{2}}} 2^{-2mN_1} \times \\ \int_{|t-k\pi| \lesssim 2^{-l}} (1 + \lambda k |t^{-1} - |x|^2 \cot t + 4|u||)^{-2N_2} dt.$$

From Lemma 8.2 (with  $n = d_2$ ,  $\Lambda = \lambda k$ ,  $\rho \lesssim k^{-1} \max\{1, 2^{2m} \lambda^{-1}\}$ )

$$(121) \quad \int_{v=0}^{\infty} \frac{(1 + \lambda k v)^{-\frac{d_2-1}{2}} v^{d_2-1}}{(1 + \lambda k |v - |x|^2 \cot t + t^{-1}|)^N} dv \\ \lesssim \lambda^{-\frac{d_2+1}{2}} k^{-d_2} \max\{1, (2^{2m} \lambda^{-1})^{\frac{d_2-1}{2}}\}.$$

We integrate in  $t$  over an interval of length  $O(2^{-l})$  and in  $x$  over the annulus  $\{x : |x| \approx 2^m (2^l \lambda k)^{-1/2}\}$ . This gives

$$(122) \quad \iint_{\substack{(x,u): \\ |x| \approx 2^m (2^l \lambda k)^{-1/2}}} |B_\lambda^{k,l}(x, u)| dx du \\ \lesssim 2^{-2mN} 2^{-l} \left( \frac{2^m}{\sqrt{2^l \lambda k}} \right)^{d_1} \mathfrak{C}_{\lambda,k,l} \lambda^{-\frac{d_2+1}{2}} k^{-d_2} \max\{1, (2^{2m} \lambda^{-1})^{\frac{d_2-1}{2}}\} \\ \lesssim (2^l k)^{-1} \lambda^{-\frac{d_1-1}{2}} 2^{-m(2N-d_1)} \max\{1, (2^{2m} \lambda^{-1})^{\frac{d_2-1}{2}}\}$$

and choosing  $N$  sufficiently large the lemma follows by summation in  $m$ .  $\square$

8.3.3.  $L^1$ -bounds for  $A_\lambda^{k,l}$ ,  $2^l k \geq 10^5 \lambda$ .

**Lemma 8.6.** For  $k \leq 8\lambda$ ,  $2^l \geq 10^5 \lambda/k$ ,

$$(123) \quad \|A_\lambda^{k,l}\|_1 \lesssim (2^l k)^{-1} \lambda^{-\frac{d_1-1}{2}}.$$

*Proof.* We use Lemma 8.4 to obtain the appropriate  $L^1$  bound in the region  $\{(x, u) : |x| \leq C_0(2^l \lambda k)^{-1/2}\}$ . Next, consider the region where

$$(124) \quad 2^m (2^l \lambda k)^{-1/2} \leq |x| \leq 2^{m+1} (2^l \lambda k)^{-1/2}$$

for large  $m$ . This region is then split into two subregions, one where  $4|u| = v \leq 10^{-2} 2^{2m+l} \lambda^{-1}$  and the complementary region.

For the region with small  $v$  we proceed as in Lemma 8.5. From formula (94b) we have  $|\psi_t| \geq kr^2 2^{2l}/20$  and hence  $|\psi_t| \geq 2^{2m+l-5} \lambda^{-1}$ . Thus if  $v \leq 10^{-2} 2^{2m+l} \lambda^{-1}$  then  $|\phi_t| \approx k 2^{2l} r^2 \approx 2^{2m+l} \lambda^{-1}$ . Moreover  $\partial_t^\nu \phi = O(2^{2m+l\nu} \lambda^{-1})$  for  $\nu \geq 2$ . Therefore, if we perform integration by parts in  $t$  several times, followed by integrations by parts on  $s$ , we obtain the bound

$$|A_\lambda^{k,l}(x, u)| \lesssim \frac{\mathfrak{E}_{\lambda,l,k}}{(1 + \lambda k |u|)^{\frac{d_2-1}{2}}} 2^{-2mN} \times \int_{|t-k\pi| \lesssim 2^{-l}} (1 + \lambda k (|x|^2 \cot t - t^{-1} - 4|u|))^{-N} dt.$$

In the present range  $|x|^2 |\cot t| \approx 2^{2m} (\lambda k)^{-1}$  and  $t^{-1} \approx k^{-1}$  and thus we see from Lemma 8.2 that inequality (121) in the proof of Lemma 8.5 holds. From this we proceed as in (122) to bound

$$\begin{aligned} & \iint_{\substack{|x| \approx 2^m (2^l \lambda k)^{-1/2} \\ 4|u| \leq 10^{-2} 2^{2m+l} \lambda^{-1}}} |A_\lambda^{k,l}(x, u)| dx du \\ & \lesssim (2^l k)^{-1} \lambda^{-\frac{d_1-1}{2}} 2^{-m(2N-d_1)} \max\{1, (2^{2m} \lambda^{-1})^{\frac{d_2-1}{2}}\}. \end{aligned}$$

For large  $N_1$  we can sum in  $m$  and obtain the bound  $C(2^l k)^{-1} \lambda^{-\frac{d_1-1}{2}}$ .

Next assume that  $v \geq 2^{2m+l} \lambda^{-1}/100$  (and still keep (124)). Then

$$(125) \quad |tv + r^2 t \cot t - 1| \geq k|v| \text{ for } t \in \text{supp}(\eta_{\lambda,k,l}).$$

Indeed, we have  $tv \geq 2^{2m} 2^l k \lambda^{-1}/100 \geq 10^3$  and

$$r^2 t |\cot t| \leq 2^{2m+2} (2^l \lambda k)^{-1} t [\sin(\frac{3\pi}{8} 2^{-l})]^{-1} \leq 2^{2m+6} \lambda^{-1} \leq \frac{2^{2m+l}}{100\lambda} 2^5 10^2 2^{-l}$$

where we used (124) and  $\sin \alpha > 2\alpha/\pi$  for  $0 \leq \alpha \leq \pi/2$ . By our assumptions  $2^l \geq 10^5 \lambda/k > 10^4$  and thus the right hand side of the display is  $\leq v/10$ . Now (125) is immediate by the triangle inequality.

We use (125) to get from an  $N_1$ -fold integration by parts in  $s$

$$|A_\lambda^{k,l}(x, u)| \lesssim 2^{-l} \mathfrak{E}_{\lambda,l,k} (\lambda k v)^{-N_1 - \frac{d_2-1}{2}}.$$

Then

$$\begin{aligned}
& \iint_{\substack{|x| \approx 2^m (2^l \lambda k)^{-1/2} \\ 4|u| \geq 10^{-2} 2^{2m+l} \lambda^{-1}}} |A_\lambda^{k,l}(x, u)| \, du \, dx \\
& \lesssim 2^{-l} \mathfrak{C}_{\lambda,l,k} \left( \frac{2^m}{\sqrt{\lambda 2^l k}} \right)^{d_1} (\lambda k)^{-N_1 - \frac{d_2-1}{2}} \left( \frac{2^{2m+l}}{\lambda} \right)^{-N_1 + \frac{d_2+1}{2}} \\
& \lesssim \lambda^{1 - \frac{d_1}{2} - \frac{d_2}{2}} 2^{-l(N_1 - \frac{d_2-1}{2})} k^{\frac{d_2-1}{2} - N_1} 2^{m(d_1 + d_2 + 1 - 2N_1)}.
\end{aligned}$$

For  $N_1$  large we may sum in  $m$  to finish the proof.  $\square$

8.3.4. *Estimates for  $A_\lambda^{k,l}$ ,  $2^l \lesssim \lambda/k$ .* In the early approaches to prove  $L^p$  boundedness for Fourier integral operators the oscillatory integral were analyzed using the method of stationary phase ([28], [18], [1]). This creates some difficulties in our case at points where  $\phi$ ,  $\phi_t$  and  $\phi_{tt}$  vanish simultaneously, namely at positive  $t$  satisfying  $\tan t = t$ . To avoid this difficulty we use a decomposition in the spirit of [30].

In what follows we assume  $k \leq 8\lambda$  and  $2^l \leq C_0 \lambda/k$  for large  $C_0$  chosen independently of  $\lambda, k, l$ . The choice  $C_0 = 10^{10}$  is suitable. We decompose the interval  $J_{k,l}$  into smaller subintervals of length  $\varepsilon \sqrt{\frac{k}{2^l \lambda}}$  (which is  $\lesssim 2^{-l}$  in the range under consideration), here  $\varepsilon \ll 10^{-100}$  (to be chosen sufficiently small but independent of  $\lambda, k, l$ ).

This decomposition is motivated by the following considerations: according to (130),  $\lambda \phi(t, r, v)$  contains the term  $-\lambda(r - r(t))^2 t \cot t$  depending entirely on  $r$  and  $t$ . For  $t \in J_{k,l}$ , this is of size  $\lambda k 2^l |r - r(t)|^2$ , hence of order  $O(1)$  if  $|r - r(t)| \lesssim (\lambda k 2^l)^{-1/2}$ . Moreover, on a subinterval  $I$  of  $J_{k,l}$  on which  $r(t)$  varies by at most a small fraction of the same size, the term  $-\lambda(r - r(t))^2 t \cot t$  is still  $O(1)$  and contributes to no oscillation in the integration with respect to  $s$ . Since  $|r'(t)| \sim 1/k$  by (111), this suggests to choose intervals  $I$  of length  $\ll k(\lambda k 2^l)^{-1/2} = \sqrt{k 2^{-l} \lambda^{-1}}$ . Similarly, the first term of  $\lambda \phi(t, r, v)$  in (130) is of size  $\lambda k |w(t, r, v)|$  and does not contribute to any oscillation in the integration with respect to  $s$  if  $|w(t, r, v)| \lesssim (\lambda k)^{-1}$ . These considerations also motivate our later definitions of the set  $\mathcal{P}_0$  and the sets  $\mathcal{P}_m, m \geq 1$ , cf. (133).

As before we denote by  $\eta_0$  a  $C_0^\infty(\mathbb{R})$  function so that  $\sum_{n \in \mathbb{Z}} \eta_0(t - \pi n) = 1$  and  $\text{supp}(\eta_0) \subset (-\pi, \pi)$ . Define, for  $b \in \pi \varepsilon \sqrt{k 2^{-l} \lambda^{-1}} \mathbb{Z}$ ,

$$(126) \quad \eta_{\lambda,k,l,b}(s, t) = \eta_{\lambda,k,l}(s, t) \eta_0(\varepsilon^{-1} \sqrt{\frac{\lambda 2^l}{k}}(t - b)).$$

Then we may split

$$(127) \quad A_\lambda^{k,l} = \sum_{b \in \mathcal{T}_{\lambda,k,l}} A_{\lambda,b}^{k,l}$$

where  $\mathcal{T}_{\lambda,k,l} \subset \pi\varepsilon\sqrt{k2^{-l}\lambda^{-1}}\mathbb{Z} \cap J_{k,l}$  (cf.(104)),  $\#\mathcal{T}_{\lambda,k,l} = O(\varepsilon^{-1}\sqrt{\lambda2^{-l}k^{-1}})$ , and

$$(128) \quad A_{\lambda,b}^{k,l}(x, u) = \mathfrak{E}_{\lambda,l,k} \iint \chi(s)\eta_{\lambda,k,l,b}(t)e^{i\lambda s(1-|x|^2t \cot t - |4u|)} \varpi_1(\lambda st|4u|) dt ds.$$

We now give some formulas relating the phase  $\phi(t, r, v) = 1 - r^2t \cot t - tv$  to the geometry of the curve  $(r(t), v(t))$  (cf.(108)). By (110) and (112),

$$\begin{aligned} \frac{\phi(t, r, v)}{t} &= \frac{\phi(t, r, v) - \phi(t, r(t), v(t))}{t} \\ &= (r(t)^2 - r^2) \cot t + v(t) - v \\ &= v(t) - v - (r - r(t))2r(t) \cot t - (r - r(t))^2 \cot t \end{aligned}$$

and, setting

$$(129) \quad w(t, r, v) = v - v(t) - \frac{v'(t)}{r'(t)}(r - r(t)),$$

we get

$$(130) \quad \frac{\phi(t, r, v)}{t} = -w(t, r, v) - (r - r(t))^2 \cot t.$$

Moreover,

$$(131) \quad \begin{aligned} \phi_t(t, r, v) &= \frac{\phi(t, r, v)}{t} + \frac{r^2t}{\sin^2 t} - \frac{1}{t} \\ &= \frac{\phi(t, r, v)}{t} + \frac{t}{\sin^2 t}(r + r(t))(r - r(t)) \end{aligned}$$

We shall need estimates describing how  $w(t, r, v)$  changes in  $t$ . Use (130) and the expansion

$$\begin{aligned} w(t, r, v) - w(b, r, v) &= -[v(t) - v(b) - \frac{v'(b)}{r'(b)}(r(t) - r(b))] \\ &\quad - \left[ \frac{v'(t)}{r'(t)} - \frac{v'(b)}{r'(b)} \right] (r - r(b)) + \left[ \frac{v'(t)}{r'(t)} - \frac{v'(b)}{r'(b)} \right] (r(t) - r(b)). \end{aligned}$$

From (114) we get  $|r''| + k|v''| \lesssim 2^{-l}k^{-1} + k^{-2}$  on  $J_{k,l}$ , thus the first term in the displayed formula is  $\lesssim (2^{-l}k^{-2} + k^{-3})|t - b|^2$ . Differentiating in (112) we also get  $(v'/r')' = O(2^{-l}k + k^{-2})$  on  $J_{k,l}$ , and see that the second term in the display is  $\lesssim (2^{-l}k^{-1} + k^{-2})|t - b||r - r(b)|$  and the third is  $\lesssim (2^{-l} + k^{-1})k^{-2}(t - b)^2$ . Hence

$$(132) \quad |w(t, r, v) - w(b, r, v)| \lesssim (2^{-l} + k^{-1})|t - b| \left( \frac{|t - b|}{k^2} + \frac{|r - r(b)|}{k} \right).$$

We now turn to the estimation of  $A_{\lambda,b}^{k,l}$  with  $k \geq 1$  and  $b \in \mathcal{T}_{\lambda,k,l}$ . Let, for  $b > 1/2$ ,  $l = 1, 2, \dots$ , and  $m = 0, 1, 2, \dots$ ,

$$(133) \quad \mathcal{P}_m \equiv \mathcal{P}_m(\lambda, l, k; b) := \{(r, v) \in (0, \infty) \times (0, \infty) : \\ v \geq (\lambda k)^{-1}, |r - r(b)| \leq 2^m (\lambda k 2^l)^{-1/2}, |w(b, r, v)| \leq 2^{2m} (\lambda k)^{-1}\}$$

and let

$$(134) \quad \Omega_m \equiv \Omega_m(\lambda, l, k; b) := \begin{cases} \{(x, u) : (|x|, 4|u|) \in \mathcal{P}_0\}, & \text{if } m = 0, \\ \{(x, u) : (|x|, 4|u|) \in \mathcal{P}_m \setminus \mathcal{P}_{m-1}\} & \text{if } m > 0. \end{cases}$$

For later reference we note that in view  $2^l \leq \lambda/k$ ,  $|t - b| \leq \varepsilon \sqrt{\frac{k}{\lambda 2^l}}$  and the upper bound  $|r'(t)| \leq 2t^{-1}$  we have  $r(t) - r(b) = O(\frac{\varepsilon}{\sqrt{k\lambda 2^l}})$ , and, by (132),

$$(135) \quad |w(t, r, v) - w(b, r, v)| \lesssim \varepsilon 2^m (\lambda k)^{-1}, \quad (r, v) \in \mathcal{P}_m.$$

Moreover it is easy to check that, still for  $|t - b| \leq \varepsilon \sqrt{\frac{k}{\lambda 2^l}}$ ,

$$(136) \quad |(r - r(t))^2 \cot t - (r - r(b))^2 \cot b| \lesssim \varepsilon 2^{2m} (\lambda k)^{-1}.$$

**Proposition 8.7.** *Assume that  $1 \leq k \leq 8\lambda$ ,  $l = 1, 2, \dots$ , and  $2^l \leq C_0 \lambda/k$  (and let  $\varepsilon$  in the definition (126) be  $\leq C_0^{-1} 10^{-100}$ ). Let  $b \geq 1$  and  $b \in \mathcal{T}_{\lambda,k,l}$ . Then*

$$(137) \quad \iint_{\Omega_0(\lambda, l, k; b)} |A_{\lambda,b}^{k,l}(x, u)| dx du \lesssim (2^l k)^{-\frac{d_1+1}{2}} \sqrt{\frac{2^l k}{\lambda}}$$

$$(138) \quad \iint_{\Omega_m(\lambda, l, k; b)} |A_{\lambda,b}^{k,l}(x, u)| dx du \lesssim_N 2^{-mN} (2^l k)^{-\frac{d_1+1}{2}} \sqrt{\frac{2^l k}{\lambda}}.$$

*Proof.* Note that, for fixed  $k \geq 1$ ,  $l \geq 1$ ,  $b \in \mathcal{T}_{\lambda,k,l}$ ,

$$(139) \quad (r, v) \in \mathcal{P}_m \implies r \lesssim 2^m (2^l k)^{-1} \text{ and } v \lesssim 2^{2m} k^{-1}.$$

This is immediate in view of  $2^l k \lesssim \lambda$ ,  $r(b) \approx (2^l k)^{-1}$ ,  $v(b) \approx k^{-1}$  and thus

$$(140) \quad \begin{aligned} r &\lesssim (2^l k)^{-1} (1 + 2^m \sqrt{\frac{k 2^l}{\lambda}}) \lesssim 2^m (2^l k)^{-1}, \\ v &\lesssim k^{-1} (1 + 2^{2m} \lambda^{-1}) \lesssim 2^{2m} k^{-1}. \end{aligned}$$

Also recall that  $v = 4|u| \geq (\lambda k)^{-1}$  for  $(x, u) \in \Omega_m(\lambda, l, k; b)$ .

A crude size estimate yields

$$(141) \quad \iint_{(|x|, 4|u|) \in \mathcal{P}_m} |A_{\lambda,b}^{k,l}(x, u)| dx du \lesssim 2^{m(d_1+d_2+1)} (2^l k)^{-(d_1+1)/2} \sqrt{\frac{2^l k}{\lambda}}.$$

Indeed, the left hand side is  $\lesssim \varepsilon \sqrt{\frac{k}{2^l \lambda}} \mathfrak{C}_{\lambda,k,l} \mathcal{I}$  where

$$\mathcal{I} := \iint_{\substack{|r-r(b)| \lesssim 2^m (2^l \lambda k)^{-1/2} \\ |w(b,r,v)| \lesssim 2^{2m} (\lambda k)^{-1}}} (\lambda k v)^{-\frac{d_2-1}{2}} v^{d_2-1} r^{d_1-1} dv dr$$



is  $\lesssim \frac{2^m}{\sqrt{\lambda 2^l k}} \left(\frac{2^m}{2^l k}\right)^{d_1-1} \frac{2^{2m}}{\lambda k} \left(\frac{2^{2m} k^{-1}}{\lambda k}\right)^{\frac{d_2-1}{2}}$ , in view of (129) and (140). This yields (141). In regard to its dependence on  $m$  this bound is nonoptimal and will be used for  $2^m \leq C(\varepsilon)$ .

We now derive an improved  $L^1$  bound for the region  $\Omega_m$  when  $m$  is large. For  $(r, v) \in \mathcal{P}_m \setminus \mathcal{P}_{m-1}$  we distinguish two cases  $I, II$  depending on the size of  $|\phi(b, r, v)|$  and define for  $m > 0$ , and fixed  $k, l, b$ ,

$$\begin{aligned} \mathcal{R}_m^I &= \{(r, v) \in \mathcal{P}_m \setminus \mathcal{P}_{m-1} : |\phi(b, r, v)| > 2^{l-100}(r - r(b))^2\}, \\ \mathcal{R}_m^{II} &= \{(r, v) \in \mathcal{P}_m \setminus \mathcal{P}_{m-1} : |\phi(b, r, v)| \leq 2^{l-100}(r - r(b))^2\}. \end{aligned}$$

We also have the corresponding decomposition  $\Omega_m = \Omega_m^I + \Omega_m^{II}$  where  $\Omega_m^I$  and  $\Omega_m^{II}$  consist of those  $(x, u)$  with  $(|x|, 4|u|) \in \mathcal{R}_m^I$  and  $(|x|, 4|u|) \in \mathcal{R}_m^{II}$ , respectively.

*Case I:*  $|\phi(b, r, v)| \geq 2^{l-100}k(r - r(b))^2$ . We shall show that

$$(142) \quad |\phi(t, r, v)| \gtrsim c 2^{2m} \lambda^{-1}, \quad \text{for } (r, v) \in \mathcal{R}_m^I, \quad |t - b| \leq \varepsilon \sqrt{\frac{k}{2^l \lambda}}.$$

with  $c > 0$  if  $0 < \varepsilon \ll 10^{-100}$  is chosen sufficiently small. Given (142) we can use an  $N_2$ -fold integration by parts in  $s$  to obtain a gain of  $2^{-2mN_2}$  over the above straightforward size estimate (141), which leads to

$$(143) \quad \iint_{\Omega_m^I} |A_{\lambda, b}^{k, l}(x, u)| dx du \lesssim_{\varepsilon, N_2} 2^{m(d_1+d_2+1-2N_2)} (2^l k)^{-\frac{d_1+1}{2}} \sqrt{\frac{2^l k}{\lambda}}.$$

It remains to show (142). We distinguish between two subcases. First if  $|r - r(b)| \geq 2^{m-5}(\lambda k 2^l)^{-1/2}$  then by the *Case I* assumption we have  $|\phi(b, r, v)| \geq 2^{l-100}k 2^{2m-10}(\lambda k 2^l)^{-1} = 2^{2m-110}\lambda^{-1}$ , and by (130), (135) and (136) we also get (142) provided that  $\varepsilon \ll 2^{-200}$ .

For the second subcase we have  $|r - r(b)| \leq 2^{m-5}(\lambda k 2^l)^{-1/2}$ . Since  $(r, v) \notin \mathcal{P}_{m-1}$  this implies that  $|w(b, r, v)| \geq 2^{2m-2}(\lambda k)^{-1}$ , and since the quantity  $b(r - r(b))^2 |\cot b|$  is bounded by  $2^{l+4}b(r - r(b))^2 \leq 2^{2m-6}(b/k)\lambda^{-1}$  we also get  $|\phi(b, r, v)| \geq 2^{2m-3}\lambda^{-1}$ , by (130). Now by (130), (135) and (136) we also get  $|\phi(t, r, v)| \geq 2^{2m-4}\lambda^{-1}$ , if  $\varepsilon$  is sufficiently small. Thus (142) is verified and (143) is proved.

*Case II:*  $|\phi(b, r, v)| \leq 2^{l-100}k(r - r(b))^2$ . We show

$$(144) \quad |\phi_t(t, r, v)| \geq 2^{m-20} 2^{3l/2} k^{1/2} (r + r(b)) \lambda^{-1/2}$$

$$\text{if } (r, v) \in \mathcal{R}_m^{II}, \quad |t - b| \leq \varepsilon \sqrt{\frac{\lambda 2^l}{k}}.$$

and this will enable us to get a gain when integrating by parts in  $t$ . To prove (144) we first establish

$$(145) \quad |r - r(b)| \geq 2^{m-10}(\lambda k 2^l)^{-1/2} \text{ for } (r, v) \in \mathcal{R}_m^{II}.$$

Note that if  $|w(b, r, v)| \leq 2^{2m-3}(\lambda k)^{-1}$  then  $|r - r(b)| \geq 2^{m-1}(\lambda k 2^l)^{-1/2}$  since  $\mathcal{R}_m^{II} \subset \mathcal{P}_{m-1}^c$ . Thus to verify (145) we may assume  $|w(b, r, v)| \geq$

$2^{2m-3}(\lambda k)^{-1}$ . In this case we get from (130),  $(r, v) \in \mathcal{P}_m$  and the *Case II* assumption

$$\begin{aligned} (r - r(b))^2 |\cot b| &\geq |w(b, r, v)| - b^{-1} |\phi(b, r, v)| \\ &\geq 2^{2m-3} (\lambda k)^{-1} - b^{-1} k 2^{l-100} 2^{2m} (\lambda k 2^l)^{-1} \geq 2^{2m-4} (\lambda k)^{-1} \end{aligned}$$

and hence  $(r - r(b))^2 2^{l+4} \geq 2^{2m-4} (\lambda k)^{-1}$  which implies (145). In order to prove (144) we use (131) and (145) to estimate

$$\begin{aligned} |\phi_t(b, r, v)| &\geq \frac{b}{\sin^2 b} (r + r(b)) |r - r(b)| - 2^{l-100} \frac{k}{b} (r - r(b))^2 \\ &\geq \frac{|r - r(b)|}{b} \left( \frac{r + r(b)}{r(b)^2} - \frac{2^l k}{2^{100}} |r - r(b)| \right) \geq \frac{(r + r(b)) |r - r(b)|}{2br(b)^2} \\ &\geq 2^{2l-4} k (r + r(b)) \frac{2^{m-10}}{\sqrt{\lambda k 2^l}} \geq 2^{m-15} k^{1/2} 2^{3l/2} (r + r(b)) \lambda^{-1/2} \end{aligned}$$

which yields (144) for  $t = b$ . We need to show the lower bound for  $|t - b| \leq \varepsilon \sqrt{k/(2^l \lambda)}$ . By (95) we have  $|\phi_{tt}(t', r, v)| \leq r^2 b 2^{3l+4}$  for  $|t' - b| \leq \varepsilon \sqrt{\frac{b}{2^l \lambda}}$  and thus

$$|\phi_t(t, r, v) - \phi_t(b, r, v)| \leq 2^6 r^2 2^{3l} k \varepsilon \sqrt{\frac{k}{2^l \lambda}} \leq 2^{m-30} 2^{3l/2} k^{1/2} \lambda^{-1/2} (r + r(b))$$

if  $\varepsilon \ll 2^{-100}$ . The second inequality in the last display is easy to check. If  $r \leq 2r(b)$  then use  $r \lesssim (2^l k)^{-1} \approx r + r(b)$  and if  $r > 2r(b)$  then use  $r - r(b) \approx r + r(b) \approx r$ . In both cases the asserted inequality holds for small  $\varepsilon$  and thus (144) holds for  $|t - b| \leq \varepsilon \sqrt{k/(2^l \lambda)}$ . We note that under the condition (145) the range  $r \leq 2r(b)$  corresponds to  $2^m \lesssim \sqrt{\lambda(2^l k)^{-1}}$  and the range  $r \geq 2r(b)$  corresponds to  $2^m \gtrsim \sqrt{\lambda(2^l k)^{-1}}$ .

We now estimate the  $L^1$  norm over the region where  $(r, v) \in \mathcal{R}_m^{II}$ . Let  $\mathcal{L}_t$  be the differential operator defined by  $\mathcal{L}_t g = \frac{\partial}{\partial t} \left( \frac{g}{\phi_t} \right)$ . By  $N_1$  integration by parts in  $t$  we get (with  $|x| = r$ ,  $4|u| = v$ )

$$\begin{aligned} A_{\lambda, b}^{k, l}(x, u) &= i^{N_1} \lambda^{-N_1} \mathfrak{C}_{\lambda, k, l} \times \\ &\quad \iint e^{i\lambda s \phi(t, |x|, 4|u|)} s^{-N_1} \mathcal{L}_t^{N_1} [\eta_{\lambda, k, l, b}(s, t) \varpi_1(\lambda s t v)] dt ds. \end{aligned}$$

To estimate the integrand use the lower bound on  $|\phi_t|$ , (144). Moreover we have the upper bounds (98) for the higher derivatives of  $\psi$  (and then  $\phi$ ) which give  $\partial_t^n \phi = O(2^{l(n+1)} b r^2)$  for  $n \geq 2$ . Each differentiation of the cutoff function produces a factor of  $(\lambda 2^l k^{-1})^{1/2}$ . By the one-dimensional version of Lemma 8.1 described in the subsequent remark the expression  $\lambda^{-N_1} (\lambda b v)^{(d_2-1)/2} |\mathcal{L}_t^{N_1} [\eta_{\lambda, k, l, b}(s, t) \varpi_1(\lambda s t v)]|$  can be estimated by a sum of  $C(N_1)$  terms of the form

(146)

$$\lambda^{-N_1} \frac{(\lambda 2^l / k)^{\alpha/2}}{(2^m 2^{3l/2} k^{1/2} (r + r(b)) \lambda^{-1/2})^\alpha} \prod_{\beta \in \mathfrak{J}} \frac{2^{l(\beta+1)} k r^2}{(2^m 2^{3l/2} k^{1/2} (r + r(b)) \lambda^{-1/2})^\beta}$$

where  $\alpha \in \{0, \dots, N_1\}$ ,  $\mathfrak{J}$  is a set of integers  $\beta \in \{2, \dots, N_1 + 1\}$  with the property that  $\sum_{\beta \in \mathfrak{J}} (\beta - 1) = N_1 - \alpha$ . If  $\mathfrak{J}$  is the empty set then we interpret the product as 1. We observe that for  $(r, v) \in \mathcal{R}_m^{II}$  we have  $|r - r(b)| \approx 2^m (\lambda k 2^l)^{-1/2}$ . Thus if  $2^m \leq \sqrt{\lambda (2^l k)^{-1}}$  we have  $r \lesssim (2^l k)^{-1}$  and  $r + r(b) \approx (2^l k)^{-1}$  while for  $2^m > \sqrt{\lambda (2^l k)^{-1}}$  we have  $r \approx r - r(b) \approx r + r(b) \approx 2^m (\lambda k 2^l)^{-1/2}$ .

A short computation which uses these observations shows that in the case  $2^m \leq \sqrt{\lambda (2^l k)^{-1}}$  the terms (146) are  $\lesssim 2^{-m\alpha} \prod_{\beta \in \mathfrak{J}} [2^{-m\beta} (2^l k / \lambda)^{\beta/2 - 1}]$ . In the case  $2^m > \sqrt{\lambda (2^l k)^{-1}}$  the terms (146) are dominated by a constant times  $(\lambda 2^{-l} k^{-1})^{\alpha/2} 2^{-2m\alpha} \prod_{\beta \in \mathfrak{J}} 2^{-m(\beta-1)}$ . In either case the terms (146) are  $\lesssim 2^{-mN_1}$  (since  $\alpha + \sum_{\beta \in \mathfrak{J}} \beta \geq N_1$ ). This means that we gain a factor of  $2^{-mN_1}$  over the size estimate (141). Consequently,

$$(147) \quad \iint_{\Omega_m^{II}} |A_{\lambda, b}^{k, l}(x, u)| dx du \lesssim 2^{m(d_1 + d_2 + 1 - N_1)} (2^l k)^{-\frac{d_1 + 1}{2}} \sqrt{\frac{2^l k}{\lambda}}.$$

The assertion of the proposition follows then from (143) and (147).  $\square$

#### 8.4. $L^1$ estimates for $T_\lambda^k$ and $W_{j, n}$ .

*Proof of (59).* Let us recall that  $k \leq 8\lambda$ . If we sum the bounds in Proposition 8.7 in  $b \in \mathcal{T}_{2^j, k, l}$  we get

$$\|A_{2^j}^{k, l}\|_{L^1} \lesssim (2^l k)^{-\frac{d_1 + 1}{2}}, \quad 2^l \lesssim \frac{2^j}{k}.$$

We also have

$$(148) \quad \|2^{-j\frac{d-1}{2}} K_{2^j}^{k, l} - A_{2^j}^{k, l}\|_1 \lesssim (2^l k)^{-1} 2^{-j\frac{d_1 - 1}{2}},$$

for the part of  $K_{2^j}^{k, l}$  where  $|u| \lesssim 1/k\lambda$  this follows from Lemma 8.3, and for the remaining part from Lemma 8.5. Combining these two estimates, we find that

$$(149) \quad \|2^{-j\frac{d-1}{2}} K_{2^j}^{k, l}\|_1 \lesssim (2^l k)^{-\frac{d_1 + 1}{2}}, \quad 2^l \lesssim \frac{2^j}{k}.$$

Moreover, by Lemma 8.5 and Lemma 8.6, we have

$$(150) \quad \|2^{-j\frac{d-1}{2}} K_{2^j}^{k, l}\|_1 \lesssim (2^l k)^{-1} 2^{-j\frac{d_1 - 1}{2}}, \quad 2^l \geq 10^6 \frac{2^j}{k}.$$

Altogether this leads to

$$(151) \quad 2^{-j(d-1)/2} \|T_{2^j}^{k, l}\|_{L^1 \rightarrow L^1} \lesssim (2^l k)^{-\frac{d_1 + 1}{2}}.$$

and (59) follows if we sum in  $l$ .  $\square$

8.5. *An estimate away from the singular support.* For later use in the proof of Theorem 1.4 we need the following observation.

**Proposition 8.8.** *Let  $\lambda \geq 1$ ,  $K_\lambda$  be the convolution kernel for the operator  $\chi(\lambda^{-1}\sqrt{L})e^{i\sqrt{L}}$ , where  $\chi \in \mathcal{S}(\mathbb{R})$ , and let  $R \geq 10$ . Then, for every  $N \in \mathbb{N}$ ,*

$$\int_{\max\{|x|, |u|\} \geq R} |K_\lambda(x, u)| dx du \leq C_N (\lambda R)^{-N}.$$

Moreover, the constants  $C_N$  depend only on  $N$  and a suitable Schwartz norm of  $\chi$ .

*Proof.* This estimate is implicit in our arguments above, but it is easier to establish it as a direct consequence of the finite propagation speed of solutions to the wave equation [17]. Indeed, write

$$\chi(\lambda^{-1}\sqrt{L})e^{i\sqrt{L}} = \chi(\lambda^{-1}\sqrt{L}) \cos \sqrt{L} + i\lambda \tilde{\chi}(\lambda^{-1}\sqrt{L}) \frac{\sin \sqrt{L}}{\sqrt{L}},$$

with  $\tilde{\chi}(s) = s\chi(s)$ , and denote by  $\varphi_\lambda$  and  $\mathcal{P}$  the convolution kernels for the operators  $\chi(\lambda^{-1}\sqrt{L})$  and  $\cos \sqrt{L}$ , respectively. Then  $\mathcal{P}$  is a compactly supported distribution (of finite order). Indeed,  $\mathcal{P}$  is supported in the unit ball with respect to the optimal control distance associated to the Hörmander system of vector fields  $X_1, \dots, X_{d_1}$ , which is contained in the Euclidean ball of radius 10. Moreover, by homogeneity,  $\varphi_\lambda(x, u) = \lambda^{d_1+2d_2} \varphi(\lambda x, \lambda^2 u)$ , with a fixed Schwartz function  $\varphi$ . Note also that by Hulanicki's theorem [12], the mapping taking  $\chi$  to  $\varphi$  is continuous in the Schwartz topologies. Since the convolution kernel  $K_\lambda^c$  for the operator  $\chi(\lambda^{-1}\sqrt{L}) \cos \sqrt{L}$  is given by  $\varphi_\lambda * \mathcal{P}$ , it is then easily seen  $K_\lambda^c(x, u)$  can be estimated by  $C_N \lambda^M (\lambda|x| + \lambda^2|u|)^{-N}$  for every  $N \in \mathbb{N}$ , with a fixed constant  $M$ . A very similar argument applies to  $\tilde{\chi}(\lambda^{-1}\sqrt{L}) \frac{\sin \sqrt{L}}{\sqrt{L}}$ , and thus we obtain the above integral estimate for  $K_\lambda$ .  $\square$

### 9. $h_{\text{iso}}^1 \rightarrow L^1$ ESTIMATES FOR THE OPERATORS $\mathcal{W}_n$

In this section we consider the operators  $\mathcal{W}_n = \sum_j W_{j,n}$  and prove the relevant estimate in Theorem 5.3. In the proof we shall use a simple  $L^2$  bound which follows from the spectral theorem, namely for  $j_0 > 0$

$$(152) \quad \left\| \sum_{j \geq j_0} W_{j,n} \right\|_{L^2 \rightarrow L^2} \lesssim 2^{-j_0(d-1)/2}.$$

*Preliminary considerations.* Let  $\rho \leq 1$  and let  $f_\rho$  be an  $L^2$ -function satisfying

$$(153) \quad \begin{aligned} \|f_\rho\|_2 &\leq \rho^{-d/2}, \\ \text{supp}(f_\rho) &\subset Q_{\rho,E} := \{(x, u) : \max\{|x|, |u|\} \leq \rho\}, \end{aligned}$$

and we also assume that

$$(154) \quad \iint f_\rho(x, u) dx du = 0, \text{ if } \rho \leq 1/2.$$

In what follows we also need notation for the expanded Euclidean "ball"

$$(155) \quad Q_{\rho,E,*} = \{(x, u) : \max\{|x|, |u|\} \leq C_*\rho\},$$

where  $C_* = 10(1 + d_2 \max_i \|J_i\|)$ .

We begin with the situation given by (154). By translation invariance and the definition of  $h_{\text{iso}}^1$  it will suffice to check that

$$(156) \quad \|\mathcal{W}_n f_\rho\|_{L^1} \lesssim (1+n)2^{-n(d_1-1)/2}.$$

We work with dyadic spectral decompositions for the operators  $|U|$  and  $\sqrt{L}$  and need to discuss how they act on the atom  $f_\rho$ .

For  $j > 0$ ,  $n \geq 0$ , let  $H_{j,n}$  be the convolution kernel defined by

$$\chi_1(2^{-2j}L)\zeta_1(2^{-j-n}|U|)f = f * H_{j,n}.$$

From (52) we see that

$$H_{j,n} = 0 \text{ when } n > j + 11.$$

**Lemma 9.1.** *Let  $\rho \leq 1$ , and  $f_\rho$  be as in (153). Then*

(i)  $\|f_\rho * H_{j,n}\|_1 \lesssim 1$  and

$$(157) \quad \|f_\rho * H_{j,n}\|_{L^1(Q_{\rho,E,*}^c)} \lesssim_N (2^j \rho)^{-N}.$$

(ii) If  $f_\rho$  satisfies (154) then

$$(158) \quad \|f_\rho * H_{j,n}\|_1 \lesssim \min\{1, 2^{j+n}\rho\}.$$

*Proof.* By Hulanicki's theorem [12] the convolution kernel of  $\chi_1(L)$  is a Schwartz function  $g_1$  on  $\mathbb{R}^{d_1+d_2}$ . The convolution kernel of  $\zeta_1(|U|)$  is  $\delta \otimes g_2$  where  $\delta$  is the Dirac measure in  $\mathbb{R}^{d_1}$  and  $g_2$  is a Schwartz function on  $\mathbb{R}^{d_2}$ . Then

$$(159) \quad H_{j,n}(x, u) = \int 2^{j(d_1+2d_2)} g_1(2^j x, 2^{2j} w) 2^{(j+n)d_2} g_2(2^{j+n}(u-w)) dw$$

Clearly  $\|H_{j,n}\|_1 = O(1)$  uniformly in  $j, n$  and since  $\|f_\rho\|_1 \lesssim 1$  we get from Minkowski's inequality  $\|f_\rho * H_{j,n}\|_1 \lesssim 1$ .

For the proof of (157) we may thus assume  $2^j \geq 1/\rho$  and it suffices to verify that for every  $(y, v) \in Q_{\rho,E}$  the  $L^1(Q_{A\rho,E}^c)$  norm of  $(x, u) \mapsto$

$$\int \frac{2^{j(d_1+2d_2)}}{(1+2^j|x-y|+2^{2j}|w|)^{N_1}} \frac{2^{(j+n)d_2}}{(1+2^{j+n}|u-v-w+\frac{1}{2}\langle \vec{J}x, y \rangle|)^{N_1}} dw$$

is bounded by  $C(2^j \rho)^{-N}$  if  $N_1 \gg N + d_1 + 2d_2$ . This is straightforward. For the proof of (158) we observe that (159) implies

$$2^{-j}\|\nabla_x H_{j,n}\|_1 + 2^{-j-n}\|\nabla_u H_{j,n}\|_1 = O(1).$$

Moreover  $2^{-n} \| |x| \nabla_u H_{j,n} \|_1 = O(1)$ . By the cancellation condition (154)

$$\begin{aligned} f * H_{j,n}(x, u) &= \int f_\rho(y, v) [H_{j,n}(x - y, u - v + \frac{1}{2} \langle \vec{J}x, y \rangle) - H_{j,n}(x, u)] dy dv \\ &= - \int f_\rho(y, v) \left( \int_0^1 \langle y, \nabla_x H_{j,n}(x - sy, u - sv + \frac{s}{2} \langle \vec{J}x, y \rangle) \rangle \right. \\ &\quad \left. + \langle v + \frac{1}{2} \langle \vec{J}x, y \rangle, \nabla_u H(x - sy, u - sv + \frac{s}{2} \langle \vec{J}x, y \rangle) \rangle ds \right) dy dv. \end{aligned}$$

We also use  $\langle \vec{J}x, y \rangle = \langle \vec{J}(x - sy), y \rangle$  and a change of variable to estimate

$$\| f_\rho * H_{j,n} \| \lesssim \| f_\rho \|_1 \rho [ \| \nabla_x H_{j,n} \|_1 + \| \nabla_u H_{j,n} \|_1 + \| |x| \nabla_u H_{j,n} \|_1 ],$$

and (158) follows.  $\square$

*Proof of (156).* For  $n > 0$  split

$$\begin{aligned} W_n f_\rho &= \sum_{\substack{j \geq n-11 \\ 2^j \rho < 2^{-10n}}} W_{j,n} f_\rho + \sum_{\substack{j \geq n-11 \\ 2^{-10n} \leq 2^j \rho \leq 2^{10n}}} W_{j,n} f_\rho + \sum_{2^{10n} < 2^j \rho} W_{j,n} f_\rho \\ &:= I_{n,\rho} + II_{n,\rho} + III_{n,\rho}. \end{aligned}$$

The main contribution comes from the middle term and by (66) and the estimate  $\| f_\rho \|_1 \lesssim 1$  we immediately get

$$(160) \quad \| II_{n,\rho} \|_1 \lesssim (1+n) 2^{-n(d_1-1)/2}.$$

Let  $\mathcal{J}_n$  be as in (64), so that  $\#(\mathcal{J}_n) = O(2^n)$ . We use the estimate (151) in conjunction with (158) and estimate

$$\begin{aligned} \| I_{n,\rho} \|_1 &\leq \sum_{2^j \rho < 2^{-10n}} \sum_{k \in \mathcal{J}_n} \sum_{l=1}^{\infty} \| 2^{-j(d-1)/2} T_{2^j}^{k,l}(f_\rho * H_{j,n}) \|_1 \\ &\lesssim \sum_{2^j \rho < 2^{-10n}} \sum_{k \in \mathcal{J}_n} \sum_{l=1}^{\infty} (2^l k)^{-\frac{d_1+1}{2}} 2^{n+j} \rho \lesssim 2^{-n(9+\frac{d_1-1}{2})}. \end{aligned}$$

We turn to the estimation of the term  $III_{n,\rho}$ . Let  $\mathfrak{X}_{\rho,n}$  be a maximal  $\sqrt{\varepsilon\rho}$  separated set of  $[2^{n-6}, 2^{n+6}]$ . For each  $\beta \in \mathfrak{X}_{\rho,n}$  let, for large  $C_1 \gg 1$ ,

$$(161) \quad \mathcal{N}_{n,\rho}(\beta) = \{(x, u) : |x| - r(\beta) \leq \sqrt{C_1\rho}, \quad |w(\beta, x, 4|u|)| \leq C_1\rho\}$$

and

$$\mathcal{N}_{n,\rho} = \bigcup_{\beta \in \mathfrak{X}_{\rho,n}} \mathcal{N}_{n,\rho}(\beta).$$

Observe that  $\text{meas}(\mathcal{N}_{n,\rho}(\beta)) \lesssim_{C_1} 2^{-n(d_1+d_2-2)} \rho^{3/2}$  (by (108) and (112)) and thus  $\text{meas}(\mathcal{N}_{n,\rho}) \lesssim_{C_1} \rho$ . We separately estimate the quantity  $III_{n,\rho}$  on  $\mathcal{N}_{n,\rho}$  and its complement. First, by the Cauchy-Schwarz inequality and (152) (with  $2^{j_0} \approx 2^{10n} \rho^{-1}$ )

$$\| III_{n,\rho} \|_{L^1(\mathcal{N}_{n,\rho})} \lesssim \rho^{1/2} \| III_{n,\rho} \|_2 \lesssim (2^{-10n} \rho)^{\frac{d-1}{2}} \rho^{1/2} \| f_\rho \|_2$$

and, since  $\rho^{d/2}\|f_\rho\|_2 \lesssim 1$ ,

$$(162) \quad \|III_{n,\rho}\|_{L^1(\mathcal{N}_{n,\rho})} \lesssim 2^{-5(d-1)n}.$$

In the complement of the exceptional set  $\mathcal{N}_{n,\rho}$  we split the term  $III_{n,\rho}$  as

$$III_{n,\rho} = \sum_{2^j \rho > 2^{10n}} \sum_{k \in \mathcal{J}_n} \sum_{l=1}^{\infty} (III_{n,\rho,j}^{k,l} + IV_{n,\rho,j}^{k,l})$$

where

$$\begin{aligned} III_{n,\rho,j}^{k,l} &= 2^{-j \frac{d-1}{2}} T_{2^j}^{k,l} [(f_\rho * H_{j,n}) \chi_{Q_{\rho,E,*}}] \\ IV_{n,\rho,j}^{k,l} &= 2^{-j \frac{d-1}{2}} T_{2^j}^{k,l} [(f_\rho * H_{j,n}) \chi_{Q_{\rho,E,*}^c}] \end{aligned}$$

and  $Q_{\rho,E,*}$  is as in (155). From (157) and (151) we immediately get  $\|IV_{n,\rho,j}^{k,l}\|_1 \lesssim_N (2^l k)^{-(d_1+1)/2} (2^j \rho)^{-N}$  and thus

$$\sum_{2^j \rho > 2^{10n}} \sum_{l=1}^{\infty} \sum_{k \in \mathcal{J}_n} \|IV_{n,\rho,j}^{k,l}\|_1 \lesssim 2^{-10nN}.$$

It remains to show that

$$(163) \quad \sum_{l=1}^{\infty} \sum_{k \in \mathcal{J}_n} \sum_{2^j \rho > 2^{10n}} \|III_{n,\rho,j}^{k,l}\|_{L^1(\mathcal{N}_{n,\rho}^c)} \lesssim 2^{-n(d_1-1)/2}.$$

Let  $F_{j,n,\rho} = (f_\rho * H_{j,n}) \chi_{Q_{\rho,E,*}}$ , so that  $\|F_{j,n,\rho}\|_1 \lesssim 1$ . We shall show that for  $k \approx 2^n$

$$(164) \quad \|F_{j,n,\rho} * A_{2^j}^{k,l}\|_{L^1(\mathcal{N}_{n,\rho}^c)} \lesssim_N (2^{j-n} \rho)^{-N} 2^{-(l+n) \frac{d_1}{2}}, \quad 2^l \leq 10^8 2^{j-n},$$

and (163) follows by combining (164) with the estimates (148) and (150).

*Proof of (164).* We split  $A_{2^j}^{k,l} = \sum_{b \in \mathcal{T}_{2^j,k,l}} A_{2^j,b}^{k,l}$  as in (127). For each  $b \in \mathcal{T}_{2^j,k,l}$  we may assign a  $\beta(b) \in \mathfrak{I}_{\rho,n}$  so that  $|\beta(b) - b| \leq \sqrt{\varepsilon \rho}$ . Let  $\mathcal{T}_{2^j,k,l}^\beta$  be the set of  $b \in \mathcal{T}_{2^j,k,l}$  with  $\beta(b) = \beta$ . Then  $\#\mathcal{T}_{2^j,k,l}^\beta \lesssim 2^{-n/2} \sqrt{2^{l+j} \rho}$ . In order to see (164) it thus suffices to show that for  $2^l \leq 10^8 2^{j-n}$ ,  $|\beta - b| \leq \rho$ ,

$$(165) \quad \|F_{j,n,\rho} * A_{2^j,b}^{k,l}\|_{L^1((\mathcal{N}_{n,\rho}(\beta))^c)} \lesssim_{N_1} (2^{j-n} \rho)^{-N_1} 2^{-(l+n) \frac{d_1+1}{2}} 2^{(n+l-j)/2}.$$

To prove this we verify the following

$$(166) \quad \begin{aligned} \text{Claim:} & \text{ If } (\tilde{x}, \tilde{u}) \in Q_{\rho,E,*}, (x, u) \in (\mathcal{N}_{n,\rho}(\beta))^c \text{ and } 2^{2m-j+n} \leq \rho \\ & \text{then } (|x - \tilde{x}|, 4|u - \tilde{u} + \frac{1}{2} \langle \vec{J}x, \tilde{x} \rangle) \notin \mathcal{P}_m(2^j, l, k; b); \end{aligned}$$

$\mathcal{P}_m(2^j, l, k; b)$  was defined in (133). Indeed the claim implies

$$\|F_{j,n,\rho} * A_{2^j,b}^{k,l}\|_{L^1((\mathcal{N}_{n,\rho}(\beta))^c)} \lesssim \int_{\substack{(|x|, 4|u|) \notin \\ \mathcal{P}_m(2^j, l, k; b)}} |A_{2^j,b}^{k,l}(x, u)| dx du$$

since  $\|F_{j,n,\rho}\|_1 = O(1)$  and (165) follows from Proposition 8.7.

To verify the claim (166) we pick  $(x, u) \notin \mathcal{N}_{n,\rho}(\beta)$  and distinguish two cases:

$$\text{Case 1: } ||x| - r(\beta)| \geq \sqrt{C_1\rho}.$$

$$\text{Case 2: } |w(\beta, |x|, 4|u|)| \geq C_1\rho \text{ and } ||x| - r(\beta)| \leq \sqrt{C_1\rho}.$$

It is clear that the conclusion of the claim holds if we can show that under the assumption that  $C_1$  in the definition (161) is chosen sufficiently large (depending only on  $\vec{J}$  and the dimension  $d$ ) we have for all  $(\tilde{x}, \tilde{u}) \in Q_{\rho,E,*}$

$$(167) \quad ||x - \tilde{x}| - r(b)| \geq \frac{\sqrt{C_1\rho}}{2} \quad \text{in Case 1,}$$

$$(168) \quad |w(b, |x - \tilde{x}|, 4|u - \tilde{u} + \frac{1}{2}\langle \vec{J}x, \tilde{x} \rangle)| \geq \frac{C_1\rho}{2} \quad \text{in Case 2.}$$

The Case 1 assumption implies for  $(\tilde{x}, \tilde{u}) \in Q_{\rho,E,*}$  (and sufficiently large  $C_1$ )

$$\begin{aligned} ||x - \tilde{x}| - r(b)| &\geq ||x| - r(\beta)| - |\tilde{x}| - |r(b) - r(\beta)| \\ &\geq C_1\rho^{1/2} - C_*\rho - C|b - \beta|2^{-n} \geq \frac{\sqrt{C_1\rho}}{2} \end{aligned}$$

which is (167).

Now assume that  $(x, u)$  satisfies the Case 2 assumption. We then have for all  $(\tilde{x}, \tilde{u}) \in Q_{\rho,E,*}$

$$\begin{aligned} &|w(b, |x - \tilde{x}|, 4|u - \tilde{u} + \frac{1}{2}\langle \vec{J}x, \tilde{x} \rangle)| - w(\beta, |x|, 4|u|)| \\ &\leq |w(b, |x|, 4|u|) - w(\beta, |x|, 4|u|)| \\ &\quad + |w(b, |x - \tilde{x}|, 4|u - \tilde{u} + \frac{1}{2}\langle \vec{J}x, \tilde{x} \rangle) - w(b, |x|, 4|u|)| \end{aligned}$$

The first term on the right hand side can be estimated using (132) (with  $(t, b)$  replaced by  $(b, \beta)$ ), and we see that it is  $\leq (C + \sqrt{C_1})\rho$  under the present Case 2 assumption. The second term on the right hand side is equal to

$$\left| 4|u| - 4|u - \tilde{u} + \frac{1}{2}\langle \vec{J}x, \tilde{x} \rangle| - \frac{v'(b)}{r'(b)}(|x| - |x - \tilde{x}|) \right|$$

and since the Case 2 assumption implies  $|x| = O(1)$  we see that the displayed expression is  $O(\rho)$ . Thus, if  $C_1$  in the definition is sufficiently large we obtain (168). This concludes the proof of the claim (166) and thus the estimate (164).

We finally consider the case where  $1/2 < \rho \leq 1$ , in which condition (154) is not required. This case can easily be handled by means of Proposition 8.8. To this end, we decompose

$$a(\sqrt{L})e^{i\sqrt{L}} = \sum_{j \geq 10} 2^{-\frac{d-1}{2}j} g_j(2^{-j}\sqrt{L})e^{i\sqrt{L}},$$

with  $g_j(s) = 2^{\frac{d-1}{2}j} a(2^j s) \chi_1(s)$ . The family of functions  $g_j$  is uniformly bounded in the Schwartz space. If  $K_j$  denotes the convolution kernel for



the operator  $g_j(2^{-j}\sqrt{L})e^{i\sqrt{L}}$ , we thus obtain from Proposition 8.8 the uniform estimates

$$\int_{\max\{|x|,|u|\}\geq 100} |K_j(x,u)| dx du \leq C_N 2^{-jN}.$$

This implies that

$$\int_{\max\{|x|,|u|\}\geq 200} |(a(\sqrt{L})e^{i\sqrt{L}}f_\rho)(x)| dx du \lesssim \|f\|_1 \lesssim 1.$$

And, by Hölder's inequality,

$$\int_{\max\{|x|,|u|\}\leq 200} |(a(\sqrt{L})e^{i\sqrt{L}}f_\rho)(x)| dx \lesssim \|(a(\sqrt{L})e^{i\sqrt{L}}f_\rho)\|_2 \lesssim \|f_\rho\|_2 \lesssim 1.$$

This concludes the proof of Theorem 5.3.  $\square$

## 10. INTERPOLATION OF HARDY SPACES

By interpolation for analytic families Theorem 1.1 can be deduced from the Hardy space estimate if we show that  $L^p(G)$  is an interpolation space for the couple  $(h_{\text{iso}}^1(G), L^2(G))$ , with respect to Calderón's complex  $[\cdot, \cdot]_\vartheta$  method.

**Theorem 10.1.** *For  $1 < p \leq 2$ ,*

$$(169) \quad [h_{\text{iso}}^1(G), L^2(G)]_\vartheta = L^p(G), \quad \vartheta = 2 - 2/p,$$

*with equivalence of norms.*

*Proof.* We deduce (169) from an analogous formula for the Euclidean local Hardy spaces  $h_E^1$ , more precisely, the vector-valued extension

$$(170) \quad [\ell^1(h_E^1), \ell^2(L^2)]_\vartheta = \ell^p(L^p), \quad \vartheta = 2 - 2/p.$$

Here  $\ell^p \equiv \ell^p(\mathbb{Z}^{d_1+d_2})$ . To do this one uses the method of retractions and coretractions (*cf.* [34]); (170) follows from the definition of the complex interpolation method if operators

$$\begin{aligned} R &: h_{\text{iso}}^1 + L^2 \rightarrow \ell^1(h_E^1) + \ell^2(L^2) \\ S &: \ell^1(h_E^1) + \ell^2(L^2) \rightarrow h_{\text{iso}}^1 + L^2 \end{aligned}$$

can be constructed such that

$$R : \begin{cases} h_{\text{iso}}^1 \rightarrow \ell^1(h_E^1) \\ L^2 \rightarrow \ell^2(L^2) \end{cases} \quad S : \begin{cases} \ell^1(h_E^1) \rightarrow h_{\text{iso}}^1 \\ \ell^2(L^2) \rightarrow L^2 \end{cases}$$

are bounded and

$$SR = I,$$

the identity operator on  $L^p$  or  $h_{\text{iso}}^1$ .

To define  $R$  and  $S$  let  $\varphi_1 \in C_0^\infty(\mathbb{R}^{d_1})$ ,  $\varphi_2 \in C_0^\infty(\mathbb{R}^{d_2})$  supported in  $(-1, 1)^{d_1}$  and  $(-1, 1)^{d_2}$  respectively and such that for all  $x \in \mathbb{R}^{d_1}$ ,  $u \in \mathbb{R}^{d_2}$

$$(171) \quad \sum_{X \in \mathbb{Z}^{d_1}} \varphi_1^2(x + X) = 1, \quad \sum_{U \in \mathbb{Z}^{d_2}} \varphi_2^2(u + U) = 1.$$

We define  $\varphi(x, u) = \varphi_1(x)\varphi_2(u)$  and set

$$Rf = \{R_{X,U}f\}_{(X,U) \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}}$$

where  $R_{X,U}f(x, u) = \varphi(-x, -u)f(x + X, u + U + \frac{1}{2}\langle \vec{J}X, x \rangle)$ ;

moreover for  $H = \{H_{X,U}\}_{(X,U) \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}} \in \ell^1(h_E^1)$  we set

$$SH(x, u) = \sum_{(X,U) \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}} \varphi(X - x, U - u - \frac{1}{2}\langle \vec{J}X, x \rangle) H_{X,U}(x - X, u - U - \frac{1}{2}\langle \vec{J}X, x \rangle)$$

One verifies quickly from (171) that  $SR$  is the identity.

We now examine the boundedness properties of  $R$  and  $S$ . For the  $h_{\text{iso}}^1 \rightarrow \ell^1(h_E^1)$  of  $R$  we consider a (Heisenberg-) $(P, \rho)$  atom  $a$  with  $P = (x_P, u_P)$  and  $\rho \leq 1$ . Note that  $\varphi(-x, -u)a(x + X, u + U + \langle \vec{J}X, x \rangle)$  is then supported on the set of  $(x, u) \in (-1, 1)^{d_1+d_2}$  such that

$$|x_P - X - x|^2 + |u_P - U - u - \langle \vec{J}(X - x_P), x \rangle|^2 \leq \rho^2.$$

Thus  $R_{X,U}f$  is not identically zero only when  $|X - x_P| + |U - u_P| \leq C_d$  some absolute constant  $C_d$ . And, since  $\langle \vec{J}(X - x_P), X - x_P \rangle = 0$  we also see that in this case the function

$$(172) \quad (x, u) \mapsto a(x + X, u + U + \frac{1}{2}\langle \vec{J}X, x \rangle)$$

is supported in a Euclidean ball of radius  $C\rho$  with center  $(x_P - X, u_P - U)$ . Since the cancellation property (if  $\rho \leq 1/2$ ) is not affected by the change of variable we see that the function (172) is equal to  $c_b b$  where  $b$  is a Euclidean atom and  $|c_b| \lesssim 1$ . Thus this function is in  $h_E^1$  with norm  $\lesssim 1$ . We also use that multiplication with  $\varphi(-x, -u)$  defines an operator which is bounded on the local Hardy-space  $h_E^1$ . Now it follows quickly that  $R$  is bounded as an operator from  $h_{\text{iso}}^1$  to  $\ell^1(h_E^1)$ . Indeed if  $f = \sum_{c_\nu} a_\nu$  where  $a_\nu$  are  $(P_\nu, r_\nu)$  atoms for suitable  $r_\nu \leq 1$  and  $P_\nu$  then

$$\begin{aligned} \|Rf\|_{\ell^1(h_E^1)} &= \sum_{X,U} \left\| R_{X,U} \sum_{\nu} c_\nu a_{P_\nu} \right\|_{h_E^1} \\ &\leq C \sum_{X,U} \sum_{\nu: |x_{P_\nu} - X| \leq C_d, |u_{P_\nu} - U| \leq C_d} |c_\nu| \leq C' \sum_{\nu} |c_\nu|. \end{aligned}$$

This completes the proof of the  $h_{\text{iso}}^1 \rightarrow \ell^1(h_E^1)$  boundedness of  $R$ .

We now show that  $S$  maps  $\ell^1(h_E^1)$  boundedly to  $h_{\text{iso}}^1$ . We first recall that the operation of multiplication with a smooth bump function maps  $h_E^1$  to itself (cf. [8]), thus

$$\|\varphi(\cdot)G_{X,U}\|_{h_E^1} \leq C\|G_{X,U}\|_{h_E^1}.$$

Using the atomic decomposition of  $h_E^1$  functions we can decompose

$$\varphi(\cdot)G_{X,U} = \sum_{\nu} c_{X,U,\nu} a_{X,U,\nu}$$

where  $\sum_{X,U,\nu} |c_{X,U,\nu}| \lesssim \|G\|_{\ell^1(h_E^1)}$  and the  $a_{X,U,\nu}$  are Euclidean atoms supported in a ball

$$\{(x, u) : |x - x_P|^2 + |u - u_P|^2 \leq r^2\} \subset [-3, 3]^{d_1+d_2};$$

with  $P = P(X, U, \nu)$  and  $r = r(X, U, \nu)$ . Fix such an atom  $a = a_{X,U,\nu}$ . The function

$$(173) \quad \tilde{a}_{X,U,\nu} : (x, u) \mapsto a_{X,U,\nu}(x - X, u - U - \frac{1}{2}\langle \vec{J}X, x \rangle)$$

is supported in

$$\{(x, u) : (|x - X - x_P|^2 + |u - U - \frac{1}{2}\langle \vec{J}X, x \rangle - u_P|^2)^{1/2} \leq r\}$$

which is contained in the set of  $(x, u)$  such that

$$(|x - (X + x_P)|^2 + |u - U - \frac{1}{2}\langle \vec{J}(X + x_P), x \rangle - u_P + \frac{1}{2}\langle \vec{J}x_P, X + x_P \rangle|^2)^{\frac{1}{2}}$$

is  $\leq (1 + \frac{3}{2}\sqrt{d_1})r$ . Here we have used that  $|\langle \vec{J}x_P, x - (X + x_P) \rangle| \leq |x_P|r$  and  $|x_P| \leq 3\sqrt{d_1}$ . The inclusion shows that there is a constant independent of  $X, U, \nu$  so that function  $\tilde{a}_{X,U,\nu}/C$  is a Heisenberg atom associated with a cube centered at  $(X + x_P, U + u_P + \frac{1}{2}\langle \vec{J}x_P, X + x_P \rangle)$ . This statement holds at least if  $r \leq 1/(4d_1)$ . If  $r$  is close to one then we can express  $\tilde{a}_{X,U}$  as a finite sum of  $6^d$  atoms supported in cubes of sidelength 1. Thus we see that the function in (173) has  $h_{\text{iso}}^1$  norm  $\lesssim 1$ . This implies the  $\ell^1(h_E^1) \rightarrow h_{\text{iso}}^1$  boundedness of  $S$ , since it follows that

$$\|SG\|_{h_{\text{iso}}^1} \lesssim \sum_{(X,U)} \sum_{\nu} |c_{X,U,\nu}| \|\tilde{a}_{X,U,\nu}\|_{h_{\text{iso}}^1} \lesssim \sum_{X,U,\nu} |c_{X,U,\nu}| \lesssim \|G\|_{\ell^1(h_E^1)}.$$

Finally the  $L^2 \rightarrow \ell^2(L^2)$  boundedness of  $R$  and the  $\ell^2(L^2) \rightarrow L^2$  boundedness of  $S$  are even more straightforward and follow by modifications of the arguments.  $\square$

*Proof of Theorem 1.1.* By duality we may assume  $1 < p < 2$ . By scaling and symmetry we may assume  $\tau = 1$ . Let  $a \in S^{-(d-1)(1/p-1/2)}$ . Consider the analytic family of operators

$$\mathcal{A}_z = e^{z^2} \sum_{j=0}^{\infty} 2^{-jz\frac{d-1}{2}} 2^{j(d-1)(\frac{1}{p}-\frac{1}{2})} \zeta_j(\sqrt{L}) a(\sqrt{L}) e^{i\sqrt{L}}.$$

We need to check that  $\mathcal{A}_z$  is bounded on  $L^p$  for  $z = (2/p - 1)$ . But for  $\text{Re}(z) = 0$  the operators  $\mathcal{A}_z$  are bounded on  $L^2$ ; and for  $\text{Re}(z) = 1$  we

have shown that  $\mathcal{A}_z$  maps  $h_{\text{iso}}^1$  boundedly to  $L^1$ , by Theorem 1.3. We apply the abstract version of the interpolation theorem for analytic families in conjunction with Theorem 10.1 and the corresponding standard version interpolation result for  $L^p$  spaces; the result is that  $\mathcal{A}_\theta$  is bounded on  $L^p$  for  $\theta = 2/p - 1$ . This proves Theorem 1.1.  $\square$

## 11. PROOF OF THEOREM 1.4

We decompose  $m = \sum_{k \in \mathbb{Z}} m_k$  where  $m_k$  is supported in  $(2^{k-1}, 2^{k+1})$  and where  $h_k = m_k(2^k \cdot)$  satisfies

$$\sum_{\ell > 1} \sup_k \int_{2^\ell}^\infty |\widehat{h}_k(\tau)| \tau^{\frac{d-1}{2}} d\tau \leq A.$$

By the translation invariance and the usual Calderón-Zygmund arguments (see, e.g., [31]) it suffices to prove that for all  $\rho > 0$  and for all  $L^1$  functions  $f_\rho$  supported in the Koranyi-ball  $Q_\rho := Q_\rho(0, 0)$  and satisfying  $\int f_\rho dx = 0$  we have

$$(174) \quad \sum_k \iint_{Q_{10\rho}^\ell} |m_k(\sqrt{L})f_\rho| dx \lesssim A + \|m\|_\infty$$

Let  $\chi_1 \in C_0^\infty$  be supported in  $(1/5, 5)$  so that  $\chi_1(s) = 1$  for  $s \in [1/4, 4]$ . Then for each  $k$  write

$$m_k(\sqrt{L}) = h_k(2^{-k}\sqrt{L})\chi_1(2^{-k}\sqrt{L}) = \int \widehat{h}_k(\tau)\chi_1(2^{-k}\sqrt{L})e^{i2^{-k}\tau\sqrt{L}}d\tau.$$

By scale invariance and Theorem 1.2, the  $L^1$  operator norm of the operator  $\chi_1(2^{-k}\sqrt{L})e^{i2^{-k}\tau\sqrt{L}}$  is  $O(1 + |\tau|)^{(d-1)/2}$  and thus

$$\|m_k(\sqrt{L})\|_{L^1 \rightarrow L^1} \lesssim \int_{-\infty}^\infty |\widehat{h}_k(\tau)|(1 + |\tau|)^{\frac{d-1}{2}} d\tau.$$

Also observe that since the convolution kernel of  $\chi_1(\sqrt{L})$  is a Schwartz kernel we can use the cancellation and support properties of  $f_\rho$  to get, with some  $\varepsilon > 0$ ,

$$\|\chi_1(2^{-k}\sqrt{L})f_\rho\|_1 \lesssim \min\{1, (2^k\rho)^\varepsilon\} \|f_\rho\|_1.$$

Thus the two preceding displayed inequalities yield

$$(175) \quad \begin{aligned} \sum_{k: 2^k\rho \leq M} \|m_k(\sqrt{L})f_\rho\|_1 &\leq C_M \sup_k \int_{-\infty}^\infty |\widehat{h}_k(\tau)|(1 + |\tau|)^{\frac{d-1}{2}} d\tau \|f_\rho\|_1 \\ &\lesssim_M (\|m\|_\infty + \mathfrak{A}_2) \|f_\rho\|_1 \end{aligned}$$

where for the last estimate we use  $|\widehat{h}_k(\tau)| \leq \|h_k\|_\infty \lesssim \|m\|_\infty$  when  $|\tau| \leq 2$ .

We now consider the terms for  $2^k\rho \geq M$  and  $M$  large, in the complement of the expanded Koranyi-ball  $Q_{\rho,*} = Q_{C\rho}$  (for suitable large  $C \gg 2$ ). By a

change of variable and an application of Proposition 8.8,

$$\begin{aligned} & \|e^{i2^{-k}\tau\sqrt{L}}\chi_1(2^{-k}\sqrt{L})f_\rho\|_{L^1(Q_{\rho,*}^c)} = \|e^{i\sqrt{L}}\chi_1(\tau^{-1}\sqrt{L})f_\rho^{2^k/\tau}\|_{L^1(Q_{C^*\tau^{-1}2^k\rho}^c)} \\ & \lesssim (2^k\rho\tau^{-1})^{-N} \quad \text{if } 2^k\rho \gg \tau, \end{aligned}$$

where  $f_\rho^{2^k/\tau}$  is a re-scaling of  $f_\rho$  such that  $\|f_\rho^{2^k/\tau}\|_1 = \|f_\rho\|_1 \lesssim 1$ .

Hence if  $M$  is sufficiently large then for  $2^k\rho > M$

$$\begin{aligned} \|m_k(\sqrt{L})f_\rho\|_{L^1(Q_{\rho,*}^c)} & \lesssim_N \|f_\rho\|_1 \left[ \int_{|\tau|>2^k\rho} |\widehat{h}_k(\tau)|(1+|\tau|)^{\frac{d-1}{2}} d\tau \right. \\ & \quad \left. + (2^k\rho)^{-N} \int_{|\tau|\leq 2^k\rho} |\widehat{h}_k(\tau)|(1+|\tau|)^{-N} d\tau \right] \end{aligned}$$

and thus

$$(176) \quad \sum_{2^k\rho>M} \|m_k(\sqrt{L})f_\rho\|_{L^1(Q_{\rho,*}^c)} \lesssim \|m\|_\infty + \sum_{k:2^k\rho>M} \mathfrak{A}_{2^k\rho}.$$

The theorem follows from (175) and (176).  $\square$

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