IMPROVEMENTS IN WOLFF'S INEQUALITY FOR DECOMPOSITIONS OF CONE MULTIPLIERS

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ABSTRACT. We prove, for suitable values of $p \gg 2$, new mixed norm $\ell^s(L^p)$ versions of an inequality introduced by Wolff in the context of local smoothing for the wave equation. Moreover, we improve on their range of p, both in the original $\ell^p(L^p)$ and also in the stronger $\ell^2(L^p)$ formulation, in all dimensions $d \geq 2$. As a consequence progress is made on a number of problems, including an L^4 bound for the cone multiplier operator in $\mathbb{R}^3$, as well as new inequalities on boundedness of Bergman projections in tubes over light-cones.

1. Introduction

Wolff's inequality for cone multipliers and its variants involve projection operators to spaces of functions which are frequency-supported in boxes adapted to thin neighborhoods of light cones.

Let $\eta \in C_c^\infty(\mathbb{R})$ be supported in $(-2, 2)$ and, for $\omega \in S^{d-1}$ define a convolution operator $T_{N,\omega}$ on the Schwartz space $\mathcal{S}(\mathbb{R}^{d+1})$ by

$$\widehat{T_{N,\omega} f}(\xi', \xi_{d+1}) = \eta\big(N(1 - \frac{|\xi'|^2}{\xi_{d+1}^2})\big)\eta\big(N^{1/2}(\frac{\xi'}{|\xi'|} - \omega)\big)\widehat{f}(\xi', \xi_{d+1}). \tag{1.1}$$

Note that the Fourier multiplier $\eta(N(1 - \frac{|\xi'|^2}{\xi_{d+1}^2}))$ localizes the Fourier transform of f to a neighborhood of the light cone which is of angular width CN^{-1}. The multiplier $\eta(N^{1/2}(\frac{\xi'}{|\xi'|} - \omega))$ localizes the Fourier transform to a sector in $\mathbb{R}^d$, of angular width $\approx N^{-1/2}$, with trivial wedge extension to $\mathbb{R}^{d+1}$. It is easy to see, by a nonisotropic scaling and standard Calderón-Zygmund theory, that the operators $T_{N,\omega}$ are bounded on L^p, $1 < p < \infty$, with operator norm independent of ω and N. Let Ω be an $N^{-1/2}$ separated subset of S^{d-1}. We are interested in efficiently bounding the L^p norm of $\sum_{\omega \in \Omega} T_{\omega,N} f_\omega$ by the $\ell^s(L^p)$ norm of $\{f_\omega\}_{\omega \in \Omega}$.

First author partially supported by grant "MTM2007-60952" and *Programa Ramón y Cajal*, MCyT (Spain). Second author partially supported by NSF grant DMS 0653841. Third author partially supported by NSF grant DMS 0652890.

The mixed norm variant of Wolff's inequality, for suitably $p < \infty$, and suitable $\varepsilon > 0$ states that

$$\Big\| \sum_{\omega\in\Omega} T_{\omega,N} f_\omega \Big\|_p \le C_\varepsilon\, N^{\beta(p,s)+\varepsilon} \Big(\sum_{\omega\in\Omega} \|f_\omega\|_p^s \Big)^{1/s} \tag{1.2}$$

holds with the exponent

$$\beta(p,s) = \frac{d-1}{2s'} - \frac{d+1}{2p}\,. \tag{1.3}$$

Here $1/s' = 1 - 1/s$. It is also possible to formulate this inequality in an equivalent integral version, namely as

$$\Big\| \int_{S^{d-1}} T_{\omega,N} f_\omega\, d\omega \Big\|_p \le C_{\varepsilon,p} N^\varepsilon N^{-\frac{d+1}{2p}} \Big(\int_{S^{d-1}} \|f_\omega\|_p^s\, d\omega \Big)^{1/s}. \tag{1.4}$$

Here $d\omega$ is the normalized rotation-invariant measure on the sphere. If in (1.4) we replace $N^{-(d+1)/2p}$ with the larger $N^{-(d-1)/2p}$ then the resulting inequality is certainly true for $p = 2$ and $s = 2$, and then, by standard arguments, also for $2 < p < \infty$ and $s = p'$. However the corresponding inequality with the improved constant seems quite deep.

For $s = p$, and functions whose Fourier transforms are supported in an annulus, (1.2) was introduced by T. Wolff in his fundamental article [22]. He showed that in this case the inequality can hold for for all $\varepsilon > 0$ only when $p \ge 2 + \frac{4}{d-1}$ which is the conjectured range. The optimal exponent $\beta(p,p) = d(\frac12 - \frac1p) - \frac12$ is the standard Bochner-Riesz critical index in d dimensions. In [22], Wolff developed an induction on scales method to prove such inequalities for large values of p, and obtained a positive answer for $s = p > 74$ when $d = 2$. This method was extended to higher dimensions in the paper by Łaba and Wolff [12], establishing the $s = p$ case for $p > 2+\min\{\frac{32}{3d-7}, \frac{8}{d-3}\}$. In both papers the authors state that improvements over these indices should be possible, although perhaps still far from the conjectured exponents. In fact, slightly better ranges for all $d \ge 2$ were already presented by the first and third authors in [7], based on the use of bilinear Fourier extension theorems in conjunction with the original proofs.

The purpose of this paper is to (i) improve the range of Wolff's original inequality, and (ii) to further strengthen these results by proving better $\ell^s(L^p)$ results for the same range of p but certain $s < p$. Note that, for given p, the validity of (1.2) for some s_0 and all $\varepsilon > 0$ implies the validity for $s_0 < s \le \infty$, by Hölder's inequality. We note that the main motivation to consider these mixed norm improvements comes from complex analysis, namely from questions on the Bergman projection for tube domains over light cones [2], [1]. See Corollary 1.6 below.

Our contribution to this problem is restricted to the case $s \geq 2$. The main result is in two dimensions. It implies that the original Wolff inequality in $\mathbb{R}^{2+1}$ holds for $p \geq 20$, and the sharp $\ell^2(L^p)$ inequality holds in a slightly smaller range.

Theorem 1.1. *Let $d = 2$, $s \geq 2$ and*

$$p_2(s) = \begin{cases} 20 & \text{if } s \geq 3 - \frac{3}{13}, \\ \frac{5(11s-6+\sqrt{65s^2-76s+36})}{6(s-1)} & \text{if } 2 < s \leq 3 - \frac{3}{13}, \\ 23 + \frac{1}{3} & \text{if } s = 2. \end{cases} \tag{1.5}$$

Then, for all $N \geq 10$ and all $N^{-1/2}$-separated sets $\Omega \in S^{d-1}$,

$$\Big\| \sum_{\omega \in \Omega} T_{\omega,N} f_\omega \Big\|_p \leq \mathcal{C}_{p,s}(N) N^{\beta(p,s)} \Big(\sum_{\omega \in \Omega} \|f_\omega\|_p^s \Big)^{1/s}; \quad p_2(s) < p < \infty \tag{1.6}$$

where

$$\log \mathcal{C}_{p,s}(N) \leq C(p,s)(\log N)^{a(p,s)} \text{ with } a(p,s) < 1. \tag{1.7}$$

It is noteworthy that for $2 \leq s \leq 36/13$ the range of p is better than what could be obtained by interpolation between $s = 2$ and $s = 36/13$ (*cf.* the figure). Also note that (1.7) implies that

$$\mathcal{C}_{p,s}(N) \leq C_{\varepsilon,p,s} N^{\varepsilon},$$

for any $\varepsilon > 0$.

In higher dimensions we obtain the sharp $\ell^2(L^p)$-result in the same p-range as the weaker $\ell^p(L^p)$-result.

Theorem 1.2. *Let $d \geq 3$, $s \geq 2$, and $p_d < p < \infty$, where*

$$p_d := 2 + \frac{8}{d-2}\Big(1 - \frac{1}{2d+2}\Big). \tag{1.8}$$

Then, for all $N \geq 10$ and all $N^{-1/2}$-separated sets $\Omega \in S^{d-1}$,

$$\Big\| \sum_{\omega \in \Omega} T_{\omega,N} f_\omega \Big\|_p \leq \mathcal{C}(N) N^{\beta(p,s)} \Big(\sum_{\omega \in \Omega} \|f_\omega\|_p^s \Big)^{1/s} \tag{1.9}$$

where

$$\log \mathcal{C}_p(N) \leq C_{d,p}(\log N)^{\alpha(d,p)} \text{ with } \alpha(d,p) < 1. \tag{1.10}$$

FIGURE 1.1. New regions for the (p, s)-inequality for $d = 2$ and $d \geq 3$

Remark 1.3. The argument also shows that $\alpha(d,p) \geq c(\frac{1}{p_d} - \frac{1}{p})^3$ as $p \searrow p_d$. Choosing p so that $\frac{1}{p_d} - \frac{1}{p} = c\frac{\log\log\log N}{\log\log N}$ one may interpolate this L^p estimate with a trivial L^2 estimate. As a consequence one finds that that the inequality (1.9) holds for $p = p_d$, with

$$\log \mathcal{C}_{p_d}(N) \leq C_d \frac{\log N}{(\log\log N)^b}$$

for some $b > 0$ (and $N \gg e^3$). A similar statement applies to Theorem 1.1 for $p = p_2(s)$.

As mentioned before, Theorems 1.1 and 1.2 can be proved by following the method [22, 12]. Our contribution lies on three points: first, a packet decomposition adapted to the $\ell^s(L^p)$ formulation of the problem. Secondly, a suitable iteration of the induction on scales method from the original proof, leading in particular to a unified exponent for all dimensions $d \geq 3$. Third, in the special (and more difficult) case $d = 2$ we additionally refine one of the combinatorial lemmas of Wolff, which in turn improves and also somewhat simplifies the results in [22].

These methods, together with the use of bilinear restriction estimates as described in [7], give the improved exponents in Theorems 1.1 and 1.2. We emphasize that the main combinatorial arguments (and specially the very deep ones for $d = 2$ involving circle tangencies) remain untouched and have been quoted from [22, 12].

We recall that the validity of (1.2) for $p = s$ implies progress on various important problems in harmonic analysis. The main original application concerns some (almost) sharp space time estimates for wave operators,

$$\Big(\int_1^2 \big\|e^{it\sqrt{-\Delta}}f\big\|^p_{L^p(\mathbb{R}^d)}dt\Big)^{1/p} \lesssim \|f\|_{L^p_\alpha(\mathbb{R}^d)}, \tag{1.11}$$

where $L^p_\alpha(\mathbb{R}^d)$ is the standard Sobolev space. One aims to prove (1.11) in the range

$$\alpha > \alpha(p) = (d-1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{p} \tag{1.12}$$

which is sharp (with the possible exception of the endpoint). Theorem 1.1 implies improvements over known results on (1.11) in low dimensions.

Corollary 1.4. *Suppose $\alpha > \alpha(p)$. The inequality* (1.11) *holds for $p \geq 20$ if $d = 2$ and $p \geq \frac{8}{d-2}\big(1 - \frac{1}{2d+2}\big)$ if $d \geq 3$.*

We remark that a better result in higher dimensions was recently obtained in [9], using different methods. There it is shown that (1.11) holds even with the endpoint $\alpha = \alpha(p)$ provided that $d \geq 4$ and $p > \frac{2d-2}{d-3}$; this improves the p-range implied by Corollary 1.4 in dimension $d \geq 5$.

It turns out that the Wolff estimates can also be used to sharpen the known estimate for the cone multiplier in $\mathbb{R}^{1+2}$ on the endpoint $L^4(\mathbb{R}^2)$ space. Recall that for $d = 2$, $p = 4$ the inequality (1.11) is conjectured to hold for all $\alpha > 0$ but this is very much open. A number of substantial papers have been written to obtain nontrivial ranges for the L^4 bounds, with incremental improvements. The initial cone multiplier result $\alpha > 1/8$ by Mockenhaupt [14] and its extension [15] on (1.11) had been first improved by Bourgain [4], using entirely new methods. Bourgain's bounds were further improved by Tao and Vargas, using bilinear Fourier extension theorems for the cone [20]. In fact combining [20] with Wolff's optimal L^2 bilinear Fourier extension results [23] one obtains L^4 boundedness for $\alpha > 5/44$. In [7] it was observed that the use of the Wolff inequality in the method of Tao and Vargas yields a better range; more precisely from Theorem 1.5 in [7] one gets $\alpha > \frac{5}{44}(\frac{p_2-4}{p_2-41/11})$ if (1.2) is known for $s = p > p_2$. Using Theorem 1.1 for $p > 20$ the range $\alpha > \frac{445}{3934}$ from [7] is thus improved to $\alpha > \frac{20}{179}$. However it is more efficient to use a mixed norm bound, namely Theorem 1.1 for p near 20 and $s = 4$. This yields:

Corollary 1.5. *Let $\alpha > 1/9$. Then* (1.11) *holds for $d = 2$ and $p = 4$.*

We shall now describe progress on a complex analysis problem, namely the L^p boundedness of Bergman projections in tube domains over full light cones, see e.g.

[2], [1]. Denote by $Q(Y) = y_0^2 - |y'|^2$ the Lorentz form in $\mathbb{R}^{d+1}$ and consider the forward light cone on which Q is positive;

$$\Lambda^{d+1} = \{Y = (y_0, y') \in \mathbb{R} \times \mathbb{R}^d : y_0^2 - |y'|^2 > 0, y_0 > 0\}.$$

Let $\mathcal{T}^{d+1} \subset \mathbb{C}^{d+1}$ be the tube domain over Λ^{d+1}, i.e.

$$\mathcal{T}^{d+1} = \mathbb{R}^{d+1} + i\Lambda^{d+1}.$$

Let $w_\gamma(Y) = Q(Y)^\gamma$ and consider the weighted space $L^p(\mathcal{T}^{d+1}, w_\gamma)$ with norm

$$\|F\|_{L^p(w_\gamma)} = \Big(\iint_{\mathcal{T}^{d+1}} |F(X+iY)|^p \Delta^\gamma(Y)\, dY dX \Big)^{1/p}.$$

Let $\mathcal{P}_\gamma$ be the orthogonal projection mapping the weighted space $L^2(\mathcal{T}^{d+1}, w_\gamma)$ to its subspace $\mathcal{A}^p_\gamma$ consisting of the holomorphic functions. Only the case $\gamma > -1$ is relevant since $\mathcal{A}^p_\gamma = \{0\}$ for $\gamma \le -1$. One is interested in the boundedness of $\mathcal{P}_\gamma$ in $L^p(\mathcal{T}^{d+1}, w_\gamma)$. A known and trivial necessary condition is

$$1 + \frac{d-1}{2(\gamma+d+1)} < p < 1 + \frac{2(\gamma+d+1)}{d-1} \tag{1.13}$$

(see e.g. [2]). In fact it has been conjectured that boundedness should hold in this range (1.13), except $d = 2$ and $\gamma \in (-1, -1/2)$, in which case there are additional counterexamples for $p \ge 8 + 4\gamma$ (see [1]). Here we obtain

Corollary 1.6. *Let $d \ge 2$ and let $p_2 = 20$ and $p_d = 2 + \frac{8}{d-2}\big(\frac{2d+1}{2d+2}\big)$ if $d \ge 3$. Then for all*

$$\gamma \ge \max\big\{-1 + \tfrac{d-1}{4}(p_d - \tfrac{2(d+1)}{d-1}), \tfrac{d-1}{2}(p_d - \tfrac{2(d+1)}{d-1} - 1)\big\}, \tag{1.14}$$

the Bergman projection $\mathcal{P}_\gamma$ is a bounded operator in $L^p(\mathcal{T}^{d+1}, w_\gamma)$ in the sharp range (1.13).

This will be a consequence of a more general mixed norm estimates stated in §7.

Remark. We point out that the range in Corollary 1.6 is a consequence of the stronger $\ell^s(L^p)$ inequalities in Theorem 1.1 and 1.2, with $s - 2$ if $d \ge 3$ and $s = 3$ with $d = 2$. The weaker $\ell^p(L^p)$ estimates only imply a solution to the problem in the smaller range $\gamma \ge \frac{d-1}{2}(p_d - \frac{2(d+1)}{d-1})$ (see Corollary 1.4. in [7]).

Finally, in [17] Wolff's $\ell^p(L^p)$ inequalities in $\mathbb{R}^3$ were employed to prove regularity of averages

$$\mathcal{A}_t f(x) = \int_I f(x - t\gamma(s))ds$$

and L^q boundedness for associated maximal functions. Here I is compact, γ is a smooth parametrization smooth of a curve in $\mathbb{R}^3$, and γ is of type $m \geq 3$. The result in [17] in combination with Theorem 1.1 yields

Corollary 1.7. *The operator $\mathcal{A}_t$ maps $L^q(\mathbb{R}^3)$ to the optimal Sobolev-space $L^q_{1/q}(\mathbb{R}^3)$ for $q > \max\{m, 11\}$. Moreover, the maximal function $Mf(x) = \sup_{t>0} |A_t f(x)|$ defines an operator bounded on $L^q(\mathbb{R}^3)$ for $q > \max\{m, 11\}$.*

Structure of the paper.

2. Notation and basic definitions

2.1. ***Plates and plate families.*** We recall the basic notation concerning plates and tubes in [22, 12], which we write with the same scaling as in [11]. For $N \geq 10^d$ {1} and $\omega \in S^{d-1}$, an (N,ω)-*plate* will be a rectangular box in $\mathbb{R}^{d+1}$ of size $1 \times \sqrt{N} \times \overset{(d-1)}{\cdots} \times \sqrt{N} \times N$, whose longest axis is parallel to $(-\omega, 1)$, whose shortest axis is parallel to $\mathfrak{n}_\omega := (\omega, 1)$; the midlength axes are parallel to $(e_{i,\omega}, 0)$, where $i = 1, \dots, d-1$ and the unit-vecors $e_{i,\omega}$ are mutually orthogonal and orthogonal to ω. We shall typically denote plates in x-space by π and families of plates by $\mathcal{P}$.
{2}

An (N,ω)-*tube* τ is a rectangular box of size $\sqrt{N} \times \overset{(d \text{ times})}{\cdots} \times \sqrt{N} \times N$, whose longest axis is parallel to $(-\omega, 1)$. For any (N,ω) plate we denote by $\tau(\pi)$ the $(5N,\omega)$ tube whose axes are parallel to the axes of π and which has the same center as π.

Separated plate families and tube densities.

Definition 2.1. A *separated* (N,ω) *plate family* is a family of parallel (N,ω) plates with the following properties.

(i) The $10d$-fold dilates of the plates are disjoint.

(ii) For each pair of plates (π, π') we have that either both $\pi \in \tau(\pi')$ and $\pi' \in \tau(\pi)$ hold or the $10d$-fold dilates of the tubes $\tau(\pi), \tau(\pi')$ are disjoint.

For each π we also define $\mu(\pi)$ be the largest integer μ for which

$$\mu \leq \log_2 \big(\#\{\pi' \in \mathcal{P} : \pi' \subset \tau(\pi)\}\big).$$

{1}modify

{2}parallel or point in $\sqrt{\delta}$-separated directions. We shall also assume that families $\mathcal{P}$ consist only of *separated* plates, meaning that for each $\pi \in \mathcal{P}$ at most C_1 plates from $\mathcal{P}$ can be contained in a fixed dilate $C_2\pi$, where C_1 and C_2 are fixed universal constants.

We note that if $\mathcal{P}$ is a separated (N,ω) then then $\mu(\pi)=\mu(\pi')$.

Definition 2.2. A separated (N,ω) plate family $\mathcal{P}$ is said to have tube constant tube density if $\mu(\pi)=\mu$ for all $\pi\in\mathcal{P}$. We then refer to μ as the tube density of $\mathcal{P}$.

Definition 2.3. (i) Given an N-cube Q and $\omega\in S^{d-1}$ an (N,ω,Q) plate family is an (N,ω) plate family consisting only of plates π with $Q\cap\pi\neq\emptyset$.

(ii) Let $E\subset S^{d-1}$ be an $N^{-1/2}d$-separated set of directions and let Q be an N-cube. We say that $\mathcal{P}$ is an (N,E,Q) *plate family* if $\mathcal{P}=\cup_{\omega\in E}\mathcal{P}_\omega$ where each $\mathcal{P}_\omega$ is an (N,ω,Q) plate family. We say that $\mathcal{P}$ has tube density μ if $\mathcal{P}_\omega$ has tube density μ for every $\omega\in E$. We say that $\mathcal{P}$ is a *stable* (N,E,Q) *plate family* if in addition

$$|\mathcal{P}_\omega|\le 2|\mathcal{P}_{\omega'}|,\quad \forall\,\omega,\omega'\in E. \tag{2.1}$$

Condition (2.1) will be crucial when dealing with' $\|\cdot\|_{p,s;\delta}$ norms, as it implies that the cardinalities of the $\mathcal{P}_\omega$'s are comparable, for all $\omega\in E$.

Lemma 2.4. *(i) If $\mathcal{P}$ is a separated (N,ω) plate family $\mathcal{P}$ then one can find a family $\mathcal{T}$ of disjoint (N,ω)-tubes so that each $\pi\in\mathcal{P}$ is contained in exactly one tube $\tau\in\Omega$.*

(ii) Let $\widetilde{\mathcal{P}}$ be a family of (N,ω) plates with the property that each point in $\mathbb{R}^{d+1}$ belongs to at most A plates in $\widetilde{\mathcal{P}}$. Then $\widetilde{\mathcal{P}}$ is a union of no more than $A(10d)^{3d}$ [3] separated plate families.

(iii) $\widetilde{\mathcal{P}}$ is the union of no more than $C_d\log N$ separated (ω,N) plate families with constant tube density.

We omit the proof.

An N-plate family with direction set $\Omega\subset S^{d-1}$ is the union of plate families $\cup_{\omega\in\Omega}\mathcal{P}_\omega$ where each $\mathcal{P}_\omega$ consists of (N,ω) plates. We say that $\mathcal{P}$ is separated if $\mathcal{P}=\cup_{\omega\in\Omega}\mathcal{P}_\omega$ and (i) the direction set is $10^dN^{-1/2}$ separated, and (ii) each $\mathcal{P}_\omega$ is a separated (N,ω) plate family.

[4]

[3] CHECK

[4] Tube families $\mathcal{T}$ will also be assumed to be separated. Finally, a *σ-cube* Δ is a cube of sidelength σ centered at some point of the grid $\sigma\mathbb{Z}^{d+1}$. By $\{\Delta\}$ we denote the tiling of $\mathbb{R}^{d+1}$ formed by all such cubes. In general, given a rectangular box R (e.g., a cube, tube or plate), we denote by cR the box obtained from R by dilating it by a factor $c>0$ about its center.

2.2. ***Bump functions.*** Set

$$w(x) = (1+|x|)^{-2d-4}, \tag{2.2}$$

and given a rectangle R we define $w_R = w \circ a_R^{-1}$, where a_R is an affine map taking the unit cube centered at 0 to the rectangle R. The function w_R behaves 'roughly' like the characteristic function of R, with a fast decaying tail off R,

==================== We fix an even nonnegative Schwartz-function $\psi_\circ$ on the real line whose Fourier transform is supported $\widehat{\psi}_\circ$ s supported in $[-1,1]$, satisfies $\int \widehat{\psi}_\circ(\sigma)d\sigma = 1$. By [], Lemma , we can choose $\psi_\circ$ so that

We shall also use a fixed non-negative Schwartz function ψ, which is defined as a tensor product $\psi(\xi) = \prod_{i=1}^{d+1} \psi_\circ(\xi_i)$ with an even Schwartz function $\psi_c irc$, with the additional property that the Fourier transform $\widehat{\psi}_c irc$ is supported in $[-(10d)^{-1}, (10d)^{-1}]$, moreover ψ_1 is even and $\int |\widehat{\psi}_\circ(\sigma)|^2 d\sigma = 1$.

We can also arrange that We apply the Poisson summation formula $\sum_{\nu\in\mathbb{Z}} g(x+\nu) = \sum_{k\in\mathbb{Z}} \widehat{g}(2\pi k)e^{2\pi ikx}$ with $g = |\psi_0|^2$ (or use [**?**], p. 50) and obtain

$$\sum_{\nu\in\mathbb{Z}} |\psi_\circ(x+\nu)|^2 = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx} \text{ with } c_k = \int \widehat{\psi}_\circ(2\pi k - y)\overline{\widehat{\psi}_\circ}(-y)dy \tag{2.3}$$

and by our assumption on $\widehat{\psi}_\circ$ we see that $c_0 = 1$ and $c_k = 0$ for $k \neq 0$. Thus the right hand side of (2.3) equals 1 everywhere and after taking tensor products we also get

$$\sum_{n\in\mathbb{Z}^{d+1}} \psi(\xi+n)^2 = 1, \text{ for all } \xi \in \mathbb{R}^{d+1}.$$

whose Fourier transform is supported in strictly positive in $B_2(0)$, with Fourier transform supported in $\{\xi : |\xi| \le (10d)^{-2}\}$, and so that

Again we set

$$\psi_R = \psi \circ a_R^{-1}. \tag{2.4}$$

In particular, if $\{R\}$ is a tiling of $\mathbb{R}^{d+1}$ by rectangles (generally cubes, plates, or tubes), then

$$\sum_R \psi_R^2 = 1. \tag{2.5}$$

2.3. ***The norms*** $\|\cdot\|_{p,s;N}$**.** Let $\eta_\circ \in C_c^\infty(\mathbb{R})$ be supported in $(-2,2)$ so that $\eta_\circ(s)=1$ for $|s|\le 3/2$. For $\omega\in S^{d-1}$ define $P_{N,\omega}$ by

$$\widehat{P_{N,\omega}f}(\xi',\xi_{d+1}) = \eta_\circ\big(N^{1/2}(\frac{\xi'}{|\xi'|}-\omega)\big)\widehat{f}(\xi',\xi_{d+1}). \tag{2.6}$$

Let Ω be a *maximal* $N^{-1/2}$ separated subset of S^{d-1}. It is easy to see that then $\sum_{\omega\in\Omega}\eta_\circ^2\big(N^{1/2}(\frac{\xi'}{|\xi'|}-\omega)\big)\ge c>0$. Define $Q_{N,\omega}$ by

$$\widehat{Q_{N,\omega}f}(\xi) = \frac{\eta_\circ\big(N^{1/2}(\frac{\xi'}{|\xi'|}-\omega)\big)}{\sum_{\omega'\in\Omega}\eta_\circ^2\big(N^{1/2}(\frac{\xi'}{|\xi'|}-\omega)\big)}\widehat{f}(\xi). \tag{2.7}$$

By a nonisotropic rescaling and standard singular integral theory it is easy to see that $P_{N,\omega}$, $Q_{N,\omega}$ are bounded operators on $L^p(\mathbb{R}^{d+1})$ for $1<p<\infty$. We also have the reproducing formula

$$\sum_{\omega\in\Omega} Q_{\omega,N}P_{\omega,N} = \mathrm{Id} \tag{2.8}$$

For $1<p<\infty$ we define a norm on L^p by setting

$$\|f\|_{p,s;N} = \Big(\sum_{\omega\in\Omega}\big\|P_{N,\omega}f\big\|_p^s\Big)^{1/s}. \tag{2.9}$$

It is easy to check that (2.9) defines a norm on the space L^p, for $1<p<\infty$. The definition depends on the choice of Ω and on the choice of the particular function $\eta_\circ$. However different choices of Ω and $\eta_\circ$ produce equivalent norms where the constants in the equivalences do not depend on N. This is easy to see using (2.8). Indeed if $\widetilde{\Omega}$ is another maximal $N^{-1/2}$-separated set on the sphere, and if $\widetilde{Q}_{N,\omega}$ is defined as in (2.7) but with respect to $\widetilde{\Omega}$ then the operators $P_{N,\omega}\widetilde{Q}_{N,\omega'}$ are uniformly bounded on L^p provided that $|\omega-\omega'|\le CN^{-1/2}$. Also notice that $P_{N,\omega}P_{N,\omega'}=0$ if $|\omega-\omega'|>8N^{-1/2}$. Thus

$$\begin{aligned}\Big(\sum_{\omega\in\Omega}\big\|P_{N,\omega}f\big\|_p^s\Big)^{1/s} &= \Big(\sum_{\omega\in\Omega}\Big\|P_{N,\omega}\sum_{\substack{\omega'\in\Omega'\\|\omega-\omega'|\le 8N^{-1/2}}}\widetilde{Q}_{N,\omega'}\widetilde{P}_{N,\omega'}f\Big\|_p^s\Big)^{1/s}\\ &\lesssim \Big(\sum_{\omega'\in\Omega'}\sum_{\substack{\omega\in\Omega\\|\omega'-\omega|\le 8N^{-1/2}}}\big\|\widetilde{P}_{N,\omega}f\big\|_p^s\Big)^{1/s} \lesssim \Big(\sum_{\omega'\in\Omega'}\big\|\widetilde{P}_{N,\omega'}f\big\|_p^s\Big)^{1/s},\end{aligned}$$

and the opposite inequality follows by reversing the roles of Ω and Ω'. We also note that the multiplier transformations defined for $A\ge 1$, by

$$\widehat{T_{NA}f}(\xi) = \eta(AN(1-\frac{|\xi'|}{|\xi_{d+1}|})\widehat{f}(\xi)$$

behave nicely with respect to the $(p,s;N)$-norms; *i.e.*, for $1<p<\infty$,

$$\|T_{NA}f\|_{p,s;N} \le C_{d,p}(1+A)^{\frac{d-1}{2}}\|f\|_{p,s;N}. \tag{2.10}$$

Here is the exponent of $1+A$ is not sharp but this is irrelevant. For the proof of (2.10) one notes that $T_N P_{N,\omega}$ is bounded on $L^p(\mathbb{R}^{d+1}$ with operator norm $\lesssim (1+A)^{\frac{d-1}{2}})$ and thus

$$\|T_{NA}f\|_{p,s;N} \lesssim \Big(\sum_{\omega\in\Omega}\Big\|\sum_{\substack{\omega'\in\Omega\\|\omega-\omega'|\le 8N^{-1/2}}} TP_{N,\omega}Q_{N,\omega'}P_{N,\omega'}f\Big\|_p^s\Big)^{1/s}$$
$$\lesssim (1+A)^{\frac{d-1}{2}}\Big(\sum_{\omega'\in\Omega}\big\|P_{N,\omega'}f\big\|_p^s\Big)^{1/s} \lesssim (1+A)^{\frac{d-1}{2}}\|f\|_{p,s;N}.$$

2.4. ***Reformulation of the theorems.*** For each s the $(p,s;N)$ norms are equivalent to the L^p norm, with dependence on N in the sense that

$$\|f\|_{p,s;N} \lesssim \|f\|_p \lesssim C_N\|f\|_{p,s;N}.$$

Our task is to prove nontrivial bounds for C_N under the additional assumption that the Fourier transforms of the functions are supported in

$$\Gamma(N) = \big\{\xi\in\mathbb{R}^{d+1} : \big||\xi'|-\xi_{d+1}\big| \le N^{-1}|\xi_{d+1}|\,\big\}. \tag{2.11}$$

By the considerations in the previous subsection it is now easy to see that Theorems 1.1 and 1.2 follow from the following statement using the $(p,s;N)$-norms.

Theorem 2.5. *Suppose that supp* $\widehat{f}\subset\Gamma(N)$. *Then*

$$\|f\|_p \le \mathcal{C}(N)N^{\beta(p,s)}\|f\|_{p,s;N} \tag{2.12}$$

holds with $\mathcal{C}(N)$ *as in* (1.7) *if*
either (i) $d\ge 2$, $s\ge 2$ *and let* p *be is as in* (1.5),
or (ii) $d\ge 3$, $s\ge 2$, $p_d<p<\infty$ *with* p_d *as in* (1.8).

2.5. ***Elementary properties of the*** $(p,s;N)$***-norms.*** In what follows we shall often consider functions whose Fourier transform is supported in the truncated conical set

$$\Gamma_h(N) = \{\xi\in\Gamma(N) : h/2\le|\xi_{d+1}|\le 2h\},$$

and by scaling arguments we shall usually reduce to the case $h=1$. We also notice that for distributions f whose Fourier transforms are supported in a compact set away from the origin it makes sense to extend the definition of the $(p,s;N)$ norms to

$p = \infty$. Note that the operators $P_{N,\omega}$ are uniformly bounded on the subspace of L^∞ functions whose Fourier transform is supported in $\{\xi : h/4 \le |\xi| \le 4h\}$.

For $\omega \in S^{d-1}$ let $u_\omega = (1, \omega)$, which determines a line on the forward light cone. Also let $\mathfrak{n}_\omega = (\omega, -1)/\sqrt{2}$ the outer unit normal vector at u_ω and let $\wp_\omega$ be the projection to the $(d-1)$-dimensional plane orthogonal to u_ω and $\mathfrak{n}_\omega$, i.e. $\wp_\omega(\eta) = \eta - \langle \eta, \mathfrak{n}_\omega\rangle \mathfrak{n}_\omega - \eta - \langle \eta, u_\omega\rangle u_\omega/2$. We note that that the intersection of $\Gamma_1(N)$ with the support of $\widehat{P_{N,\omega} f}$ is contained in a "Fourier-plate" consisting of all $\xi \in \mathbb{R}^{d+1}$ for which

$$C^{-1}\frac{1}{2} \le \langle \xi, u_\omega\rangle \le 2C \quad |\wp_\omega(\xi - u_\omega)| \le CN^{-1/2}, \quad |\langle \xi - u_\omega, \mathfrak{n}_\omega\rangle| \le CN^{-1}.$$

This is a slightly expanded versions of the region Π_ω^N which we define as the set of all $\xi = (\xi', \xi_{d+1}) \in \mathbb{R}^{d+1}$ satisfying the inequalities

$$\Big|1 - \frac{|\xi'|}{\xi_{d+1}}\Big| \le \frac{1}{10N}, \qquad \Big|\frac{\xi'}{|\xi'|} - \omega\Big| \le \frac{1}{10\sqrt{N}}, \qquad \frac{9}{10} \le \xi_{d+1} \le \frac{11}{10}. \tag{2.13}$$

The following elementary observation will be frequently used.

Lemma 2.6. *Let $f \in L^p$ with $\widehat{f}$ supported in $\Gamma_1(N)$, $N \ge 10^{10}$. Then there is a set $\mathcal{G}_f$ consisting of no more than 100^d functions, for which the following properties hold.*

(i) For each function $g \in \mathcal{G}_f$ there is a $10N^{-1/2}$-separated subset Ω_g of S^{d-1} so that $\widehat{g}$ is supported in $\bigcup_{\omega\in\Omega_g} \Pi_\omega^N$.

(ii) $\|g\|_{p,s;N} \lesssim \|f\|_{p,s;N}$ for every $g \in \mathcal{G}_f$.

(iii) For every $g \in \mathcal{G}_f$ there are $\tau_{1,g}$, $\tau_{2,g}$ in $(1/2, 2)$ so that for all $x \in \mathbb{R}^{d+1}$

$$f(x', x_{d+1}) = \sum_{g\in\mathcal{G}_f} \tau_{1,g}^d \tau_{2,g} g(\tau_{1,g}x', \tau_{2,g}x_{d+1}).$$

(iii) so that

Proof. Let η be supported in $(-1, 1)$ so that $\sum_{k\in\mathbb{Z}} \eta^2(s + n) = 1$ for all $s \in \mathbb{R}$. Split

$$f = \sum_{k=(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}} f_k \text{ where } \widehat{f_k}(\xi) = \eta(20N(1 - \tfrac{|\xi'|}{\xi_{d+1}}) - k_2)\eta(20\xi_{d+1} - k_2)\widehat{f}.$$

Since supp $\widehat{f} \subset \Gamma_1(N)$ we see that only terms with $|k_1| \le 40$ and with $1/4 \le k_2/20 \le 3$ contribute to the sum. We scale and see that for suitable $(\tau_{1,k}, \tau_{2,k}) \in (1/2, 2)$ the function $\widehat{g}_k(\xi) := \widehat{f}_k(\tau_{1,k}^{-1}\xi', \tau_{2,k}^{-1}\xi_{d+1})$ is supported in the region where $|1 - \frac{|\xi'|}{\xi_{d+1}}| \le (10N)^{-1}$ and $\frac{9}{10} \le \xi_{d+1} \le \frac{11}{10}$. Moreover $f(x) = \sum_k \tau_{1,k}^d \tau_{2,k} g_k(\tau_{1,k}x', \tau_{2,k}x_{d+1})$.

We now make a further spherical decomposition and let $\widetilde{\Omega}$ be a maximal $10^{-2}N^{-1/2}$-separated set. Define operators $\widetilde{P}_\omega$, $\widetilde{Q}_\omega$ similarly as in (2.6), (2.7) (except that N is replaced with $10^4 N$ and the definition involves $\widetilde{\Omega}$).

Now split $\widetilde{\Omega}$ into $O(10^{10d})$ disjoint families $\widetilde{\Omega}_\nu$ of $10N^{-1/2}$ separated vectors and define $g_{k,\nu} = \sum_{\omega\in\Omega_\nu} \widetilde{P}_\omega \widetilde{Q}_\omega g_k$. Then $f(x) = \sum_{k_1,k_2}\sum_\nu \tau_{1,k}^d \tau_{2,k} g_{k,\nu}(\tau_{1,k}x', \tau_{2,k}x_{d+1})$ and the family $\mathcal{G}_f$ consists of the functions $g_{k,\nu}$ with $|k_1| \le 40$ and $1/4 \le k_2/20 \le 3$. The inequality $\|g_{k,\nu}\|_{p,s;N} \lesssim \|f\|_{p,s;N}$ follows quickly using the various separatedness conditions, and the fact that the operators $T_{NA}\widetilde{P}_\omega\widetilde{Q}_\omega$ are uniformly bounded on L^p (*cf.* also (2.10)). □

We first state an elementary Sobolev embedding result for the $(p, s; N)$ norms.

Lemma 2.7. *Let $p \ge 2$, $1 \le s \le \infty$ and let $\widehat{f}$ be supported in $\Gamma_1(N)$. Then*

$$\|f\|_{\infty,s;N} \lesssim N^{-\frac{d+1}{2p}} \|f\|_{p,s;N} \,, \tag{2.14}$$

and

$$\|f\|_\infty \lesssim N^{\beta(p,s)}\|f\|_{p,s;N}. \tag{2.15}$$

Proof. Observe that, by Young's inequality,

$$\|g\|_\infty \lesssim N^{-\frac{d+1}{2p}} \|g\|_p \,, \quad \text{when supp } \hat{g} \subset \Pi_\omega^N. \tag{2.16}$$

This yields (2.14). If $f = \sum_{\omega\in\Omega} f_\omega$ with $\widehat{f_\omega}$ supported in Π_ω^N, and Ω is $N^{-1/2}$ separated, then, using (2.14),

$$\|f\|_\infty \lesssim \sum_\omega \big\|f_\omega\big\|_\infty \lesssim N^{\frac{d-1}{2s'}}\Big(\sum_\omega \big\|f_\omega\big\|_\infty^s\Big)^{1/s} \lesssim N^{\frac{d-1}{2s'}-\frac{d+1}{2p}}\Big(\sum_{\omega\in\Omega} \big\|f_\omega\big\|_p^s\Big)^{1/s}$$

which gives (2.15). □

Lemma 2.8. *Let $p \ge 2$, $s \ge 2$ and let $\widehat{f}$ be supported in $\Gamma_1(N)$. Then*

$$\|f\|_{p,2;N} \lesssim \|f\|_2^{2/p}\|f\|_{\infty,2;N}^{1-2/p} \,. \tag{2.17}$$

Moreover, for every $p \in (2,\infty)$, $s \in [2,p]$, we have

$$\|f\|_{p,s;N} \lesssim \|f\|_2^{2/p}\|f\|_{\infty,r;N}^{1-2/p} \,, \tag{2.18}$$

where $r = r(p,s)$ is defined by

$$\frac{2}{r} = \Big(\frac{1}{s} - \frac{1}{p}\Big)\Big/\Big(\frac{1}{2} - \frac{1}{p}\Big). \tag{2.19}$$

Proof. We fix $2 \le r \le \infty$. Then by convexity of $\ell^s(L^p)$ norms

$$\Big(\sum_{\omega\in\Omega}\|f_\omega\|_p^s\Big)^{\frac1s} \le \Big(\sum_{\omega\in\Omega}\|f_\omega\|_2^2\Big)^{\frac{1-\vartheta}{2}}\Big(\sum_{\omega\in\Omega}\|f_\omega\|_\infty^r\Big)^{\frac{\vartheta}{r}}$$

where $0 \le \vartheta \le 1$ and $(1/p, 1/s) = (1-\vartheta)(1/2, 1/2) + \vartheta(0, 1/r)$. Thus $\vartheta = 1 - 2/p$.

If we let Ω to be an $N^{-1/2}$ separated subset of S^{d-1} and apply the last inequality with $f_\omega = P_{N,\omega} f$ we have of course $\sum_{\omega\in\Omega}\|P_{N,\omega}f\|_2 \lesssim \|f\|_2^2$ and therefore we get

$$\|f\|_{p,2;N} \lesssim \|f\|_2^{2/p}\|f\|_{\infty,r;N}^{1-2/p}$$

for $s^{-1} = p^{-1} + r^{-1}(1-2p^{-1})$. For $r = 2$ this yields (2.17). For $p > 2$ the relation between r and s is equivalent with (2.19) and thus (2.18) is proven. □

We shall also use the following localization estimate.

Lemma 2.9. *Let $1 \le s \le p \le \infty$ and let $f \in L^p$ so that $\widehat{f}$ be supported in $\{\xi : 1/4 \le |\xi| \le 4\}$. Let $\mathcal{Q} = \{Q\}$ be a grid of N-cubes and ψ_Q be as in* (2.4) *(so that $\widehat{\psi}_Q$ is supported in $\{|\xi| \le 10^{-2}N^{-1}\}$. Then*

$$\Big(\sum_{Q\in\mathcal{Q}}\|\psi_Q f\|_{p,s;N}^p\Big)^{1/p} \lesssim \|f\|_{p,s;N}\ . \tag{2.20}$$

Proof. Let $F_\omega = Q_{N,\omega'}P_{N,\omega'}f$ then by the support property of $\widehat{\psi_Q}$ and $\widehat{f}$

$$\|P_{N,\omega}(\psi_Q f)\|_p \lesssim \sum_{\substack{\omega'\in\Omega\\ |\omega-\omega'|\le CN^{-1/2}}}\big\|P_{N,\omega}[\psi_Q F_{\omega'}]\big\|_p.$$

Thus by Minkowski's inequality (since $p/s \ge 1$) we obtain with $G_\omega := \sum_{\omega'\in\mathcal{I}_\omega} F_\omega$

$$\begin{aligned}\Big(\sum_Q\Big[\sum_\omega\big\|P_{N,\omega}(\psi_Q f)\big\|_p^s\Big]^{\frac ps}\Big)^{\frac1p} &\le \Big(\sum_\omega\Big[\sum_Q\big\|P_{N,\omega}(\psi_Q G_\omega)\big\|_p^p\Big]^{\frac sp}\Big)^{\frac1s}\\ &\lesssim \Big(\sum_{om}\Big[\sum_Q\big\|\psi_Q G_\omega\big\|_p^p\Big]^{\frac sp}\Big)^{\frac1s} \lesssim \Big(\sum_\omega\big\|G_\omega\big\|_p^s\Big)^{\frac1s}.\end{aligned}$$

Note that the cardinality of $\mathcal{I}_\omega = \{\omega' \in \Omega\colon |\omega-\omega'| \le CN^{-1/2}\}$ is bounded above independent of N. Thus the last expression is bounded by

$$\Big(\sum_\omega\Big(\sum_{\omega'\in\mathcal{I}_\omega}\big\|F_{\omega'}\big\|_p\Big)^s\Big)^{\frac1s} \lesssim \Big(\sum_{\omega'}\sum_{\omega:\omega'\in\mathcal{I}_\omega}\big\|F_{\omega'}\big\|_p^s\Big)^{\frac1s} \lesssim \Big(\sum_{\omega'}\big\|P_{N,\omega'}f\big\|_p^s\Big)^{\frac1s}.$$

which yields the assertion. □

2.6. ***Packets.*** We shall define the concept of packets which are special cases of "N-functions", as defined in [22, 12] (however with the scaling used in [11].

Definition 2.10. (i) Let $\omega \in S^{d-1}$ and let $\mathcal{P} \equiv \mathcal{P}_\omega$ be a finite separated (N, ω) plate family. f is called an (N, ω)-*packet* associated with $\mathcal{P}$ if it can be written as $f = \sum_{\pi \in \mathcal{P}} f_\pi$ with functions f_π satisfying

$$|f_\pi| \le w_\pi \qquad \text{and} \qquad \operatorname{supp} \widehat{f_\pi} \subset \Pi_\omega^N . \tag{2.21}$$

If $\mathcal{P}_\omega$ is an (N, ω, Q) plate family then we refer to f as an (N, ω, Q)-*packet.*

(ii) Let E be an $N^{-1/2}d$ separated set of directions and let Q be an N-cube. We say that f is an (N, E, Q)-*packet* if for every $\omega \in \mathcal{E}$ there is a finite separated (N, ω) plate family $\mathcal{P}_\omega$ consisting of plates intersecting Q so that f can be written as

$$f = \sum_{\omega \in E} f_\omega \tag{2.22}$$

where each f_ω is an (N, ω, Q)-packet.

If for every $\omega \in E$ the family $\mathcal{P}_\omega$ is of constant tube density μ then we refer to f as an (N, E, Q)-packet with tube density μ. We also say that f is a *stable* (N, E, Q)-*packet* if $\mathcal{P}$ is a stable (N, E, Q) plate family.

(iii) Let f be an (N, E, Q)-packet with plate family $\mathcal{P} = \cup_{k \in E} \mathcal{P}_\omega$ and with the representation (2.22). A *subpacket of* f is a function $\tilde{f}$ of the form

$$\tilde{f} = \sum_{\omega \in E} \sum_{\pi \in \widetilde{\mathcal{P}}_\omega} f_\pi, \tag{2.23}$$

where each $\widetilde{\mathcal{P}}_\omega$ is a subset of $\mathcal{P}_\omega$.

Observe that every subpacket of an (N, E, Q)-packet is again an (N, E, Q)-packet. However, subpackets of stable (N, E, Q)-packets are not necessarily stable. Moreover subpackets of [ackets with constant tube density do not necessarily have constant tube density.

Lemma 2.11. *Let f be an (N, E, Q)-packet with plate family $\mathcal{P} = \cup_{\omega \in E} \mathcal{P}_\omega$. Then*

$$\|f\|_{\infty, r} \lesssim |E|^{\frac{1}{r}}, \quad 1 \le r \le \infty, \tag{2.24}$$

$$\|f\|_\infty \lesssim |E| \lesssim N^{\frac{d-1}{2}}, \tag{2.25}$$

$$\|f\|_2^2 \lesssim N^{(d+1)/2} |\mathcal{P}|, \tag{2.26}$$

Moreover, if $p \geq 2$ and $1 \leq s \leq p$ we also have

$$\|f\|_{p,s} \lesssim \left(N^{\frac{d+1}{2}}|\mathcal{P}|\right)^{\frac{1}{p}}|E|^{\frac{1}{s}-\frac{1}{p}}. \tag{2.27}$$

If in addition f is a stable (N, E, Q)-packet, then (2.27) *holds for all $p, s \geq 1$.*

Proof. (2.24) and (2.25) are immediate. We next prove (2.27) and estimate

$$\|f\|_{p,s} \lesssim \Big(\sum_{\omega\in E}\Big\|\sum_{\pi\in\mathcal{P}_\omega} f_\pi\Big\|_p^s\Big)^{1/s} \lesssim \Big(\sum_{\omega\in E}\Big\|\sum_{\pi\in\mathcal{P}_\omega} w_\pi\Big\|_p^s\Big)^{1/s} \lesssim \Big(\sum_{\omega\in E}\big[N^{\frac{d+1}{2}}|\mathcal{P}_\omega|\big]^{s/p}\Big)^{1/p}.$$

If $s \leq p$ we have

$$\Big(\sum_{\omega\in E}|\mathcal{P}_\omega|^{s/p}\Big)^{1/p} \leq |E|^{1/s-1/p}\Big(\sum_{\omega}|\mathcal{P}_\omega|\Big)^{1/p} = |E|^{1/s-1/p}|\mathcal{P}|,$$

by Hölder's inequality. If $\mathcal{P}$ is stable then the last inequality remains true (up to a constant) for all $p, s \geq 1$ since $|\mathcal{P}_\omega| \approx |\mathcal{P}|/|E|$ by (2.1).

Finally, to obtain (2.27) for general packets when $p \geq 2$ and $1 \leq s \leq p$, one uses the interpolation estimate in (2.18), together with the previous (2.24) and (2.26):

$$\|f\|_{p,s} \lesssim \|f\|_2^{\frac{2}{p}}\,\|f\|_{\infty,r}^{1-\frac{2}{p}} \lesssim \left(N^{\frac{d+1}{2}}\,|\mathcal{P}(f)|\right)^{1/p}|E|^{(1-\frac{2}{p})\frac{1}{r}},$$

where $r = r(p, s)$ is defined in (2.19). However, from the definition of $r(p, s)$ one sees that $(1-\frac{2}{p})\frac{1}{r} = \frac{1}{s}-\frac{1}{p}$, which establishes (2.27). □

2.7. ***Decomposing functions into packets.*** The main lemma in this section concerns decompositions of functions with Fourier support in $\Gamma_\delta(c)$ into stable N-packets. The stability condition on the packets is crucial to obtain the inequality in (2.31) below, which is a sort of converse to the inequality in (2.27). This estimate will be strongly used in the proof of Proposition 3.3 and in the iteration process which starts with Lemma 5.2.

Lemma 2.12. *Let $f = \sum f_k$ with $\widehat{f_k}$ supported in Π_ω^N and assume that*

$$\sup_k \|f_k\|_\infty \leq A. \tag{2.28}$$

Then, for every N-cube Q, we may decompose

$$f(x) = \sum_{AN^{-10d}\leq 2^j\lesssim A}\ \sum_{\ell=1}^{n_j} 2^j\, f^{[j,\ell]}(x) + g(x), \quad x \in Q \tag{2.29}$$

for some constant integers $n_j \lesssim \log N$, and where *(i) for each j, ℓ, the functions $f^{[j,\ell]}$ are stable $(N, E_{j,\ell}, Q)$-packets, for certain sets of directions $E_{j,\ell} \subset \Omega$. The corresponding plate families $\mathcal{P}^{[j,\ell]}$ consist only of plates $\pi \subset 2N^{\varepsilon_0}Q$. Also, we can write*

$$f^{[j,\ell]} = \sum_{k \in E_{j,\ell}} \sum_{\pi \in \mathcal{P}_k^{[j,\ell]}} f_\pi,$$

for plate families $\mathcal{P}_k^{[j,\ell]}$ consisting of plates $\parallel k$, and so that $\mathcal{P}^{[j,\ell]} = \cup_{k \in E_{j,\ell}} \mathcal{P}_k^{[j,\ell]}$.

(ii) The function $g(x)$ satisfies

$$\|g\|_{L^\infty(Q)} \lesssim N^{-8d} A. \tag{2.30}$$

(iii) For every $s, p \geq 1$ and every j, ℓ

$$2^j \left(N^{\frac{d+1}{2}} |\mathcal{P}^{[j,\ell]}| \right)^{\frac{1}{p}} |E_{j,\ell}|^{\frac{1}{s} - \frac{1}{p}} \lesssim \|f\|_{p,s;\delta}\,. \tag{2.31}$$

Proof. Fix k, and consider a tiling $\{\pi\}$ of $\mathbb{R}^{d+1}$ by plates $\pi \parallel k$. Write

$$f_k \;=\; \sum_{\pi \parallel k} f_k \psi_\pi^2 \;=\; \sum_{\pi \parallel k:\ \pi \cap (N^{\varepsilon_0} Q) \neq \emptyset} f_k \psi_\pi^2 \;+\; g_k. \tag{2.32}$$

For each $j \in \mathbb{Z}$, let

$$\mathcal{P}_k^{[j]} = \left\{ \pi \;:\; \pi \parallel k,\ \pi \cap (N^{\varepsilon_0} Q) \neq \emptyset \quad \text{and} \quad 2^j \leq \|f_k \psi_\pi\|_\infty < 2^{j+1} \right\}.$$

Observe that $\|f_k \psi_\pi\|_\infty \lesssim \|f_k\|_\infty \leq A$, and therefore these plate sets are non-empty only for $2^j \lesssim A$. Next, fix j, and for every positive integer ℓ define

$$E_{j,\ell} = \left\{ k \;:\; 2^{\ell-1} \leq \# \mathcal{P}_k^{[j]} < 2^\ell \right\}.$$

Since $\# \mathcal{P}_k^{[j]} \leq \# \{ \pi \;:\; \pi \subset 2N^{\varepsilon_0} Q \} \lesssim N^{(d+1)(1+\varepsilon_0)} / N^{\frac{d+1}{2}}$, the sets $E_{j,\ell}$ are non-empty only for $\ell \lesssim \log N$.

Call $f_\pi := 2^{-j} f_k \psi_\pi^2$, when $\pi \in \mathcal{P}_k^{[j]}$. Clearly, these are N-packets associated with Π_ω^N. Define the functions

$$f^{[j,\ell]} = \sum_{k \in E_{j,\ell}} \sum_{\pi \in \mathcal{P}_k^{[j]}} f_\pi,$$

so that from (2.32) we see that

$$f(x) \;=\; \sum_{2^j \lesssim A} 2^j \sum_{\ell=1}^{C \log N} f^{[j,\ell]}(x) \;+\; \sum_k g_k(x). \tag{2.33}$$

By construction it is easy to see that, for each j and ℓ, the function $f^{[j,\ell]}$ is an $(N, E_{j,\ell}, Q)$-packet. The stability condition in (2.1) is immediate since

$$|\mathcal{P}^{[j]}_{k_0}| \,<\, 2^\ell \,\leq\, 2\,|\mathcal{P}^{[j]}_{k_1}|, \quad \forall\, k_0, k_1 \in |E_{j,\ell}|.$$

To pass from (2.33) to the decomposition in (2.29), define the function

$$g = \sum_k g_k \;+\; \sum_{2^j < AN^{-10d}} 2^j \sum_\ell f^{[j,\ell]}.$$

Observe that, from (2.32) and the Schwartz decay of ψ,

$$\begin{aligned}\big\|\sum_k g_k\big\|_{L^\infty(Q)} \;&\leq\; \sum_k \sum_{\pi \| k:\ \pi\cap(N^{\varepsilon_0}Q)=\emptyset} \big\|f_k \psi_\pi^2\big\|_{L^\infty(Q)} \\ &\lesssim\; N^{\frac{d-1}{2}} \sup_k \|f_k\|_\infty\, C_L\, N^{-\varepsilon_0 L} \,\lesssim\, N^{-9d} A,\end{aligned}$$

if we choose L sufficiently large (depending on ε_0). On the other hand, using (2.25),

$$\sum_{2^j \leq AN^{-10d}} 2^j \sum_\ell \|f^{[j,\ell]}\|_\infty \lesssim (\log N) \sum_{2^j < AN^{-10d}} 2^j\, N^{\frac{d-1}{2}} \,\lesssim\, AN^{-9d}.$$

Putting the last two estimates together we obtain (2.30).

Finally, we must verify (2.31), for every j, ℓ. Fix $k_0 \in E_{j,\ell}$, and use that $|\mathcal{P}^{[j]}_k| \approx |\mathcal{P}^{[j]}_{k_0}| \approx 2^\ell$, for all $k \in E_{j,\ell}$, which implies

$$\begin{aligned}2^{jp}\, N^{\frac{d+1}{2}}\, |\mathcal{P}^{[j,\ell]}| \;&\leq\; 2^{jp}\, N^{\frac{d+1}{2}} |E_{j,\ell}|\, 2^\ell \,\lesssim\, |E_{j,\ell}| \sum_{\pi\in\mathcal{P}^{[j]}_{k_0}} 2^{jp}\, |\pi| \\ &\lesssim\; |E_{j,\ell}| \sum_{\pi\in\mathcal{P}^{[j]}_{k_0}} \|f_{k_0}\psi_\pi\|_\infty^p\, |\pi| \\ &\lesssim\; |E_{j,\ell}| \sum_{\pi\in\mathcal{P}^{[j]}_{k_0}} \|f_{k_0}\psi_\pi\|_p^p \,\lesssim\, |E_{j,\ell}|\, \|f_{k_0}\|_p^p\,,\end{aligned}$$

where in the last two inequalities we have used (2.16) and $\sum_{n\in\mathbb{Z}^{d+1}} \psi(\cdot+n)^\rho \lesssim 1$. Thus, we have

$$\begin{aligned}\|f\|_{p,s} \;&\geq\; \Big(\sum_{k_0\in E_{j,\ell}} \|f_{k_0}\|_p^s\Big)^{\frac1s} \\ &\gtrsim\; \Big(\sum_{k_0\in E_{j,\ell}} \Big[\frac{2^{jp}\, N^{\frac{d+1}{2}}\, |\mathcal{P}^{[j,\ell]}|}{|E_{j,\ell}|}\Big]^{s/p}\Big)^{\frac1s} \;=\; 2^j\, (N^{\frac{d+1}{2}}\, |\mathcal{P}^{[j,\ell]}|)^{\frac1p}\, |E_{j,\ell}|^{\frac1s-\frac1p},\end{aligned}$$

as we wished to prove. □

3. Equivalent formulations of the problem

Definition 3.1. Given $p \geq 2$, $s \in [1,p]$ and $\gamma > 0$, we say that *hypothesis* $\mathcal{H}^{str}(p,s,\gamma)$ *holds* if there exists $C_\gamma > 0$ so that for any $\delta = N^{-1} \leq \delta_0$ and any $f = \sum_k f_k$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$

$$\|f\|_p \leq C_\gamma N^{\beta(p,s)+\gamma} \Big(\sum_k \|f_k\|_p^s \Big)^{1/s}, \tag{3.1}$$

where $\beta(p,s) = \frac{d-1}{2s'} - \frac{d+1}{2p}$.

It is our objective to prove $\mathcal{H}^{str}(p,2,\gamma)$ for all $\gamma > 0$, in the asserted range of p's in (1.8) (and likewise for $\mathcal{H}^{str}(p,s,\gamma)$ when $d = 2$ in the range in (**??**)). We formulate a slightly weaker condition which can be seen as an analogue of a restricted weak type inequality.

Definition 3.2. Given $p \geq 2$, $s \in [1,p]$ and $\gamma > 0$, we say that *hypothesis* $\mathcal{H}(p,s,\gamma)$ *holds* if there exists $C_\gamma > 0$ so that for all $\delta = N^{-1} \leq \delta_0$, for all N-cubes Q, all $E \subset \Omega$ and all stable (N,E,Q)-packets f with plate family $\mathcal{P}(f)$ the following estimate holds

$$\big|\{x \in Q : |f(x)| > \lambda\}\big| \;\leq\; C_\gamma \, \lambda^{-p} \, N^{(\beta(p,s)+\gamma)p} \, N^{\frac{d+1}{2}} \, |\mathcal{P}(f)| \, |E|^{\frac{p}{s}-1}, \tag{3.2}$$

for all positive real number $\lambda > 0$.

The main result in this section is the following.

Proposition 3.3. *Let* $p \geq 2$, $s \in [1,p]$ *and* $0 < \gamma < \gamma_1$. *Then*

$$\mathcal{H}^{str}(p,s,\gamma) \implies \mathcal{H}(p,s,\gamma) \implies \mathcal{H}^{str}(p,s,\gamma_1). \tag{3.3}$$

The first implication follows by Čebyšev's inequality and the estimate (2.27) for the (p,s)-norm of stable plates. The second implication is less trivial and will be proved below. Observe that one always has the trivial bound $\mathcal{H}^{str}(p,s,\gamma = \frac{d+1}{2p})$, since $\|\sum_k f_k\|_p \leq \sum_k \|f_k\|_p \lesssim N^{\frac{d-1}{2s'}} \|f\|_{p,s} = N^{\beta(p,s)+\frac{d+1}{2p}} \|f\|_{p,s}$. Thus, assuming Proposition 3.3, Theorem 1.1 is reduced to prove the following

Theorem 3.4. *Let* p *and* s *be as in* (1.8) *and* (**??**). *Then, there exists* $\epsilon_0' = \epsilon_0'(p,s)$ *so that if hypothesis* $\mathcal{H}^{str}(p,s,\gamma_0)$ *holds for some* $\gamma_0 > 0$, *then hypothesis* $\mathcal{H}(p,s,\gamma)$ *holds for all* $\gamma > (1-\epsilon_0')\gamma_0$.

Indeed, if Theorem 3.4 holds, then Proposition 3.3 together with an iteration gives the validity of the strong type estimate $\mathcal{H}^{str}(p,s,\epsilon)$ for all $\epsilon > 0$, thus establishing Theorem 1.1.

In the proof of Proposition 3.3 we shall also use the following localization lemma.

Lemma 3.5. *Let $1 \le s \le p < \infty$ and $\alpha > 0$. Assume that for all N-cubes Q and all $f = \sum_k f_k$ with $\widehat{f_k} \subset \Pi_\omega^N$ we have*

$$\|f\|_{L^p(Q)} \le C\, N^\alpha\, \|f\|_{p,s;\delta}. \tag{3.4}$$

Then,

$$\|f\|_{L^p(\mathbb{R}^{d+1})} \lesssim C\, N^\alpha\, \|f\|_{p,s;\delta}. \tag{3.5}$$

Proof. Write $f = \sum_{Q\in\mathcal{Q}} \psi_Q^2 f$, where $\mathcal{Q}$ is a tiling of $\mathbb{R}^{d+1}$ by N-cubes and ψ_Q is as in (2.4). Then, using the Schwartz decay of ψ,

$$\begin{aligned}\|f\|^p_{L^p(\mathbb{R}^{d+1})} &= \sum_{Q'} \Big\|\sum_{Q} \psi_Q^2 f\Big\|^p_{L^p(Q')} \lesssim \sum_{Q,Q'} \big\|\psi_Q^{3/2} f\big\|^p_{L^p(Q')} \\ &\lesssim \sum_{Q,Q'} \big\|\psi_Q f\big\|^p_{L^p(Q')} \big(1 + \operatorname{dist}(Q,Q')/N\big)^{-10d}.\end{aligned}$$

Since each $\psi_Q f_k$ has spectrum contained in $\Pi_k^{(2\delta)}$, we can apply (3.4) with f replaced by $\psi_Q f$ (and δ by 2δ) to obtain

$$\|f\|^p_{L^p(\mathbb{R}^{d+1})} \lesssim C^p\, N^{\alpha p} \sum_{Q,Q'} \big\|\psi_Q f\big\|^p_{p,s;2\delta} \big(1 + \operatorname{dist}(Q,Q')/N\big)^{-10d},$$

which by Lemma 2.9 is controlled by $\|f\|^p_{p,s;\delta}$. □

Proof of Proposition 3.3. We show the proof of the main implication

$$\mathcal{H}(p,s,\gamma) \implies \mathcal{H}^{str}(p,s,\gamma_1), \text{ for } \gamma_1 > \gamma. \tag{3.6}$$

By the previous lemma it suffices to show

$$\|f\|_{L^p(Q)} \le C_\varepsilon\, N^{(\beta(p,s)+\gamma+\varepsilon)} \|f\|_{p,s;\delta} \tag{3.7}$$

for all $\varepsilon > 0$, all N-cubes Q and all $f = \sum_k f_k$ with $\widehat{f_k} \subset \Pi_\omega^N$. To do so we may assume

$$\|f\|_{p,s;\delta} = 1. \tag{3.8}$$

Fix an N-cube Q. Then

$$\|f\|^p_{L^p(Q)} \lesssim \sum_{m\in\mathbb{Z}} 2^{mp} \operatorname{meas}\big(\{x \in Q : |f| > 2^m\}\big). \tag{3.9}$$

By Lemma 2.7 we have $\|f\|_\infty \lesssim N^{\beta(p,s)}$, and thus we may assume $m \lesssim \log N$ in (3.9). Also, if $2^m \le N^{-(d+1)}$, then the right hand side of (3.9) is controlled by

$$\sum_{2^m \le N^{-(d+1)}} 2^{mp}\,|Q| \lesssim N^{-(d+1)(p-1)} \le 1.$$

Thus, only a logarithmic number of m's are relevant in (3.9), so by a pigeonhole argument we can find m_* so that

$$\|f\|^p_{L^p(Q)} \lesssim (\log N)\, 2^{m_* p} \operatorname{meas}\big(\{x \in Q : |f| > 2^{m_*}\}\big) + 1. \tag{3.10}$$

Using $\sup_k \|f_k\|_\infty \lesssim \sup_k N^{-\frac{d+1}{2p}}\|f_k\|_p \le N^{-\frac{d+1}{2p}}$, we can apply to f the packet decomposition in Lemma 2.12, with $A = N^{-\frac{d+1}{2p}}$ and the N-cube Q fixed above. By (2.30), the function g in (2.29) is then $\lesssim N^{-8d}$ which in turn is $\ll 2^{m_*}$. By the pidgeonhole principle applied to the $O((\log N)^2)$ terms in the sum in (2.29), there are integers j_* and ℓ_*, so that the set of directions $E^* = E_{j_*,\ell_*}$ and the stable (N, E^*, Q)-packet $f^* = f^{[j_*,\ell_*]}$ satisfy

$$\operatorname{meas}\{x \in Q : |f| > 2^{m_*}\} \lesssim (\log N)^2 \operatorname{meas}\Big(\Big\{x \in Q \ : \ 2^{j^*}|f_*| > \tfrac{2^{m_*}}{C(\log N)^2}\Big\}\Big).$$

By Hypothesis $\mathcal{H}(p, s, \gamma)$ the right hand side of (3.10) is then estimated by

$$C_\gamma\, (\log N)^{3+2p}\, N^{(\beta(p,s)+\gamma)p}\, 2^{j_* p}\, N^{\frac{d+1}{2}}|\mathcal{P}(f_*)|\,|E_*|^{\frac{p}{s}-1} \lessapprox C_\gamma\, N^{(\beta(p,s)+\gamma)p},$$

where the last inequality follows from the crucial estimate (2.31) and the assumption $\|f\|_{p,s;\delta} = 1$. Since the powers of $\log N$ are controlled by $C_\varepsilon N^\varepsilon$, for any $\varepsilon > 0$, this finishes the proof of (3.7) and thus the proposition. □

There are some situations in which the inequality in (3.2) is trivial to verify, namely when either $|E|$ or λ are sufficiently small.

Lemma 3.6. *Let $p > 2$ and $1 \le s \le p$. Then the inequality (3.2) is true for every $\gamma > 0$ and every (N, E, Q)-packet when either*

$$\lambda \le N^{\frac{\beta(p,s)\,p}{p-2}}\,|E|^{(\frac{p}{s}-1)/(p-2)}, \tag{3.11}$$

or when

$$|E|^{\frac{p}{s'}-1} \le N^{\beta(p,s)\,p}. \tag{3.12}$$

Proof. By Čebyšev's inequality and Lemma 2.11

$$\operatorname{meas}\big(\{x : |f(x)| > \lambda\}\big) \le \lambda^{-2}\|f\|_2^2 \lesssim \lambda^{-2}\, N^{(d+1)/2}\,|\mathcal{P}(f)|\,,$$

and therefore

$$\text{meas}\,\big(\{x:|f(x)|>\lambda\}\big)\le\;\lambda^{-p}\,N^{\beta(p,s)\,p}\,N^{\frac{d+1}{2}}\,|\mathcal{P}(f)||E|^{\frac{p}{s}-1}, \tag{3.13}$$

if $\lambda^{-2}\;\le\;\lambda^{-p}\,N^{\beta(p,s)\,p}|E|^{\frac{p}{s}-1}$, which is easily seen to the same as (3.11). On the other hand, for packets f we have $\|f\|_\infty\;\lesssim\;|E|$ (by (2.25)), so that (3.2) only needs to be verified when $\lambda\lesssim|E|$. But in this range (3.11) always holds if $|E|\lesssim N^{\beta(p,s)p/(p-2)}|E|^{(\frac{p}{s}-1)/(p-2)}$, which is the same as (3.12). □

The previous lemma can be slightly improved using the following known (although probably non optimal) square function estimate: *for all* $f=\sum f_k$ *with supp* $\hat f_k\subset\Pi^N_\omega$ *and for all* $\varepsilon>0$ *it holds*

$$\|\sum_k f_k\|_q\;\le\;C_\varepsilon\,N^{\frac{d-1}{4(d+3)}+\varepsilon}\,\big\|\big(\sum_k|f_k|^2\big)^{\frac12}\big\|_q,\qquad\text{where}\quad q=\tfrac{2(d+3)}{d+1}. \tag{3.14}$$

This inequality follows from the bilinear methods of Tao and Vargas [20], combined with Wolff's bilinear restriction theorem for the cone [23]. See e.g. [7, Prop. 2.3] for a detailed proof.

Lemma 3.7. *Let* $q=2(d+3)/(d+1)$, $p>q$ *and* $1\le s\le p$. *Then the inequality* (3.2) *is true for every* $\gamma>0$ *and every* (N,E,Q)*-packet when either*

$$\lambda\;\lessapprox\;N^{(\beta(p,s)p-\frac{d-1}{2(d+1)})/(p-q)}\,|E|^{(\frac{p}{s}-\frac{q}{2})/(p-q)}, \tag{3.15}$$

or when

$$|E|^{\frac{p}{s'}-\frac{q}{2}}\;\lessapprox\;N^{\beta(p,s)\,p-\frac{d-1}{2(d+1)}}. \tag{3.16}$$

Proof. Using Chebichev's inequality and (3.14) we see that

$$\text{meas}\,\big(\{x:|f(x)|>\lambda\}\big)\le\lambda^{-q}\|f\|_q^q\le\,C_\varepsilon\,\lambda^{-q}\,N^{(\frac{d-1}{4(d+3)}+\varepsilon)q}\,\big\|\big(\sum_k|f_k|^2\big)^{\frac12}\big\|_q^q.$$

Since $q>2$, Minkowski's inequality gives $\big\|\big(\sum_k|f_k|^2\big)^{\frac12}\big\|_q\le\|f\|_{q,2;\delta}$, while for (N,E,Q)-packets we have $\|f\|^q_{q,2;\delta}\lesssim N^{\frac{d+1}{2}}\,|\mathcal{P}(f)|\,|E|^{\frac{q}{2}-1}$, by Lemma 2.11. Thus, choosing $\varepsilon<\gamma$, (3.2) will hold for all λ so that

$$\lambda^{-q}\,N^{\frac{d-1}{4(d+3)}q}\,|E|^{\frac{q}{2}}\;\lessapprox\;\lambda^{-p}\,N^{\beta(p,s)\,p}\,|E|^{\frac{p}{s}},$$

or equivalently when (3.15) holds. On the other hand, since we only consider $\lambda\le\|f\|_\infty\lesssim|E|$, we see that (3.15) is always true when $|E|\lessapprox\big(N^{\beta(p,s)p-\frac{d-1}{2(d+1)}}\,|E|^{\frac{p}{s}-\frac{q}{2}}\big)^{1/(p-q)}$, which after easy arithmetics gives the condition in (3.16). □

Remark 3.8. Thus, in the proof of Theorem 3.4 below we only need to consider the validity of (3.2) for (N, E, Q)-packets f whose associated direction sets E have cardinality

$$N^{(\beta(p,s)\,p-\frac{d-1}{2(d+1)})/(\frac{p}{s'}-\frac{q}{2})}\,(\log N)^C \leq |E| \lesssim N^{\frac{d-1}{2}}, \tag{3.17}$$

and for real numbers λ in the range

$$N^{(\beta(p,s)p-\frac{d-1}{2(d+1)})/(p-q)}\,|E|^{(\frac{p}{s}-\frac{q}{2})/(p-q)}\,(\log N)^C \leq \lambda \lesssim |E|, \tag{3.18}$$

where C can be a suitably large constant.

4. Sufficient conditions for Theorem 3.4

The purpose of this section is identify properties of packets so that the improvement in Theorem 3.4 holds. As in [22, 12] these can be phrased via localization of the level sets $\{|f| > \lambda\}$ using grids of slightly smaller cubes. Also, such localization assumptions will hold when the cardinality of the involved plate families is suitably controlled in terms of λ.

4.1. **Localization.** We begin with an easy (but crucial) localization estimate.

Lemma 4.1. *Let $\widehat{f}$ be supported in $\Gamma_\delta(c)$, let R be a cube of diameter tN, where $t \leq 1$. Then*

$$\|\psi_R f\|_2 \lesssim t^{1/2}\|f\|_2 \tag{4.1}$$

Proof. By Plancherel this is equivalent with a statement about the integral operator $TF(\xi) = \int K_\delta(\xi,\eta)F(\eta)d\eta$ with kernel

$$K_\delta(\xi,\eta) = \widehat{\psi_R}(\xi-\eta)\chi_{\Gamma_\delta(c)}(\eta).$$

The L^2 operator norm is $\leq \sqrt{A_1 A_2}$ where

$$A_1 = \sup_\xi \int |K_\delta(\xi,\eta)|d\eta$$

$$A_2 = \sup_\eta \int |K_\delta(\xi,\eta)|d\xi.$$

Now clearly $A_2 = O(1)$ while the smaller η-support yields $A_1 = O(t)$. This implies the assertion. $\square$

We shall also use the following result.

Lemma 4.2. *Let* $f = \sum_k f_k$ *with supp* $\hat f_k \subset \Pi_\omega^N$, $t \in [\sqrt{\delta}, 1]$ *and* R *a* tN*-cube. Then the function* $f\psi_R$ *has Fourier transform supported in* $\Gamma_{\delta/t}(C)$ *and*[5]

$$\|f\psi_R\|_{\infty,r;\delta/t} \lesssim t^{-\frac{d-1}{2r'}} \|f\|_{\infty,r;\delta}, \quad \forall\, r \geq 1. \tag{4.2}$$

Proof. Since $\widehat{\psi_R}$ is supported in $B_{\delta/(100t)}(0)$, it follows immediately that $\widehat{f\psi_R} = \hat f * \widehat{\psi_R}$ is supported in $\Gamma_{\delta/t}(C)$, for a sufficiently large constant $C > 0$. Next, denote by $P_{k'}^{(\delta/t)}$ the projections adapted to the plates $\Pi_{k'}^{(\delta/t)}$ as in (**??**), and for each k' let $\Omega_{k'} = \{k \ : \ \big(\Pi_k^{(\delta)} + B_{\delta/(100t)}(0)\big) \cap \Pi_{k'}^{(2\delta/t)} \neq \emptyset\}$. Observe that $t \in [\sqrt{\delta}, 1]$ implies $\#\Omega_{k'} \lesssim t^{-(d-1)/2}$, $\forall\, k'$, and $\#\{k' \ : \ k \in \Omega_{k'}\} \lesssim 1$, $\forall\, k$. Also we have

$$\big\|P_{k'}^{(\delta/t)}(f\psi_R)\big\|_\infty = \big\|P_{k'}^{(\delta/t)}\big(\sum_{k\in\Omega_{k'}} f_k\psi_R\big)\big\|_\infty \lesssim \sum_{k\in\Omega_{k'}} \|f_k\|_\infty.$$

Then, (4.2) follows from the above observations and Hölder's inequality. □

We now state a definition of λ-localization using tN-cubes. Below, $\mathcal{Q}(t) = \{B\}$ denotes a fixed partition of $\mathbb{R}^{d+1}$ by tN-cubes[6].

Definition 4.3. *Let* f *be an* (N, E, Q)*-packet, let* $\lambda > 0$ *and as before* $t = \delta^{\epsilon_0} = N^{-\epsilon_0}$*. We say that* f ***localizes at height*** λ *if there are subpackets* f^B *of* f*, where* B *runs over* tN*-cubes in a grid* $\mathcal{Q}(t)$*, such that*

$$\sum_B \#\mathcal{P}(f^B) \lessapprox \#\mathcal{P}(f) \tag{4.3}$$

and

$$\text{meas}\,\big(\{x : |f(x)| > \lambda\}\big) \lessapprox \sum_B \text{meas}\,\big(B \cap \{x : |f^B| \gtrapprox \lambda\}\big). \tag{4.4}$$

The next lemma gives, under the localization assumption, the crucial gain in the exponent γ asserted in Theorem 3.4. The statement is just a straightforward modification of [12, Lemma 6.2], but we sketch the proof below for completeness.

Lemma 4.4. *Let* $p \geq 2$, $s \in [1, p]$ *and suppose that* $\mathcal{H}^{str}(p, s, \gamma_0)$ *holds for a fixed* $\gamma_0 > 0$*. Let* $\lambda > 0$ *and suppose that* f *is an* (N, E, Q)*-packet which localizes at height*

[5] One could prove here a more general inequality $\|f\psi_R\|_{p,s;\delta/t} \lesssim t^{-\beta(p,s)}\|f\|_{p,s;\delta}$, for $2 \leq p \leq \infty$ and $s \in [p', \infty]$. However, this is not used later (except for a weaker version at the beginning of Lemma 4.4).

[6] Below B will always denote a tN-cube, while we keep the notation Q for N-cubes, and Δ for $\sqrt{N}$-cubes.

λ (with respect to tN-cubes). Then, the estimate (3.2), i.e.

$$\left|\{x \in Q : |f(x)| > \lambda\}\right| \le C_\gamma \, \lambda^{-p} \, N^{(\beta(p,s)+\gamma)p} \, N^{(d+1)/2} |\mathcal{P}(f)| \, |E|^{\frac{p}{s}-1}$$

holds for such f, Q and λ, and for all $\gamma > \gamma_0(1 - \epsilon_0/2)$.

Proof. For each tN-cube $B \in \mathcal{Q}(t)$, the function $f^B \psi_B$ has Fourier transform supported in $\Gamma_{\delta/t}(C)$. We claim that

$$\left\| f^B \psi_B \right\|_{p,s;\delta/t} \lesssim t^{-\beta(p,s)} \, \|f^B\|_2^{2/p} \, |E|^{\frac{1}{s}-\frac{1}{p}} . \tag{4.5}$$

Indeed, using the convexity inequality in (2.18) (with $r = r(s,p)$ as in (2.19)), followed by Lemmas 4.1 and 4.2, we have

$$\begin{aligned} \left\| f^B \psi_B \right\|_{p,s;\delta/t}^p &\lesssim \left\| f^B \psi_B \right\|_2^2 \left\| f^B \psi_B \right\|_{\infty,r;\delta/t}^{p-2} \\ &\lesssim t \left\| f^B \right\|_2^2 t^{-\frac{d-1}{2r'}(p-2)} \left\| f^B \right\|_{\infty,r;\delta}^{p-2} . \end{aligned} \tag{4.6}$$

Now, $\|f^B\|_{\infty,r} \lesssim |E|^{1/r}$, while by the definition of $r = r(s,p)$ in (2.19) we can write $\frac{p-2}{r'} = \frac{p}{s'} - 1$ and $\frac{p-2}{r} = \frac{p}{s} - 1$. Inserting these estimates in the right hand side of (4.6), the claimed inequality (4.5) follows easily.

Thus, using the localization condition and the hypothesis $\mathcal{H}^{str}(p, s, \gamma_0)$ (with δ replaced by δ/t) we obtain

$$\begin{aligned} \left|\{|f| > \lambda\}\right| &\lessapprox \sum_B \left| \left\{ |f^B \psi_B| \gtrapprox \lambda \right\} \right| \qquad \text{(by (4.4))} \\ &\lessapprox \lambda^{-p} \sum_B (tN)^{(\beta(p,s)+\gamma_0)p} \, \|f^B \psi_B\|_{p,s;\delta/t}^p \\ &\lesssim \lambda^{-p} \sum_B N^{(\beta(p,s)+\gamma_0)p} \, t^{\gamma_0 p} \, \|f^B\|_2^2 \, |E|^{\frac{p}{s}-1}, \end{aligned}$$

where in the last step we have used (4.5). Since by (4.3)

$$\sum_B \|f^B\|_2^2 \lesssim N^{\frac{d+1}{2}} \sum_B \#\mathcal{P}(f^B) \lessapprox N^{\frac{d+1}{2}} \#\mathcal{P}(f),$$

the lemma follows. □

4.2. **Sufficient conditions for λ-localization.** It is now important to identify situations in which the localization conditions of Definition 4.3 apply and thus the improvement of Lemma 4.4 holds. In [22, 12] a number of sufficient conditions are given, when the cardinality of $\mathcal{P}(f)$ is controlled by a power of λ. The simplest one is the following.

Proposition 4.5. [12, Lemma 5.2]. *Let f be an (N,E,Q)-packet and $\lambda>0$ such that*

$$\#\mathcal{P}(f)\,\le\,t^{14d}\,\lambda^2\,. \tag{4.7}$$

Then f localizes at height λ with tN-cubes. In particular, if $p\ge 2$, $s\in[1,p]$ and we assume $\mathcal{H}^{str}(p,s,\gamma_0)$ for some $\gamma_0>0$, then the inequality (3.2), i.e.

$$\Big|\{x\in Q:|f(x)|>\lambda\}\Big|\,\le\,C\,\lambda^{-p}\,N^{(\beta(p,s)+\gamma)p}\,N^{\frac{d+1}{2}}\,|\mathcal{P}(f)|\,|E|^{\frac{p}{s}-1}$$

holds for such f, λ and Q, and for all $\gamma>\,(1-\epsilon_0/2)\,\gamma_0$.

We refer to [12] for details about the proof, which involves only simple combinatorial arguments.

`I have included the proof in small print, since it is already written, but we do not need to include it in the last version.`

The main geometrical argument behind Proposition 4.5 is in the following result which (in a slightly more complicated version) will be applied to $W=\{|f|>\lambda\}$. For a proof we refer to [12, Lemma 4.2]. Below, $\mathcal{Q}(t)=\{B\}$ denotes a grid of tN-cubes, and for $x\in\mathbb{R}^{d+1}$ we define $B(x)$ as the cube B in the grid containing x (which is well defined apart from a null set).

Lemma 4.6. *Let W be a measurable subset of $\mathbb{R}^{d+1}$ and let $\mathcal{P}$ be a plate family, whose elements are contained in a fixed cube of diameter $CN^{1+\epsilon_0}$. As before, let $t=\delta^{\epsilon_0}=N^{-\epsilon_0}$. Consider the following relation "$\sim$" between plates $\pi\in\mathcal{P}$ and3 cubes $B\in\mathcal{Q}(t)$: we say that $\pi\sim B$ if B intersects the 9-fold dilate of B_π, where B_π is a tN-cube in $\mathcal{Q}(t)$ for which the quantity $|W\cap\pi\cap B_\pi|$ is maximal. . Then*

$$\#\{B:\pi\sim B\}\le 10^d,\quad \textit{for every } \pi\in\mathcal{P} \tag{4.8}$$

and

$$\mathcal{I}\,:=\,\int_W\sum_{\pi\in\mathcal{P},\pi\not\sim B(x)}\chi_\pi(x)dx\,\lessapprox\,t^{-5d}|W|\sqrt{\#\mathcal{P}}. \tag{4.9}$$

Proof. The condition that all plates in $\mathcal{P}$ are contained in a fixed $CN^{1+\epsilon_0}$-cube, and the separation property of the plates implies $\#\mathcal{P}=O(t^{-d-1}N^d)$. Note that (4.8) is trivial from the definition of the relation. To prove (4.9) we first note that

$$\mathcal{I}=\sum_\pi\nu(\pi)$$

where

$$\nu(\pi)=\Big|\{x\in W\cap\pi:B(x)\not\sim\pi\}\Big|.$$

There is the trivial estimate

$$\int_W \sum_{\substack{\pi\in\mathcal{P}:\pi\not\sim B(x)\\ \nu(\pi)\le |W|N^{-d}}} \chi_\pi(x)dx \;\lesssim\; \#\mathcal{P}|W|N^{-d} \lesssim t^{-d-1}\,|W|.$$

Thus we only need to bound

$$\widetilde{\mathcal{I}} = \sum_{\substack{\pi\in\mathcal{P}:\\ N^{-d}|W|\le\nu(\pi)\le|W|}} \nu(\pi)$$

As there are $O(\log N)$ dyadic intervals between $N^{-d}|W|$ and $|W|$ we can use a pidgeonhole argument to get a subfamily $\mathcal{P}'\subset\mathcal{P}$ and a value of ν between $N^{-d}|W|$ and $|W|$ so that

$$|\widetilde{\mathcal{I}}| \;\lessapprox\; \nu\,\mathrm{card}(\mathcal{P}') \tag{4.10}$$

and

$$\nu\le\nu(\pi)\le 2\nu \quad\text{ for each } \pi\in\mathcal{P}'.$$

Since every plate can be covered with $O(t^{-1})$ cubes, for each $\pi\in\mathcal{P}'$ there must be a cube $B'(\pi)$ *not related to* π so that

$$|W\cap B'(\pi)\cap\pi|\gtrsim t\nu.$$

By the maximality condition in the definition of B_π we must then also have

$$|W\cap B_\pi\cap\pi|\gtrsim t\nu \text{ for each } \pi\in\mathcal{P}'.$$

Clearly the number of all possible pairs of tN cubes is $O(t^{-4(d+1)})$. This means that we can find two tN-cubes B,B' in $\mathcal{Q}(t)$ and a subfamily $\mathcal{P}''$ of $\mathcal{P}'$ which has cardinality $\gtrsim t^{4(d+1)}\#\mathcal{P}'$ so that for all $\pi\in\mathcal{P}''$ we have $B_\pi=B$ and $B'(\pi)=B'$.

We now fix these two tN-cubes B and B' and consider the auxiliary expression

$$\mathcal{A}=\sum_{\pi\in\mathcal{P}''}|W\cap B\cap\pi||W\cap B'\cap\pi|.$$

Then we have the lower bound

$$\mathcal{A}\gtrsim(t\nu)^2\mathrm{card}(\mathcal{P}'')\gtrsim t^{4d+6}\mathrm{card}(\mathcal{P}')\nu^2.$$

We can also derive an upper bound by rewriting

$$\mathcal{A}=\int_{W\cap B}\int_{W\cap B'}\sum_{\pi\in\mathcal{P}''}\chi_\pi(x)\chi_\pi(x')dxdx'$$

If $\pi\cap B\neq\emptyset$ and $\pi\cap B'\neq\emptyset$ for some $\pi\in\mathcal{P}''$ then π is related to B but not to B', thus the distance of B to B' is at least tN. This means that for each pair of points $(x,x')\in B\times B'$ there are $\lesssim t^{-d+1}$ separated plates which go to both x and x'. This means that the integrand $\sum_{\pi\in\mathcal{P}''}\chi_\pi(x)\chi_\pi(x)$ is $O(t^{-d+1})$ and hence we get the upper bound

$$\mathcal{A}\lesssim t^{-d+1}|W\cap B||W\cap B'|\lesssim t^{-d+1}|W|^2.$$

Comparing the upper and the lower bounds for $\mathcal{A}$ we find that

$$\nu\le t^{-d-1}(\#\mathcal{P}')^{-1/2}\sqrt{\mathcal{A}}\le t^{-5(d+1)/2}|W|(\#\mathcal{P}')^{-1/2}$$

and thus using (4.10) (i.e. $\widetilde{\mathcal{I}} \lessapprox \nu\,\mathrm{card}(\mathcal{P}'))$ and we obtain

$$\widetilde{\mathcal{I}} \lessapprox t^{-5(d+1)/2}|W|\sqrt{\#\mathcal{P}'}.$$

□

For technical reasons Lemma 4.6 is not quite enough for us since we wish to replace the characteristic functions χ_π by the similar weights w_π with "Schwartz-tails". This is fairly straightforward and requires adjustments in the definition of the relation $\sim$ between plates and tN-cubes and some additional pidgeonholing. We state the required estimate and refer to Lemma 4.3 in the paper by Łaba and Wolff [12] for details of the proof.

Lemma 4.7. *Let W be a measurable subset of $\mathbb{R}^{d+1}$ and let $\mathcal{P}$ be a plate family, whose elements are contained in a fixed cube of diameter $CN^{1+\epsilon_0}$. Let M_0 be a large constant, and assume the constant M in the definition of w (see (2.2)) is so large that $M \geq 10M_0 d$. Let $t = N^{-\varepsilon_0}$ and $\mathcal{Q}(t) = \{B\}$ be a grid of tN cubes as before. Then, there is a relation "$\sim$" between plates in $\mathcal{P}$ and tN-cubes in $\mathcal{Q}(t)$ so that*

$$\#\{B : \pi \sim B\} \lessapprox 1, \quad \text{for every } \pi \in \mathcal{P} \tag{4.11}$$

and if

$$\mathfrak{W}_{\mathcal{P}}(x) = \sum_{\substack{\pi\in\mathcal{P}\\ \pi\not\sim B(x)}} w_\pi(x)$$

then

$$\int_W \mathfrak{W}_{\mathcal{P}}(x)\,dx \lessapprox t^{-5d}\,|W|\,\sqrt{\#\mathcal{P}} + N^{-M_0}|W|.$$

Proof of Proposition 4.5. We wish to apply Lemma 4.4 and therefore have to show that with $\mathcal{P} \equiv \mathcal{P}(f)$ under the assumption $\#\mathcal{P} \leq t^{14d}\lambda^2$ the localization condition in Definition 4.3 holds.

We proceed applying Lemma 4.7 to $W = \{x : |f| \geq \lambda\}$ and $\mathcal{P}$, and let $\sim$ be the relation between N-plates and tN-cubes from Lemma 4.7. Recall that

$$f(x) = \sum_{\pi\in\mathcal{P}} f_\pi$$

with $|f_\pi| \lesssim w_\pi$. For every tN-cube $B \in \mathcal{Q}(t)$ define

$$f^B(x) = \sum_{\pi\sim B} f_\pi.$$

By condition (4.11) we have $\sum_B |\mathcal{P}(f^B)| \lessapprox |\mathcal{P}(f)|$, i.e. (4.3). Moreover with $\mathcal{P} \equiv \mathcal{P}(f)$

$$\int_W \mathfrak{W}_{\mathcal{P}}(x)dx \lessapprox t^{-5d}|W|\sqrt{\#\mathcal{P}} \lesssim t^{-5d}|W|\sqrt{t^{14d}\,\lambda^2} \lesssim t^{2d}\,|W|\,\lambda\,.$$

This means that there is a subset W^* of W so that $|W^*| \geq |W|/2$ on which we have the pointwise bound

$$\mathfrak{W}_{\mathcal{P}}(x) \lesssim t\lambda, \quad x \in W^*$$

Also if $x \in W^* \cap B$ we have

$$|f(x) - f^B(x)| = \Big| \sum_{\pi : \pi \not\sim B} f_\pi(x) \Big| \lesssim \mathfrak{W}_{\mathcal{P}}(x) \lesssim t\lambda$$

and hence

$$|f^B(x)| \geq \lambda/2, \qquad x \in W^* \cup B.$$

This implies the localization condition (4.4). □

A second sufficient condition, which also appears in [12] can be described as follows. Following [12, §4], to every plate family $\mathcal{P}$ we can associate a (separated) N-tube family $\mathcal{T} = \mathcal{T}(\mathcal{P})$ of minimal cardinality so that each $\pi \in \mathcal{P}$ is contained in a 10-fold dilate of some $\tau \in \mathcal{T}$. For each $\tau \in \mathcal{T}(\mathcal{P})$ we call

$$\mathcal{P}(\tau) = \{\pi \in \mathcal{P} \;\;:\;\; \pi \subset 10\tau\}$$

and for every positive integer $\mu \in \mathbb{N}$ we define a subfamily of $\mathcal{P}$ by

$$\mathcal{P}^{(\mu)} = \bigcup_{\tau \in \mathcal{T}(\mathcal{P})} \big\{\mathcal{P}(\tau) \;\;:\;\; 2^{\mu-1} \leq \#\mathcal{P}(\tau) < 2^\mu\big\}, \tag{4.12}$$

and a corresponding subfamily of tubes

$$\mathcal{T}(\mathcal{P}^{(\mu)}) = \big\{\tau \in \mathcal{T}(\mathcal{P}) \;\;:\;\; 2^{\mu-1} \leq \#\mathcal{P}(\tau) < 2^\mu\big\}. \tag{4.13}$$

Observe that the families $\mathcal{P}^{(\mu)}$ are nonempty only for $\mu \lesssim \log N$, since we always have

$$\#\{\pi \;\;:\;\; \pi \subset 10\tau\} \lesssim \sqrt{N}, \quad \forall\, \tau.$$

It is also clear that

$$\#\mathcal{T}(\mathcal{P}^{(\mu)}) \lesssim \frac{\#\mathcal{P}}{2^\mu}. \tag{4.14}$$

Definition 4.8. Given an (N, E, Q)-packet $f = \sum_{\pi \in \mathcal{P}} f_\pi$ and a real number $\lambda > 0$, we define $\mu_* = \mu_*(f, \lambda)$ as a positive integer at random among those for which the subpacket $f^* = \sum_{\pi \in \mathcal{P}^*} f_\pi$ with plate family $\mathcal{P}^* = [\mathcal{P}(f)]^{(\mu_*)}$ (defined as in (4.12)), satisfies

$$\big|\{|f| > \lambda\}\big| \;\leq\; C_0 \log N \,\Big|\Big\{|f^*| > \frac{\lambda}{C_0 \log N}\Big\}\Big|, \tag{4.15}$$

for a fixed constant $C_0 > 0$. Observe that by an elementary pigeonhole argument at least one such μ_* exists provided C_0 is chosen large enough.

The second sufficient condition for λ-localization can now be written as follows (see [12, Lemma 5.3]).

Proposition 4.9. *Let f be an (N, E, Q)-packet and $\lambda > 0$, and assume that for $\mu_* = \mu_*(f, \lambda)$ defined as above we have*

$$\frac{|\mathcal{P}(f)|}{2^{\mu_*}} \leq t^{14d} \lambda^2 . \tag{4.16}$$

Then, f localizes at height λ with tN-cubes. In particular, if $p \geq 2$, $s \in [1, p]$ and we assume $\mathcal{H}^{str}(p, s, \gamma_0)$ for some $\gamma_0 > 0$, then the inequality (3.2), i.e.

$$\left|\{x \in Q : |f(x)| > \lambda\}\right| \leq C \lambda^{-p} N^{(\beta(p,s)+\gamma)p} N^{\frac{d+1}{2}} |\mathcal{P}(f)| \, |E|^{\frac{p}{s}-1}$$

holds for such f, Q and λ, and for all $\gamma > \gamma_0(1 - \epsilon_0/2)$.

Moreover, if (4.16) does not hold, then for every $\sqrt{N}$-cube Δ the subpacket f^* in Definition 4.8 satisfies

$$\left\|f^* \psi_\Delta\right\|_2^2 \lesssim t^{-14d} \frac{|\mathcal{P}(f)|}{\lambda^2 \sqrt{N}} |\Delta||E|. \tag{4.17}$$

Proof. The first part of Proposition 4.9 is precisely the statement of [12, Lemma 5.3, Case 1], so we refer to this paper for a detailed proof.

We now establish the second part of the proposition, that is the inequality (4.17). Write $f^* = \sum_k f_k^*$ with supp $\widehat{f_k^*} \subset \Pi_\omega^N$. Since for each $\sqrt{N}$-cube Δ the functions in $\{f_k^* \psi_\Delta\}_k$ are essentially orthogonal, by Plancherel we have

$$\left\|f^* \psi_\Delta\right\|_2^2 \lesssim \sum_k \int |f_k^* \psi_\Delta|^2 \lesssim \int \sum_{\pi \in \mathcal{P}^*} w_\pi \, w_\Delta .$$

By Lemma 4.1 in [12], we can estimate

$$\int \sum_{\pi \in \mathcal{P}^*} w_\pi \, w_\Delta \lesssim \frac{2^{\mu_*}}{\sqrt{N}} \int \sum_{\tau \in \mathcal{T}(\mathcal{P}^*)} w_\tau \, w_\Delta \lesssim \frac{2^{\mu_*} |E| \, |\Delta|}{\sqrt{N}} .$$

Then, (4.17) follows using the upper bound for 2^{μ_*} obtained when (4.16) does not hold. ☐

4.3. **Sufficient conditions for $d = 2$.** One cannot expect the sufficient conditions (4.7) or (4.16) to hold for general packets f, since $|\mathcal{P}(f)|$ can be as large as N^d while λ^2 is at most N^{d-1}. As explained below, one can go over this difficulty localizing the problem with $\sqrt{N}$-cubes, and reconsidering the above sufficient conditions at scale $\sqrt{N}$. As noticed in [12], this idea turns out to work well for dimensions $d \geq 3$, but does not give anything when $d = 2$. Fortunately in the latter case much better sufficient conditions hold, as was proved by Wolff in [22].

Proposition 4.10. : [22, Lemma 3.1]**.** *Let f be an (N, E, Q)-packet in $\mathbb{R}^{2+1}$ and $\lambda \geq 1$ so that*

$$|\mathcal{P}(f)| \leq t^{300}\,\lambda^3\,. \tag{4.18}$$

Then f localizes at λ with tN-cubes. In particular, if $p \geq 2$, $s \in [1, p]$ and we assume $\mathcal{H}^{str}(p, s, \gamma_0)$ for some $\gamma_0 > 0$, then the inequality (3.2) *holds for such f, Q and λ, and for all $\gamma > \gamma_0(1 - \epsilon_0/2)$.*

Proposition 4.11. *Let f be an (N, E, Q)-packet in $\mathbb{R}^{2+1}$ and $\lambda \geq 1$, and assume that for $\mu_* = \mu_*(f, \lambda)$ as in Definition 4.8 we have*

$$\frac{|\mathcal{P}(f)|}{2^{\mu_*}} \leq t^{3000}\,\delta^{1/4}\,\lambda^3\,, \tag{4.19}$$

then f localizes at height λ. In particular, if $p \geq 2$, $s \in [1, p]$ and we assume $\mathcal{H}^{str}(p, s, \gamma_0)$ for some $\gamma_0 > 0$, then the inequality (3.2) *holds for all $\gamma > \gamma_0(1 - \epsilon_0/2)$.*

Moreover, if (4.19) *does not hold, then for every $\sqrt{N}$-cube Δ we have*

$$\left\| f^* \psi_\Delta \right\|_2^2 \lesssim t^{-C}\,\frac{N^{5/4}}{\lambda^3}\,|\mathcal{P}(f)||E|. \tag{4.20}$$

We refer to [22] for the deep proof of Proposition 4.10, which among other things relies on combinatorial methods of Clarkson et al [5] for counting tangencies in arrangements of circles.

Proposition 4.11 is new, and improves over Lemma 3.2 in [22], which (essentially) requires the stronger sufficient condition

$$\frac{\#\mathcal{P}(f)}{2^{r_*}} \leq t^C\,\delta^{7/4}\,\lambda^6. \tag{4.21}$$

It is straightforward to verify that (4.21) implies (4.19) since $\lambda \leq N^{\frac{1}{2}}$. We will give a complete proof of Proposition 4.11 in §6.

5. The proof of theorem 3.4

5.1. **A parabolic rescaling.** The next lemma is an analogue and consequence of Wolff's inequality for Fourier plates contained in an angular sector of length $\sqrt{\sigma} \gg \sqrt{\delta}$.

Lemma 5.1. *Let $\delta < \sigma < 1$ and consider a fixed σ-plate $\Pi^{(\sigma)}$ contained in $\Gamma_\sigma(C)$. Suppose that Hypothesis $\mathcal{H}^{str}(p,s,\gamma)$ holds for some $p, s \geq 1$ and $\gamma > 0$. Then*

(5.1)
$$\Big\| \sum_{\substack{k:\\ \Pi_k^{(\delta)} \subset \Pi^{(\sigma)}}} P_k^{(\delta)}(h_k) \Big\|_p \lesssim (\delta/\sigma)^{-\beta(p,s)-\gamma} \Big(\sum_k \|h_k\|_p^s \Big)^{1/s}, \qquad \forall \, \{h_k\} \subset L^p(\mathbb{R}^{d+1}).$$

Proof. The lemma follows by rescaling the problem with a suitable Lorentz transformation and using hypothesis $\mathcal{H}^{str}(p,s,\gamma)$ (see e.g. [12, p. 167]). For completeness, we describe the argument here.

Let $\{\eta_1, \dots, \eta_d\}$ be an orthonormal basis of $\mathbb{R}^d$, where η_1 is chosen so that $(1,\eta_1)$ is the center of the plate $\Pi^{(\sigma)}$. Then $\{(1,\eta_1), (-1,\eta_1), (0,\eta_2), \dots, (0,\eta_d)\}$ is a basis of $\mathbb{R}^{d+1}$. Define a linear operator $L \in Gl_{d+1}(\mathbb{R})$ preserving the cone and acting on this basis by

$$L(1,\eta_1) = (1,\eta_1), \quad L(-1,\eta_1) = \tfrac{1}{\sigma}(-1,\eta_1) \quad \text{and} \quad L(0,\eta_\ell) = \tfrac{1}{\sqrt{\sigma}}(0,\eta_\ell), \ \ell = 2, ..., d.$$

Set $f_k = P_k^{(\delta)}(h_k)$, so that the functions $f_k \circ L$ have now spectrum in (perhaps a multiple) of the plates $\Pi_k^{(\delta/\sigma)}$ corresponding to the $\sqrt{\delta/\sigma}$-separated centers $\{L(1,\omega_k)\}$. Thus, hypothesis $\mathcal{H}^{str}(p,s,\gamma)$ can be applied at scale δ/σ giving

$$\big\|\sum_k f_k \circ L\big\|_p \lesssim (\delta/\sigma)^{-\beta(p,s)-\gamma} \Big(\sum_k \big\|f_k \circ L\big\|_p^s \Big)^{\frac{1}{s}},$$

which after a change of variables yields (5.1). □

5.2. **The two main lemmas.** To prove Theorem 3.4 we must show that for every (N,E,Q)-packet f and λ as in (3.18) the inequality (3.2) holds in the improved range $\gamma > \gamma_0(1-\epsilon_0')$, under the assumption $\mathcal{H}^{str}(p,s,\gamma_0)$. This will be done by repeatedly localizing at smaller scales, and then using the induction hypothesis at the lowest scale. In this section we prove the main two lemmas which show how this process works at each step. Proofs are similar to [12, Lemma 6.1].

Below, we let $N_1 = \sqrt{N}$ (hence $\delta_1 = \sqrt{\delta}$) and denote by $\mathcal{Q}_1 = \{\Delta\}$ a tiling of $\mathbb{R}^{d+1}$ by N_1-cubes. Then, for every (N,E,Q_0)-packet f we can write

$$\big|\{x \in Q_0 \ : \ |f(x)| > \lambda\}\big| \leq \sum_{\Delta \subset Q_0} \big|\{ x \in \Delta \ : \ |f\psi_\Delta(x)| > c\lambda\}\big| \tag{5.2}$$

for some constant $c > 0$. Observe that, for a fixed Δ, the function $f\psi_\Delta$ has Fourier transform supported in $\Gamma_{\delta_1}(C)$, but in general is not a packet. However, by Lemma

2.12, $f\psi_\Delta$ can be decomposed on Δ in terms of (N_1, E_1, Δ) packets. Below we denote by Ω_1 a $\sqrt{\delta_1}$-separated set in S^{d-1}, and by $\{\Pi_k^{(\delta_1)}\}_{k\in\Omega_1}$ the corresponding plate decomposition of $\Gamma_{\delta_1}(C)$.

Lemma 5.2. *Let Q_0 be an N-cube, f be an (N, E, Q_0)-packet, and let $\lambda \geq 1$. Then there exists $\lambda_1 > 0$ so that for every N_1-cube $\Delta \subset Q_0$ there is a plate family $\mathcal{P}_{(1,\Delta)}$, a set $E_{(1,\Delta)} \subset \Omega_1$, and a stable $(N_1, E_{(1,\Delta)}, \Delta)$-packet $f_{(1,\Delta)}$ with plate set $\mathcal{P}_{(1,\Delta)}$ so that*

$$\big|\{\, x \in Q_0 \ : \ |f(x)| > \lambda \,\}\big| \lessapprox \sum_{\Delta\subset Q_0} \big|\{\, x \in \Delta \ : \ |f_{(1,\Delta)}(x)| \geq \lambda_1 \,\}\big| \tag{5.3}$$

and

$$|\mathcal{P}_{(1,\Delta)}| \lessapprox \frac{\lambda_1^2}{\lambda^2} \frac{\|f\psi_\Delta\|_2^2}{N_1^{\frac{d+1}{2}}} \lessapprox \frac{\lambda_1^2}{\lambda^2} N_1^{\frac{d+1}{2}} |E|. \tag{5.4}$$

Moreover, for all $p, s \geq 1$ we have

$$|\mathcal{P}_{(1,\Delta)}|\, |E_{(1,\Delta)}|^{\frac{p}{s}-1} \lessapprox \frac{\lambda_1^p}{\lambda^p} \frac{\|f\psi_\Delta\|_{p,s;\delta_1}^p}{N_1^{\frac{d+1}{2}}}. \tag{5.5}$$

Proof. Fix $\Delta \subset Q_0$ and let $g^\Delta \equiv f\psi_\Delta$, which has Fourier transform supported in $\Gamma^{\delta_1}(C)$ and satisfies

$$\|g^\Delta\|_{\infty,\infty;\delta_1} \lesssim (N/N_1)^{(d-1)/2} = N_1^{\frac{d-1}{2}}$$

(by Lemma 4.2). Applying Lemma 2.12 with $A = N_1^{\frac{d-1}{2}}$ and $Q = \Delta$, we can write

$$g^\Delta(x) = \sum_{N_1^{-10d} A \lesssim 2^j \lesssim A N_1^d} 2^j \sum_{\ell=1}^{n_{j,\Delta}} g^\Delta_{[j,\ell]}(x) + h^\Delta(x), \quad x \in \Delta, \tag{5.6}$$

where

$$n_{j,\Delta} \lesssim \log N_1, \tag{5.7}$$

$$\sup_{x\in\Delta} |h_\Delta(x)| \lesssim N_1^{-8d} A \leq N_1^{-7d}; \tag{5.8}$$

moreover, for each (j, ℓ, Δ) there is a subset $E^\Delta_{j,\ell}$ of Ω_1 so that $g^\Delta_{[j,\ell]}$ is a stable $(N_1, E^\Delta_{j,\ell}, \Delta)$-packet, with associated plate family $\mathcal{P}^\Delta_{j,\ell}$, consisting only of N_1-plates π contained in $2N_1^{1+\epsilon_0}\Delta$, and more importantly satisfying

$$2^{jp} N_1^{\frac{d+1}{2}} |\mathcal{P}^\Delta_{j,\ell}|\, |E^\Delta_{j,\ell}|^{\frac{p}{s}-1} \lesssim \|f\psi_\Delta\|^p_{p,s;\delta_1}, \qquad \forall\, p, s \geq 1. \tag{5.9}$$

As there are only $O(\log N)$ values of j and $O(\log N)$ values of ℓ a simple pidgeonhole argument and (5.8) show that, for $\lambda \geq 1$,

$$\Big|\big\{x \in \Delta \ : \ |g^{\Delta}| > c\lambda\big\}\Big| \leq \Big|\big\{x \in \Delta \ : \ \big|\textstyle\sum_{N_1^{-10d}A \lesssim 2^j \lesssim N_1^d A} 2^j \sum_{\ell=1}^{n_{j,\Delta}} g^{\Delta}_{[j,\ell]}(x)\big| > \frac{c\lambda}{2}\big\}\Big|$$
$$\leq C\,(\log N)^2 \Big|\big\{x \in \Delta \ : \ |2^{j_\Delta} g^{\Delta}_{[j_\Delta,\ell_\Delta]}(x)\big| > \tfrac{\lambda}{C(\log N)^2}\big\}\Big|$$

for some fixed j_Δ, ℓ_Δ. Pigeonholing once again we can find, among the (j_Δ, ℓ_Δ)'s, a fixed pair $j_*, \ell_* \in \mathbb{Z}$ (independent of Δ) so that

$$\sum_{\Delta}\Big|\big\{x \in \Delta \ : \ |g^{\Delta}| > c\lambda\big\}\Big| \lessapprox \sum_{\Delta} \Big|\big\{x \in \Delta \ : \ |2^{j_*} g^{\Delta}_{[j_*,\ell_*]}(x)\big| > \tfrac{\lambda}{C(\log N)^2}\big\}\Big|.$$

Using (5.2) this means that (5.3) holds with $\lambda_1 = 2^{-j_*}\lambda/(C\log N)^2$ and $f_{(1,\Delta)} = g^{\Delta}_{[j_*,\ell_*]}$, and hence that $E_{(1,\Delta)} = E^{\Delta}_{j_*,\ell_*}$ and $\mathcal{P}_{(1,\Delta)} = \mathcal{P}(g^{\Delta}_{[j_*,\ell_*]})$. Observe also that (5.5) follows immediately from (5.9) and the definition of λ_1.

The first inequality in (5.4) follows from the case $p = s = 2$ of (5.9) in the same fashion. For the second inequality in (5.4) we observe that if $f = \sum_k f_k$ with $\operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)}$ then the Fourier transforms $\widehat{f_k\psi_\Delta}$ are supported in essentially disjoint sets. Thus we have the crucial orthogonality estimate

$$\|f\psi_\Delta\|_2^2 \lesssim \sum_k \|f_k\psi_\Delta\|_2^2 \lesssim |\Delta| \sum_{k\in E} \|f_k\|_\infty^2 \lesssim N_1^{d+1}\,|E|, \tag{5.10}$$

the last step following from the fact that $\|f_k\|_\infty \lesssim 1$ for N-packets. This establishes the second inequality in (5.4) and hence the lemma. $\square$

Below we shall use the bound in (5.4) to argue that at least one of the sufficient conditions, Proposition 4.5 or Proposition 4.9, can be applied to the triplet $(f_{(1,\Delta)}, \lambda_1, \Delta)$. The next lemma, shows how to conclude the theorem for the original packet f in such case.

Lemma 5.3. *Let $p \geq 2$, $s \in [1,p]$ and assume that $\mathcal{H}^{str}(p,s,\gamma_0)$ holds for some $\gamma_0 > 0$. Consider an (N,E,Q_0)-packet f and a real number $\lambda \geq 1$. Suppose we are given a number $\lambda_1 > 0$ and a collection $\{f_{(1,\Delta)}\}_\Delta$, where Δ runs over a grid of N_1-cubes contained in Q_0, where each $f_{(1,\Delta)}$ is an $(N_1, E_{(1,\Delta)}, \Delta)$-packet with plate family $\mathcal{P}_{(1,\Delta)}$ satisfying* (5.3), (5.4) *and* (5.5) *(e.g., when $f_{(1,\Delta)}$ are generated as in Lemma 5.2). Assume in addition that there is a real number $\alpha > 0$ so that, for every $\Delta \subset Q_0$, the pairs $(f_{(1,\Delta)}, \lambda_1)$ satisfy the inequality:*

$$\big|\{\, x \in \Delta \ : \ |f_{(1,\Delta)}(x)| > \lambda_1 \,\}\big| \lessapprox \frac{N_1^{(\beta(p,s)+\alpha)p}}{\lambda_1^p}\, N_1^{\frac{d+1}{2}}\, |\mathcal{P}_{(1,\Delta)}|\, |E_{(1,\Delta)}|^{\frac{p}{s}-1}, \tag{5.11}$$

Then, we also have

$$\big|\{\, x\in Q_0 \;:\; |f(x)|>\lambda\,\}\big| \lessapprox \lambda^{-p}\, N^{(\beta(p,s)+\gamma)p}\, N^{\frac{d+1}{2}}\, |\mathcal{P}(f)|\, |E|^{\frac{p}{s}-1}, \tag{5.12}$$

with $\gamma=(\gamma_0+\alpha)/2$.

In particular, (5.12) *holds with* $\gamma=\gamma_0(1-\epsilon_0/4)$ *at least when one of the following conditions is satisfied for every* N_1*-cube* $\Delta\subset Q_0$*:*

(i) $\lambda_1 \lessapprox N_1^{(\beta(p,s)p-\frac{d-1}{2(d+1)})/(p-q)}\, |E_{(1,\Delta)}|^{(\frac{p}{s}-\frac{q}{2})/(p-q)}$*;*

(ii) $|E_{(1,\Delta)}|^{\frac{p}{s'}-\frac{q}{2}} \lessapprox N_1^{\beta(p,s)\,p-\frac{d-1}{2(d+1)}}$*;*

(iii) $(f_{(1,\Delta)},\lambda_1)$ *satisfies any of the sufficient conditions* (4.7) *or* (4.16)*;*

(iv) when $d=2$*,* $(f_{(1,\Delta)},\lambda_1)$ *satisfies any of the sufficient conditions* (4.18) *or* (4.19)*.*

Remark 5.4. We observe that the previous two lemmas already give Theorem 3.4 for p and d sufficiently large. For instance, to verify that (5.11) holds, say with $s=2$, by Proposition 4.5 we only need to check that the plate families $\mathcal{P}_{(1,\Delta)}$ satisfy

$$|\mathcal{P}_{(1,\Delta)}| \lesssim t^{7d}\lambda_1^2.$$

By the inequality (5.4) and the fact that we only consider $\lambda \geq N^{\frac{\beta(p,2)\,p}{p-2}}\,|E|^{1/2}$ (by Lemma 3.6 with $s=2$), we obtain (after some arithmetics)

$$\lambda_1^{-2}\,|\mathcal{P}_{(1,\Delta)}| \lessapprox \frac{N_1^{(d+1)/2}\,|E|}{\lambda^2} \leq N^{\frac{d+1}{4}-\frac{2p\,\beta(p,2)}{p-2}} = N^{\frac{2}{p-2}-\frac{d-3}{4}}, \tag{5.13}$$

which is $\lesssim t^{7d}=N^{-7d\varepsilon_0}$ if $d>3$ and $p>2+\frac{8}{d-3-4d\varepsilon_0}$. Thus, choosing $\varepsilon_0=\varepsilon_0(p)$ small we can exhaust the range $p>2+8/(d-3)$, which is one of the indices obtained in [12] for the validity of (**??**). To improve over this index one must iterate the process with successive $N^{1/4}, N^{1/8}$,... localizations, as described in the next subsection.

Proof of Lemma 5.3. By (5.3) and (5.11) we have

$$\begin{aligned}\big|\{\, x\in Q_0 \;:\; |f|>\lambda\,\}\big| &\lessapprox \sum_{\Delta\subset Q_0} \big|\{\, x\in\Delta \;:\; |f_{(1,\Delta)}|>\lambda_1\,\}\big| \\ &\lessapprox \sum_{\Delta\subset Q_0} \lambda_1^{-p}\, N_1^{(\beta(p,s)+\alpha)p}\, N_1^{\frac{d+1}{2}}\, |\mathcal{P}_{(1,\Delta)}|\, |E_{(1,\Delta)}|^{\frac{p}{s}-1}.\end{aligned}$$

Thus, the result will be established if we can show

$$\sum_{\Delta\subset Q_0} N_1^{\frac{d+1}{2}}\, |\mathcal{P}_{(1,\Delta)}|\, |E_{(1,\Delta)}|^{\frac{p}{s}-1} \lesssim \frac{{\lambda_1}^p}{\lambda^p}\, N_1^{(\beta(p,s)+\gamma_0)p}\, N^{\frac{d+1}{2}}\, |\mathcal{P}(f)|\, |E|^{\frac{p}{s}-1}. \tag{5.14}$$

To do so, recall that $f\psi_\Delta$ are functions with spectrum in $\Gamma_{\delta_1}(C)$, and denote by $P_\ell^{(\delta_1)}$ the projections as in (**??**) associated with the usual partition of $\Gamma_{\delta_1}(C)$ by $1 \times \delta_1 \times \sqrt{\delta_1} \times ... \times \sqrt{\delta_1}$ plates: $\{\Pi_\ell^{(\delta_1)}\}_\ell$. Then by (5.5) we have for each Δ,

$$\begin{aligned} N_1^{\frac{d+1}{2}} \, |\mathcal{P}_{(1,\Delta)}| \, |E_{(1,\Delta)}|^{\frac{p}{s}-1} &\lessapprox \frac{{\lambda_1}^p}{\lambda^p} \, \|f\psi_\Delta\|_{p,s;\delta_1}^p \\ &\lesssim \frac{{\lambda_1}^p}{\lambda^p} \Big(\sum_\ell \big\| P_\ell^{(\delta_1)}(f\psi_\Delta) \big\|_p^s \Big)^{p/s} \\ &= \frac{{\lambda_1}^p}{\lambda^p} \Big(\sum_\ell \Big\| \, P_\ell^{(\delta_1)} \big[\, \psi_\Delta \, (\sum_{k \,:\, \Pi_k^{(\delta)} \subset C\Pi_\ell^{(\delta_1)}} f_k) \big] \, \Big\|_p^s \Big)^{p/s} \\ &\lesssim \frac{{\lambda_1}^p}{\lambda^p} \Big(\sum_\ell \Big\| \psi_\Delta (\sum_{\Pi_k^{(\delta)} \subset C\Pi_\ell^{(\delta_1)}} f_k) \Big\|_p^s \Big)^{p/s}. \end{aligned}$$

We sum in Δ and apply Minkowski's inequality (since $p \ge s$) to obtain

$$\begin{aligned} \sum_\Delta N_1^{\frac{d+1}{2}} \, |\mathcal{P}_{(1,\Delta)}| \, |E_{(1,\Delta)}|^{\frac{p}{s}-1} &\lessapprox \frac{\lambda_1^p}{\lambda^p} \sum_\Delta \Big(\sum_\ell \Big\| \psi_\Delta (\sum_{\Pi_k^{(\delta)} \subset C\Pi_\ell^{(\delta_1)}} f_k) \Big\|_p^s \Big)^{p/s} \\ &\lesssim \frac{\lambda_1^p}{\lambda^p} \Big(\sum_\ell \Big[\sum_\Delta \Big\| \psi_\Delta (\sum_{\Pi_k^{(\delta)} \subset C\Pi_\ell^{(\delta_1)}} f_k) \Big\|_p^p \Big]^{s/p} \Big)^{p/s} \\ &\lesssim \frac{\lambda_1^p}{\lambda^p} \Big(\sum_\ell \Big\| \sum_{\Pi_k^{(\delta)} \subset C\Pi_\ell^{(\delta_1)}} f_k \Big\|_p^s \Big)^{p/s}. \end{aligned}$$

Now, we apply Hypothesis $\mathcal{H}^{str}(p, s, \gamma_0)$ in the rescaled version of Lemma 5.1 and bound for each ℓ

$$\Big\| \sum_{k:\Pi_k^{(\delta)} \subset C\Pi_\ell^{(\delta_1)}} f_k \Big\|_p \lesssim (N/N_1)^{\beta(p,s)+\gamma_0} \Big(\sum_{k:\Pi_k^{(\delta)} \subset C\Pi_\ell^{(\delta_1)}} \|f_k\|_p^s \Big)^{1/s}.$$

This yields

$$\begin{aligned} \Big(\sum_\ell \Big\| \sum_{\Pi_k^{(\delta)} \subset c\Pi_\ell^{(\delta_1)}} f_k \Big\|_p^s \Big)^{p/s} &\lessapprox N_1^{(\beta(p,s)+\gamma_0)p} \Big(\sum_\ell \sum_{\Pi_k^{(\delta)} \subset c\Pi_\ell^{(\delta_1)}} \big\| f_k \big\|_p^s \Big)^{p/s} \\ &\lesssim N_1^{(\beta(p,s)+\gamma_0)p} \, \big\| f \big\|_{p,s}^p \\ &\lesssim N_1^{(\beta(p,s)+\gamma_0)p} \, N^{\frac{d+1}{2}} |\mathcal{P}(f)| \, |E|^{\frac{p}{s}-1}, \end{aligned}$$

where the last inequality follows from (2.27). This proves (5.14) and establishes the lemma. ∎

5.3. **Iteration.** We are now ready to describe the iteration. Here we fix $p > p_d$ as in (**??**) and $s \in [1,p]$. We let $N_j = N^{1/2^j}$ for $j = 0,1,2,\dots$ Starting with an (N,E,Q_0)-packet $f = f_0$ and $\lambda = \lambda_0$ as in (3.18), at step j we shall define, for each N_{j-1}-cube Δ_{j-1}, a real number $\lambda_j > 0$ and a collection of functions $\{f_{(j,\Delta)}\}_{\Delta\subset\Delta_{j-1}}$, where Δ runs in a grid $\mathcal{Q}_j$ of N_j-cubes and each $f_{(j,\Delta)}$ is an $(N_j, E_{(j,\Delta)}, \Delta)$-packet with plate family $\mathcal{P}_{(j,\Delta)}$, and so that the pair $(f_{(j,\Delta)}, \lambda_j)$ satisfies

$$(a) \qquad \big|\{\, x \in \Delta_{j-1} \ : \ |f_{(j-1,\Delta_{j-1})}| > \lambda_{j-1} \,\}\big| \lessapprox \sum_{\substack{\Delta\in\mathcal{Q}_j\\ \Delta\subset\Delta_{j-1}}} \big|\{\, x \in \Delta \ : \ |f_{(j,\Delta)}| > \lambda_j \,\}\big|$$

$$(b) \qquad |\mathcal{P}_{(j,\Delta)}| \lessapprox \frac{\lambda_j^2}{\lambda_{j-1}^2}\, \frac{\|f_{(j-1,\Delta_{j-1})}\psi_\Delta\|_2^2}{N_j^{\frac{d+1}{2}}};$$

$$(c) \qquad |\mathcal{P}_{(j,\Delta)}| \lessapprox \frac{\lambda_j^p}{\lambda_{j-1}^p}\, \frac{\|f_{(j-1,\Delta_{j-1})}\psi_\Delta\|_{p,s;\delta_j}}{N_j^{\frac{d+1}{2}}}.$$

It is clear from Lemma 5.2 that this is possible for $j = 1$. We next show how to pass from step j to step $j+1$.

Suppose we are at step j. Then we stop the process for the N_j-cubes $\Delta \in \mathcal{Q}_j$ for which the pair $(f_{(j,\Delta)}, \lambda_j)$ already satisfies the improved inequality in (5.11); in particular when at least one of the conditions (i)-(iv) in Lemma 5.3 holds (with the subindex "1" replaced by "j"). Observe that when for *all* cubes $\Delta \subset \Delta_{j-1}$ the inequality (5.11) is satisfied, then a direct application of Lemma 5.3 gives the improved estimate at the next scale, i.e.

$$\big|\{\, x \in \Delta_{j-1} \ : \ |f_{(j-1,\Delta_{j-1})}| > \lambda_{j-1} \,\}\big| \lessapprox \lambda_{j-1}^{-p}\, N_{j-1}^{(\alpha+\gamma_0(1-\frac{\epsilon_0}{4}))p}\, N_{j-1}^{\frac{d+1}{2}}\, |\mathcal{P}_{(j-1,\Delta_{j-1})}|^{\frac{p}{s}-1},$$

which after $j-1$ more applications of the lemma leads to (5.12) with $\gamma = \gamma_0(1 - \epsilon_0/2^{j+1})$, hence establishing Theorem 3.4 with $\epsilon_0' = \epsilon_0/2^{j+1}$.

Assume therefore that we are dealing with cubes $\Delta \in \mathcal{Q}_j$ for which $(f_{(j,\Delta)}, \lambda_j)$ does not satisfy any of the conditions (i)-(iv) in Lemma 5.3. That is, we are only considering

$$\lambda_j \ \geq\ (\log N)^C N^{(\beta(p,s)p-\frac{d-1}{2(d+1)})/(p-q)}\, |E_{(j,\Delta)}|^{(\frac{p}{s}-\frac{q}{2})/(p-q)} \tag{5.15}$$

and

$$|E_{(j,\Delta)}| \geq\ N^{(\beta(p,s)\,p-\frac{d-1}{2(d+1)})/(\frac{p}{s'}-\frac{q}{2})}. \tag{5.16}$$

Also, since (iii) fails, by Proposition 4.9, there must exist a subpacket $f^*_{(j,\Delta)}$ of $f_{(j,\Delta)}$ so that

$$\big|\{|f_{(j,\Delta)}| > \lambda_j\}\big| \lessapprox \big|\{|f^*_{(j,\Delta)}| > c\lambda_j/\log N\}\big| \tag{5.17}$$

and moreover, for every N_{j+1}-cube Δ_{j+1}

$$\big\|f^*_{(j,\Delta)}\psi_{\Delta_{j+1}}\big\|_2^2 \lesssim t^{-14d}\,\frac{N_j^{d/2}}{\lambda_j^2}\,|\mathcal{P}_{(j,\Delta)}|\,|E_{(j,\Delta)}|. \tag{5.18}$$

Then, we can replace the original $(f_{(j,\Delta)}, \mathcal{P}_{(j,\Delta)}, \lambda_j)$ by $(f^*_{(j,\Delta)}, \mathcal{P}^*_{(j,\Delta)}, \lambda^*_j = c\lambda_j/\log N)$, which also satisfies (a), (b), (c) and (5.15), (5.16). Next, we apply Lemma 5.2 to each pair $(f^*_{(j,\Delta)}, \lambda^*_j)$ to obtain new quadruplets $(f_{(j+1,\Delta_{j+1})}, \mathcal{P}_{(j+1,\Delta_{j+1})}, E_{(j+1,\Delta_{j+1})}, \lambda_{j+1})$ with the required conditions, i.e.

$$\big|\{\,x\in\Delta \;:\; |f^*_{(j,\Delta)}| > \lambda^*_j\,\}\big| \lessapprox \sum_{\substack{\Delta_{j+1}\in\mathcal{Q}_{j+1}\\ \Delta_{j+1}\subset\Delta}} \big|\{\,x\in\Delta_{j+1} \;:\; |f_{(j+1,\Delta_{j+1})}| > \lambda_{j+1}\,\}\big|$$

$$|\mathcal{P}_{(j+1,\Delta_{j+1})}| \lessapprox \frac{\lambda_{j+1}^2}{\lambda_j^2}\,\frac{\|f^*_{(j,\Delta_j)}\psi_{\Delta_{j+1}}\|_2^2}{N_{j+1}^{(d+1)/2}}; \tag{5.19}$$

$$|\mathcal{P}_{(j+1,\Delta_{j+1})}| \lessapprox \frac{\lambda_{j+1}^p}{\lambda_j^p}\,\frac{\|f^*_{(j,\Delta_j)}\psi_{\Delta_{j+1}}\|_{p,s;\delta_{j+1}}^p}{N_{j+1}^{(d+1)/2}}.$$

Observe that, with this construction, if we combine (5.19) and (5.18) we obtain in addition the inequality

$$|\mathcal{P}_{(j+1,\Delta_{j+1})}| \lessapprox t^{-14d}\,\frac{\lambda_{j+1}^2}{\lambda_j^4}\,N_j^{\frac{d-1}{4}}\,|\mathcal{P}_{(j,\Delta_j)}|\,|E_{(j,\Delta_j)}|, \quad j = 1, 2, \ldots \tag{d}$$

The case $d \geq 3$ and $s = 2$.

Claim. *If $d \geq 3$, $p > p_d$ and $s = 2$, then the above process will stop after a finite number of iterations. More precisely, there exists $\ell = \ell(p) \in \mathbb{N}$ so that the quadruplets $(f_{(\ell,\Delta)}, \mathcal{P}_{(\ell,\Delta)}, E_{(\ell,\Delta)}, \lambda_\ell)$ satisfy the sufficient condition (4.7) in Lemma 4.5 for all $\Delta \in \mathcal{Q}_\ell$.*

For simplicity, denote $A_j = |\mathcal{P}_{(j,\Delta_j)}|$ and $E_j = |E_{(j,\Delta_j)}|$. Then, from (d) above one obtains

$$\lambda_\ell^{-2}\,A_\ell \lessapprox t^{-14d(\ell-1)}\,\frac{E_{\ell-1}\cdots E_1}{\lambda_{\ell-1}^2\cdots\lambda_2^2\lambda_1^4}\,(N_{\ell-1}\ldots N_1)^{\frac{d-1}{4}}\,A_1. \tag{5.20}$$

Now, to estimate A_1 we use (5.4), that is

$$A_1 \lessapprox \frac{\lambda_1^2}{\lambda^2} N^{\frac{d+1}{4}} E_0. \tag{5.21}$$

Inserting this into (5.20) leads to

$$\lambda_\ell^{-2} A_\ell \lessapprox t^{-14d(\ell-1)} \frac{E_{\ell-1}\cdots E_1 E_0}{\lambda_{\ell-1}^2 \ldots \lambda_1^2 \lambda^2} (N_{\ell-1} \ldots N_1)^{\frac{d-1}{4}} N^{\frac{d+1}{4}}. \tag{5.22}$$

We need to show that the right hand side of this expression is smaller than t^{14d}. Observe that we can replace the symbol "$\lessapprox$" in (5.22) by "$\le t^{-1}$", provided $N \ge N_0(\epsilon_0)$. Thus it will suffice to prove the inequality

$$\left(N^{1-\frac{1}{2^{\ell-1}}}\right)^{\frac{d-1}{4}} N^{\frac{d+1}{4}} \le t^{30d\ell} \frac{\lambda_{\ell-1}^2}{E_{\ell-1}} \ldots \frac{\lambda_1^2}{E_1} \frac{\lambda^2}{E_0}. \tag{5.23}$$

By (5.15) (with $s = 2$) we know that

$$\lambda_j \ge N^{(\beta(p,s)p - \frac{d-1}{2(d+1)})/(p-q)} E_j^{1/2} = N_j^{\frac{d-1}{4} - \frac{q}{4(p-q)}} E_j^{1/2},$$

and therefore it is enough to show that

$$N^{\frac{d-1}{4}} N^{\frac{d+1}{4}} \le t^{30d\ell} \left(N^{\frac{d-1}{2} - \frac{q}{2(p-q)}}\right)^{2-\frac{1}{2^{\ell-1}}}. \tag{5.24}$$

Since $t = N^{-\epsilon_0}$, the previous is equivalent to

$$2(1 - \tfrac{1}{2^\ell})(\tfrac{d-1}{2} - \tfrac{q}{2(p-q)}) - \tfrac{d}{2} \ge 30d\ell\epsilon_0. \tag{5.25}$$

It is now easy to verify that this holds when $p > p_d$, for a sufficiently large integer $\ell = \ell(p)$, and a suitable choice of $\epsilon_0 = \epsilon_0(p)$. More precisely, condition $p > p_d$ (for $d \ge 3$) can be read as $\frac{q}{4(p-q)} < \frac{d-2}{8}$, which is equivalent to

$$\epsilon_p := 2(\tfrac{d-1}{2} - \tfrac{q}{2(p-q)}) - \tfrac{d}{2} > 0.$$

Thus, we only need to choose $\ell = \ell(p)$ so that $2^{-\ell+1}(\frac{d-1}{2} - \frac{q}{2(p-q)}) < \epsilon_p/2$, and next choose $\epsilon_0 = \epsilon_0(\ell, p) = \epsilon_0(p)$ so that $30d\ell\epsilon_0 < \epsilon_p/2$. This will satisfy (5.25) and establish the claim. Thus letting $\epsilon_0' = \epsilon_0/2^{\ell+1}$, one obtains Theorem 3.4 for $d \ge 3$.

Replace the previous argument by induction

We claim that for $j = 1, \ldots, \ell$

$$\frac{|\mathcal{P}_{(j,\Delta_j)}|}{\lambda_j^2} \lessapprox t^{-14d(j-1)} N^{\frac{q}{p-q} - \frac{d-2}{2}} N_{j-1}^{\frac{d-1}{4} - \frac{q}{2(p-q)}} \tag{5.26}$$

{7}

We first note that for $j = 0, 1, \ldots$

{7}Since t could also be allowed to change with j we do not need that much powers ...

Notice that

$$\lambda_j \ge N^{(\beta(p,s)p-\frac{d-1}{2(d+1)})/(p-q)} E_j^{1/2} = N_j^{\frac{d-1}{4}-\frac{q}{4(p-q)}} E_j^{1/2}. \qquad \text{if } s=2. \tag{5.27}$$

For $j=1$ this says $\frac{|\mathcal{P}_{(1,\Delta_1)}|}{\lambda_1^2} \lessapprox N^{\frac{q}{2(p-q)}-\frac{d-3}{4}}$. To see this we have by (5.4)

$$\frac{|\mathcal{P}_{(1,\Delta_1)}|}{\lambda_1^2} \le N^{\frac{d+1}{4}} |E|\lambda^{-2}$$

and since $|E|\lambda^{-2} \lesssim N^{\frac{q}{2(p-q)}-\frac{d-1}{2}}$ the assertion holds for $j=1$.

Assuming (5.26) for a $j\ge 1$ we estimate

$$\frac{|\mathcal{P}_{(j+1,\Delta_{j+1})}|}{\lambda_{j+1}^2} \lessapprox t^{-14d} N_j^{\frac{d-1}{4}} \frac{|E_{j,\Delta_j}|}{\lambda_j^2} \frac{|\mathcal{P}_{(j,\Delta_j)}|}{\lambda_j^2} \tag{5.28}$$

by "(d)". By (5.27) the right hand side of (5.28) is estimated by

$$t^{-14d} N_j^{\frac{d-1}{4}} N_j^{-\frac{d-1}{2}+\frac{q}{2(p-q)}} \frac{|\mathcal{P}_{(j,\Delta_j)}|}{\lambda_j^2}$$

By the induction hypothesis this is

$$\begin{aligned}
&\lessapprox t^{-14d} N_j^{\frac{d-1}{4}} N_j^{-\frac{d-1}{2}+\frac{q}{2(p-q)}} t^{-14d(j-1)} N^{\frac{q}{p-q}-\frac{d-2}{2}} N_{j-1}^{\frac{d-1}{4}-\frac{q}{2(p-q)}} \\
&= t^{-14dj} N^{\frac{q}{p-q}-\frac{d-2}{2}} (N_{j-1}/N_j)^{\frac{d-1}{4}-\frac{q}{2(p-q)}} \\
&\lessapprox t^{-14dj} N^{\frac{q}{p-q}-\frac{d-2}{2}} N_j^{\frac{d-1}{4}-\frac{q}{2(p-q)}}.
\end{aligned}$$

The case $d=2$. In this case the previous scheme does not give anything. One must use in (5.18) above Proposition 4.11, rather than the weaker Proposition 4.9. In such case the inequality in (5.18) can be replaced with the improved version

$$\big\| f^*_{(j,\Delta)} \psi_{\Delta_{j+1}} \big\|_2^2 \lessapprox t^{-C} \frac{N_j^{5/4}}{\lambda_j^3} |\mathcal{P}_{(j,\Delta)}|\, |E_{(j,\Delta_j)}| \tag{5.29}$$

(which follows from (4.20)). Thus (d) will take the form

$$|\mathcal{P}_{(j+1,\Delta_{j+1})}| \lessapprox t^{-C} \frac{\lambda_{j+1}^2}{\lambda_j^5} N_{j+1} |\mathcal{P}_{(j,\Delta)}|\, |E_{(j,\Delta_j)}|, \quad j=1,2,\dots \tag{5.30}$$

Then calling $A_j = |\mathcal{P}_{(j,\Delta_j)}|$, $E_j = |E_{(j,\Delta_j)}|$ and iterating as in the proof of the claim we are led to

$$\begin{aligned}
\lambda_\ell^{-2} A_\ell &\le t^{-C} \frac{E_{\ell-1}\dots E_1}{\lambda_{\ell-1}^3 \dots \lambda_2^3 \lambda_1^5} \big(N_\ell \dots N_2\big) A_1 \\
\text{(by (5.21))} \qquad &\le t^{-C} \frac{E_{\ell-1}}{\lambda_{\ell-1}^3} \dots \frac{E_1}{\lambda_1^3} \frac{E_0}{\lambda^2} \big(N_\ell \dots N_2\big) N^{\frac{3}{4}}.
\end{aligned} \tag{5.31}$$

The lower bound for λ_j from (5.15) gives

$$\frac{|E|}{\lambda^2} \le \frac{|E|^{1-2(\frac{p}{s}-\frac{q}{2})/(p-q)}}{N^{2(\beta(p,s)-\frac{1}{6})/(p-q)}} = \frac{|E|^{\frac{2p}{p-q}\left(\frac{1}{2}-\frac{1}{s}\right)}}{N^{\frac{p}{p-q}\left(\frac{1}{s'}-\frac{q}{p}\right)}}$$

$$\le \frac{N^{\frac{p}{p-q}\left(\frac{1}{2}-\frac{1}{s}\right)}}{N^{\frac{p}{p-q}\left(\frac{1}{s'}-\frac{q}{p}\right)}} = N^{-\frac{p-2q}{2(p-q)}}, \tag{5.32}$$

where in the last inequality we have used that $|E| \lesssim N^{1/2}$ (since we only consider $s \ge 2$). On the other hand the same bound for λ_j from (5.15) gives

$$\frac{E_j}{\lambda_j^3} \le \frac{E_j^{1-3(\frac{p}{s}-\frac{q}{2})/(p-q)}}{N_j^{3(\beta(p,s)-\frac{1}{6})/(p-q)}} = \frac{E_j^{\frac{3p}{p-q}\left(\frac{2p+q}{6p}-\frac{1}{s}\right)}}{N_j^{\frac{3p}{2(p-q)}\left(\frac{1}{s'}-\frac{q}{p}\right)}} \tag{5.33}$$

To estimate further this quantity we must distinguish cases.

Case 1: $s \ge \frac{6p}{2p+q} = 3 - \frac{15}{3p+5}$. Then the exponent of E_j in (5.33) is positive and we can use again the trivial bound $E_j \lesssim N_j^{1/2}$, which leads to

$$\frac{E_j}{\lambda_j^3} \le \frac{N_j^{\frac{3p}{2(p-q)}\left(\frac{2p+q}{6p}-\frac{1}{s}\right)}}{N_j^{\frac{3p}{2(p-q)}\left(\frac{1}{s'}-\frac{q}{p}\right)}} = N_j^{-\frac{4p-7q}{4(p-q)}}. \tag{5.34}$$

Inserting (5.32) and (5.34) into (5.31) we obtain

$$\lambda_\ell^{-2} A_\ell \le t^{-C} N^{-\frac{4p-7q}{4(p-q)}} N^{-\frac{p-2q}{2(p-q)}} N^{1/2} N^{3/4}.$$

Since by Lemma 4.5 it suffices to show [8] that $\lambda_\ell^{-2} A_\ell \le t^{28}$, we will be done when

$$\tfrac{5}{4} < \tfrac{4p-7q}{4(p-q)} + \tfrac{p-2q}{2(p-q)} = \tfrac{6p-11q}{4(p-q)},$$

or equivalently when

$$p > 6q = 20.$$

This establishes Theorem 1.1 in this case.

Case 2: $2 \le s \le \frac{6p}{2p+q} = 3 - \frac{15}{3p+5}$. Then the exponent of E_j in (5.33) is negative and

[8] We could also require the weaker estimate $A_\ell \le t^{C'} \lambda_\ell^3 |E_\ell|^{\frac{1}{2}}$ (by Lemma 4.10), but this makes no difference at this point.

we must use instead the lower bound $E_j \gtrsim N_j^{(\beta(p,s)\,p-\frac{1}{6})/(\frac{p}{s'}-\frac{q}{2})}$ in (5.16), which after a simple but tedious computation leads to

$$\frac{E_j}{\lambda_j^3} \le N_j^{-(\frac{p}{s'}-q)/(\frac{p}{s'}-\frac{q}{2})}. \tag{5.35}$$

Inserting this expression together with (5.32) in (5.31) we obtain

$$\lambda_\ell^{-2}\,A_\ell \le t^{-C}\,N^{-(\frac{p}{s'}-q)/(\frac{p}{s'}-\frac{q}{2})}\,N^{-\frac{p-2q}{2(p-q)}}\,N^{1/2}\,N^{3/4},$$

so that we will have $\lambda_\ell^{-2}A_\ell \le t^{28}$ at least when

$$\tfrac{5}{4} < \tfrac{2p-2s'q}{2p-s'q} + \tfrac{p-2q}{2(p-q)}\,. \tag{5.36}$$

When $s=2$ this is easily seen to be equivalent to

$$p > 7q = \tfrac{70}{3} = 23.333...$$

as asserted in Theorem 1.1. When $2 < s < 3-\frac{15}{3p+5}$, then solving for p in (5.36) leads to the range

$$p > p(s) = \big(11s - 6 + \sqrt{65s^2 - 76s + 36}\big)\,q/(4(s-1)), \tag{5.37}$$

which therefore completes the proof of Theorem 1.1.

Remark 5.5. We point out that the range of p obtained in (5.37) when $2 < s < 3-\frac{15}{3p+5}$ and $d=2$ is slightly better than the interpolated line between $(\frac{1}{p}=\frac{3}{70}, \frac{1}{s}=\frac{1}{2})$ and $(\frac{1}{p}=\frac{1}{20}, \frac{1}{s}=\frac{13}{36})$ (see Figure 1.2).

Remark 5.6. One can do similar computations to establish a range for $s<2$, however the region that comes out corresponds precisely to interpolating the case $s=2$ with the trivial $p=\infty, s=1$, and therefore no new result appears in this case (again, see Figure 1.2).

6. Proof of Proposition 4.11

The main result in this section is Lemma 6.2, which gives an improvement over Lemma 2.5 in [22]. The rest of the proof of Proposition 4.11 follows from exactly the same reasoning as in [22], replacing at each occurrence Wolff's Lemma 2.5 by its improved version; we sketch the argument in §6.2. Recall that throughout this section $d=2$.

6.1. **The combinatorial lemma.** In this subsection it will be convenient to follow the notation in [22, §2]. Namely, $\mathcal{P} = \{\pi\}$ will denote a collection of $1 \times \sqrt{\delta} \times \delta$ plates, and $\mathcal{T} = \{\tau\}$ a collection of $1 \times \sqrt{\delta} \times \sqrt{\delta}$ tubes. As usual the longest axes of π and τ point in $\sqrt{\delta}$-separated light rays. We shall also use a collection $\mathbb{P} = \{\Pi\}$ of much larger plates with dimensions $1 \times \delta^{\frac{1}{4}} \times \sqrt{\delta}$, and longest axes pointing in $\delta^{\frac{1}{4}}$-separated directions. All such families are assumed to consist of *separated* plates or tubes, meaning that $C_1\pi$ contains less than C_2 plates from $\mathcal{P}$, and similarly with $\mathcal{T}$ and $\mathbb{P}$.

Fix $t = \delta^{\epsilon_0}$ and consider a tiling $\{B\}$ of $\mathbb{R}^3$ by t-cubes. If $w \in \mathbb{R}^3$, we denote by $B(w)$ the t-cube containing w. Given a finite set $\mathcal{W}$ consisting of $\sqrt{\delta}$-separated points in $\mathbb{R}^3$ and a tube family $\mathcal{T}$, we wish to define relations "$\sim$" between tubes and t-cubes which keep as small as possible the cardinality of the *bad incidence set*

$$I_b(\mathcal{W}, \mathcal{T}) \;=\; \big\{(w,\tau) \in \mathcal{W} \times \mathcal{T} \;:\; w \in \tau,\; \tau \not\sim B(w)\big\}.$$

These relations will be *admissible* if they satisfy the property

$$\text{for every } \tau \in \mathcal{T} \qquad \text{Card}\,\{B \;:\; \tau \sim B\} \lessapprox 1. \tag{6.1}$$

One defines likewise the concept of admissible relation between t-cubes and $\mathbb{P}$-plates, as well as the bad incidence set $I_b(\mathcal{W}, \mathbb{P})$.

As a special example consider the relation $\tau \sim B$ if B is equal or adjacent to a fixed cube maximizing $|\mathcal{W} \cap \tau \cap B|$, and likewise for $\mathbb{P}$-plates. Using this relation, Wolff proves the following result.

Lemma 6.1. : (see [22, Lemma 2.3]**).** *Let $\mathcal{W}$ be a $\sqrt{\delta}$-separated set in $\mathbb{R}^3$.*
(i) *Given a plate family $\mathbb{P}$, there exists an admissible relation $\sim$ so that, for every $\varepsilon > 0$*

$$\textit{Card}\; I_b(\mathcal{W}, \mathbb{P}) \le C_\varepsilon\, \delta^{-\varepsilon}\, t^{-6}\, |\mathbb{P}|^{1/3}\, |\mathcal{W}|. \tag{6.2}$$

(ii) *Given a tube family $\mathcal{T}$, there exists an admissible relation $\sim$ so that*

$$\textit{Card}\; I_b(\mathcal{W}, \mathcal{T}) \lessapprox t^{-5}\, |\mathcal{T}|^{1/2}\, |\mathcal{W}|. \tag{6.3}$$

The statement in (i) is by far much deeper than its counterpart in (ii), relying on highly non trivial bounds for circle tangencies. In his paper, Wolff improves the bound in (6.3) by combining it with (6.2) (see [22, Lemma 2.5]). It seems, though, that both his statement and proof can be simplified. Below, given a set $\mathcal{T}$, we denote by $\mathbb{P}(\mathcal{T})$ a plate family of minimal cardinality so that each $\tau \in \mathcal{T}$ is contained in some $\Pi \in \mathbb{P}(\mathcal{T})$ (as in [22, p. 1255]).

Lemma 6.2. *Let $\mathcal{W}$ be a $\sqrt{\delta}$-separated set in $\mathbb{R}^3$, and $\mathcal{T}$ a tube family so that every $\Pi \in \mathbb{P}(\mathcal{T})$ contains at most m tubes. Then, there exists an admissible relation $\sim$ so that, for every $\varepsilon > 0$*

$$\text{Card } I_b(\mathcal{W},\mathcal{T}) \le C_\varepsilon\, \delta^{-\varepsilon}\, t^{-11}\, m^{1/6}\, |\mathcal{T}|^{1/3}\, |\mathcal{W}|. \tag{6.4}$$

Proof. Assume first that every plate $\Pi \in \mathbb{P}(\mathcal{T})$ contains between $m/2$ and m tubes from $\mathcal{T}$. Given $\tau \in \mathcal{T}$, let Π be the plate in $\mathbb{P}(\mathcal{T})$ containing τ, and $\mathcal{T}_\Pi$ the subset of all tubes from $\mathcal{T}$ contained in Π. Define the relation $\tau \sim B$ when one of the following holds:

(a) $\Pi \sim B$, as in (i) of Lemma 6.1, with respect to the set $\mathcal{W}$ and the plate family $\mathbb{P}(\mathcal{T})$;

(b) $\tau \sim B$, as in (ii) of Lemma 6.1, with respect to the set $\mathcal{W} \cap \Pi \cap [\cup_{B \not\sim \Pi} B]$ and the tube family $\mathcal{T}_\Pi$.

More precisely, if we denote by B_Π the union of the t-cubes B which are equal or adjacent to the cube maximizing $|\mathcal{W} \cap \Pi \cap B|$, and denote by B_τ the union of the t-cubes B which are equal or adjacent to the cube maximizing $|[\mathcal{W} \cap B_\Pi^c] \cap \tau \cap B|$, then

$$\tau \sim B \quad \text{iff} \quad B \subset B_\Pi \cup B_\tau.$$

Clearly $\sim$ is an admissible relation. Moreover,

$$\begin{aligned}
\text{Card } I_b(\mathcal{W},\mathcal{T}) &= \sum_{\tau\in\mathcal{T}} \big|\mathcal{W} \cap \tau \cap \big[\cup_{B\not\sim\tau} B\big]\big| = \sum_{\Pi\in\Pi(\mathcal{T})} \sum_{\substack{\tau\in\mathcal{T}\\ \tau\subset\Pi}} \big|\mathcal{W}\cap\tau\cap B_\Pi^c \cap B_\tau^c\big| \\
\text{(by (6.3))} \quad &\lessapprox \sum_{\Pi\in\Pi(\mathcal{T})} t^{-5}\, m^{1/2}\, \big|\mathcal{W}\cap\Pi\cap B_\Pi^c\big| \\
\text{(by (6.2))} \quad &\le C_\varepsilon\, \delta^{-\varepsilon}\, t^{-11}\, m^{1/2}\, \big|\mathbb{P}(\mathcal{T})\big|^{1/3}\, \big|\mathcal{W}\big| \\
&\lesssim C_\varepsilon\, \delta^{-\varepsilon}\, t^{-11}\, m^{\frac12-\frac13}\, \big|\mathcal{T}\big|^{1/3}\, \big|\mathcal{W}\big|,
\end{aligned}$$

since by assumption $|\mathbb{P}(\mathcal{T})| \approx |\mathcal{T}|/m$. Finally, to remove the condition that each Π contains at least $m/2$ tubes, simply partition $\mathcal{T}$ into the subfamilies $\mathcal{T}_j = \cup\{\mathcal{T}_\Pi \ :\ 2^{j-1} \le |\mathcal{T}_\Pi| < 2^j\}$, and apply the above reasoning to each $\mathcal{T}_j$. □

Remark 6.3. Observe that m in the statement of the lemma is always $m \lesssim N^{\frac12}$, since each Π may contain at most $N^{\frac14}$ parallel tubes pointing in each of $N^{\frac14}$ different directions. In fact, below we shall only use (6.4) with $m = N^{\frac12}$.

Remark 6.4. From Lemma 6.2 it is easy to derive a version with "Schwartz tails" as in [22, Lemma 2.7]. Namely, letting

$$\mathcal{I}_b(\mathcal{T},\mathcal{W}) = \sum_{w\in\mathcal{W}} \sum_{\substack{\tau\in\mathcal{T}\\ \tau\not\sim B(w)}} w_\tau(w),$$

then with the same conditions as in Lemma 6.2 there is an admissible relation $\sim$ so that for all $\varepsilon > 0$

$$\mathcal{I}_b(\mathcal{T},\mathcal{W}) \le C_\varepsilon\, \delta^{-\varepsilon}\, t^{-11}\, N^{1/12}\, |\mathcal{T}|^{1/3}\, |\mathcal{W}| + \delta^{100}|\mathcal{W}|. \tag{6.5}$$

Remark 6.5. We point out that, according to the scaling we have adopted in the paper, we will use the results in this subsection with families $\mathcal{T}$ of $N \times \sqrt{N} \times \sqrt{N}$-tubes and sets $\mathcal{W}$ of $\sqrt{N}$-separated points. Of course, all the results remain valid with this scaling, by a simple change of variables.

6.2. **Proof of Proposition 4.11.** We only sketch the proof of Proposition 4.11, since it is essentially the same as in [22, Lemma 3.2] or [12, Lemma 5.3].

We are given an (N, E, Q)-packet f, and consider the subpacket $f^* = \sum_{\pi\in\mathcal{P}^*} f_\pi$ in Definition 4.8 and $\lambda \ge 1$ so that (4.19) holds. Reasoning as in [22, p. 1267] one can find a finite set of $N^{\frac{1}{2}}$-separated points $\mathcal{W} \subset \{|f^*| > c\lambda/\log N\}$ and a real number $a = a(N) > 0$ so that the set

$$\widetilde{W} := \bigcup_{w\in\mathcal{W}} \Delta(w) \cap \left\{|f^*| > \tfrac{c\lambda}{\log N}\right\}$$

(with $\Delta(w)$ denoting the $\sqrt{N}$-cube containing w) satisfies

$$\text{meas}\,\left\{|f^*| > c\lambda/\log N\right\} \lessapprox \text{meas}\,(\widetilde{W})$$

and

$$\text{meas}\,\left(\Delta(w) \cap \left\{|f^*| > \tfrac{c\lambda}{\log N}\right\}\right) \approx a\, N^{\frac{3}{2}}, \quad \forall\, w \in \mathcal{W}. \tag{6.6}$$

Let $\sim$ denote the equivalence relation relative to $(\mathcal{W}, \mathcal{T}(\mathcal{P}^*))$ obtained in Remark 6.4, and given $\pi \in \mathcal{P}^*$, define $\pi \sim B$ when the tube $\tau \in \mathcal{T}(\mathcal{P}^*)$ whose 10-fold dilate contains π satisfies $\tau \sim B$. Define the plate families $\mathcal{P}_B = \{\pi \in \mathcal{P}^* \ :\ \pi \sim B\}$, which satisfy

$$\sum_B |\mathcal{P}_B| \lessapprox |\mathcal{P}^*|$$

by property (6.1) from the previous subsection. By (4.15), to obtain the λ-localization of f as in Definition 4.3 it suffices to show that

$$\big|\{|f^*| > c\lambda/\log N\}\big| \lessapprox \sum_B \big|B \cap \{|f^B| \gtrapprox \lambda\}\big|, \tag{6.7}$$

where $f^B = \sum_{\pi \in \mathcal{P}_B} f_\pi$. To prove (6.7) we use the crude estimate

$$|f^*(x) - f^B(x)| \lesssim \sum_{\tau \not\sim B} w_\tau(x)$$

and show that the right hand side is $\ll \lambda/\log N$ when $x \in B \cap \widetilde{W}$. Indeed, by Lemma 6.2 (in its version with Schwartz tails; see Remark 6.4) and the fact that w_τ is essentially constant in $\sqrt{N}$-cubes we have

$$\begin{aligned}
\int_{\widetilde{W}} \sum_{\tau \not\sim B(x)} w_\tau(x)\,dx \;&\lesssim\; a\,N^{\frac{3}{2}} \sum_{w \in \mathcal{W}} \sum_{\tau \not\sim B(w)} w_\tau(w) \;=\; a\,N^{\frac{3}{2}}\,\mathcal{I}_b(\mathcal{T}(\mathcal{P}^*), \mathcal{W}) \\
&\le\; C_\varepsilon\,N^\varepsilon\,t^{-11}\,N^{\frac{1}{12}}\,\big|\mathcal{T}(\mathcal{P}_r)\big|^{\frac{1}{3}}\,\#(\mathcal{W})\,a\,N^{\frac{3}{2}} \\
&\lesssim\; C_\varepsilon\,N^\varepsilon\,t^{-11}\,N^{\frac{1}{12}}\left[\tfrac{|\mathcal{P}_f|}{2^{\mu_*}}\right]^{\frac{1}{3}}\big|\widetilde{W}\big|,
\end{aligned}$$

which is smaller than $c|\widetilde{W}|\lambda/(4\log N)$ if the sufficient condition (4.19) holds (choosing $\varepsilon \ll \epsilon_0$ and $N \ge N_0(\epsilon_0)$). Thus, there exists a subset W^* of $\widetilde{W}$ with proportional measure so that

$$\sum_{\tau \not\sim B(x)} w_\tau(x) < c\lambda/(4\log N), \quad x \in W^*.$$

Therefore, if $x \in B \cap W^*$ we have $|f^*(x) - f^B(x)| \le c\lambda/(4\log N)$, which implies $|f^B(x)| > c\lambda/(4\log N)$. Thus,

$$\big|\{|f^*| > c\lambda/\log N\}\big| \lessapprox \big|\widetilde{W}\big| \lesssim \big|W^*\big| \le \sum_B \big|B \cap \{|f^B| \gtrapprox \lambda\}\big|,$$

as we wished to prove. Finally, to obtain (4.20) when the condition (4.19) does not hold, one repeats the same argument as at the end of the proof of Proposition 4.9. We leave details to the reader.

7. Boundedness of Bergman projections

Corollary 1.6 follows from Theorem 1.1 and the arguments in [1, §5]. To be more precise, one has the following (stronger) result:

Proposition 7.1. *Let* $2 \leq s \leq w < \infty$ *and suppose that* $\mathcal{H}^{str}(w, s, \varepsilon)$ *holds for all* $\varepsilon > 0$. *Let*

$$\gamma(w, s) = -1 + 2s\beta(w, s). \tag{7.1}$$

Then, for every $\gamma > -1$, *the Bergman projection* P_γ *is bounded in the mixed-norm space* $L^{p,u}_\gamma(\mathcal{T}^{d+1}) = L^u(\Delta^\gamma(Y)dY; L^p(dX))$ *in the optimal range* $2 \leq u < \tilde{u}_{\gamma,p} = (\gamma + d)/(\frac{d+1}{2p'} - 1)$ *whenever*

$$p \geq p_{w,s,\gamma} := w + \tfrac{w}{s}\,\frac{(\gamma_{w,s} - \gamma)_+}{\gamma + 1}. \tag{7.2}$$

Proof. The result follows from [1, Prop. 5.5] by using a similar reasoning as in [1, Corol. 5.11]. Namely, assuming first $\gamma \geq \gamma(w, s)$, then $\mathcal{H}^{str}(w, s)$ implies [1, (5.6)] for all $p \geq w$ and all $\mu > s\beta(p, s)$, which in turn by [1, Prop. 5.5] implies (after some arithmetics) the boundedness of P_γ in $L^{p,u}_\gamma(\mathcal{T}^{d+1})$ in the optimal range $2 \leq u < \tilde{u}_{\gamma,p}$.

When $\gamma < \gamma(w, s)$ one must find (ρ, σ) so that $\mathcal{H}^{str}(\rho, \sigma)$ holds and $\gamma = \gamma(\rho, \sigma)$. By interpolation with the trivial $(\infty, 1)$-estimate, $\mathcal{H}^{str}(\rho, \sigma)$ holds when $\rho \geq w$ and $\sigma' = s'\rho/w$. Since with this choice $\gamma(\rho, \sigma) \searrow -1$ as $\rho \to \infty$, one can always find a (unique) ρ so that $\gamma = \gamma(\rho, \sigma)$. In fact, a simple computation shows that $\rho = p_{w,s,\gamma}$ as in (7.2). Thus, by the first part of the proof P_γ is bounded in $L^{p,u}_\gamma(\mathcal{T}^{d+1})$ in the optimal range $2 \leq u < \tilde{u}_{\gamma,p}$, for all $p \geq p_{w,s,\gamma}$. □

To obtain Corollary 1.6 from Proposition 7.1 one must specialize to the diagonal case $p = u$. First, an easy computation shows that $2 \leq p < \tilde{u}_{\gamma,p}$ is equivalent to $2 \leq p < 1 + 2(\gamma + d + 1)/(d - 1)$, which gives the conjectured range of L^p_γ-boundedness for P_γ in (1.13) (by duality); thus, it suffices to find all γ's so that the endpoint $p = \tilde{u}_{\gamma,p}$ is $\geq p_{w,s,\gamma}$ as in (7.2). Straightforward arithmetics show that this is the case for

$$\gamma \geq \frac{d-1}{2}\Big(w - \tfrac{2(d+1)}{d-1} - 1 + \tfrac{w}{s}\,\frac{(\gamma(w, s) - \gamma)_+}{\gamma + 1}\Big).$$

When $d \geq 3$, we let $w = p_d$ and $s = 2$, so that considering the two cases $\gamma > \gamma(w, s)$ and $\gamma \leq \gamma(w, s)$, one obtains the conditions in (1.14). When $d = 2$, one may use $w = p_2 = 20$ and $s = 3$, which leads to the same conditions on γ (namely to $\gamma \geq (w - 7)/2 = 6.5$). This establishes Corollary 1.6.

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