THE HAAR SYSTEM AS A SCHAUDER BASIS IN SPACES OF HARDY-SOBOLEV TYPE

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Abstract. We show that, for suitable enumerations, the Haar system is a Schauder basis in the classical Sobolev spaces in $\mathbb{R}^d$ with integrability $1 < p < \infty$ and smoothness $1/p-1 < s < 1/p$. This complements earlier work by the last two authors on the unconditionality of the Haar system and implies that it is a conditional Schauder basis for a nonempty open subset of the $(1/p, s)$-diagram. The results extend to (quasi-)Banach spaces of Hardy-Sobolev and Triebel-Lizorkin type in the range of parameters $\frac{d}{d+1} < p < \infty$ and $\max\{d(1/p-1), 1/p-1\} < s < \min\{1, 1/p\}$, which is optimal except perhaps at the end-points.

1. Introduction

We recall the definition of the (inhomogeneous) Haar system in $\mathbb{R}^d$. Consider the 1-variable functions

$$h^{(0)} = \mathbb{1}_{[0,1)} \quad \text{and} \quad h^{(1)} = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}.$$

For every $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_d) \in \{0,1\}^d$ one defines

$$h^{(\boldsymbol{\varepsilon})}(x_1, \dots, x_d) \;=\; h^{(\varepsilon_1)}(x_1) \cdots h^{(\varepsilon_d)}(x_d).$$

Finally, one sets

$$h^{(\boldsymbol{\varepsilon})}_{k,\ell}(x) = h^{(\boldsymbol{\varepsilon})}(2^k x - \ell), \quad k \in \mathbb{N}_0,\ \ell \in \mathbb{Z}^d,$$

Denoting $\Upsilon = \{0,1\}^d \setminus \{\vec{0}\}$, the Haar system is then given by

$$\mathcal{H}_d = \Big\{ h^{(\vec{0})}_{0,\ell} \Big\}_{\ell \in \mathbb{Z}^d} \cup \Big\{ h^{(\boldsymbol{\varepsilon})}_{k,\ell} \ :\ k \in \mathbb{N}_0,\ \ell \in \mathbb{Z}^d,\ \boldsymbol{\varepsilon} \in \Upsilon \Big\}.$$

Observe that supp $h^{(\boldsymbol{\varepsilon})}_{k,\ell}$ is the dyadic cube $I_{k,\ell} := 2^{-k}(\ell + [0,1]^d)$.

In this paper we consider basis properties of $\mathcal{H}_d$ in Besov spaces $B^s_{p,q}$, and Triebel-Lizorkin spaces $F^s_{p,q}$ in $\mathbb{R}^d$. We refer to [13], [14] for definitions and

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properties of these spaces and to [1] for terminology and general facts about bases in Banach spaces.

In the 1970's, Triebel [11, 12] proved that the Haar system $\mathcal{H}_d$ is a Schauder basis on $B^s_{p,q}(\mathbb{R}^d)$ if

$$\frac{d}{d+1} < p < \infty, \quad 0 < q < \infty, \quad \max\left\{d(\tfrac{1}{p}-1), \tfrac{1}{p}-1\right\} < s < \min\left\{1, \tfrac{1}{p}\right\}, \tag{1}$$

and that this range is maximal, except perhaps at the endpoints. Moreover, the basis is unconditional when (1) holds; see [15, Theorem 2.21]. Concerning $F^s_{p,q}$ spaces, however, in [15] it is only shown that $\mathcal{H}_d$ is an unconditional basis for $F^s_{p,q}(\mathbb{R}^d)$ when, besides (1), the additional assumption

$$\max\left\{d(\tfrac{1}{q}-1), \tfrac{1}{q}-1\right\} < s < \tfrac{1}{q} \tag{2}$$

is satisfied. Recently, two of the authors showed in [9, 10] that the additional restriction (2) is in fact necessary, at least when $d = 1$. It was left open whether suitable enumerations of the Haar system can form a Schauder basis in $F^s_{p,q}$ in the larger range (1). We shall answer this question affirmatively.

Given an enumeration $\{u_1, u_2, \dots\}$ of the system $\mathcal{H}_d$, we let P_N be the orthogonal projection onto the subspace spanned by $u_1, \dots, u_N$, i.e.

$$P_N f = \sum_{n=1}^{N} \|u_n\|_2^{-2} \langle f, u_n\rangle u_n\,. \tag{3}$$

The sequence $\{u_n\}_{n=1}^\infty$ is a Schauder basis on $F^s_{p,q}$ if

$$\lim_{N\to\infty} \|P_N f - f\|_{F^s_{p,q}} = 0, \quad \text{for all } f \in F^s_{p,q}. \tag{4}$$

In view of the uniform boundedness principle, density theorems and the result for Besov spaces, (4) follows if we can show that the operators P_N have uniform $F^s_{p,q} \to F^s_{p,q}$ operator norms. Note, that the condition $s < 1/p$ is necessary since the Haar functions need to belong to $F^s_{p,q}$. By duality, if $1 < p < \infty$, the condition $s > 1/p - 1$ becomes also necessary, so the range in (1) is optimal in this case. If $p \le 1$, then an interpolation argument shows that (1) is also a maximal range, except perhaps at the end-points; see §4 below.

Definition. An enumeration $\mathcal{U} = \{u_1, u_2, ...\}$ of the Haar system $\mathcal{H}_d$ is *admissible* if the following condition holds for each cube $I_\nu = \nu + [0,1]^d, \nu \in \mathbb{Z}^d$. If u_n and $u_{n'}$ are both supported in I_ν and $|\operatorname{supp}(u_n)| > |\operatorname{supp}(u_{n'})|$, then necessarily $n < n'$.

The table in the figure shows how to obtain an admissible (natural) enumeration of $\mathcal{H}_d$ via a diagonalization of the intervals I_ν versus the levels k. We first label the set $\mathbb{Z}^d = \{\nu_1, \nu_2, \dots\}$. Then, we follow the order indicated by the table, where being at position (ν_i, k) means to pick all the Haar functions with support contained in I_{ν_i} and size 2^{-kd}, arbitrarily enumerated, before going to the subsequent table entry.

Our main result reads as follows.

$k \backslash I_\nu$	I_{ν_0}	I_{ν_1}	I_{ν_2}	I_{ν_3}	I_{ν_4}	$\dots$
0	1	2	4	7	11	
1	3	5	8	12		
2	6	9	13			
3	10	14				
4	15					

FIGURE 1. An admissible enumeration of $\mathcal{H}_d$.

Theorem 1.1. *Let $\mathcal{U} = \{u_n\}_{n=1}^\infty$ be an admissible enumeration of the Haar system $\mathcal{H}_d$. Assume that*

(i) $\frac{d}{d+1} < p < \infty$,

(ii) $0 < q < \infty$,

(iii) $\max\{d(\frac{1}{p} - 1), \frac{1}{p} - 1\} < s < \min\{1, \frac{1}{p}\}$.

Then $\mathcal{U}$ is a Schauder basis in $F^s_{p,q}(\mathbb{R}^d)$.

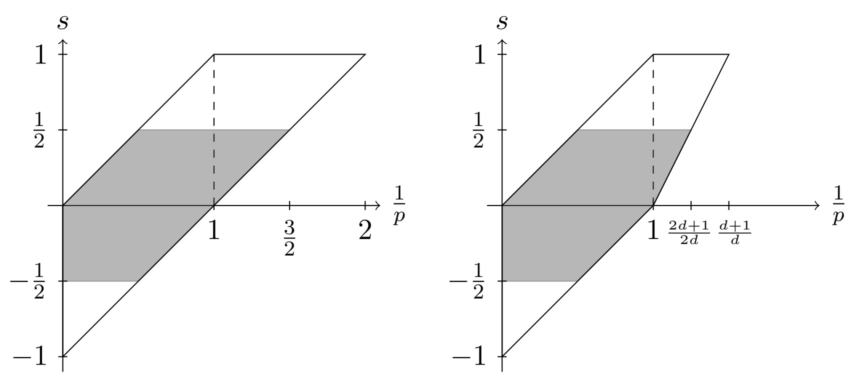

FIGURE 2. Unconditionality of the Haar system in Hardy-Sobolev spaces in $\mathbb{R}$ and $\mathbb{R}^d$

In the left part of Figure 2, the trapezoid is the parameter domain for which the Haar system is a Schauder basis in the Hardy-Sobolev space $H^s_p(\mathbb{R})$ ($= F^s_{p,2}(\mathbb{R})$) while the shaded part represents the parameter domain for which the Haar system is an unconditional basis in $H^s_p(\mathbb{R})$. The right figure shows the respective parameter domain for $H^s_p(\mathbb{R}^d)$.

The heart of the matter is a boundedness result for the dyadic averaging operators $\mathbb{E}_N$ given by

$$\mathbb{E}_N f(x) = \sum_{\mu \in \mathbb{Z}^d} \mathbb{1}_{I_{N,\mu}}(x)\, 2^N \int_{I_{N,\mu}} f(t)dt \tag{5}$$

with

$$I_{N,\mu} = 2^{-N}(\mu + [0,1)^d), \quad \mu \in \mathbb{Z}^d,\ N = 0, 1, 2, \dots$$

Note that $\mathbb{E}_N f$ is just the conditional expectation of f with respect to the σ-algebra generated by the set $\mathcal{D}_N$ of all dyadic cubes of side length 2^{-N}. There is a well known relation between the Haar system and the dyadic averaging operators, namely for $N = 0, 1, 2, \dots,$

$$\mathbb{E}_{N+1} f - \mathbb{E}_N f = \sum_{\varepsilon \in \Upsilon} \sum_{\mu \in \mathbb{Z}^d} 2^{Nd} \langle f, h_{N,\mu}^{(\varepsilon)} \rangle h_{N,\mu}^{(\varepsilon)}, \tag{6}$$

i.e. $\mathbb{E}_{N+1} - \mathbb{E}_N$ is the orthogonal projection onto the space generated by the Haar functions with Haar frequency 2^N.

Now let η_0 be a Schwartz function on $\mathbb{R}^d$, supported in $\{|\xi| < 3/8\}$ and so that $\eta_0(\xi) = 1$ for $|\xi| \le 1/4$. Let Π_N be defined by

$$\widehat{\Pi_N f}(\xi) = \eta_0(2^{-N}\xi) \widehat{f}(\xi). \tag{7}$$

There is a basic standard inequality (almost immediate from the definition of Triebel-Lizorkin spaces)

$$\sup_N \|\Pi_N f\|_{F^s_{p,q}} \le C(p,q,s) \|f\|_{F^s_{p,q}} \tag{8}$$

which is valid for all $s \in \mathbb{R}$ and for $0 < p < \infty$, $0 < q \le \infty$. Moreover, (8) and the fact that $\|\Pi_N g - g\|_{F^s_{p,q}} \to 0$ for Schwartz functions g gives

$$\lim_{N \to \infty} \|\Pi_N f - f\|_{F^s_{p,q}} = 0 \tag{9}$$

if $f \in F^s_{p,q}$ and $0 < p, q < \infty$. The main tool in proving Theorem 1.1 is a similar bound for the operators $\mathbb{E}_N$ which of course follows from the corresponding bound for $\mathbb{E}_N - \Pi_N$. It turns out that the operators $\mathbb{E}_N - \Pi_N$ enjoy better mapping properties in Besov spaces.

Similar bounds are also satisfied by projection operators into sets of Haar functions with fixed Haar frequency. Namely, for $N \in \mathbb{N}$ and functions $a \in \ell^\infty(\mathbb{Z}^d \times \Upsilon)$, we define

$$T_N[f,a] = \sum_{\varepsilon \in \Upsilon} \sum_{\mu \in \mathbb{Z}^d} a_{\mu,\varepsilon} 2^{Nd} \langle f, h_{N,\mu}^{(\varepsilon)} \rangle h_{N,\mu}^{(\varepsilon)}. \tag{10}$$

Observe that the choice $a_{\mu,\varepsilon} \equiv 1$ recovers the operator $\mathbb{E}_{N+1} - \mathbb{E}_N$. Then, we shall prove the following.

Theorem 1.2. *Let $d/(d+1) < p \le \infty$, $0 < r \le \infty$, and*

$$\max\{d(1/p - 1), 1/p - 1\} < s < \min\{1, 1/p\}. \tag{11}$$

Then there is a constant $C := C(p,r,s) > 0$ such that for all $f \in B^s_{p,\infty}$

$$\sup_N \|\mathbb{E}_N f - \Pi_N f\|_{B^s_{p,r}} \le C \|f\|_{B^s_{p,\infty}}. \tag{12}$$

Moreover,

$$\sup_N \|T_N[f,a]\|_{B^s_{p,r}} \lesssim \|a\|_\infty \|f\|_{B^s_{p,\infty}}. \tag{13}$$

We have the embedding $F^s_{p,q} \subset F^s_{p,\infty} \subset B^s_{p,\infty}$ which we use on the function side. For $r \le p$ we have the embedding $B^s_{p,r} \subset F^s_{p,r}$ (by Minkowski's inequality in $L^{p/r}$) and if also $r < q$ we have $F^s_{p,r} \subset F^s_{p,q}$; these two are used for $\mathbb{E}_N f - \Pi_N f$, or $T_N[f, a]$. In particular we conclude from Theorem 1.2 that $\mathbb{E}_N - \Pi_N$ is bounded on $F^s_{p,q}$, uniformly in N. Hence

Corollary 1.3. *Let p, s be as in* (11) *and $0 < q \le \infty$. Then*

$$\sup_N \|\mathbb{E}_N f\|_{F^s_{p,q}} + \sup_N \sup_{\|a\|_{\ell^\infty} \le 1} \|T_N[f, a]\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}}. \tag{14}$$

The proofs in this paper use basic principles in the theory of function spaces, such as L^p inequalities for the Peetre maximal functions. A different approach to Corollary 1.3 via wavelet theory is presented in the subsequent paper [3]. The main arguments and the proof of Theorem 1.2 are contained in §2. In §3 we show how estimates in the proof of Theorem 1.2 are used to deduce Theorem 1.1. Finally, in §4 we discuss the optimality of the results.

2. Proof of Theorem 1.2

We start with some preliminaries on convolution kernels which are used in Littlewood-Paley type decompositions. Let β_0, β be Schwartz functions on $\mathbb{R}^d$, compactly supported in $(-1/2, 1/2)^d$ such that $|\widehat{\beta}_0(\xi)| > 0$ when $|\xi| \le 1$ and $|\widehat{\beta}(\xi)| > 0$ when $1/8 \le |\xi| \le 1$. Moreover assume β has vanishing moments up to a large order

$$M > \frac{d}{p} + |s|, \tag{15}$$

that is,

$$\int_{\mathbb{R}^d} \beta(x)\, x_1^{m_1} \cdots x_d^{m_d}\, dx = 0 \quad \text{when} \quad m_1 + \ldots + m_d < M\,. \tag{16}$$

For $k = 1, 2, \ldots$ let $\beta_k := 2^{kd}\beta(2^k\cdot)$ and $L_k f = \beta_k * f$. We shall use the inequality

$$\|g\|_{B^s_{p,r}} \lesssim \Big(\sum_{k=0}^{\infty} 2^{ksr} \|L_k g\|_p^r\Big)^{1/r} \tag{17}$$

and apply it to $g = \mathbb{E}_N f - \Pi_N f$. Inequality (17) is of course just one part of a characterization of $B^s_{p,r}$ spaces by sequences of compactly supported kernels (or 'local means'), with sufficient cancellation assumptions, see for example [14, §2.5.3].

Let $\eta_0 \in C_c^\infty(\mathbb{R}^d)$ be as in (7), that is, supported on $\{|\xi| < 3/8\}$ and such that $\eta_0(\xi) = 1$ when $|\xi| \le 1/4$. Define Λ_0, and Λ_k for $k \ge 1$ by

$$\widehat{\Lambda_0 f}(\xi) = \frac{\eta_0(\xi)}{\widehat{\beta}_0(\xi)} \widehat{f}(\xi)$$
$$\widehat{\Lambda_k f}(\xi) = \frac{\eta_0(2^{-k}\xi) - \eta_0(2^{-k+1}\xi)}{\widehat{\beta}(2^{-k}\xi)} \widehat{f}(\xi), \quad k \ge 1.$$

Then $\sum_{j=0}^\infty L_j \Lambda_j = Id$ with convergence in $\mathcal{S}'$, and

$$\sup_{j \ge 0} 2^{js} \|\Lambda_j f\|_p \lesssim \|f\|_{B^s_{p,\infty}} .$$

Moreover $\mathit{\Pi}_N = \sum_{j=0}^N L_j \Lambda_j$, and therefore

$$\mathbb{E}_N f - \mathit{\Pi}_N f = \sum_{j=0}^N (\mathbb{E}_N L_j \Lambda_j f - L_j \Lambda_j f) + \sum_{j=N+1}^\infty \mathbb{E}_N L_j \Lambda_j f. \tag{18}$$

If we use the convenient notation

$$\mathbb{E}_N^\perp := I - \mathbb{E}_N,$$

then the asserted estimate (12) will follow from

$$\Big(\sum_{k=0}^\infty 2^{ksr} \Big\| \sum_{j=N+1}^\infty L_k \mathbb{E}_N L_j \Lambda_j f \Big\|_p^r \Big)^{1/r} \lesssim \sup_j 2^{js} \|\Lambda_j f\|_p . \tag{19}$$

and

$$\Big(\sum_{k=0}^\infty 2^{ksr} \Big\| \sum_{j=0}^N L_k \mathbb{E}_N^\perp L_j \Lambda_j f \Big\|_p^r \Big)^{1/r} \lesssim \sup_j 2^{js} \|\Lambda_j f\|_p . \tag{20}$$

Below we shall use variants of the Peetre maximal functions, which are a standard tool in the study of Besov and Triebel-Lizorkin spaces. We define

$$\mathfrak{M}_j g(x) = \sup_{|h|_\infty \le 2^{-j+1}} |g(x+h)| , \tag{21a}$$

$$\mathfrak{M}_j^* g(x) = \sup_{|h|_\infty \le 2^{-j+5}} |g(x+h)| , \tag{21b}$$

$$\mathfrak{M}_{A,j}^{**} g(x) = \sup_{h \in \mathbb{R}^d} \frac{|g(x+h)|}{(1+2^j|h|)^A} , \tag{21c}$$

where $|h|_\infty = \max\{|h_1|, \dots, |h_d|\}$, $h = (h_1, \dots, h_d) \in \mathbb{R}^d$. These different versions are introduced for technical purposes in the proofs. They satisfy obvious pointwise inequalities,

$$\mathfrak{M}_j g(x) \le \mathfrak{M}_j^* g(x) \le C_A \mathfrak{M}_{A,j}^{**} g(x),$$

and

$$\begin{aligned}\mathfrak{M}_j g(x) &\le \inf_{|h|_\infty \le 2^{-j+4}} \mathfrak{M}_j^* g(x+h) \\ &\le \Big(2^{(j-4)d} \int_{|h|_\infty \le 2^{-j+4}} [\mathfrak{M}_j^* g(x+h)]^r dh\Big)^{1/r}, \quad 0<r\le\infty.\end{aligned} \tag{22}$$

Below we shall use Peetre's inequality ([6], see also [13, §1.3.1])

$$\|\mathfrak{M}^{**}_{A,j} f\|_p \le C_{p,A}\|f\|_p, \quad 0<p\le\infty, \quad A>d/p, \tag{23}$$

for $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfying

$$\operatorname{supp}(\widehat{f}) \subset \{\xi : |\xi| \le 2^{j+1}\}. \tag{24}$$

Throughout we shall assume that $M \gg A$; we require specifically

$$d/p < A < M - |s|\,.$$

The main estimates needed in the proof of (19) and (20) are summarized in

Proposition 2.1. *Let $0<p\le\infty$ and*

$$B(j,k,N) = \begin{cases} 2^{N-j}\, 2^{\frac{j-k}{p}}\, 2^{(j-N)(d-1)(\frac1p-1)_+} & \text{if } j,k \ge N+1, \\ 2^{\frac{N-k}{p}} 2^{j-N} & \text{if } j\le N,\ k\ge N+1, \\ 2^{k-N} 2^{j-N} 2^{(N-k)d(\frac1p -1)_+} & \text{if } 0\le j,k\le N, \\ 2^{k-j+\frac{j-N}{p}+[N-k+(j-k)(d-1)](\frac1p-1)_+} & \text{if } j\ge N+1,\ k\le N. \end{cases} \tag{25}$$

Then the following inequalities hold for all $f \in \mathcal{S}'(\mathbb{R}^d)$ whose Fourier transform is supported in $\{|\xi|\le 2^{j+1}\}$.

(i) For $j \ge N+1$,

$$\|L_k \mathbb{E}_N [L_j f]\|_p \lesssim \begin{cases} B(j,k,N)\|f\|_p & \text{if } k\ge N+1, \\ [B(j,k,N) + 2^{-|j-k|(M-A)}]\|f\|_p & \text{if } 0\le k\le N. \end{cases} \tag{26}$$

(ii) For $0\le j\le N$,

$$\|L_k \mathbb{E}_N^\perp [L_j f]\|_p \lesssim \begin{cases} \big[B(j,k,N) + 2^{-|j-k|(M-A)}\big]\|f\|_p & \text{if } k\ge N+1, \\ B(j,k,N)\|f\|_p & \text{if } 0\le k\le N. \end{cases} \tag{27}$$

(iii) The same bounds hold if the operators $\mathbb{E}_N$ in (i) and $\mathbb{E}_N^\perp$ in (ii) are replaced by $T_N[\cdot,a]$, uniformly in $\|a\|_\infty \le 1$.

We begin with two preliminary lemmata, the first a straightforward estimate for $L_k L_j$.

Lemma 2.2. *Let $k,j\ge 0$ and suppose that f is locally integrable. Let M be as in* (16) *with $M>A>d/p$. Then*

$$|L_k L_j f(x)| \lesssim 2^{-|k-j|(M-A)} \mathfrak{M}^{**}_{A,\max\{j,k\}} f(x). \tag{28}$$

If $f \in \mathcal{S}'(\mathbb{R}^d)$ with $\widehat{f}(\xi) = 0$ for $|\xi| \ge 2^{j+1}$ then

$$\|L_k L_j f\|_p \lesssim 2^{-|k-j|(M-A)} \|f\|_p\,.$$

Proof. The second assertion is an immediate consequence of (28), by (23). We have $L_k L_j f = \gamma_{j,k} * f$ where $\gamma_{j,k} = \beta_k * \beta_j$. By symmetry we may assume $k \le j$. Using the cancellation assumption (16) on the β_j we get

$$\begin{aligned}
|\gamma_{j,k}(x)| &= \Big| \int 2^{kd} \Big[\beta(2^k(x-y) - \sum_{m=0}^{M-1} \frac{1}{m!} \langle -2^k y, \nabla\rangle^m \beta(2^k x) \Big] 2^{jd} \beta(2^j y) dy \Big| \\
&= \Big| \int 2^{kd} \int_0^1 \frac{(1-s)^{M-1}}{(M-1)!} \langle -2^k y, \nabla\rangle^M \beta(2^k x - s 2^k y)\, ds\, 2^{jd} \beta(2^j y) dy \Big| \\
&\lesssim 2^{(k-j)M}\, 2^{kd}\, \mathbb{1}_{[-2^{-k}, 2^{-k}]^d}(x),
\end{aligned}$$

and thus

$$\begin{aligned}
2^{(j-k)M} |\gamma_{j,k} * f(x)| &\lesssim 2^{kd} \int_{|h|_\infty \le 2^{-k}} |f(x-h)|\, dh \\
&\lesssim 2^{kd} \int_{|h|_\infty \le 2^{-k}} \frac{2^{(j-k)A} |f(x-h)|}{(1+2^j|h|)^A} dh \lesssim 2^{(j-k)A}\, \mathfrak{M}^{**}_{A,j} f(x).
\end{aligned}$$

Hence (28) holds. □

Some notation. (i) Below, when $j > N$ we use the notation

$$\mathcal{U}_{N,j} = \Big\{ (y_1, \dots, y_d) \in \mathbb{R}^d \;:\; \min_{1 \le i \le d} \operatorname{dist}(y_i, 2^{-N}\mathbb{Z}) \le 2^{-j-1} \Big\}.$$

That is, $\mathcal{U}_{N,j}$ is a 2^{-j-1}-neighborhood of the set $\cup_{I \in \mathcal{D}_N} \partial I$.

(ii) For a dyadic cube I of side length 2^{-N} and $l > N$ we shall denote by $\mathcal{D}_l[\partial I]$ the set of dyadic cubes $J \in \mathcal{D}_l$ such that $\bar{J} \cap \partial I \ne \emptyset$.

(iii) For a dyadic cube I of side length 2^{-N} denote by $\mathcal{D}_N(I)$ the neighboring cubes of I, that is, the cubes $I' \in \mathcal{D}_N$ with $\bar{I} \cap \bar{I}' \ne \emptyset$.

Lemma 2.3. *(i) Let $k > N \ge 1$ and g be locally integrable. Then*

$$L_k(\mathbb{E}_N g)(x) = 0, \quad \text{for all } x \in \mathcal{U}^{\complement}_{N,k} = \mathbb{R}^d \setminus \mathcal{U}_{N,k}\,. \tag{29}$$

(ii) Let $j > N \ge 1$, and f locally integrable. Then

$$\mathbb{E}_N[L_j f] = \mathbb{E}_N[L_j(\mathbb{1}_{\mathcal{U}_{N,j}} f)]. \tag{30}$$

Moreover,

$$\big|\mathbb{E}_N(L_j f)\big| \lesssim 2^{(N-j)d} \sum_{I \in \mathcal{D}_N} \sum_{J \in \mathcal{D}_{j+1}[\partial I]} \|f\|_{L^\infty(J)}\, \mathbb{1}_I. \tag{31}$$

Proof. (i) We use the support and cancellation properties of β_k. Note that

$$L_k(\mathbb{E}_N g)(x) = \int \beta_k(x-y)\,\mathbb{E}_N g(y)\,dy,$$

and $\operatorname{supp}\beta_k(x-\cdot) \subset x + 2^{-k}[-1/2, 1/2]^d$. So, if $I \in \mathcal{D}_N$ and $x \in I \cap \mathcal{U}_{N,k}^{\complement}$, then $\operatorname{supp}\beta_k(x-\cdot) \subset I$, and hence

$$L_k(\mathbb{E}_N g)(x) = (\mathbb{E}_N g)_{|I}(x) \int_I \beta_k(x-y)\,dy \;=\; 0.$$

(ii) One argues similarly. First note that, changing the order of integration,

$$\mathbb{E}_N(L_j f) = \sum_{I\in\mathcal{D}_N} \int_{\mathbb{R}^d} f(y)\Big[\int_I \beta_j(x-y)\,dx\Big]\,dy\;\frac{\mathbb{1}_I}{|I|}. \tag{32}$$

Now if $J \in \mathcal{D}_N$ and $y \in J \cap \mathcal{U}_{N,k}^{\complement}$ then $\operatorname{supp}\beta_j(\cdot - y) \subset J$, and hence $\int_I \beta_j(x-y)\,dx = 0$. Thus $\mathbb{E}_N[L_j(\mathbb{1}_{\mathcal{U}_{N,j}^{\complement}} f)] = 0$. Finally, to prove (31) note that, if $I \in \mathcal{D}_N$ and $x \in I$, then from (32) it follows

$$\begin{aligned}
\big|\mathbb{E}_N(L_j f)(x)\big| \;&=\; |I|^{-1}\Big|\sum_{J\in\mathcal{D}_{j+1}[\partial I]}\int_J f(y)\Big[\int_I \beta_j(x-y)\,dx\Big]\,dy\Big| \\
&\le\; 2^{Nd}\sum_{J\in\mathcal{D}_{j+1}[\partial I]} \|f\|_{L^\infty(J)} 2^{-(j+1)d}\|\beta_j\|_1,
\end{aligned}$$

which gives the asserted (31). $\square$

Proof of Proposition 2.1.

Proof of (26) *in the case* $j, k \ge N+1$. By Lemma 2.3.i, $L_k\mathbb{E}_N[L_j f](x) = 0$ if $x \in \mathcal{U}_{N,K}^{\complement}$, so we assume that $x \in \mathcal{U}_{N,k} \cap I$, for some $I \in \mathcal{D}_N$. Recall that $\mathcal{D}_N(I)$ consists of the neighboring cubes of I. Then (31) and the support property of β_k give

$$\begin{aligned}
|L_k\mathbb{E}_N[L_j f](x)| \;&\le\; \int |\beta_k(x-y)|\,\big|\mathbb{E}_N(L_j f)(y)\big|\,dy \\
&\lesssim\; 2^{(N-j)d}\sum_{I'\in\mathcal{D}_N(I)}\;\sum_{J\in\mathcal{D}_{j+1}[\partial I']} \|f\|_{L^\infty(J)}\,\|\beta_k\|_1.
\end{aligned}$$

Hence

$$\begin{aligned}
\|L_k\mathbb{E}_N[L_j f]\|_p \;&=\; \Big[\sum_{I\in\mathcal{D}_N}\int_{I\cap\mathcal{U}_{N,k}} |L_k(\mathbb{E}_N L_j f)|^p\,dx\Big]^{\frac{1}{p}} \\
&\lesssim\; 2^{(N-j)d}\Big[\sum_{I\in\mathcal{D}_N}\Big(\sum_{J\in\mathcal{D}_{j+1}[\partial I]}\|f\|_{L^\infty(J)}\Big)^p\,|I\cap\mathcal{U}_{N,k}|\Big]^{\frac{1}{p}}. \qquad (33)
\end{aligned}$$

Now, $|I \cap \mathcal{U}_{N,k}| \approx 2^{-k} 2^{-N(d-1)}$, and $\operatorname{card} \mathcal{D}_{j+1}[\partial I] \approx 2^{(j-N)(d-1)}$. Also, if we write $J = 2^{-j-1}(\ell_J + [0,1]^d)$, then

$$\|f\|_{L^\infty(J)} \le \inf_{|h|_\infty \le 2^{-j-1}} \mathfrak{M}_j^* f(\ell_J + h) \le \Big[2^{jd} \int_{|h|_\infty \le 2^{-j-1}} \mathfrak{M}_j^* f(\ell_J + h)^p \, dh \Big]^{\frac{1}{p}}.$$

Therefore, using either Hölder's inequality (if $p > 1$), or the embedding $\ell^p \hookrightarrow \ell^1$ (if $p \le 1$), we have

$$\begin{aligned}
&\Big[\sum_{I \in \mathcal{D}_N} \Big(\sum_{J \in \mathcal{D}_{j+1}[\partial I]} \|f\|_{L^\infty(J)} \Big)^p \Big]^{\frac{1}{p}} \\
&\lesssim 2^{(j-N)(d-1)(1-\frac{1}{p})_+} \Big[\sum_{I \in \mathcal{D}_N} \sum_{J \in \mathcal{D}_{j+1}[\partial I]} \|f\|_{L^\infty(J)}^p \Big]^{\frac{1}{p}} \\
&\lesssim 2^{(j-N)(d-1)(1-\frac{1}{p})_+} \Big[\sum_{J \in \mathcal{D}_{j+1}} 2^{jd} \int_{|h|_\infty \le 2^{-j-1}} \mathfrak{M}_j^* f(\ell_J + h)^p \, dh \Big]^{\frac{1}{p}} \\
&\lesssim 2^{(j-N)(d-1)(1-\frac{1}{p})_+} \, 2^{\frac{jd}{p}} \left\| \mathfrak{M}_j^* f \right\|_{L^p(\mathbb{R}^d)} . \qquad (34)
\end{aligned}$$

Finally, inserting (34) into (33), and using (23), yields

$$\begin{aligned}
\|L_k \mathbb{E}_N [L_j f]\|_p &\lesssim 2^{(N-j)d} 2^{(j-N)(d-1)(1-\frac{1}{p})_+} \, 2^{\frac{jd}{p}} \, \|f\|_p \, 2^{-\frac{k}{p}} 2^{-\frac{N(d-1)}{p}} \\
&= 2^{N-j} \, 2^{\frac{j-k}{p}} \, 2^{(j-N)(d-1)(\frac{1}{p}-1)_+} \, \|f\|_p \, ,
\end{aligned}$$

using in the last step the trivial identity $(1 - \frac{1}{p})_+ = (\frac{1}{p} - 1)_+ - (\frac{1}{p} - 1)$. This establishes (26) for $j, k \ge N+1$. $\square$

Proof of (27) *in the case* $j \le N$, $k \ge N+1$. For $w \in I$ with $I \in \mathcal{D}_N$ we have

$$\begin{aligned}
&\left| \mathbb{E}_N^\perp (L_j f)(w) \right| = \left| \mathbb{E}_N [L_j f](w) - L_j f(w) \right| \\
&\quad = 2^{Nd} \Big| \int_I \int_{\mathbb{R}^d} 2^{jd} \big[\beta(2^j(v-y)) - \beta(2^j(w-y)) \big] f(y) dy \, dv \Big| \\
&\quad = 2^{(N+j)d} \Big| \int_I \int_{\mathbb{R}^d} \int_0^1 \nabla\beta(2^j[(1-t)w + tv - y]) \cdot 2^j (v-w) \, dt \, f(y) \, dy \, dv \Big| \\
&\quad \le 2^{(N+j)d} 2^{j-N} \int_I \int_0^1 \int_{\mathbb{R}^d} |f(y)| \, |\nabla\beta(2^j[(1-t)w + tv - y])| \, dy \, dt \, dv \\
&\quad \lesssim 2^{j-N} \, \mathfrak{M}_j f(w),
\end{aligned}$$

since for fixed w, v, t the expression involving $\nabla\beta$ is supported in the set $\{y \;:\; |y - w|_\infty \le 2^{-j-1} + 2^{-N}\}$. Moreover, since $k > N$, when $w \in I_{N,\mu}$ and $|z|_\infty \le 2^{-k-1}$ we have

$$\left| \mathbb{E}_N^\perp [L_j f](w - z) \right| \lesssim 2^{j-N} \inf_{|h|_\infty \le 2^{-j}} \mathfrak{M}_j^* f(2^{-N}\mu + h) \, , \qquad (35)$$

and therefore,

$$\begin{aligned}\big|L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)(w)\big| &\le \int |\mathbb{E}_N^{\perp}(w-z)|\,|\beta_k(z)|\,dz \\ &\lesssim 2^{j-N}\Big[⨍_{|h|_\infty\le 2^{-j}} \mathfrak{M}_j^* f(2^{-N}\mu+h)^p\,dh\Big]^{\frac{1}{p}}. \end{aligned} \tag{36}$$

Now Lemma 2.3.i gives

$$\begin{aligned} &\big\|L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)\big\|_p \\ &\qquad \lesssim \|L_k L_j f\|_{L^p(\mathcal{U}_{N,k}^{\complement})} + \Big[\sum_{\mu\in\mathbb{Z}^d} \big\|L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)\big\|^p_{L^p(\mathcal{U}_{N,k}\cap I_{N,\mu})}\Big]^{\frac{1}{p}}. \end{aligned} \tag{37}$$

Using (36), the last term is controlled by

$$\begin{aligned} &2^{j-N}\Big[\sum_{\mu\in\mathbb{Z}^d} |I_{N,\mu}\cap \mathcal{U}_{N,k}|⨍_{|h|_\infty\le 2^{-j}} \mathfrak{M}_j^* f(2^{-N}\mu+h)^p\,dh\Big]^{\frac{1}{p}} \\ &\lesssim 2^{j-N}\,[2^{-k}2^{-N(d-1)}]^{\frac{1}{p}}\,2^{\frac{Nd}{p}}\,\|\mathfrak{M}_j^* f\|_p \;\lesssim\; 2^{j-N}\,2^{\frac{N-k}{p}}\,\|f\|_p. \end{aligned}$$

Finally, the first term in (37) is controlled by Lemma 2.2, so overall one obtains

$$\big\|L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)\big\|_p \lesssim [2^{-(M-A)|k-j|} + 2^{j-N}\,2^{\frac{N-k}{p}}]\,\|f\|_p,$$

establishing (27) in the case $j\le N$, $k\ge N+1$. □

Proof of (27) *in the case* $0\le j,k\le N$. We use

$$\int_I \mathbb{E}_N^{\perp}[L_j f](y)\,dy = 0, \quad I\in\mathcal{D}_N,$$

to write

$$L_k\big(\mathbb{E}_N^{\perp}[L_j f]\big)(x) = \sum_{\mu}\int_{I_{N,\mu}} \big(\beta_k(x-y)-\beta_k(x-2^{-N}\mu)\big)\,\mathbb{E}_N^{\perp}[L_j f](y)\,dy\,.$$

For fixed x, we say that

$$\mu\in\Lambda(x) \text{ if } |x-2^{-N}\mu|_\infty \le 2^{-N}+2^{-k-1}. \tag{38}$$

Observe that only these μ's contribute to the above sum. Notice also that

$$|\beta_k(x-y)-\beta_k(x-2^{-N}\mu)| \lesssim 2^{kd}\,2^{k-N}, \qquad \text{if } y\in I_{N,\mu},$$

and since $j\le N$, the estimate in (35) gives

$$\big|\mathbb{E}_N^{\perp}[L_j f](y)\big| \lesssim 2^{j-N}\inf_{|h|_\infty\le 2^{-j}} \mathfrak{M}_j^* f(2^{-N}\mu+h)\,, \quad y\in I_{N,\mu}.$$

Combining all these bounds we obtain

$$\begin{aligned}|L_k\big(\mathbb{E}_N^{\perp}[L_jf]\big)(x)| &\lesssim 2^{(k-N)(d+1)}2^{j-N}\sum_{\mu\in\Lambda(x)}\Big(\fint_{\frac{\mu}{2^N}+[-\frac{1}{2^j},\frac{1}{2^j}]^d}[\mathfrak{M}_j^*f]^p\Big)^{\frac1p}\\ &\lesssim 2^{(k-N)(d+1)}2^{j-N}2^{(N-k)d(1-\frac1p)_+}\Big(\sum_{\mu\in\Lambda(x)}\fint_{\frac{\mu}{2^N}+[-\frac{1}{2^j},\frac{1}{2^j}]^d}[\mathfrak{M}_j^*f]^p\Big)^{\frac1p},\end{aligned}$$

using in the last step Hölder's inequality (or $\ell^p\hookrightarrow\ell^1$ if $p\le1$) and the fact that $\operatorname{card}\Lambda(x)\approx 2^{(N-k)d}$. Observe also that the L^p-quasinorm of the last bracketed expression satisfies

$$\begin{aligned}\Big(\int_{\mathbb{R}^d}\sum_{\mu\in\Lambda(x)}\fint_{\frac{\mu}{2^N}+[-\frac{1}{2^j},\frac{1}{2^j}]^d}[\mathfrak{M}_j^*f]^p\Big)^{\frac1p} &\approx \Big(\sum_{\mu\in\mathbb{Z}^d}2^{-kd}\fint_{\frac{\mu}{2^N}+[-\frac{1}{2^j},\frac{1}{2^j}]^d}[\mathfrak{M}_j^*f]^p\Big)^{\frac1p}\\ &\lesssim 2^{(N-k)d/p}\,\|\mathfrak{M}_j^*f\|_p.\end{aligned}$$

Thus, overall we obtain

$$\begin{aligned}\big\|L_k\mathbb{E}_N^{\perp}[L_jf]\big\|_p &\lesssim 2^{(k-N)(d+1)}2^{j-N}2^{(N-k)d(1-\frac1p)_+}\,2^{(N-k)d/p}\,\|f\|_p\\ &= 2^{k-N}2^{j-N}2^{(N-k)d(\frac1p-1)_+}\,\|f\|_p,\end{aligned}$$

after simplifying the indices in the last step. This establishes (27) in the case $0\le j,k\le N$. □

Proof of (26) *in the case $j\ge N+1$, $k\le N$.* This condition and (30) in Lemma 2.3 imply that $\mathbb{E}_N[L_jf]=\mathbb{E}_N[L_j(f\mathbb{1}_{\mathcal{U}_{N,j}})]$. For simplicity, we denote $\widetilde{f}=f\mathbb{1}_{\mathcal{U}_{N,j}}$, and write

$$L_k\mathbb{E}_N[L_jf]=L_k(\mathbb{E}_N[L_j\widetilde{f}]-L_j\widetilde{f})+L_kL_j\widetilde{f}. \tag{39}$$

Observe that, by Lemma 2.2,

$$\|L_kL_j\widetilde{f}\|_p\lesssim 2^{-(M-A)|j-k|}\,\big\|\mathfrak{M}_{A,j}^{**}f(x)\big\|_p\lesssim 2^{-(M-A)|j-k|}\,\|f\|_p.$$

So, we only need to estimate $\|L_k\mathbb{E}_N^{\perp}[L_j\widetilde{f}]\|_p$. Proceeding as in the proof of the case $j,k\le N$, we write (with $\Lambda(x)$ as in (38))

$$\begin{aligned}&|L_k\big(\mathbb{E}_N^{\perp}[L_j\widetilde{f}]\big)(x)|\\ &\quad\le\sum_{\mu\in\Lambda(x)}\int_{I_{N,\mu}}\big|\beta_k(x-y)-\beta_k(x-2^{-N}\mu)\big|\,\big|\mathbb{E}_N^{\perp}[L_j\widetilde{f}](y)\big|\,dy\\ &\quad\lesssim 2^{kd}2^{k-N}\sum_{\mu\in\Lambda(x)}\int_{I_{N,\mu}}\Big(|\mathbb{E}_N[L_j\widetilde{f}]|\,+\,|L_j(\widetilde{f})|\Big)\\ &\quad=\mathcal{A}_1(x)+\mathcal{A}_2(x).\end{aligned}\tag{40}$$

Now, using again (31), we have

$$|\mathcal{A}_1(x)| \lesssim 2^{(k-N)(d+1)}2^{(N-j)d} \sum_{\mu\in\Lambda(x)} \sum_{J\in\mathcal{D}_{j+1}[\partial I_{N,\mu}]} \|f\|_{L^\infty(J)}$$

$$\lesssim 2^{k-N}2^{(k-j)d}\, 2^{(N-k)d(1-\frac{1}{p})_+} \Big[\sum_{\mu\in\Lambda(x)} \Big(\sum_{J\in\mathcal{D}_{j+1}[\partial I_{N,\mu}]} \|f\|_{L^\infty(J)}\Big)^p\Big]^{\frac{1}{p}}, \tag{41}$$

since $\operatorname{card}\Lambda(x) \approx 2^{(N-k)d}$. Taking the L^p-quasinorm of the last bracketed expression gives

$$\Big[\int_{x\in\mathbb{R}^d} \sum_{\mu\in\Lambda(x)} \Big(\sum_{J\in\mathcal{D}_{j+1}[\partial I_{N,\mu}]} \|f\|_{L^\infty(J)}\Big)^p\, dx\Big]^{\frac{1}{p}}$$

$$\lesssim \Big[\sum_{I\in\mathcal{D}_N} 2^{-kd}\Big(\sum_{J\in\mathcal{D}_{j+1}[\partial I]} \|f\|_{L^\infty(J)}\Big)^p\Big]^{\frac{1}{p}}$$

$$\lesssim 2^{\frac{(j-k)d}{p}}\, 2^{(j-N)(d-1)(1-\frac{1}{p})_+} \big\|\mathfrak{M}_j^* f\big\|_{L^p(\mathbb{R}^d)} \quad \text{by (34)}. \tag{42}$$

Therefore, combining exponents in (41) and (42) one obtains

$$\|\mathcal{A}_1\|_p \lesssim 2^{k-N}2^{(k-j)d}\, 2^{(N-k)d(1-\frac{1}{p})_+}\, 2^{\frac{(j-k)d}{p}}\, 2^{(j-N)(d-1)(1-\frac{1}{p})_+}\, \|f\|_p$$

$$= 2^{k-j}\, 2^{\frac{j-N}{p}}\, 2^{(N-k)(\frac{1}{p}-1)_+}\, 2^{(j-k)(d-1)(\frac{1}{p}-1)_+}\, \|f\|_p. \tag{43}$$

Finally, we estimate the term $\mathcal{A}_2(x)$ in (40). First notice that

$$|L_j(\widetilde{f})(y)| \le \int_{\mathcal{U}_{N,j}} |\beta_j(y-z)||f(z)|\, dz = 0, \quad \text{if } y \in \mathcal{U}_{N,j-1}^{\complement},$$

since $\operatorname{supp}\beta_j(y-\cdot) \subset y + 2^{-j}[-\frac{1}{2},\frac{1}{2}]^d \subset \mathcal{U}_{N,j}^{\complement}$. Moreover, if $I\in\mathcal{D}_N$, then for every cube $J\in\mathcal{D}_j$ such that $J\subset I\cap\mathcal{U}_{N,j-1}$ we have

$$|L_j(\widetilde{f})(y)| \le \int |\beta_j(z)||f(y-z)|\, dz \lesssim \|f\|_{L^\infty(J^*)}, \quad \text{if } y\in J$$

where $J^* = J + 2^{-j}[-\frac{1}{2},\frac{1}{2}]^d$. Therefore,

$$\int_I |L_j(\widetilde{f})(y)| \lesssim \sum_{J\in\mathcal{D}_j[\partial I]} \|f\|_{L^\infty(J^*)}|J|,$$

and overall we obtain

$$|\mathcal{A}_2(x)| \lesssim 2^{(k-j)d}2^{k-N} \sum_{\mu\in\Lambda(x)} \sum_{J\in\mathcal{D}_{j-1}[\partial I_{N,\mu}]} \|f\|_{L^\infty(J)}.$$

But this is essentially the same expression we obtained in (41) for the term $|\mathcal{A}_1(x)|$, so the same argument will give an estimate of $\|\mathcal{A}_2\|_p$ in terms of the quantity in (43). This concludes the proof of (26) in the case $j\ge N+1$, $k\le N$.

Finally, concerning (iii) in Proposition 2.1, we remark that the previous proofs can easily be adapted replacing the operators $\mathbb{E}_N$ and $\mathbb{E}_N^\perp$ by $T_N[\cdot,a]$, keeping in mind that $T_N[g,a]$ is now constant in cubes $I \in \mathcal{D}_{N+1}$, and enjoys an additional cancellation, $\int_{I_{N,\mu}} T_N[g,a](x)dx = 0$, which simplifies some of the previous steps. $\square$

Proof of Theorem 1.2, conclusion. It remains to prove inequalities (19) and (20). By the embedding properties for the sequence spaces ℓ^r it suffices to verify both inequalities for very small r, say

$$r \le \min\{p, 1\}. \tag{44}$$

In view of the embedding $\ell^r \hookrightarrow \ell^1$ and Minkowski's inequality (in $L^{p/r}$) it suffices then to prove

$$\sup_N \Big(\sum_{k=0}^{\infty} 2^{ksr} \sum_{j=N+1}^{\infty} \big\| L_k \mathbb{E}_N L_j \Lambda_j f \big\|_p^r \Big)^{1/r} \lesssim \sup_j 2^{js} \|\Lambda_j f\|_p \tag{45}$$

and

$$\sup_N \Big(\sum_{k=0}^{\infty} 2^{ksr} \sum_{j=0}^{N} \big\| L_k (\mathbb{E}_N^{\perp} L_j \Lambda_j f) \big\|_p^r \Big)^{1/r} \lesssim \sup_j 2^{js} \|\Lambda_j f\|_p \,. \tag{46}$$

If we apply Proposition 2.1 to each of the functions $\Lambda_j f$, we reduce matters to observe that

$$\sup_N \sum_{k=0}^{\infty} 2^{ksr} \sum_{j=0}^{\infty} \big[2^{-js} B(j,k,N) \big]^r < \infty, \tag{47}$$

with $B(j,k,N)$ as in (25), and that

$$\Big(\sum_{j=N+1}^{\infty} \sum_{k=0}^{N} + \sum_{k=N+1}^{\infty} \sum_{j=0}^{N} \Big) 2^{-|j-k|(M-A)} < \infty$$

which is trivial. The verification of (47) under the assumptions in (11) is also elementary, but we carry out some details to clarify how the conditions on p and s are used.

When $j,k > N$, we have $B(j,k,N) = 2^{N-j}\, 2^{\frac{j-k}{p}}\, 2^{(j-N)(d-1)(\frac{1}{p}-1)_+}$ and thus

$$\begin{aligned} &\sum_{k>N} 2^{ksr} \sum_{j>N} \big[2^{-js} B(j,k,N) \big]^r \\ &= \Big(\sum_{k>N} 2^{-kr(\frac{1}{p}-s)} \Big) \Big(\sum_{j>N} 2^{-rj[s+1-\frac{1}{p}-(d-1)(\frac{1}{p}-1)_+]} \Big)\, 2^{Nr[1-(d-1)(\frac{1}{p}-1)_+]}, \end{aligned} \tag{48}$$

and the series converge provided $s < 1/p$ and

$$s > \tfrac{1}{p} - 1 + (d-1)(\tfrac{1}{p}-1)_+ \;=\; \max\Big\{ d(\tfrac{1}{p}-1), \tfrac{1}{p}-1 \Big\}. \tag{49}$$

Further, being geometric sums, the final outcome in (48) is bounded uniformly in N.

Next assume $j \le N < k$, then $B(j,k,N) = 2^{\frac{N-k}{p}} 2^{j-N}$ and hence

$$\sum_{k>N} 2^{ksr} \sum_{j\le N} \big[2^{-js} B(j,k,N)\big]^r = \Big(\sum_{k>N} 2^{-kr(\frac1p - s)}\Big)\Big(\sum_{j\le N} 2^{rj(1-s)}\Big) 2^{Nr(\frac1p-1)},$$

which are finite expressions provided $s < \min\{1, 1/p\}$.

Consider $j,k \le N$, with $B(j,k,N) = 2^{k-N}2^{j-N}2^{(N-k)d(\frac1p-1)_+}$. Then

$$\sum_{k\le N} 2^{ksr} \sum_{j\le N} \big[2^{-js} B(j,k,N)\big]^r =$$

$$= \Big(\sum_{k\le N} 2^{kr[s+1-d(\frac1p-1)_+]}\Big)\Big(\sum_{j\le N} 2^{rj(1-s)}\Big) 2^{-Nr[2-d(\frac1p-1)_+]},$$

which leads to uniform expressions in N under the assumptions $s<1$ and

$$s > d(\tfrac1p - 1)_+ - 1, \tag{50}$$

the latter being weaker than (49).

When $k \le N < j$ we have $B(j,k,N) = 2^{k-j+\frac{j-N}{p}+[N-k+(j-k)(d-1)](\frac1p-1)_+}$ and

$$\sum_{k\le N} 2^{ksr} \sum_{j>N} \big[2^{-js} B(j,k,N)\big]^r =$$

$$= \Big(\sum_{k\le N} 2^{kr[s+1-d(\frac1p-1)_+]}\Big)\Big(\sum_{j>N} 2^{-rj[s+1-\frac1p-(d-1)(\frac1p-1)_+]}\Big) 2^{-Nr[\frac1p-(\frac1p-1)_+]},$$

where in the first series we would use (50) and in the second series (49). We have verified (47) in all cases. This finishes the proof of Theorem 1.2. □

3. Schauder bases

Let P_N be defined as in (3). For the proof of Theorem 1.1 we need to prove that $\|P_N f - f\|_{F^s_{p,q}} \to 0$ for every $f \in F^s_{p,q}$, with (p,s) as in (11) and $0<q<\infty$. We first discuss some preliminaries about localization and pointwise multiplication by characteristic functions of cubes, then prove uniform bounds for the $F^s_{p,q} \to F^s_{p,q}$ operator norms of the P_N and then establish the asserted limiting property.

Preliminaries. For $\nu \in \mathbb{Z}^d$ let χ_ν be the characteristic function of $\nu + [0,1)^d$.

Lemma 3.1. *Assume that*

$$\tfrac{d-1}{d} < p < \infty, \quad 0 < q \le \infty, \quad \textit{and} \quad \max\{d(\tfrac1p - 1), \tfrac1p - 1\} < s < \tfrac1p. \tag{51}$$

Then, the following holds for all g_ν and $f \in F^s_{p,q}$:

$$\Big\|\sum_{\nu\in\mathbb{Z}^d} \chi_\nu g_\nu\Big\|_{F^s_{p,q}} \lesssim \Big(\sum_{\nu\in\mathbb{Z}^d} \|g_\nu\|^p_{F^s_{p,q}}\Big)^{1/p}$$

and

$$\Big(\sum_{\nu\in\mathbb{Z}^d}\|f\chi_\nu\|^p_{F^s_{p,q}}\Big)^{1/p}\lesssim\|f\|_{F^s_{p,q}}\,.$$

Proof. Let $\varsigma\in C^\infty_c(\mathbb{R}^d)$ so that $\mathrm{supp}(\varsigma)\subset(-1,1)^d$ and $\sum_{\nu\in\mathbb{Z}^d}\varsigma(x-\nu)=1$ for all $x\in\mathbb{R}^d$. Let $\varsigma_\nu=\varsigma(\cdot-\nu)$. We have, for all $s\in\mathbb{R}$,

$$\|g\|_{F^s_{p,q}}\asymp\Big(\sum_\nu\big\|\varsigma_\nu g\big\|^p_{F^s_{p,q}}\Big)^{1/p}; \tag{52}$$

see [14, 2.4.7]. Hence

$$\Big\|\sum_{\nu\in\mathbb{Z}^d}\chi_\nu g_\nu\Big\|_{F^s_{p,q}}=\Big\|\sum_{\nu'}\varsigma_{\nu'}\sum_\nu\chi_\nu g_\nu\Big\|_{F^s_{p,q}}\lesssim\Big(\sum_{\nu'}\Big\|\varsigma_{\nu'}\sum_{|\nu-\nu'|_\infty\le1}\chi_\nu g_\nu\Big\|^p_{F^s_{p,q}}\Big)^{1/p}$$
$$\lesssim\Big(\sum_{\nu'}\sum_{|\nu-\nu'|_\infty\le1}\|g_\nu\|^p_{F^s_{p,q}}\Big)^{1/p}\lesssim\Big(\sum_\nu\|g_\nu\|^p_{F^s_{p,q}}\Big)^{1/p}.$$

Here we have used that $\varsigma_{\nu'}\chi_\nu$ are pointwise multipliers of $F^s_{p,q}$, with uniform bounds in (ν,ν'), in the range given by (51); see [7, Thm. 4.6.3/1]. This proves the first inequality.

For the second inequality we first observe that, by (52),

$$\|f\chi_\nu\|_{F^s_{p,q}}\lesssim\Big(\sum_{\nu'}\|f\chi_\nu\varsigma_{\nu'}\|^p_{F^s_{p,q}}\Big)^{1/p},\quad \nu\in\mathbb{Z}^d,$$

which yields

$$\begin{aligned}\Big(\sum_\nu\|f\chi_\nu\|^p_{F^s_{p,q}})^{1/p}&\lesssim\Big(\sum_\nu\sum_{\nu'}\|f\chi_\nu\varsigma_{\nu'}\|^p_{F^s_{p,q}}\Big)^{1/p}\\&\lesssim\Big(\sum_{\nu'}\sum_{|\nu-\nu'|_\infty\le1}\|f\chi_\nu\varsigma_{\nu'}\|^p_{F^s_{p,q}}\Big)^{1/p}\\&\lesssim\Big(\sum_{\nu'}\|f\varsigma_{\nu'}\|^p_{F^s_{p,q}}\Big)^{1/p}\lesssim\|f\|_{F^s_{p,q}}\,,\end{aligned}$$

where we have used the pointwise multiplier assertion [7, Thm. 4.6.3/1] and then again (52) in the last step. ☐

Uniform boundedness of the P_N. Observe that by the localization property of the Haar functions we have $P_Nf=\sum_{\nu\in\mathbb{Z}^d}\chi_\nu P_Nf=\sum_\nu\chi_\nu P_N[f\chi_\nu]$. Thus by Lemma 3.1

$$\|P_Nf\|_{F^s_{p,q}}\lesssim\Big(\sum_\nu\big\|P_N[f\chi_\nu]\big\|^p_{F^s_{p,q}}\Big)^{1/p}.$$

Since the enumeration of the Haar system is assumed to be admissible we have

$$P_N[f\chi_\nu]=\mathbb{E}_{N_\nu}[f\chi_\nu]+T_{N_\nu}[f\chi_\nu,a^{N,\nu}] \tag{53}$$

for some $N_\nu \in \mathbb{N}$, with $N_\nu \le N$ and appropriate sequences $a^{N,\nu}$ assuming only the values 1 and 0. We remark that for each ν, $N_\nu = N_\nu(N)$ with

$$\lim_{N\to\infty} N_\nu(N) = \infty\,. \tag{54}$$

By Theorem 1.2

$$\begin{aligned}
&\Big(\sum_\nu \|P_N[f\chi_\nu]\|^p_{F^s_{p,q}}\Big)^{1/p}\\
\lesssim&\Big(\sum_\nu \|\mathbb{E}_{N_\nu}[f\chi_\nu]\|^p_{F^s_{p,q}}\Big)^{1/p} + \Big(\sum_\nu \|T_{N_\nu}[f\chi_\nu, a^{N,\nu}]\|^p_{F^s_{p,q}}\Big)^{1/p}\\
\lesssim&\Big(\sum_\nu \|f\chi_\nu\|^p_{F^s_{p,q}}\Big)^{1/p} \lesssim \|f\|_{F^s_{p,q}}\,,
\end{aligned}$$

where for the last inequality we have used Lemma 3.1 again.

Proof of Theorem 1.1, conclusion. Let $f \in F^s_{p,q}$, with (p,s) as in (11) and $0 < q < \infty$. Let $C = \max\{1, \sup_N \|P_N\|_{F^s_{p,q}\to F^s_{p,q}}\}$. Since Schwartz functions are dense in $F^s_{p,q}$ when $0 < p, q < \infty$ there is $\tilde f \in \mathcal{S}(\mathbb{R})$ such that $\|f - \tilde f\|_{F^s_{p,q}} < (3C)^{-1}\epsilon$ and hence $\|P_N f - P_N \tilde f\|_{F^s_{p,q}} < \epsilon/3$. Choose s_1 so that $s < s_1 < \max\{1/p, 1\}$ then $\tilde f \in B^{s_1}_{p,q} \hookrightarrow F^s_{p,q}$. Since the Haar system is an unconditional basis on $B^{s_1}_{p,q}$ ([15]) we have $\lim_{N\to\infty} \|P_N \tilde f - \tilde f\|_{B^{s_1}_{p,q}} = 0$ and therefore $\lim_{N\to\infty} \|P_N \tilde f - \tilde f\|_{F^s_{p,q}} = 0$. Combining these facts we get $\|P_N f - f\|_{F^s_{p,q}} < \epsilon$ for sufficiently large N which shows that $P_N f \to f$ in $F^s_{p,q}$. ☐

4. Optimality away from the end-points

Proposition 4.1. *Let $0 < q < \infty$. Then, the Haar system $\mathcal{H}_d$ is not a Schauder basis of $F^s_{p,q}(\mathbb{R}^d)$ in each of the following cases:*

(i) if $1 < p < \infty$ and $s \ge 1/p$ or $s \le 1/p - 1$,

(ii) if $d/(d+1) \le p \le 1$ and $s > 1$ or $s < d(1/p - 1)$,

(iii) if $0 < p < d/(d+1)$ and $s \in \mathbb{R}$.

The same result for the spaces $B^s_{p,q}(\mathbb{R}^d)$ was proved by Triebel in [12]; see also [15, Proposition 2.24]. Proposition 4.1 can be obtained from this and Theorem 1.1 by suitable interpolation.

Indeed, assertion (i) was already discussed in the paragraph following (4), so we restrict to $p \le 1$. Assume next that $\mathcal{H}_d$ is a basis for $F^s_{p,q}$ for some $d/(d+1) < p < 1$ and $s > 1$ or $s < d(1/p - 1)$. By Theorem 1.1, $\mathcal{H}_d$ is also a basis for $F^{s_0}_{p,q}$, for any $d(1/p - 1) < s_0 < 1$. By real interpolation, see e.g. [13, Thm. 2.4.2(ii)], for all $0 < \theta < 1$, the system $\mathcal{H}_d$ will then be a basis of

$$\big(F^{s_0}_{p,q}, F^s_{p,q}\big)_{\theta,q} = B^{s_\theta}_{p,q}, \quad \text{with } s_\theta = (1-\theta)s_0 + \theta s.$$

But when θ is close to 1 this would contradict Triebel's result. The remaining cases, $p = 1$ and $p \ge d/(d+1)$ can be proved similarly using complex interpolation of F-spaces; see [14, 1.6.7].

We remark that, in the paper [12], the failure of the Schauder basis property in the B-spaces is sometimes due to the fact that $\operatorname{span} \mathcal{H}_d$ fails to be dense in $B^s_{p,q}$. This is the case, for instance, in the region

$$(d-1)/d < p < 1 \quad \text{and} \quad \max\big\{1, d(1/p-1)\big\} < s < 1/p; \tag{55}$$

see [12, Corollary 2]. Here we show that also a quantitative bound holds, therefore ruling out the possibility that $\mathcal{H}_d$ could be a basic sequence.

Proposition 4.2. *Let $0 < q \le \infty$, and (p,s) be as in (55). Then,*

$$\|\mathbb{E}_N\|_{B^s_{p,q}\to B^s_{p,q}} \gtrsim 2^{(s-1)N}.$$

Proof. Let $\eta \in C^\infty_c(\mathbb{R}^d)$ such that $\eta \equiv 1$ on $[-2,2]^d$, and consider the Schwartz function $f(x) = x_1\,\eta(x)$. It suffices to show that

$$\big\|\mathbb{E}_N f\big\|_{B^s_{p,q}} \gtrsim 2^{(s-1)N}. \tag{56}$$

Under (55) we have $s > \sigma_p := d(1/p-1)_+$. Assume first that $s < 2$ (which is always the case if $d > 1$). Then we can use the equivalence of quasi-norms

$$\|g\|_{B^s_{p,q}(\mathbb{R}^d)} \approx \|g\|_p + \sum_{j=1}^d \Big(\int_0^1 \frac{\|\Delta^2_{he_j} g\|_p^q}{h^{sq}}\,\frac{dh}{h}\Big)^{1/q},$$

with the usual modification in the case $q = \infty$, see [14, 2.6.1]. In particular

$$\big\|\mathbb{E}_N f\big\|_{B^s_{p,q}} \gtrsim \left(\int_0^{2^{-N-1}} \frac{\big\|\Delta^2_{he_1}\big(\mathbb{E}_N f\big)\big\|^q_{L^p([0,1]^d)}}{h^{sq}}\,\frac{dh}{h}\right)^{1/q}. \tag{57}$$

Now, it is easily checked that, when $x \in [0,1)^d$, one has

$$\mathbb{E}_N f = \sum_{0\le k<2^N} \tfrac{k+1/2}{2^N}\,\mathbb{1}_{[\frac{k}{2^N},\frac{k+1}{2^N})\times[0,1)^{d-1}},$$

and likewise, if we additionally assume $0 < h < 2^{-N-1}$, then

$$\Delta_{he_1}\big(\mathbb{E}_N f\big) = 2^{-N-1}\sum_{k=1}^{2^N} \mathbb{1}_{[\frac{k}{2^N}-h,\frac{k}{2^N})\times[0,1)^{d-1}}.$$

and

$$\Delta^2_{he_1}\big(\mathbb{E}_N f\big) = 2^{-N-1}\sum_{k=1}^{2^N}\Big[\mathbb{1}_{[\frac{k}{2^N}-2h,\frac{k}{2^N}-h)\times[0,1)^{d-1}} - \mathbb{1}_{[\frac{k}{2^N}-h,\frac{k}{2^N})\times[0,1)^{d-1}}\Big].$$

Therefore,

$$\|\Delta^2_{he_1}\mathbb{E}_N f\|_{L^p([0,1]^d)} = 2^{(N+1)(1/p-1)}\,h^{1/p},$$

which, inserted into (57), gives (56). If $d = 1$ and $s \ge 2$, one applies a similar argument to the functions $\Delta^L_{he_1}(\mathbb{E}_N f)$ with $L = \lfloor s\rfloor + 1$ and $h < 2^{-N}/L$. $\square$

By interpolation one obtains as well a quantitative bound for the relevant cases in Proposition 4.1(ii).

Corollary 4.3. *Let* $0 < q \le \infty$, $d/(d+1) < p < 1$ *and* $1 < s < 1/p$. *Then, for all* $\varepsilon > 0$,

$$\|\mathbb{E}_N\|_{F^s_{p,q}\to F^s_{p,q}} \gtrsim c_\varepsilon\, 2^{(s-1-\varepsilon)N}. \tag{58}$$

Proof. If $d(1/p-1) < s_0 < 1$ and $\theta \in (0,1)$, then the real interpolation inequalities give

$$\big\|\mathbb{E}_N\big\|^{1-\theta}_{F^{s_0}_{p,q}\to F^{s_0}_{p,q}}\,\big\|\mathbb{E}_N\big\|^{\theta}_{F^{s}_{p,q}\to F^{s}_{p,q}} \;\ge\; c_\theta\,\big\|\mathbb{E}_N\big\|_{B^{s_\theta}_{p,q}\to B^{s_\theta}_{p,q}},$$

with $s_\theta = (1-\theta)s_0+\theta s$. By Proposition 4.2 the right hand side is larger than a constant times $2^{N(s_\theta-1)}$, while by Corollary 1.3 we have $\big\|\mathbb{E}_N\big\|_{F^{s_0}_{p,q}\to F^{s_0}_{p,q}} \approx 1$. Choosing θ sufficiently close to 1 one derives (58). □

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