ON UNIFORM BOUNDEDNESS OF DYADIC AVERAGING OPERATORS IN SPACES OF HARDY-SOBOLEV TYPE

GUSTAVO GARRIGÓS   ANDREAS SEEGER   TINO ULLRICH

Abstract. We give an alternative proof of recent results by the authors of uniform boundedness of dyadic averaging operators in (quasi-)Banach spaces of Hardy-Sobolev and Triebel-Lizorkin type in the largest possible range of parameters. The proof here is based on characterizations of the respective spaces in terms of compactly supported Daubechies wavelets.

1. Introduction

Consider the dyadic averaging operators $\mathbb{E}_N$ on the real line given by

condexp (1)
$$\mathbb{E}_N f(x) = \sum_{\mu\in\mathbb{Z}} \mathbb{1}_{I_{N,\mu}}(x)\, 2^N \int_{I_{N,\mu}} f(t)dt \tag{1}$$

with $I_{N,\mu} = [2^{-N}\mu, 2^{-N}(\mu+1))$. $\mathbb{E}_N f$ is the conditional expectation of f with respect to the σ-algebra generated by the dyadic intervals of length 2^{-N}. The following theorem on uniform boundedness in Triebel-Lizorkin spaces $F^s_{p,q}$ was proved by the authors in [6], and the purpose of the present note is to give an alternative proof.

expthm **Theorem 1.1.** [6] *Let $1/2 < p < \infty$, $0 < q \le \infty$, and $1/p - 1 < s < \min\{1/p, 1\}$. Then there is a constant $C := C(p,q,s) > 0$ such that for all $f \in F^s_{p,q}$*

E_N (2)
$$\sup_{N\in\mathbb{N}} \|\mathbb{E}_N f\|_{F^s_{p,q}} \le C\|f\|_{F^s_{p,q}}. \tag{2}$$

This result served as the main tool to establish that suitably regular enumerations of the Haar system form a Schauder basis for the spaces $F^s_{p,q}$ in the parameter ranges of the theorem, see §3. The relation to the Haar system is via the martingale difference operators $\mathbb{D}_N = \mathbb{E}_{N+1} - \mathbb{E}_N$ which are the orthogonal projections to the spaces generated by Haar functions with fixed Haar frequency 2^N.

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In previous works stronger notions of convergence have been examined, such as the validity of unconditional convergence in the martingale difference series, i.e. the inequality

mult
$$\Big\| \sum_n b_n \mathbb{D}_n f \Big\|_{F^s_{p,q}} \lesssim \|b\|_{\ell^\infty(\mathbb{N})} \|f\|_{F^s_{p,q}}. \tag{3}$$

It follows from the results in Triebel [17] that inequality (3) holds when we add the condition $1/q - 1 < s < 1/1$ to the hypothesis in the theorem. For the case $q = 2$ this corresponds to the shaded region in Figure 1.

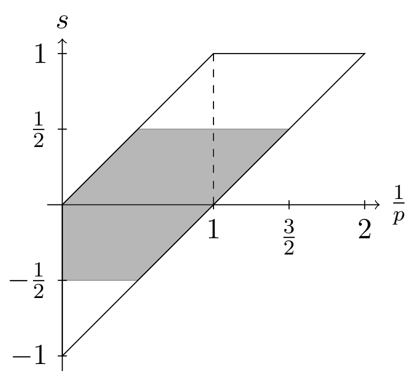

FIGURE 1. Unconditional convergence in Hardy-Sobolev spaces fig2

It was shown in [10], [11] that the additional restriction on the q-parameter is necessary for (3) to hold. If we drop it then Theorem 1.1 implies that (3) holds with the larger norm $\|b\|_\infty + \|b\|_{BV}$. It would be interesting to show sharp results involving sequence spaces that are intermediate between $\ell^\infty(\mathbb{N})$ and $BV(\mathbb{N})$.

In §2 we give a proof of Theorem 1.1 using wavelet decompositions. In §3 we apply these methods to get an additional result needed to obtain the Schauder basis property of the Haar system.

proofsect

2. Proof of Theorems 1.1

We start with some preliminaries on convolution kernels which are used in Littlewood-Paley type decompositions. We use a characterization of the Triebel-Lizorkin spaces via Littlewood-Paley operators defined by so-called "local means". Let φ_0, φ be Schwartz functions on the real line, compactly supported in $(-1/2, 1/2)$ such that $|\hat{\varphi}_0(\xi)| > 0$ on $(-1/2, 1/2)$ and $|\hat{\varphi}(\xi)| > 0$ on $\{\xi \in \mathbb{R} : 1/8 < |\xi| < 1\}$. Moreover φ has vanishing moments up to large order M, i.e.,

$$\int \varphi(x) x^n \, dx = 0 \quad \text{for} \quad n = 0, 1, ..., M\,.$$

Let $\varphi_j := 2^j\varphi(2^j\cdot)$ and $L_j f = \varphi_j * f$. We then have

$$\|f\|_{F^s_{p,q}} \asymp \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j * f)(\cdot)|^q \Big)^{1/q} \Big\|_p.$$

This result is closely related to the classical theorem of Benedek, Calderón and Panzone [1] on vector-valued singular integrals (at least for $1 < p, q < \infty$). For the quasi-Banach case and further refinements we refer to Triebel's book [15, §2.4.6], Rychkov [9] and the references therein.

In addition to the characterization via "local means" we will use a characterization via compactly supported Daubechies wavelets [2], [18, Sect. 4]. Let ψ_0 and ψ be the orthogonal scaling function and corresponding wavelet of Daubechies type such that ψ_0, ψ being sufficiently smooth (C^K) and ψ having sufficiently many vanishing moments (L). We denote

$$\psi_{j,\nu}(\cdot) := \frac{1}{\sqrt{2}}\psi(2^{j-1}\cdot - \nu) \quad , \quad j \in \mathbb{N}, \nu \in \mathbb{Z}\,,$$

and $\psi_{0,\nu}(\cdot) := \psi_0(\cdot - \nu)$ for $\nu \in \mathbb{Z}$. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If K and L are large enough (depending on p, q and s) then we have the equivalent characterization, see Triebel [16, Thm. 1.64] and the references therein,

wave_char (4)

$$\|f\|_{F^s_{p,q}} \asymp \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big| \sum_{\nu\in\mathbb{Z}} \lambda_{j,\nu}(f) \mathbb{1}_{j,\nu}(\cdot) \Big|^q \Big)^{1/q} \Big\|_p\,, \tag{4}$$

where $\mathbb{1}_{j,\nu}$ denotes the characteristic function of the interval $I_{j,\nu} := [2^{-j}\nu, 2^{-j}(\nu+1)]$ and $\lambda_{j,\nu}(f) := 2^j\langle f, \psi_{j,\nu}\rangle$. A corresponding characterization also holds true for Besov spaces $B^s_{p,q}$. Since we also deal with distributions which are not locally integrable, the inner product $\langle f, \psi_{j,\nu}\rangle$ has to be interpreted in the usual way. Clearly, f can be decomposed into wavelet building blocks, i.e.

decomp (5)

$$f = \sum_{j\in\mathbb{Z}} f_j := \sum_{j\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}} \lambda_{j,\nu}(f)\psi_{j,\nu}\,, \tag{5}$$

where we simply put $f \equiv 0$ if $j < 0$. Note, that the f_j represent K times continuously differentiable functions due to the regularity assumption on the wavelet.

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2.1. **Proof in the case $1/2 < p \leq 1$.** Now $1/p - 1 < s < 1$.

Step 1. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ denote the local mean kernels from above. Using the decomposition (5) we can write with $\theta := \min\{1,p,q\}$

$$\|\mathbb{E}_N(f)\|_{F^s_{p,q}} \asymp \Big\|\Big(\sum_{j=0}^{\infty}|2^{js}\mathbb{E}_N(f)*\varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p$$

$$\lesssim \Big(\sum_{\ell\in\mathbb{Z}}\Big\|\Big(\sum_{j=0}^{\infty}|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p^\theta\Big)^{1/\theta}$$

eq1_a

$$\lesssim \Big(\sum_{\ell\in\mathbb{Z}}\Big\|\Big(\sum_{j+\ell\le N}^{\infty}|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p^\theta \tag{6}$$

eq1_a1

$$+\Big\|\Big(\sum_{j+\ell\ge N}^{\infty}|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p^\theta\Big)^{1/\theta}. \tag{7}$$

Let us estimate the p-norms in (6) and (7) via distinguishing several cases.

Step 2. Here we restrict to $j+\ell\ge N$. We deal with (7) and use that

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$$\Big\|\Big(\sum_{j}^{\infty}|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p^\theta \le \sum_{j}^{\infty}\|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j\|_p^\theta. \tag{8}$$

We continue estimating $\|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j\|_p$. Note first that due to $p\le 1$

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$$\|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j\|_p \le \Big(\sum_{\nu\in\mathbb{Z}}|\lambda_{j+\ell,\nu}(f)|^p\|2^{js}\mathbb{E}_N(\psi_{j+\ell,\nu})*\varphi_j\|_p^p\Big)^{1/p}. \tag{9}$$

So it remains to deal with $\|2^{js}\mathbb{E}_N(\psi_{j+\ell,\nu})*\varphi_j\|_p$. Note, that due to $j+\ell\ge N$ the function $\mathbb{E}_N(\psi_{j+\ell,\nu})$ is a step function consisting of $O(1)$ non-vanishing steps. These steps have length 2^{-N} and magnitude bounded by $O(2^{N-(j+\ell)})$.

Case 2.1 Assume $j\ge N$.
Due to the cancellation of φ_j and $j\ge N$ we have that $|\mathbb{E}_N(\psi_{j+\ell,\nu})*\varphi_j(x)|$ is supported on a union of intervals of total measure $O(2^{-j})$ and bounded from above by $O(2^{N-(j+\ell)})$. This gives

eq11

$$\|2^{js}\mathbb{E}_N(\psi_{j+\ell,\nu})*\varphi_j\|_p \lesssim 2^{js}2^{-j/p}2^{N-(j+\ell)}. \tag{10}$$

Case 2.2. Assume $j\le N$.
Clearly, we have $\ell\ge 0$ since $j+\ell\ge N$. Since $j\le N$ we can not make use of the cancellation properties of φ_j. We still have that $|\mathbb{E}_N(\psi_{j+\ell,\nu})*\varphi_j(x)|$ is supported on an interval of size $O(2^{-j})$. However, due to the fact that $\mathbb{E}_N(\psi_{j+\ell,\nu})$ consists of $O(1)$ steps of length 2^{-N} each and $N\ge j$ the convolution produces an additional factor $O(2^{j-N})$. Hence,

eq12

$$\|2^{js}\mathbb{E}_N(\psi_{j+\ell,\nu})*\varphi_j\|_p \lesssim 2^{js}2^{-j/p}2^{N-(j+\ell)}2^{j-N}. \tag{11}$$

Step 3. Let us consider $j+\ell\le N$. In fact, we need a different strategy to estimate (6).

Case 3.1. Let us first deal with the case $j \le N$. We estimate as follows

eq31

$$\begin{aligned}
&\Big\| \Big(\sum_j |2^{js} \mathbb{E}_N(f_{j+\ell}) * \varphi_j(\cdot)|^q \Big)^{1/q} \Big\|_p^\theta \\
&\lesssim \Big\| \Big(\sum_j |2^{js} [\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}] * \varphi_j(\cdot)|^q \Big)^{1/q} \Big\|_p^\theta \\
&\quad + \Big\| \Big(\sum_j |2^{js} f_{j+\ell} * \varphi_j(\cdot)|^q \Big)^{1/q} \Big\|_p^\theta .
\end{aligned} \tag{12}$$

Similar as in (8) we estimate the first term on the right-hand side of (12) via

eq31a

$$\begin{aligned}
&\Big\| \Big(\sum_j^\infty |2^{js} [\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}] * \varphi_j(\cdot)|^q \Big)^{1/q} \Big\|_p^\theta \\
&\quad \le \sum_j \| 2^{js} [\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}] * \varphi_j \|_p^\theta .
\end{aligned} \tag{13}$$

Again, analogously to (9) we have

eq31b

$$\begin{aligned}
&\| 2^{js} [\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}] * \varphi_j \|_p \\
&\quad \lesssim \Big(\sum_{\nu \in \mathbb{Z}} |\lambda_{j+\ell,\nu}(f)|^p \| 2^{js} [\mathbb{E}_N(\psi_{j+\ell,\nu}) - \psi_{j+\ell,\nu}] * \varphi_j \|_p^p \Big)^{1/p} .
\end{aligned} \tag{14}$$

Using the mean value theorem of calculus together with (1) we see for all $x \in \mathbb{R}$ that

$$|(\mathbb{E}_N(\psi_{j+\ell,\nu}) - \psi_{j+\ell,\nu})(x)| \le 2^{j+\ell-N} .$$

Due to $j + \ell \le N$, its support has length $O(2^{-(j+\ell)})$ around $\nu 2^{-(j+\ell)}$. We continue distinguishing two more cases.

Case 3.1.1. Let $\ell \ge 0$. Since $j + \ell \ge j$ the convolution with φ_j gives an additional factor $2^{-\ell}$ and increases the support to an interval of size $O(2^{-j})$. Hence, we get

eq311

$$\| 2^{js} [\mathbb{E}_N(\psi_{j+\ell,\nu}) - \psi_{j+\ell,\nu}] * \varphi_j \|_p \lesssim 2^{js} 2^{j+\ell-N} 2^{-\ell} 2^{-j/p} . \tag{15}$$

Case 3.1.2. Assume $\ell \le 0$. This time the convolution with φ_j does not give an extra factor and the support of $[\mathbb{E}_N(\psi_{j+\ell,\nu}) - \psi_{j+\ell,\nu}] * \varphi_j$ has length $2^{-(j+\ell)}$. Thus, we have in this case

eq312

$$\| 2^{js} [\mathbb{E}_N(\psi_{j+\ell,\nu}) - \psi_{j+\ell,\nu}] * \varphi_j \|_p \lesssim 2^{js} 2^{j+\ell-N} 2^{-(j+\ell)/p} . \tag{16}$$

It remains to deal with the second term on the right-hand side of (12). Since $\psi_{j+\ell,\nu}$ and φ_j are sufficiently smooth and have sufficiently many vanishing moments well-known convolution inequalities, see for instance [7, p. 466] for the most general version or Frazier, Jawerth [4, Lem. 3.3], [5, Lem. B.1, B.2], yield

conv

$$|(f_{j+\ell} * \varphi_j)(x)| \lesssim 2^{-|\ell| M} \sum_{\nu \in \mathbb{Z}} |\lambda_{j+\ell,\nu}(f)| (1 + 2^{\min\{j, j+\ell\}} |x - \nu 2^{-(j+\ell)}|)^{-R} , \tag{17}$$

where M depends on the number of vanishing moments of the wavelet ψ and R can be chosen arbitrary large due to the compact support of $\psi_{j+\ell}$ and φ. Next we apply [8, Lem. 7.1] to (17) which yields that for any $0 < r < 1$ and $R > 1/r$

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$$\begin{aligned} &\sum_{\nu\in\mathbb{Z}} |\lambda_{j+\ell,\nu}(f)|(1+2^{\min\{j,j+\ell\}}|x-\nu 2^{-(j+\ell)}|)^{-R} \\ &\quad\lesssim 2^{|\ell_+|/r}\Big[M_{\mathrm{HL}}\Big|\sum_{\nu\in\mathbb{Z}}\lambda_{j+\ell,\nu}(f)\mathbb{1}_{j+\ell,\nu}\Big|^r\Big]^{1/r}(x) \quad , \quad x\in\mathbb{R}\,. \end{aligned} \tag{18}$$

If the order of the Daubechies wavelet system (resulting in smoothness and vanishing moments) is now chosen such that M in (17) is larger than $1/r+1$ there is a positive $\delta > 0$ such that

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$$\begin{aligned} &\Big\|\Big(\sum_j |2^{js} f_{j+\ell} * \varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p \\ &\quad\lesssim 2^{-|\ell|\delta}\Big\|\Big(\sum_j \Big[M_{\mathrm{HL}}\Big|2^{(j+\ell)s}\sum_{\nu\in\mathbb{Z}}\lambda_{j+\ell,\nu}(f)\mathbb{1}_{j+\ell,\nu}\Big|^r\Big]^{q/r}(\cdot)\Big)^{1/q}\Big\|_p\,. \end{aligned} \tag{19}$$

Choosing $r < \min\{p,q,1\}$ we can apply Fefferman-Stein maximal inequality [3] which, together with (4), yields

eq-fs

$$\Big\|\Big(\sum_j |2^{js} f_{j+\ell} * \varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p \lesssim 2^{-|\ell|\delta}\|f\|_{F^s_{p,q}}\,. \tag{20}$$

Case 3.2. Assume $j \geq N \geq j+\ell$ which means implicitly that $\ell \leq 0$. Using (8) and (9) again we reduce everything to estimating $\|2^{js}\mathbb{E}_N(\psi_{j+\ell,\nu}) * \varphi_j\|_p$. Due to the step function $\mathbb{E}_N(\psi_{j+\ell,\nu})$ and the cancellation of the φ_j we have the following identity

eq32

$$\begin{aligned} &2^{js}\|\mathbb{E}_N(\psi_{j+\ell,\nu}) * \varphi_j\|_p \\ &\quad\lesssim 2^{js}\Big(\sum_{\mu\in\mathbb{Z}}\int_{|x-2^{-N}\mu|\lesssim 2^{-j}} |\mathbb{E}_N(\psi_{j+\ell,\nu}) * \varphi_j(x)|^p\,dx\Big)^{1/p} \\ &\quad\lesssim 2^{js}\Big(\sum_{\mu\in\mathbb{Z}}\int_{|x-2^{-N}\mu|\lesssim 2^{-j}} |[\mathbb{E}_N(\psi_{j+\ell,\nu}) - \psi_{j+\ell,\nu}] * \varphi_j(x)|^p\,dx\Big)^{1/p} \\ &\qquad + 2^{js}\Big(\sum_{\mu\in\mathbb{Z}}\int_{|x-2^{-N}\mu|\lesssim 2^{-j}} |\psi_{j+\ell,\nu} * \varphi_j(x)|^p\,dx\Big)^{1/p} \\ &\quad\lesssim 2^{js}2^{j+\ell-N}2^{[N-(j+\ell)-j]/p} + \|2^{js}\psi_{j+\ell,\nu} * \varphi_j\|_p\,, \end{aligned} \tag{21}$$

where we took into account that the μ-sum consists of $O(2^{N-(j+\ell)})$ summands. It remains to deal with the quantity $\|2^{js}\psi_{j+\ell,\nu} * \varphi_j\|_p$ in (21). By the same convolution inequality as used in (17) we obtain $\|2^{js}\psi_{j+\ell,\nu} * \varphi_j\|_p \lesssim$

$2^{-|\ell|\delta}$ if the wavelet system has enough smoothness and vanishing moments. Hence,

eqB.3
$$2^{js}\|\mathbb{E}_N(\psi_{j+\ell,\nu}) * \varphi_j\|_p \lesssim 2^{js}2^{j+\ell-N}2^{[N-(j+\ell)-j]/p} + 2^{-\delta|\ell|}\,. \tag{22}$$

Step 4. (Estimation of (7). Plugging (8), (9) and (10) into (7) yields

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$$\begin{aligned}&\Big(\sum_{\ell\in\mathbb{Z}}\Big\|\Big(\sum_{j\geq\max\{N-\ell,N\}}^{\infty}|2^{js}\mathbb{E}_N(f_{j+\ell}) * \varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p^\theta\Big)^{1/\theta}\\ &\quad\lesssim \Big(\sum_{j\geq N}2^{(N-j)\theta}\sum_{\ell\geq N-j}2^{\theta\ell(1/p-1-s)}\Big)^{1/\theta}\sup_{j,\ell}\Big(\sum_{\nu\in\mathbb{Z}}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}.\end{aligned} \tag{23}$$

If $1/p-1<s<1/p$ then the sums are uniformly bounded (in N). Furthermore,

sup
$$\begin{aligned}\sup_{j,\ell}\Big(\sum_{\nu\in\mathbb{Z}}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p} &= \sup_j\Big\|2^{js}\sum_{\nu\in\mathbb{Z}}\lambda_{j,\nu}(f)\mathbb{1}_{j,\nu}(\cdot)\Big\|_p\\ &\leq \Big\|\sup_j 2^{js}\Big|\sum_{\nu\in\mathbb{Z}}\lambda_{j,\nu}(f)\mathbb{1}_{j,\nu}(\cdot)\Big|\Big\|_p\\ &\lesssim \|f\|_{F^s_{p,q}}\,,\end{aligned} \tag{24}$$

where we used (4) in the last estimate.

Plugging (8), (9) and (11) into (7) leads to a similar estimate as above, only the sums over j and ℓ change to

$$\sum_{j\leq N}\sum_{\ell\geq N-j}2^{\theta\ell(1/p-1-s)}$$

which is uniformly bounded (in N) if $s>1/p-1$.

Step 5. (Estimation of (6)*).* Replacing (6) by (12) we observe (using (20)) that the second summand in (12) after summing over ℓ already yields the desired bound. It remains to deal with the first summand in (12). Combining (13), (14), (15) and (24) we find

$$\begin{aligned}&\sum_{\ell=-\infty}^{N}\Big\|\Big(\sum_{j=0}^{\min\{N-\ell,N\}}|2^{js}[\mathbb{E}_N(f_{j+\ell})-f_{j+\ell}] * \varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p^\theta\\ &\quad\lesssim \Big(\sum_{j\leq N}2^{(j-N)\theta}\sum_{\ell=-\infty}^{N-j}2^{\theta\ell(1/p-s)}\Big)^{1/\theta}\|f\|_{F^s_{p,q}}\,.\end{aligned}$$

The sums are finite and uniformly bounded if $1/p-1<s<1/p$. Together with (24) we obtain the desired bound in case $\ell\geq 0$ (see Case 3.1.1).

Combining (13), (14) and (16) leads to a similar calculation where the sums over j and ℓ change to

$$\sum_{j\le N} 2^{(j-N)\theta} \sum_{\ell\le 0} 2^{\theta\ell(1-s)}\,,$$

which is uniformly bounded if $s<1$.

Finally, we combine (8), (9), (22) and (24) to obtain

$$\Big(\sum_{\ell\le 0}\Big\|\Big(\sum_{j=N}^{N-\ell}|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j(\cdot)|^q\Big)^{1/q}\Big\|_p^\theta\Big)^{1/\theta}$$
$$\lesssim \Big(\sum_{j\ge N} 2^{(j-N)\theta}2^{\theta(N-j)/p}\sum_{\ell=-\infty}^{N-j}2^{\theta\ell(1-s)} + \sum_{j\ge N}\sum_{\ell=-\infty}^{N-j}2^{-\theta\delta|\ell|}\Big)^{1/\theta}\|f\|_{F^s_{p,q}}\,,$$

which is uniformly bounded if $s<1$. This concludes the proof. □

2.2. **Proof in the case $1<p<\infty$.** We use the method in the proof of §2.1 and follow that proof until (8) and (13), respectively. Then we have to proceed differently.

Step 1. Assume $N\le j, j+\ell$. We replace (9) by

p>1

$$\begin{aligned}\|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j\|_p^p &\le \int\Big[\sum_{\nu\in\mathbb{Z}}|2^{js}\lambda_{j+\ell,\nu}(f)|\cdot|\mathbb{E}_N(\psi_{j+\ell,\nu})*\varphi_j(x)|\Big]^p\,dx\\ &\lesssim \sum_{\nu\in\mathbb{Z}}|2^{js}\lambda_{j+\ell,\nu(f)}|^p 2^{-j}\,.\end{aligned}\tag{25}$$

Indeed, since $\mathbb{E}_N(\psi_{j+\ell,\nu})=0$ if $\operatorname{supp}\psi_{j+\ell,\nu}\subset I_{N,\mu}$ the sum on the right-hand side of (25) is lacunary and the appearing functions $\mathbb{E}_N(\psi_{j+\ell,\nu})*\varphi_j(x)$ have essentially disjoint support. Hence, we get

20_1

$$\begin{aligned}&\|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j\|_p\\ &\quad\lesssim 2^{-\ell s}2^{[N-(j+\ell)]}2^{\ell/p}\Big(\sum_{\nu\in\mathbb{Z}}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p2^{-(j+\ell)}\Big)^{1/p}.\end{aligned}\tag{26}$$

The sum over the respective range of j and ℓ is uniformly bounded if $1/p-1<s<1/p$.

Step 2. Let us now deal with $j+\ell\ge N\ge j$. Here we estimate

20_2

$$\|2^{js}\mathbb{E}_N(f_{j+\ell})*\varphi_j\|_p \lesssim \|2^{js}(\mathbb{E}_N(f_{j+\ell})-f_{j+\ell})*\varphi_j\|_p + \|2^{js}f_{j+\ell}*\varphi_j\|_p\,.\tag{27}$$

The second summand is estimated using (17) and (18). This results in

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$$\|2^{js}f_{j+\ell}*\varphi_j\|_p \lesssim 2^{-\delta\ell}\Big(\sum_\nu|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p2^{-(j+\ell)}\Big)^{1/p}.\tag{28}$$

To estimate the first summand on the right-hand side of (27) we are going to exploit the cancellation property

canc (29) $$0 = \mathbb{E}_N(\mathbb{E}_N(f) - f)(x) = \int_{I_{N,\mu(x)}} \mathbb{E}_N(f)(y) - f(y)\,dy\,.$$

We continue estimating the first summand on the right-hand side of (27). Using (29) we obtain the pointwise estimate

mu-conv (30) $$\begin{aligned}&\Big|2^{js}\int \varphi_j(x-y)(\mathbb{E}_N(f_{j+\ell})(y) - f_{j+\ell}(y))\,dy\Big|\\ &\quad= \Big|2^{js}\sum_{\mu:|2^{-N}\mu - x|\lesssim 2^{-j}} \int_{I_{N,\mu}} \varphi_j(x-y)(\mathbb{E}_N(f_{j+\ell})(y) - f_{j+\ell}(y))\,dy\Big|\\ &\quad= \Big|2^{js}\sum_{\mu:|2^{-N}\mu - x|\lesssim 2^{-j}} \int_{I_{N,\mu}} (\varphi_j(x-y) - \varphi_j(x))\times\\ &\qquad\qquad \times(\mathbb{E}_N(f_{j+\ell})(y) - f_{j+\ell}(y))\,dy\Big|\,.\end{aligned}$$

We continue estimating

$$\begin{aligned}&\lesssim 2^{js}\sum_{\mu:|2^{-N}\mu - x|\lesssim 2^{-j}} \int_{I_{N,\mu}} |(\varphi_j(x-y) - \varphi_j(x))\cdot \mathbb{E}_N(f_{j+\ell})(y)|\,dy\\ &\quad+ \Big|2^{js}\sum_{\mu:|2^{-N}\mu - x|\lesssim 2^{-j}} \int_{I_{N,\mu}} (\varphi_j(x-y) - \varphi_j(x))\cdot f^{\mu,1}_{j+\ell}(y)\,dy\Big|\\ &\quad+ \Big|2^{js}\sum_{\mu:|2^{-N}\mu - x|\lesssim 2^{-j}} \int_{I_{N,\mu}} (\varphi_j(x-y) - \varphi_j(x))\cdot f^{\mu,2}_{j+\ell}(y)\,dy\Big|\\ &=: F_0(x) + F_1(x) + F_2(x)\,,\end{aligned}$$

where

$$\begin{aligned} f^{\mu}_{j+\ell} &:= \sum_{\nu:\operatorname{supp}\psi_{j+\ell,\nu}\cap I_{N,\mu}\neq\emptyset} \lambda_{j+\ell,\nu}(f)\psi_{j+\ell,\nu}\,,\\ f^{\mu,1}_{j+\ell} &:= \sum_{\nu:\operatorname{supp}\psi_{j+\ell,\nu}\subset I_{N,\mu}} \lambda_{j+\ell,\nu}(f)\psi_{j+\ell,\nu}\,,\end{aligned}$$

$f^{\mu,2}_{j+\ell} := f^{\mu}_{j+\ell} - f^{\mu,1}_{j+\ell}$, and $g^1_{j+\ell} := \sum_\mu f^{\mu,1}_{j+\ell}$. The function $F_0(x)$ can be pointwise estimated from above by

$$2^{js}\sum_{\mu:|2^{-N}\mu - x|\lesssim 2^{-j}} 2^{2j-2N} \sup_{y\in I_{N,\mu}} \sum_{\nu:\operatorname{supp}\psi_{j+\ell,\nu}\cap I_{N,\mu}\neq\emptyset} |\lambda_{j+\ell,\nu}(f)\mathbb{E}_N(\psi_{j+\ell,\nu})(y)|\,.$$

Here $\mathbb{E}_N(\psi_{j+\ell,\nu})$ is mostly vanishing, namely if $\operatorname{supp}\psi_{j+\ell,\nu}\subset I_{N,\mu}$. If it does not vanish then $I_{N,\mu}$ is contained in its support and $|\mathbb{E}_N(\psi_{j+\ell,\nu})| \lesssim 2^{N-(j+\ell)}$. This happens only boundedly many times (indep. of j,ℓ) with respect to ν.

For a fixed y there are only finitely many coefficients contributing. Hence, we have

eq27a

$$F_0(x) \lesssim 2^{js}2^{2j-2N}2^{N-(j+\ell)}\times \\ \times \sum_{\mu:|2^{-N}\mu-x|\lesssim 2^{-j}} \sup_{\nu:\operatorname{supp}\psi_{j+\ell,\nu}\cap I_{N,\mu}\neq\emptyset} |\lambda_{j+\ell,\nu}(f)|\,. \tag{31}$$

Taking the L_p-norm and using Hölder's inequality with $1/p+1/p'=1$ yields

F0

$$\|F_0\|_p \lesssim 2^{-\ell s}2^{2j-2N}2^{N-(j+\ell)}2^{(N-j)/p'}2^{\ell/p}\times \\ \times\Big(\sum_{\nu}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}. \tag{32}$$

To estimate $F_1(x)$ we observe

$$F_1(x) = \Big|2^{js}\int \varphi_j(x-y)\Big(\sum_{\mu:|2^{-N}\mu-x|\lesssim 2^{-j}} f^{\mu,1}_{j+\ell}(y)\Big)\,dy\Big| = |2^{js}\varphi_j * g^1_{j+\ell}(x)|\,.$$

With a similar reasoning as in (28) and a monotonicity argument we achieve

$$\|F_1\|_p \lesssim 2^{-\delta\ell}\Big(2^{(j+\ell)s}\sum_{\nu\in\mathbb{Z}}|\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}.$$

Finally, we deal with $F_2(x)$. Since to $f^{\mu,2}_{j+\ell}$ only a uniformly bounded number of coefficients $\lambda_{j+\ell,\nu}$ contribute to the sum and the integrals are taken over an interval of length $O(2^{-(j+\ell)})$ we obtain, similar as above, by Hölder's inequality

F_3

$$\|F_2\|_p \lesssim 2^{-\ell s}2^{-2\ell}2^{(N-j)/p'}2^{\ell/p}\Big(\sum_{\nu}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}. \tag{33}$$

Putting the estimates from (27) to (33) together we observe that the sum over the respective range of j and ℓ (see (7)) is uniformly bounded with respect to N if $s>1/p-1$.

Step 3. Here we deal with $j+\ell, j\le N$. We return to (12) and estimate the first summand as done in (13). We continue similarly as after (29) and obtain the pointwise estimate (30). Since $j+\ell\le N$ there is only a bounded number of coefficients $\lambda_{j+\ell,\nu}(f)$ contributing to $f_{j+\ell}$ on $I_{N,\mu}$. Using the mean value theorem in both factors of the integral in (30) we obtain

$$|2^{js}[\mathbb{E}_N(f_{j+\ell})-f_{j+\ell}]*\varphi_j(x)| \lesssim 2^{js}2^{2j-2N}2^{j+\ell-N}\times \\ \times \sum_{\mu:|2^{-N}\mu-x|\lesssim 2^{-j}} \sup_{|\nu 2^{-(j+\ell)}-2^N\mu|\lesssim 1}|\lambda_{j+\ell,\nu}(f)|\,,$$

which yields

$$\|2^{js}[\mathbb{E}_N(f_{j+\ell})-f_{j+\ell}]*\varphi_j\|_p \\ \lesssim 2^{-\ell s}2^{j+\ell-N}2^{2j-2N}2^{(N-j)/p'}2^{\ell/p}\Big(\sum_{\nu\in\mathbb{Z}}|2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}.$$

The sum over the respective j and ℓ is uniformly bounded in N whenever $-1 < s < 1 + 1/p$. To estimate the second term on the right-hand side of (12) we literally follow the arguments in (17) and below to end up with (20).

Step 4. Let us finally proceed with the case $j + \ell \le N \le j$. Instead of (21) we estimate as follows.

final
$$\begin{aligned} &\|2^{js}\mathbb{E}_N(f_{j+\ell}) * \varphi_j\|_p \\ &\le 2^{js}\Big(\sum_{\mu\in\mathbb{Z}} \int_{|x-2^{-N}\mu|\lesssim 2^{-j}} |[\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}] * \varphi_j(x)|^p \,dx\Big)^{1/p} \\ &\quad + 2^{js}\Big(\sum_{\mu\in\mathbb{Z}} \int_{|x-2^{-N}\mu|\lesssim 2^{-j}} |f_{j+\ell} * \varphi_j(x)|^p \,dx\Big)^{1/p}. \end{aligned} \tag{34}$$

The second summand on the right-hand side can be estimated by

jgeqN
$$2^{-|\ell|\delta}\Big(\sum_{\nu\in\mathbb{Z}} |2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}, \tag{35}$$

whereas, similar to (21), the first summand in (34) is bounded by

first
$$2^{-\ell s}2^{j+\ell-N}2^{(N-j)/p}\Big(\sum_{\nu\in\mathbb{Z}} |2^{(j+\ell)s}\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)}\Big)^{1/p}. \tag{36}$$

Altogether we encounter the condition $1/p-1 < s < 1/p$ for any $0 < q \le \infty$ for the uniform boundedness of $\mathbb{E}_N : F^s_{p,q} \to F^s_{p,q}$ in case $1 \le p < \infty$. □

Schauder

3. On the Schauder basis property for the Haar system.

Let $\{h_{N,\mu} : \mu \in \mathbb{Z}\}$ be the set of Haar functions with Haar frequency 2^{-N} and define for $N \in \mathbb{N}$ and sequences $a \in \ell^\infty(\mathbb{Z})$,

TNdef
$$T_N[f,a] = \sum_{\mu\in\mathbb{Z}} a_\mu 2^N \langle f, h_{N,\mu}\rangle h_{N,\mu}. \tag{37}$$

In particular for the choice of $a = (1,1,1,\dots)$ one recovers the operator $\mathbb{E}_{N+1} - \mathbb{E}_N$. It was shown in [6] that Theorem 1.1 together with the inequality

T_N
$$\begin{aligned} &\sup_{N\in\mathbb{N}} \sup_{\|a\|_\infty\le 1} \|T_N[f,a]\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,\infty}}, \\ &\qquad 1/2 < p \le \infty,\ 0 < q \le \infty, \text{ and } 1/p - 1 < s < \min\{1/p, 1\}. \end{aligned} \tag{38}$$

implies Schauder basis properties for suitable enumerations of the Haar system. We give a sketch of proof which relies on the arguments in the previous section.

Proof of (38). The crucial modification of the proof of Theorem 1.1 is the fact that, due to the cancellation properties of the Haar functions participating in (37), we do not need the splittings in (12), (21), (27), and (34) and the subsequent considerations like (17) – (20). Therefore, we may start with a Besov norm $\|\cdot\|_{B^s_{p,q}}$ on the left-hand side (see (6), (7), (8)) and always end up with the Besov norm $\|\cdot\|_{B^s_{p,\infty}}$ on the right-hand side, see (23), (26), (32) and the comments below. Clearly, the described method allows for pulling out the a_μ on the expense of $\|a\|_\infty$.

Step 1. Suppose $j+\ell, j \geq N$. The estimates in (25), (26) apply almost literally to $\|2^{js}T_N[\psi_{j+\ell,\nu}, a] * \varphi_j)\|_p$ producing the additional factor $\|a\|_\infty$ on the right-hand side. Note, that we did not yet need any cancellation of the Haar functions.

Step 2. Suppose $j+\ell \geq N \geq j$. The splitting in (27) is not necessary anymore, we can work directly with $\|2^{js}T_N[f_{j+\ell}, a]\|_p$. An analogous identity to (30) holds true with $\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}$ replaced by $T_N[f_{j+\ell}, a]$ due to the cancellation of the Haar functions $h_{N,\mu}$. In what follows we only have to care for a counterpart of F_0 since F_1 and F_2 do not show up. We end up with a counterpart of (32) for $\|2^{js}T_N[f_{j+\ell}, a]\|_p$ with an additional $\|a\|_\infty$ on the right-hand side.

Step 3. Suppose $N \geq j+\ell, j$. Again, due to the cancellation of the Haar function, a splitting as in (12) is not necessary and we obtain a version of (30) as in Step 2. The mean value theorem applied to the first factor in the integral gives the factor 2^{2j-2N}, whereas the cancellation of $h_{N,\mu}$ gives $|T_N(\psi_{j+\ell,\nu})(x)| \lesssim 2^{j+\ell-N}$. We continue as in the proof of Theorem 1.1.

Step 4. The remaining case $j+\ell \leq N \leq j$ goes analogously to Step 4 in the proof of Theorem 1.1. Note, that also here the splitting in (34) and the subsequent consideration for the second summand on the right-hand side is not necessary. □

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References

BCP [1] A. Benedek, A.-P. Calderón and R. Panzone. Convolution operators on Banach space valued functions. Proc. Nat. Acad. Sci. U.S.A., 48 (1962), 356–365.

Dau92 [2] I. Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

fs [3] C. Fefferman, E.M. Stein. Some maximal inequalities. Amer. J. Math. 93 (1971), 107–115.

FrJa86 [4] M. Frazier, B. Jawerth. Decomposition of Besov spaces. Indiana Univ. Math. J., 34(4):777–799, 1985.

FrJa90 [5] M. Frazier, B. Jawerth. A discrete transform and decompositions of distribution spaces. J. Funct. Anal., 93(1):34–170, 1990.

gsu-basic [6] G. Garrigós, A. Seeger, T. Ullrich. The Haar system as a Schauder basis in spaces of Hardy-Sobolev type. Preprint 2016.

Gr08 [7] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

Ky03 [8] G. Kyriazis. Decomposition systems for function spaces. Stud. Math., 157(2):133–169, 2003.

Ry99a [9] V. S. Rychkov. On a theorem of Bui, Paluszyński and Taibleson. *Proc. Steklov Inst.*, 227:280–292, 1999.

su [10] A. Seeger, T. Ullrich. Haar projection numbers and failure of unconditional convergence in Sobolev spaces. Math. Zeitschrift, 2016, published online. See also arXiv:1507.0121.

sudet [11] ———, Lower bounds for Haar projections: Deterministic Examples. To appear in Constructive Approximation. See also arXiv:1511.01470.

triebel73 [12] H. Triebel. Über die Existenz von Schauderbasen in Sobolev-Besov-Räumen. Isomorphiebeziehungen. Studia Math. 46 (1973), 83–100.

triebel78 [13] ———. On Haar bases in Besov spaces. Serdica 4 (1978), no. 4, 330–343.

Tr83 [14] ———. *Theory of function spaces*, volume 38 of *Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology]*. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983.

triebel2 [15] ———. *Theory of function spaces II.* Monographs in Mathematics, 84. Birkhäuser Verlag, Basel, 1992.

Tr06 [16] ———. *Theory of function spaces. III*, volume 100 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006.

triebel-bases [17] ———. *Bases in function spaces, sampling, discrepancy, numerical integration.* EMS Tracts in Mathematics, 11. European Mathematical Society (EMS), Zürich, 2010.

Woj97 [18] P. Wojtaszczyk. *A mathematical introduction to wavelets*, volume 37 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

Gustavo Garrigós, Department of Mathematics, University of Murcia, 30100 Espinardo, Murcia, Spain

E-mail address: gustavo.garrigos@um.es

Andreas Seeger, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI, 53706, USA

E-mail address: seeger@math.wisc.edu

Tino Ullrich, Hausdorff Center for Mathematics, Endenicher Allee 62, 53115 Bonn, Germany

E-mail address: tino.ullrich@hcm.uni-bonn.de