A MIXED NORM VARIANT OF WOLFF'S INEQUALITY FOR PARABOLOIDS

GUSTAVO GARRIGÓS AND ANDREAS SEEGER

Abstract. We adapt the proof for $\ell^p(L^p)$ Wolff inequalities in the case of plate decompositions of paraboloids, to obtain stronger $\ell^2(L^p)$ versions. These are motivated by the study of Bergman projections for tube domains.

1. Introduction and statement of results

For small $\delta > 0$, let Σ^δ denote a truncated δ-neighborhood of the paraboloid in $\mathbb{R}^d$,

$$\Sigma^\delta \equiv \big\{\xi = (\xi', \xi_d) \in \mathbb{R}^d \ : \ \big|\xi_d - |\xi'|^2/2\big| \le \delta,\ |\xi'| \le 1\big\}. \tag{1.1}$$

Consider the usual covering of Σ^δ by $C(\delta^{1/2} \times \overset{(d-1)}{\cdots} \times \delta^{1/2} \times \delta)$-plates, $\Pi_k^{(\delta)}$, subordinated to a $\sqrt{\delta}$-*separated* sequence $\{y_k\} \subset \mathbb{R}^{d-1}$; namely $\operatorname{dist}(y_k, y_{k'}) \ge \sqrt{\delta}$ if $k \ne k'$, and

$$\Pi_k^{(\delta)} = \Big\{(\xi', \xi_d) \in \Sigma_\delta \ : \ \big|\xi' - y_k\big| \le C'\sqrt{\delta}\Big\}. \tag{1.2}$$

Typically $y_k = k\sqrt{\delta}$ for $k \in \mathbb{Z}^{d-1}$ with $|k| \le \delta^{-1/2}$.

In this paper we are interested in the validity of the inequality

$$\Big\|\sum_k f_k\Big\|_p \le C_\varepsilon\, \delta^{-\beta(p)-\varepsilon}\Big(\sum_k \|f_k\|_p^2\Big)^{1/2}, \quad \text{for all } \{f_k\} \text{ with } \operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)}, \tag{1.3}$$
$$\text{where } \beta(p) = \frac{d-1}{4} - \frac{d+1}{2p}.$$

Theorem 1.1. *Let $d \ge 2$. Then, for all $\varepsilon > 0$ the mixed norm inequality (1.3) holds when $p \ge p_{d,*} = 2 + \frac{8}{d-1} - \frac{4}{d(d-1)}$.*

The power $-\beta(p) - \varepsilon$ is best possible (except perhaps for $\varepsilon > 0$) but the range is not, indeed (1.3) is conjectured to hold for all $p \ge 2 + \frac{4}{d-1}$. The problem is motivated by questions on the Bergman projection for tube domains over light cones [1] where a similar inequality for plate decomposition of neighborhoods of cones plays a crucial role. This harder inequality is considered in [4].

Inequality (1.3) is a mixed norm variant of a *Wolff inequality for paraboloids* which itself can be considered as a model problem simplifying the corresponding harder problem for

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decompositions of cone multipliers in $\mathbb{R}^{d+1}$ (see [13], [7], [5], [4]). Let $\alpha(p) := d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$, the standard Bochner-Riesz critical index in d dimensions. Then Wolff's inequality for paraboloids asserts that for all $\varepsilon > 0$

$$\Big\| \sum_k f_k \Big\|_p \le C_\varepsilon\, \delta^{-\alpha(p)-\varepsilon} \Big(\sum_k \|f_k\|_p^p \Big)^{1/p}, \quad \text{for all } \{f_k\} \text{ with supp } \widehat{f_k} \subset \Pi_k^{(\delta)}\,. \tag{1.4}$$

As before, the power $\alpha(p)$ is optimal for each p (except for $\varepsilon > 0$), and the inequality is conjectured to hold for all $p > 2 + \frac{4}{d-1}$. By an interpolation argument the inequality (1.4) for some $\widetilde{p}$ implies the mixed norm variant (1.3) in the smaller range $p > 2(\widetilde{p} - 1)$ only. On the other hand, inequality (1.4) for fixed p is implied by (1.4) for the same p, by Hölder's inequality, since $\alpha(p) - \beta(p) = \frac{d-1}{2}(\frac{1}{2} - \frac{1}{p})$. Theorem 1.1 states that the stronger mixed norm inequality holds in the same range as the currently known range for the Wolff inequality (1.4) (*cf.* [5]), that is for $p \ge 2 + \frac{8}{d-1} - \frac{4}{d(d-1)}$. We also remark that the resolution of the problem for the paraboloid is necessary for the corresponding problems for cones in $\mathbb{R}^{d+1}$.

By a randomization argument it is easy to see that the conjectured range $p \ge 2 + 4/(d-1)$ is sharp for (1.4) (and *a fortiori* for (1.3)). Let $\{r_k\}$ be the sequence of Rademacher-functions on $[0,1]$ and define h_k by $\widehat{h}_k(\xi) = \varphi(\delta^{-1}(\xi - \omega_k))$ for a C^∞ function φ supported in $\{|\xi| \le 1/10\}$, and where $\omega_k = (y_k, |y_k|^2/2)$. Let $h_{k,t}(x) = r_k(t) h_k(x)$ for $t \in [0,1]$. Then the validity of Wolff's inequality implies that

$$\Big(\int_0^1 \Big\| \sum h_{k,t} \Big\|_p^p dt \Big)^{1/p} \lesssim \delta^{-\alpha(p)+\epsilon} \Big(\sum_k \|h_k\|_p^p \Big)^{1/p}$$

and by Fubini's theorem and the familiar inequality for Rademacher functions ([10])

$$\Big\| \Big(\sum_k |h_k|^2 \Big)^{1/2} \Big\|_p \lesssim \delta^{-\alpha(p)-\epsilon} \Big(\sum_k \|h_k\|_p^p \Big)^{1/p}.$$

This leads to $\delta^{-(d-1)/4} \lesssim \delta^{-\alpha(p)-\varepsilon} \delta^{-(d-1)/(2p)}$ and consequently to the restriction $p \ge \frac{2(d+1)}{d-1}$.

Returning to (1.3), there is a square-function variant with a larger exponent,

$$\Big\| \sum_k h_k \Big\|_{L^q(\mathbb{R}^d)} \le C_{q,\varepsilon} \delta^{-(\varepsilon + \alpha(q)/2)} \Big\| \Big(\sum_k |h_k|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^d)}, \quad \forall\, \{h_k\} \;:\; \text{supp } \widehat{h_k} \subset \Pi_k^{(\delta)}, \tag{1.5}$$

which is known to hold in some range of $q \le 2(d+1)/(d-1)$. For $q \ge 2(d+1)/(d-1)$ inequality (1.5) with $\varepsilon = 0$ is a consequence of the Stein-Tomas adjoint restriction theorem as was shown by Bourgain [2]. In two dimensions the inequality in the optimal range $q \ge 4$, again with $\varepsilon = 0$, is due to Fefferman [3], and the proof of the crucial L^4 bound is based on the observation that in two dimensions the algebraic sums of plates $\Pi_k^{(\delta)} + \Pi_{k'}^{(\delta)}$ are essentially disjoint as (k, k') run over integers with $|k|, |k'| \lesssim \delta^{-1/2}$. In dimensions $d \ge 3$ it is conjectured, but not known, that (1.5) holds on $L^{q_0}(\mathbb{R}^d)$ with $q_0 = 2d/(d-1)$. Partial results in higher dimensions follow from the bilinear adjoint restriction theorem of Tao [11] using arguments in [8], [5]; indeed (1.5) is known to hold for $q > 2 + 4/d$.

By Minkowski's inequality the square function bound (1.5) also implies the weaker (and possibly non optimal)

$$\Big\| \sum_k h_k \Big\|_{L^q(\mathbb{R}^d)} \le C_\varepsilon \delta^{-(\alpha(q)+\varepsilon)/2} \Big(\sum_k \|h_k\|^2_{L^q(\mathbb{R}^d)} \Big)^{1/2}, \quad \operatorname{supp} \widehat{h_k} \subset \Pi_k^{(\delta)}. \tag{1.6}$$

We use inequality (1.6) as a hypothesis:

Definition 1.2. Suppose $2d/(d-1) < q \le 2(d+1)/(d-1)$. We say that *Hypothesis* $S(2,q)$ holds if (1.6) holds for all $\varepsilon > 0$.

Under this hypothesis we show

Theorem 1.3. *Suppose $d \ge 2$ and $2d/(d-1) < q \le 2(d+1)/(d-1)$ and Hypothesis $S(2,q)$ holds. Then the inequality* (1.3) *holds for all $p \ge p_d := q + 4/(d-1)$.*

Theorem 1.1 follows from Theorem 1.3 since, as pointed out above, $S(2,q)$ holds for $q > 2 + 4/d$. If one could prove the above square function estimate in the optimal range $q > 2d/(d-1)$ (and therefore $S(2,q)$ in the same range) then the range of (1.3) would improve to $p \ge 2 + 6/(d-1)$.

A reformulation. Let ζ be a function in $C_c^\infty(\mathbb{R}^{d-1})$ which is identically 1 in the cube $\{\xi' : |\xi_i| \le 1, i = 1, \dots, d-1\}$, and let ζ_0 be a Schwartz function on $\mathbb{R}$ with compact support in $(-2,2)$ so that $\zeta_0(\tau) = 1$ for $|\tau| \le 1$. For $k \in \mathbb{Z}^{d-1}$, $|k| \lesssim \delta^{-1/2}$ define operators $P_k = P_k^{(\delta)}$ by

$$\widehat{P_k f}(\xi) = \zeta(\delta^{-1/2}\xi' - k)\, \zeta_0\big(\delta^{-1}(\xi_d - |\xi'|^2/2)\big)\, \widehat{f}(\xi).$$

Note that with the choice of $y_k = \delta^{1/2}k$ the supports of the functions $\widehat{P_k f}$ are essentially the plates $\Pi_k^{(\delta)}$ (actually slightly expanded plates).

The operators P_k are uniformly bounded on all L^p (as long as $|k| \lesssim \delta^{-1/2}$) and (1.3) is equivalent with the statement that for all families of L^p functions $\{h_k\}$

$$\Big\| \sum_{|k| \lesssim \delta^{-1/2}} P_k h_k \Big\|_p \le C_\varepsilon \delta^{-\beta(p)-\varepsilon} \Big(\sum_k \|h_k\|_p^2 \Big)^{1/2}. \tag{1.7}$$

For functions with Fourier transform supported in Σ^δ we may define a norm

$$\|f\|_{p,2;\delta} = \Big(\sum_k \|P_k f\|_p^2 \Big)^{1/2}. \tag{1.8}$$

Note that if $f = \sum f_k$ with $\operatorname{supp} \widehat{f_k} \subset \Pi_k^\delta$ we have

$$\|f\|_{p,2;\delta} \approx \Big(\sum_k \|f_k\|_p^2 \Big)^{1/2}. \tag{1.9}$$

More general surfaces. Theorem 1.1 may be extended to convex surfaces with nonvanishing Gaussian curvature, using arguments in §2 of [9]. Namely, one notes that on sets of diameter $\gamma^{1/3} \ll 1$ the surface can be approximated by paraboloids with accuracy $O(\gamma)$ and uses the scaled estimate in §5 below, together with an induction on scales argument. One could also modify the proof for paraboloids using arguments in [6] (which apply to more general situations).

Acknowledgements. The main ideas can be traced back to the pioneering work by Wolff [13], see also the subsequent articles [7], [9], [6], [5] and [4]. An earlier version of this paper was originally written as class notes intended to give an expository account of some of the material in [13] and [7]. For self-containedness and in order to retain the expository nature of the notes we have included in §4 material from Łaba-Wolff [7] which could have been quoted. We are indebted to Wilhelm Schlag for comments and for collaboration on [4] and to Detlef Müller for useful remarks on an earlier version of this paper.

2. Notation and basic definitions

We note that because of the appearance of ε in (1.3) we may assume that $p > p_d = q + 4/(d-1)$ since we can then interpolate with a trivial $\ell^2(L^2)$ inequality to get the result for $p = p_d$.

Throughout we fix $p > p_d$. We also fix a positive but very small ε_0, which may depend on p and q and will be determined later. We remark that for the proof of Theorem 1.3 in the range $p > p_d = q + 4/(d-1)$ the choice

$$\varepsilon_0 = 10^{-3} d^{-1}(d - 1 - 4/(p - q)) \tag{2.1}$$

is admissible. Statements involving the parameter δ are assumed to hold for all $\delta \in (0, \delta_0]$, for some fixed $\delta_0 \ll 1$. For each such δ we set

$$N = 1/\delta \quad \text{ and } t = \delta^{\epsilon_0} = N^{-\epsilon_0}. \tag{2.2}$$

The constants $C, c_0, c_1, ...$ appearing below may depend on $p, d, \varepsilon_0, \delta_0$ and also on other constants appearing below, but will be independent of δ, f_k, $\{y_k\}$, and parameters such as λ or ε. Otherwise we will indicate it by C_ε, etc... By $A \lesssim B$ we will mean $A \le C\, B$ for some C as above, and by $A \lessapprox B$ we mean $A \le C\,(\log N)^C\, B$, for some $C > 0$. We shall write either card$(\mathcal{P})$ or $\#\mathcal{P}$ for the cardinality of a finite set $\mathcal{P}$, and meas (A) or $|A|$ for the Lebesgue measure of a set in $\mathbb{R}^d$.

Plates and plate families. A rectangular box in $\mathbb{R}^d$ of size $\sqrt{N} \times \overset{(d-1)}{\cdots} \times \sqrt{N} \times N$. will be referred to as an *N-plate.* We typically denote plates in x-space by π and plate families by $\mathcal{P}$. We shall always assume that N-plates are essentially dual to some $\Pi_k^{(\delta)}$. In this case we use the notation

$$\pi \parallel k$$

to indicate that π is an N-plate, whose long side is parallel to $\mathfrak{n}_k = (y_k, -1) = (k\sqrt{\delta}, -1)$. Observe that, for different k's, plate directions are $\sqrt{\delta}$-separated, since so are the directions of $\{\mathfrak{n}_k\}$. The integer vectors k will be taken in

$$\mathcal{Z}(\sqrt{N}) = \{k \in \mathbb{Z}^{d-1} : |k_i| \leq \sqrt{N}, i = 1, \dots, d-1\}.$$

We shall also assume that families $\mathcal{P}$ consist only of *separated* plates, meaning that for each $\pi \in \mathcal{P}$ at most C_1 plates from $\mathcal{P}$ can be contained in a fixed dilate $C_2\pi$, where C_1 and C_2 are fixed universal constants. This means that for fixed k, plates $\pi \| k$ are essentially disjoint.

We recall that the cardinality of $\mathcal{Z}(\sqrt{N})$, and thus the number of essentially different directions that plates can achieve at scale N, is approximately $N^{\frac{d-1}{2}}$. Finally, a *σ-cube* Δ is a cube of sidelength σ centered at some point of the grid $\sigma\mathbb{Z}^d$.

Localizing weight functions. Given a fixed large M we let

$$w(x) = (1 + |x|^2)^{-M/2}, \tag{2.3}$$

and given a rectangle R we denote $w_R = w \circ a_R^{-1}$, where a_R is an affine map taking the unit cube centered at 0 to the rectangle R. Thus w_R is roughly the characteristic function of R with "Schwartz tails" (with an abuse of language as for fixed M the function w is not a Schwartz-function).

We shall also use a fixed Schwartz function ψ, strictly positive in $B_2(0)$, with Fourier transform supported in $B_{\frac{1}{100}}(0)$, and so that $\sum_{n \in \mathbb{Z}^d} \psi^2(\cdot + n) = 1$. Again we set

$$\psi_R = \psi \circ a_R^{-1}. \tag{2.4}$$

In particular, if $\{\Delta\}$ is a tiling of $\mathbb{R}^d$ by σ-cubes with centers c_Δ in $\sigma\mathbb{Z}^d$, then $\sum_\Delta \psi_\Delta^2 = 1$, where $\psi_\Delta(x) = \psi((x - c_\Delta)/\sigma)$.

Elementary properties of $\| \cdot \|_{p,2;\delta}$.

Lemma 2.1. *Let* $2 \leq p \leq \infty$ *and* $\widehat{f}$ *be supported in* Σ^δ. *Then*

$$\|f\|_{\infty,2;\delta} \lesssim N^{-(d+1)/2p} \|f\|_{p,2;\delta}, \tag{2.5}$$

$$\|f\|_\infty \lesssim N^{\beta(p)} \|f\|_{p,2;\delta}, \tag{2.6}$$

$$\|f\|_{p,2;\delta} \lesssim \|f\|_2^{2/p} \|f\|_{\infty,2;\delta}^{1-2/p}. \tag{2.7}$$

Proof. If $\widehat{g}$ is supported in $\Pi_k^{(\delta)}$ then by Young's inequality $\|g\|_\infty \lesssim N^{-(d+1)/2p} \|g\|_p$; this yields (2.5). If $f = \sum f_k$ with $\widehat{f_k}$ supported in $\Pi_k^{(\delta)}$ then

$$\|f\|_\infty \lesssim \sum_k \|f_k\|_\infty \lesssim N^{\frac{d-1}{4}} \Big(\sum_k \|f_k\|_\infty^2 \Big)^{1/2} \lesssim N^{\frac{d-1}{4} - \frac{d+1}{2p}} \Big(\sum_k \|f_k\|_p^2 \Big)^{1/2}$$

which is (2.6). Inequality (2.7) follows from a corresponding interpolation inequality for the projection operators P_k, namely for $\vartheta = 1 - 2/p$,

$$\Big(\sum_k \|P_k h_k\|_p^2\Big)^{1/2} \lesssim \Big(\sum_k \|h_k\|_2^2\Big)^{(1-\vartheta)/2}\Big(\sum_k \|h_k\|_\infty^2\Big)^{\vartheta/2}.$$

This follows by convexity from the obvious cases $p = 2$ and $p = \infty$. □

We also need the following localization estimate.

Lemma 2.2. *Let $\widehat{f}$ be supported in Σ^δ. Let $\mathcal{Q} = \{Q\}$ be a grid of N-cubes and let ψ_Q be as in* (2.4) *(so that $\widehat{\psi}_Q$ is supported in $|\xi| \le (100N)^{-1}$). Then*

$$\Big(\sum_Q \|\psi_Q f\|_{p,2;2\delta}^p\Big)^{1/p} \lesssim \|f\|_{p,2;\delta}\,. \tag{2.8}$$

Proof. Note that $\widehat{\psi_Q} * \widehat{f}$ is supported in $\Sigma^{2\delta}$. The case $p = \infty$ is immediate and the case $p = 2$ follows by orthogonality. One uses the projection operators P_k to set up an interpolation argument showing the inequality for $2 < p < \infty$. □

Packets.

Definition 2.3. (i) f is called an *N-packet associated with* $\Pi_k^{(\delta)}$ if it can be written as $f = \sum_{\pi\in\mathcal{P}} f_\pi$ for some family $\mathcal{P} = \mathcal{P}_k(f)$ of separated N-plates with $\pi \parallel k$, in such a way that every f_π, $\pi \in \mathcal{P}_k$, satisfies

$$|f_\pi| \le c_1\, w_\pi \qquad \text{and} \qquad \operatorname{supp} \widehat{f_\pi} \subset\, c_2\, \Pi_k^{(\delta)}. \tag{2.9}$$

(ii) Let R be a cube of diameter $\ge N$ and let $E \subset \mathcal{Z}(\sqrt{N})$. An *$(N, R, E)$-packet* f is a function that can be written as

$$f = \frac{1}{\sqrt{\#E}} \sum_{k\in E} \sum_{\pi\in\mathcal{P}_k} f_\pi \tag{2.10}$$

where $\mathcal{P}_k$ consists of plates $\pi \parallel k$ which have nonempty intersection with R, and f_π are functions so that (2.9) holds for all $\pi \in \mathcal{P}_k$ and all $k \in E$. We denote by $\mathcal{P}(f) = \cup_{k\in E}\mathcal{P}_k$ the plate family of f.

(iii) For f as in (ii), we say that *g is a subpacket of f* if $g = \frac{1}{\sqrt{\#E}} \sum_{k\in E} \sum_{\pi\in\mathcal{P}'_k} f_\pi$, with $\mathcal{P}'_k \subset \mathcal{P}_k$.

(iv) An (N, R, E)-packet f as in (2.10) is called *stable* if it satisfies

$$\frac{1}{2}\#\mathcal{P}_k \le \#\mathcal{P}_{k'} \le 2\#\mathcal{P}_k \quad \text{whenever } k, k' \in E\,. \tag{2.11}$$

Elementary properties of packets are listed in

Lemma 2.4. *Let f be an (N,Q,E)-packet. Then*

$$\|f\|_{\infty,2;\delta} \lesssim 1, \tag{2.12}$$

$$\|f\|_\infty \lesssim \sqrt{\#E} \lesssim N^{(d-1)/4}, \tag{2.13}$$

and, for $2 \le p < \infty$

$$\|f\|_{p,2;\delta}^p \le C_p \frac{N^{(d+1)/2}\#\mathcal{P}(f)}{\#E}\,. \tag{2.14}$$

Proof. The bounds (2.12) and (2.13) are immediate. We can use the interpolation inequality $\|G\|_{\ell^2(L^p)} \le \|G\|_{\ell^2(L^2)}^{2/p}\|G\|_{\ell^2(L^\infty)}^{1-2/p}$ to see that (2.14) follows from the (2.12) and the case $p=2$ of (2.14). Observe that $\|f\|_{2,2;\delta}^2$ is dominated by

$$\sum_{k\in E}\Big\|\sum_{\pi\in\mathcal{P}_k}\frac{f_\pi}{\sqrt{\#E}}\Big\|_2^2 \lesssim \frac{1}{\#E}\sum_{k\in E}\Big\|\sum_{\pi\in\mathcal{P}_k} w_\pi\Big\|_2^2 \lesssim \frac{1}{\#E}\sum_{k\in E} N^{\frac{d+1}{2}}\#\mathcal{P}_k$$

and the last expression is equal to $N^{(d+1)/2}\#\mathcal{P}(f)/\#E$. □

Another preparatory result concerns decompositions of functions with Fourier support in Σ^δ into (stable) N-packets. The stability property (2.11) gives estimate (2.18) in the following lemma (a sort of converse to (2.14)), which will be crucial in the induction on scales argument, *cf.* Lemma 6.1 below.

Lemma 2.5. *Let $\widehat{f}$ be supported in Σ^δ and assume that*

$$\|f\|_{\infty,2;\delta} \le A. \tag{2.15}$$

Let Q be an N-cube, let $0<\varepsilon\le 1$, and let R be the cube of sidelength $N^{1+\varepsilon}$ with the same center as Q. Then on Q we may decompose

$$f(x) = \sum_{AN^{-10d}\le 2^j\le C_\varepsilon A} 2^j \sum_{\ell=1}^{n_j} f_{j,\ell}(x) \,+\, g(x), \quad x\in Q, \tag{2.16}$$

for some integers $n_j \le C_\varepsilon(\log N)^2$, and where

(i) the function g satisfies

$$\sup_{x\in Q}|g(x)| \le C_\varepsilon N^{-8d}A; \tag{2.17}$$

(ii) for each j,ℓ the function $f_{j,\ell}$ is a stable $(N,R,E_{j,\ell})$-packet, for some subset $E_{j,\ell}$ of $\mathcal{Z}(N^{1/2})$, and with associated plate family $\mathcal{P}^{j,\ell}$ containing only plates π with dist$(Q,\pi)\le N^{1+\varepsilon}$;

(iii) for every $2 \le p \le \infty$ and every j, ℓ it holds

$$2^j (N^{\frac{d+1}{2}} \# \mathcal{P}^{j,\ell})^{1/p} \lesssim \|f\|_{p,2;\delta} (\# E_{j,\ell})^{1/p}. \tag{2.18}$$

Proof. We decompose $f = \sum_k f_k$ where $\widehat{f_k}$ is supported in $\Pi_k^{(\delta)}$. By a pidgeonhole argument we may immediately reduce to the case where the k are strongly separated, in the sense that, if k and k' occur in the sum and are different then $|k - k'| \ge 10d$. Note that then

$$\|f\|_{p,2;\delta} \approx (\sum_k \|f_k\|_p^2)^{1/2}.$$

Next, we fix k, and further decompose f_k as

$$f_k = \sum_{\pi \| k} f_k \psi_\pi^2.$$

We let $\mathcal{P}_k \equiv \mathcal{P}_{k,R}(f)$ be the family of all π with $\pi \parallel k$ and which intersect R. Notice that there are at most $O(N^{d(1+\varepsilon)})$ in $\cup_k \mathcal{P}_{k,R}(f)$.

We first discard the terms involving plates that do not intersect R. Let

$$g_{compl}(x) = \sum_k \sum_{\substack{\pi \| k \\ \pi \cap R = \emptyset}} f_k \psi_\pi^2. \tag{2.19}$$

Using the rapid decay of the functions w_π away from π we get

$$\|g_{compl}\|_{L^\infty(Q)} \lesssim N^{(d-1)/4} \|f\|_{\infty,2;\delta} \sup_{x \in Q} \Big(\sum_k \sum_{\substack{\pi \| k \\ \pi \cap R = \emptyset}} |w_\pi(x)|^2 \Big)^{1/2} \le C_\varepsilon A N^{\frac{d-1}{2} - M\varepsilon}$$

and here M (in the definition of (2.3)) may be chosen so large that $M\varepsilon > 10d$.

Secondly we discard terms for which π intersects R but $\|f_k \psi_\pi\|_\infty$ is very small. Define

$$g_{small}(x) = \sum_k \sum_{\substack{\pi \in \mathcal{P}_{k,R} \\ \|f_k \psi_\pi\|_\infty \le A N^{-10d}}} f_k \psi_\pi^2 \tag{2.20}$$

As the cardinality of all plates intersecting R is $O(N^{d(1+\varepsilon)})$ we trivially get

$$\|g_{small}\|_{L^\infty(Q)} \lesssim N^{2d} A N^{-10d}$$

and if we set $g = g_{small} + g_{compl}$ the bound (2.17) follows.

It remains to decompose the function

$$f - g_{small} - g_{compl} = \sum_k \sum_{\substack{\pi \in \mathcal{P}_{k,R} \\ \|f_k \psi_\pi\|_\infty > A N^{-10d}}} f_k \psi_\pi^2\,. \tag{2.21}$$

Note that $\|f\|_\infty \lesssim A N^{(d-1)/4}$ (by (2.6) for $p = \infty$) so that there are only $O(\log N)$ relevant dyadic scales for the possible size of $\|f \psi_\pi\|_\infty$.

For each k define

$$\mathcal{P}^m_{k,R} = \big\{\pi \in \mathcal{P}_{k,R} \ : \ 2^m < \|f_k\psi_\pi\|_\infty \le 2^{m+1}\big\}. \tag{2.22}$$

Next, for $i = 0, 1, 2, \dots,$ define

$$E(i,m) = \big\{k \in \mathcal{Z}(\sqrt{N}) \ : \ 2^i \le \#\mathcal{P}^m_{k,R} < 2^{i+1}\big\}; \tag{2.23}$$

clearly these sets are disjoint subsets of $\mathcal{Z}(\sqrt{N})$. Set

$$F^{i,m} = \sum_{k\in E(i,m)} \sum_{\pi\in\mathcal{P}^m_{k,R}} f_k\psi_\pi^2. \tag{2.24}$$

Notice that by definition the cardinalities of $\mathcal{P}^m_{k,R}$ are comparable for $k \in E(i,m)$. If we divide $F^{i,m}$ by $C2^m\sqrt{\#E(i,m)}$, for suitably large C, then the new function will be a *stable* $(N, R, E(i,m))$ packet.

Recall $AN^{-10d} \le C2^m\sqrt{\#E(i,m)} \lesssim AN^d$. Now for each j with $AN^{-10d} \le C2^j \lesssim AN^d$ there are $n_j = O((\log N)^2)$ pairs (i,m) with

$$2^{j-1} < C2^m\sqrt{\#E(i,m)} \le 2^j; \tag{2.25}$$

for these (i,m) the functions $2^{-j}F^{i,m}$ are also stable $(N, R, E(i,m))$-packets.

We relabel these n_j functions as $f_{j,\ell}$, $\ell = 1, \dots, n_j$, the associated plate families as $\mathcal{P}^{j,\ell}$ and the associated sets $E(i,m)$ of directions as $E_{j,\ell}$ and then obtain the decomposition $f = \sum_{2^j\ge AN^{-10d}} \sum_{\ell=1}^{n_j} 2^j f_{j,\ell} + g$, for $x \in Q$.

If $\mathcal{P}^{j,\ell}_k = \{\pi \in \mathcal{P}^{j,\ell} : \pi \parallel k\}$ then by construction

$$\#\mathcal{P}^{j,\ell} \approx (\#E_{j,\ell})(\#\mathcal{P}^{j,\ell}_k).$$

We use this to verify (2.18). Fix (j,ℓ), and with the above notation assume $E_{j,\ell} = E(i,m)$. Then we observe

$$\begin{aligned}
\|f\|_{p,2,\delta} &\gtrsim \Big(\sum_k \|f_k\|_p^2\Big)^{1/2} \gtrsim \Big(\sum_k \Big(\sum_{\pi\|k} \|f_k\psi_\pi\|_p^p\Big)^{2/p}\Big)^{1/2} \\
&\ge \Big(\sum_{k\in E(i,m)} \Big(\sum_{\pi\in\mathcal{P}^m_{k,R}} \|f_k\psi_\pi\|_p^p\Big)^{2/p}\Big)^{1/2} \gtrsim \Big(\sum_{k\in E(i,m)} \Big(\sum_{\pi\in\mathcal{P}^m_{k,R}} \|f_k\psi_\pi\|_\infty^p N^{(d+1)/2}\Big)^{2/p}\Big)^{1/2} \\
&\gtrsim \Big(\sum_{k\in E(i,m)} \big(2^{mp}\#\mathcal{P}^m_{k,R} N^{(d+1)/2}\big)^{2/p}\Big)^{1/2} \ge \Big(\sum_{k\in E_{j,\ell}} \Big(2^{mp}\frac{\#\mathcal{P}^{j,\ell}}{\#E_{j,\ell}} N^{(d+1)/2}\Big)^{2/p}\Big)^{1/2} \\
&\gtrsim 2^m (\#E_{j,\ell})^{1/2-1/p} (\#\mathcal{P}^{j,\ell})^{1/p} N^{(d+1)/2p},
\end{aligned}$$

and from (2.25) we obtain (2.18). We note that (2.18) for $p = \infty$ also shows that the sum in j in (2.16) is restricted to the range $2^j \le C_\varepsilon A$. □

3. Equivalent formulations of the problem

We continue to assume that always $p > \frac{2(d+1)}{d-1}$ and that $S(2,q)$ holds for some $q \in [\frac{2d}{d-1}, \frac{2(d+1)}{d-1}]$.

Definition 3.1. Given $p > 2$ and $\gamma > 0$, we say that *hypothesis* $\mathcal{H}^{str}(p,\gamma)$ *holds* if there exists $C_\gamma > 0$ so that for any $\delta \le \delta_0$ and any $f = \sum_k f_k$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$

$$\|f\|_p \le C_\gamma N^{\beta(p)+\gamma} \Big(\sum_k \|f_k\|_p^2 \Big)^{1/2}. \tag{3.1}$$

It is our objective to prove this 'strong' inequality $\mathcal{H}(p,\gamma)$ for all $\gamma > 0$, in the asserted range $p \ge q + 4/(d-1)$. We formulate a weaker condition which can be seen as an analogue of a restricted weak type inequality.

Definition 3.2. Given $p > 2$ and $\gamma > 0$, we say that *hypothesis* $\mathcal{H}(p,\gamma)$ *holds* if there exists $C_\gamma > 0$ so that for all $\delta = N^{-1} \le \delta_0$, for all pairs of N-cubes Q_0, Q_0', for all $E \subset \mathcal{Z}(\delta^{1/2})$, for all stable (N, Q_0, E)-packets f with plate family $\mathcal{P}(f)$, and for all $\lambda \in (N^{\frac{d-1}{4} - \frac{1}{2(p-q)}}, N^{\frac{d-1}{4}})$

$$\big|\{x \in Q_0' : |f(x)| > \lambda\}\big| \le C_\gamma \lambda^{-p} N^{(\beta(p)+\gamma)p} \frac{N^{(d+1)/2} \#\mathcal{P}(f)}{\#E}. \tag{3.2}$$

Proposition 3.3. *Let* $0 < \gamma < \gamma_1$. *Then*

$$\mathcal{H}^{str}(p,\gamma) \implies \mathcal{H}(p,\gamma) \implies \mathcal{H}^{str}(p,\gamma_1). \tag{3.3}$$

The *main task* in Wolff's bootstrapping procedure will then be to prove the following

Theorem 3.4. *Let* $d \ge 2$, $p > p_d = q + 4/(d-1)$ *and* $\gamma_0 > 0$. *Let* ε_0 *be as in* (2.1). *If hypothesis* $\mathcal{H}^{str}(p,\gamma_0)$ *holds, then hypothesis* $\mathcal{H}(p,\gamma)$ *holds for all* $\gamma > (1 - \frac{\epsilon_0}{4})\gamma_0$.

Indeed, if Theorem 3.4 holds, then Proposition 3.3 together with an iteration gives the validity of the strong type estimate $\mathcal{H}^{str}(p,\epsilon)$ for all $\epsilon > 0$. The proof of Theorem 3.4 is given in §6, after preparation in §4 and §5.

Proof of Proposition 3.3. Note that implication $\mathcal{H}^{str}(p,\gamma) \implies \mathcal{H}(p,\gamma)$ is immediate by Čebyšev's inequality and the convexity bound (2.7) (together with Lemma 2.4). We now show the proof of the main implication $\mathcal{H}(p,\gamma) \implies \mathcal{H}^{str}(p,\gamma_1)$ for $\gamma_1 > \gamma$.

We first establish that the restriction on λ is superfluous. First, for an (N, Q_0, E) packet f we have $\|f\|_\infty \lesssim N^{(d-1)/4}$ and by decomposing into a bounded number of subpackets we may assume that $\|f\|_\infty < N^{(d-1)/4}$. In this case the set $\{x : |f(x)| > \lambda\}$ has measure zero if $\lambda \ge N^{(d-1)/4}$.

Next, by Čebyšev's inequality and hypothesis $S(2,q)$

$$\text{meas}\,\big(\{x : |f(x)| > \lambda\}\big) \le \lambda^{-q} \|f\|_q^q \lesssim C_\varepsilon \lambda^{-q} N^{(\frac{\alpha(q)}{2}+\varepsilon)q} \|f\|_{q,2;\delta}^q,$$

and by Lemma 2.4, we have $\|f\|_{q,2;\delta}^q \le N^{(d+1)/2}\#\mathcal{P}(f)/\#\mathcal{E}$ since f is an (N, Q_0, E)-packet. Notice that $\lambda^{-q}N^{\frac{\alpha(q)}{2}q} \le \lambda^{-p}N^{\beta(p)p}$ if $\lambda \le N^{\frac{d-1}{4}-\frac{1}{2(p-q)}}$. Thus, under $S(2,q)$, hypothesis $\mathcal{H}(p,\gamma)$ implies the inequality (3.2) for all $\lambda > 0$, provided γ is replaced by $\gamma+\varepsilon$ for any $\varepsilon > 0$.

We now argue that assuming $\mathcal{H}(p,\gamma)$ it suffices to show

$$\Big(\int_{Q'} |f(x)|^p dx\Big)^{1/p} \le C_\epsilon N^{\beta(p)+\gamma+\epsilon}\|f\|_{p,2;\delta} \tag{3.4}$$

for all $\epsilon > 0$. Indeed once (3.4) is shown uniformly for all cubes we choose a grid $\mathcal{Q}$ of N-cubes and decompose $f = \sum \psi_Q^2 f$. Notice that $\|\psi_Q f\|_{p,2;2\delta} \lesssim \|f\|_{p,2;\delta}$. If $Q, Q' \in \mathcal{Q}$ for any $M_1 > 0$ then we use the estimate

$$\|f\psi_Q^2\|_{L^p(Q')} \le C(M_1)\big((1+\mathrm{dist}(Q,Q')\big)^{-M_1}\|f\psi_{Q'}\|_p.$$

From this it is straightforward to deduce (with $N^\varepsilon Q$ denoting the cube dilated by N^ε with respect to its center) that

$$\begin{aligned}\Big(\sum_{Q'}\Big\|\sum_{Q\in\mathcal{Q}} f\psi_Q\Big\|_{L^p(Q')}^p\Big)^{1/p} &\lesssim \Big(\sum_Q \|f\psi_Q\|_{L^p(N^{\epsilon/2d}Q)}^p\Big)^{1/p} + C(M_2,\epsilon)N^{-M_2}\|f\|_p \\ &\le \Big(\sum_Q \sum_{\substack{Q' \\ \mathrm{dist}(Q,Q')\lesssim N^{1+\epsilon/3d}}} \|f\psi_Q\|_{L^p(Q')}^p\Big)^{1/p} + C(M_2,\epsilon)N^{-\frac{M_2\epsilon}{2d}+d}\|f\|_{p,2;\delta}.\end{aligned}$$

We apply (3.4) to $\psi_Q f$ and cubes Q' with distance $\le N^{1+\epsilon/2d}$ to Q and estimate the first term on the right hand side by a constant times

$$N^{\beta(p)+\gamma+\epsilon}\Big(\sum_Q \|f\psi_Q\|_{p,2;2\delta}^p\Big)^{1/p} \lesssim N^{\beta(p)+\gamma+\epsilon}\|f\|_{p,2;\delta}. \tag{3.5}$$

For the last estimate we have used Lemma 2.2.

We now proceed to show (3.4). To do this we may assume

$$\|f\|_{p,2;\delta} = 1. \tag{3.6}$$

Fix an N-cube Q. Then

$$\|f\|_{L^p(Q)}^p \lesssim p\sum_\ell 2^{\ell p}\mathrm{meas}\,\big(\{x\in Q : |f| > 2^\ell\}\big).$$

By (3.6) and (2.6) for $p=\infty$ we have that $\|f\|_\infty \lesssim N^{(d-1)/4}$ so that the set where $|f| > 2^\ell$ is empty when $2^\ell \gg N^{(d-1)/4}$. Moreover, as the measure of Q is $O(N^d)$ we have

$$\sum_{2^\ell \le N^{-d}} 2^{\ell p}\mathrm{meas}\,\big(\{x\in Q : |f| > 2^\ell\}\big) \lesssim N^{-d(p-1)};$$

thus only the $O(\log N)$ terms with $N^{-d} \lesssim 2^\ell \lesssim N^d$ have to be estimated.

This means that it suffices to show, for $N^{-d} \le \lambda \le N^d$,

$$\text{meas}\,\big(\{x \in Q : |f| > \lambda\}\big) \lesssim \lambda^{-p} N^{(\beta(p)+\gamma+\varepsilon)p}; \tag{3.7}$$

cf. the normalization (3.6). This normalization (together with (2.5)) also implies $\|f\|_{\infty,2;\delta} \lesssim N^{-(d+1)/2p}$. We now use the decomposition in Lemma 2.5 with $A \approx N^{-(d+1)/2p}$. The function g in (2.16) is then $\lesssim N^{-9d} \ll \lambda$. By the pidgeonhole principle applied to the $O((\log N)^3)$ terms in the sum in (2.16) there is a set $E_* \subset \mathcal{Z}(N^{1/2})$, a stable (N, Q, E_*) packet f_*, a number j_* with $N^{-11d} \lesssim 2^{j_*} \lesssim 1$ and a constant C so that $2^{j_* p} N^{\frac{d+1}{2}} \#\mathcal{P}(f_*) \lesssim \#E_*$, and

$$\text{meas}\,\{x \in Q : |f| > \lambda\} \lesssim (\log N)^3 \text{meas}\,\Big(\Big\{x \in Q : 2^{j^*}|f_*| > \lambda(\log N)^{-3}C^{-1}\Big\}\Big).$$

By Hypothesis $\mathcal{H}(p,\gamma)$ (and our initial observation that the restriction on λ in this hypothesis is superfluous) the right hand side is estimated by a constant times

$$(\log N)^3 N^{(\beta(p)+\gamma)p}(\lambda 2^{-j_*}(\log N)^{-3})^{-p} \frac{N^{(d+1)/2}\#\mathcal{P}(f_*)}{\#E_*} \le C_\varepsilon \lambda^{-p} N^{(\beta(p)+\gamma+\varepsilon)p},$$

where in the last step we have used the key inequality (2.18) and $\|f\|_{p,2;\delta} = 1$. This finishes the proof of (3.7) and thus the proposition. □

4. Localization

This section is included for expository reasons; it is essentially taken from [7], with minor modifications. The purpose is to identify, for given λ, properties of specific plate families so that the improvement in Theorem 3.4 holds.

We begin with an easy localization estimate which will later give a crucial gain in the induction on scales argument.

Lemma 4.1. *Let $\widehat{f}$ be supported in Σ^δ and let Q be a cube of diameter $\rho\delta^{-1}$ (here $\rho \le 1$). Then*

$$\|\psi_Q f\|_2 \lesssim \rho^{1/2}\|f\|_2 \tag{4.1}$$

Proof. By Plancherel's theorem this is equivalent with a statement about the integral operator T with kernel $K_\delta(\xi,\eta) = \widehat{\psi_Q}(\xi-\eta)\chi_{\Sigma^\delta}(\eta)$. Let $A_1 = \sup_\xi \int |K_\delta(\xi,\eta)|d\eta$ and $A_2 = \sup_\eta \int |K_\delta(\xi,\eta)|d\xi$. Then the L^2 operator norm of T is $\le \sqrt{A_1 A_2}$. Now clearly $A_2 = O(1)$ while the smaller η-support yields $A_1 = O(\rho)$. This implies the assertion. □

We now state a definition of localization for packets.

Definition 4.2. *Let R be an N-cube and let f be an (N, R, E)-packet and $t = \delta^{\varepsilon_0}$ with $0 < \varepsilon_0 \ll 1/2$. We say that f localizes at height λ (with respect to tN cubes) if there are* *subpackets f^Q of f where Q runs over tN-cubes in a grid $\mathcal{Q}$, such that*

$$\sum_Q \#\mathcal{P}(f^Q) \lessapprox \#\mathcal{P}(f) \tag{4.2}$$

and

$$meas\,\big(\{x : |f(x)| > \lambda\}\big) \lessapprox \sum_Q meas\,\big(Q \cap \{x : |f^Q| \gtrapprox \lambda\}\big). \tag{4.3}$$

Lemma 4.3. *Let $p > 2$ and suppose that $\mathcal{H}^{str}(p, \gamma_0)$ holds. Let f be a stable (N, R, E)-packet and assume that f localizes at height λ (with respect to $tN = \delta^{\varepsilon_0 - 1}$ cubes), and let the f^Q be as in Definition 4.2. Then for any N-cube Q_0 the estimate (3.2), i.e.*

$$\big|\{x \in Q_0 : |f(x)| > \lambda\}\big| \,\le\, C_\gamma\, \lambda^{-p}\, N^{(\beta(p)+\gamma)p}\, \frac{N^{(d+1)/2}\#\mathcal{P}(f)}{\#E}$$

holds for this f, R and λ, and for all $\gamma > \gamma_0(1 - \varepsilon_0/2)$.

Proof. For each tN cube Q, the function $t^{\frac{d-1}{4}}\, f^Q \psi_Q$ has Fourier transform supported in $\Sigma^{\delta/t}$, and

$$\big\|t^{\frac{d-1}{4}}\, f^Q \psi_Q\big\|_{\infty,2;C\delta/t} \lesssim 1.$$

Thus, we may apply $\mathcal{H}^{str}(p, \gamma_0)$, with δ replaced by δ/t, and the convexity inequality (2.7) to obtain

$$\begin{aligned}
\text{meas}\,\big(\{x : |f(x)| > \lambda\}\big) \;&\lessapprox\; \sum_Q \text{meas}\,\big(\{|t^{\frac{d-1}{4}} f^Q \psi_Q| \gtrapprox t^{\frac{d-1}{4}}\lambda\}\big) \qquad \text{(by (4.3))}\\
&\lessapprox\; \sum_Q (t^{\frac{d-1}{4}}\lambda)^{-p} (tN)^{(\beta(p)+\gamma_0)p}\, \|t^{\frac{d-1}{4}}\, f^Q \psi_Q\|_2^2\\
&=\; \sum_Q \lambda^{-p} N^{(\beta(p)+\gamma_0)p}\, t^{\gamma_0 p}\, t^{-1} \|f^Q \psi_Q\|_2^2. \qquad (4.4)
\end{aligned}$$

By Lemma 4.1 we have $t^{-1}\|f^Q\psi_Q\|_2^2 \lesssim \|f^Q\|_2^2 \lesssim N^{\frac{d+1}{2}} \frac{\#\mathcal{P}(f^Q)}{\#E}$, and therefore, summing in Q and using (4.2) we see that (4.4) $\lessapprox \lambda^{-p} N^{(\beta(p)+\gamma_0(1-\varepsilon_0))p}\, N^{\frac{d+1}{2}}\, \#\mathcal{P}(f)/\#E$, which yields the assertion. □

It is now important to identify situations in which the localization conditions of Definition 4.2 apply and thus the improvement of Lemma 4.3 holds. Such a situation is described in the following proposition.

Proposition 4.4. *Let $p \ge 2$ and assume $\mathcal{H}(p, \gamma_0)$. Let f be a stable (N, R, E)-packet so that for some $\lambda > 0$*

$$\#\mathcal{P}(f) \,\le\, t^{10d}\, \lambda^2 \#E. \tag{4.5}$$

Then f localizes at height λ to tN-cubes and hence (3.2) *for any N-cube Q_0, i.e.*

$$\big|\{x \in Q_0 : |f(x)| > \lambda\}\big| \le C\,\lambda^{-p}\,N^{(\beta(p)+\gamma)p}\,\frac{N^{\frac{d+1}{2}}\,\#\mathcal{P}(f)}{\#E}$$

holds for such f and λ, and all $\gamma > \gamma_0(1-\varepsilon_0/2)$.

It will be clear from the proof that the exponent $10d$ of t in (4.5) may be substantially lowered; this however seems to be of no consequence to the range of p in Theorem 1.1.

The main geometrical argument behind Proposition 4.4 is in the following result from [7] which (in a slightly more complicated version) will be applied to $W = \{x : |f(x)| > \lambda\}$.

Lemma 4.5. *Let $\mathcal{P}$ be a family of N-plates intersecting a fixed cube of diameter CN and let W be a measurable subset of $\mathbb{R}^d$. Let $t = \delta^{\varepsilon_0}$ and let $\mathcal{Q}$ be a grid of tN-cubes; we write $Q = Q(x)$ if $x \in Q$ (this is well defined apart from a set of measure 0). For each $\pi \in \mathcal{P}$ choose a tN-cube $Q_\pi \in \mathcal{Q}$ for which the quantity $|W \cap \pi \cap Q|$ is maximal. For a plate π and a cube $Q \in \mathcal{Q}$ we say that $\pi \sim Q$ if Q intersects the 9-fold dilate of Q_π. Then*

$$\#\{Q : \pi \sim Q\} \le 10^d \text{ for every } \pi \in \mathcal{P} \tag{4.6}$$

and for $\mathcal{I} = \int_W \sum_{\pi\in\mathcal{P},\pi\not\sim Q(x)} \chi_\pi(x)dx$ there is the estimate

$$\mathcal{I} \lessapprox t^{-3d}|W|\sqrt{\#\mathcal{P}}\,. \tag{4.7}$$

Proof. The condition that all plates in $\mathcal{P}$ intersect a fixed N cube, and the separation property of the plates implies $\#\mathcal{P} = O(N^d)$.

Note that (4.6) is trivial from the definition of the relation. To prove (4.7) we first note that $\mathcal{I} = \sum_\pi \nu(\pi)$ where $\nu(\pi) = \Big|\{x \in W \cap \pi : Q(x) \not\sim \pi\}\Big|$. We only need to bound

$$\widetilde{\mathcal{I}} = \sum_{\substack{\pi\in\mathcal{P}:\\ N^{-d}|W|\le\nu(\pi)\le|W|}} \nu(\pi) \tag{4.8}$$

since the analogous sum involving plates $\pi \in \mathcal{P}$ with $\nu(\pi) \le |W|N^{-d}$ is trivially bounded by $\#\mathcal{P}|W|N^{-d} \lesssim |W|$.

In (4.8) there are $O(\log N)$ relevant dyadic scales between $N^{-d}|W|$ and $|W|$ and thus we can use a pidgeonhole argument to get a subfamily $\mathcal{P}' \subset \mathcal{P}$ and a value of ν between $N^{-d}|W|$ and $|W|$ so that

$$|\widetilde{\mathcal{I}}| \lessapprox \nu\,\mathrm{card}(\mathcal{P}') \quad \text{and} \quad \nu \le \nu(\pi) \le 2\nu \quad \text{for each } \pi \in \mathcal{P}'. \tag{4.9}$$

Hence for each $\pi \in \mathcal{P}'$ there is a cube $Q'(\pi)$ *not related to* π so that

$$|W \cap Q'(\pi) \cap \pi| \gtrsim t\nu.$$

By the maximality condition in the definition of Q_π we must then also have

$$|W \cap Q_\pi \cap \pi| \gtrsim t\nu \text{ for each } \pi \in \mathcal{P}'.$$

Clearly the number of all possible pairs of tN cubes is $O(t^{-2d})$. This means that we can find two tN cubes Q, Q' in $\mathcal{Q}$ and a subfamily $\mathcal{P}''$ of $\mathcal{P}'$ which has cardinality $\gtrsim t^{2d}\#\mathcal{P}'$ so that for all $\pi \in \mathcal{P}''$ we have $Q_\pi = Q$ and $Q'(\pi) = Q'$.

We now fix these two tN cubes Q and Q' and consider the auxiliary expression

$$\mathcal{A} = \sum_{\pi \in \mathcal{P}''} |W \cap Q \cap \pi||W \cap Q' \cap \pi|.$$

Then we have the lower bound

$$\mathcal{A} \gtrsim (t\nu)^2 \text{card}(\mathcal{P}'') \gtrsim t^{2d+2} \text{card}(\mathcal{P}')\nu^2.$$

We can also derive an upper bound by rewriting

$$\mathcal{A} = \int_{W \cap Q} \int_{W \cap Q'} \sum_{\pi \in \mathcal{P}''} \chi_\pi(x)\chi_\pi(x')dxdx'$$

If $\pi \cap Q \neq \emptyset$ and $\pi \cap Q' \neq \emptyset$ for some $\pi \in \mathcal{P}''$ then π is related to Q but not to Q', thus the distance of Q to Q' is at least tN. This means that for each pair of points $(x, x') \in Q \times Q'$ there are no more than Ct^{-d+1} separated plates which go through both x and x'. Therefore the integrand $\sum_{\pi \in \mathcal{P}''} \chi_\pi(x)\chi_\pi(x)$ is $O(t^{-d+1})$, and hence we get the upper bound

$$\mathcal{A} \lesssim t^{-d+1}|W \cap Q||W \cap Q'| \lesssim t^{-d+1}|W|^2.$$

Comparing the upper and the lower bounds for $\mathcal{A}$ we find that

$$\nu \leq t^{-d-1}(\#\mathcal{P}')^{-1/2}\sqrt{\mathcal{A}} \leq t^{-(3d+1)/2}|W|(\#\mathcal{P}')^{-1/2}$$

and thus using (4.9) we obtain

$$\widetilde{\mathcal{I}} \lessapprox t^{-(3d+1)/2}|W|\sqrt{\#\mathcal{P}'}.$$

□

Unfortunately, for technical reasons Lemma 4.5 is not quite enough since we need to replace the characteristic functions χ_π by the similar weights w_π with "Schwartz-tails"). This is fairly straightforward and requires adjustments in the definition of the relation $\sim$ between plates and tN-cubes and some additional pidgeonholing. We state the required estimate and refer to Lemma 4.4 in the paper by Łaba and Wolff [7] for the details of the proof.

Lemma 4.6. *Let $\mathcal{P}$ be a family of N-plates intersecting a fixed cube of diameter CN and let W be a measurable subset of $\mathbb{R}^d$. Let M_0 be a large constant and assume that the constant M in the definition of $w(x)$ is large (see (2.3)), so that $M \geq 10M_0 d$. Let $t = \delta^{\varepsilon_0}$ and let $\mathcal{Q}$ be a grid of tN-cubes, where again we write $Q = Q(x)$ if $x \in Q$. There is a relation $\sim$ between plates in $\mathcal{P}$ and tN-cubes in $\mathcal{Q}$ so that*

$$\#\{Q : \pi \sim Q\} \lessapprox 1 \text{ for every } \pi \in \mathcal{P} \tag{4.10}$$

and if

$$\mathfrak{W}_{\mathcal{P}}(x) = \sum_{\substack{\pi \in \mathcal{P} \\ \pi \not\sim Q(x)}} w_\pi(x)$$

then

$$\int_W \mathfrak{W}_{\mathcal{P}}(x) dx \lessapprox t^{-3d} |W| \sqrt{\#\mathcal{P}} + \delta^{M_0} |W|.$$

Proof of Proposition 4.4. We wish to apply Lemma 4.3 and therefore have to show that with $\mathcal{P} \equiv \mathcal{P}(f)$ under the assumption $\#\mathcal{P} \le ct^{10d}\lambda^2 \#E$ the localization condition in Definition 4.2 holds.

We proceed applying Lemma 4.6 to $W = \{x : |f| \ge \lambda\}$, and let $\sim$ be the relation between N-plates and tN-cubes from Lemma 4.6. Recall that $f(x) = (\#E)^{-1/2} \sum_{\pi \in \mathcal{P}} f_\pi$ with $|f_\pi| \lesssim w_\pi$. For every tN-cube $Q \in \mathcal{Q}$ define $f^Q(x) = (\#E)^{-1/2} \sum_{\pi \sim Q} f_\pi$.

By condition (4.10) we have $\sum_Q \#\mathcal{P}(f^Q) \lessapprox \#\mathcal{P}(f)$, *i.e.* (4.2). Moreover with $\mathcal{P} \equiv \mathcal{P}(f)$

$$\int_W \mathfrak{W}_{\mathcal{P}}(x) dx \lessapprox t^{-3d} |W| \sqrt{\#\mathcal{P}} \lesssim t^{-3d} |W| \sqrt{t^{10d} \lambda^2 \#E} \lesssim t^{2d} |W| \lambda \sqrt{\#E}.$$

This means that there is a subset W^* of W so that $|W^*| \ge |W|/2$ so that the pointwise bound $\mathfrak{W}_{\mathcal{P}}(x) \lessapprox t\lambda\sqrt{\#E}$ for $x \in W^*$. Also if $x \in W^* \cap Q$ we have

$$|f(x) - f^Q(x)| = \Big| \frac{1}{\sqrt{\#E}} \sum_{\pi : \pi \not\sim Q} f_\pi(x) \Big| \lesssim \frac{\mathfrak{W}_{\mathcal{P}}(x)}{\sqrt{\#E}} \lessapprox t\lambda$$

and hence $|f^Q(x)| \ge \lambda$ for $x \in W^* \cap Q$. This implies the localization condition (4.3). □

5. A parabolic rescaling

We first note that the paraboloid in Wolff's theorem can be replaced by $\{\xi : \xi_d = c + (\xi' - a')^t A (\xi' - a')\}$ for any positive definite matrix A, by a linear transformation. We also may rotate the paraboloid in $\mathbb{R}^d$ and obtain a similar result.

More useful is the following Lemma which is an analogue and consequence of Wolff's inequality for Fourier plates in an angular sector of angle $\sqrt{\sigma} \gg \sqrt{\delta}$ (or equivalently, for δ-Fourier plates contained in a fixed σ-Fourier plate).

Lemma 5.1. *Let $\delta < \sigma < 1$ and consider a σ-plate $\Pi^{(\sigma)}$ contained in Σ^σ. Suppose that Hypothesis $\mathcal{H}^{str}(p,\gamma)$ holds. Then for all functions $h_k \in L^p(\mathbb{R}^d)$*

$$\Big\| \sum_{k : \Pi_k^{(\delta)} \subset \Pi^{(\sigma)}} P_k^{(\delta)} h_k \Big\|_p \lesssim (\sigma/\delta)^{\beta(p)+\gamma} \Big(\sum_k \|h_k\|_p^2 \Big)^{1/2}.$$

Proof. By a rotation and translation we may assume that we are working with the standard paraboloid and the σ-plate $\Pi^{(\sigma)} = \{\xi : |\xi_i| \le \sqrt{\sigma},\ i = 1, \dots, d-1;\ |\xi_d| \le \sigma\}$. Let $f_k = P_k^{(\delta)} h_k$, $L_\sigma(\xi) = (\sigma^{1/2}\xi', \sigma\xi_d)$ and let $f_k^\sigma(x) := \sigma^{-(d+1)/2} f_k(L_\sigma^{-1}x)$ so that $\widehat{f_k^\sigma}(\xi) = \widehat{f}_k(\sigma^{1/2}\xi', \sigma\xi_d)$. The functions $\widehat{f_k^\sigma}$ are supported in $(\delta/\sigma)^{1/2} \times \cdots \times (\delta/\sigma)^{1/2} \times \delta/\sigma$ plates tangential to the paraboloid and Hypothesis $\mathcal{H}^{str}(p,\gamma)$ yields

$$\Big\| \sum_{|k| \lesssim \sqrt{\sigma/\delta}} f_k^\sigma \Big\|_p \lesssim (\delta/\sigma)^{-\beta(p)-\gamma} \Big(\sum_k \|f_k^\sigma\|_p^2 \Big)^{1/2}.$$

Changing variables $y = L_\sigma^{-1}x$ on both sides yields the assertion. □

6. Proof of Theorem 3.4

Let R be an N-cube, let $p > q + 4/(d-1)$ and ε_0 be as in (2.1). We also fix $0 < \varepsilon_1 \le 10^{-2}\varepsilon_0$. Assuming that $\mathcal{H}^{str}(p,\gamma_0)$ holds we need to show for any stable (N,R,E)-packet f and any fixed N-cube Q_0 that

$$\text{meas}\,(\{x \in Q_0 : |f(x)| > \lambda\}) \le C_\gamma \lambda^{-p} N^{(\beta(p)+\gamma)p} N^{(d+1)/2} \frac{\#\mathcal{P}(f)}{\#E} \tag{6.1}$$

for all $\gamma > \gamma_0(1 - \varepsilon_0/4)$ and all λ in the range

$$N^{\frac{d-1}{4} - \frac{1}{2(p-q)}} \lesssim \lambda \lesssim N^{\frac{d-1}{4}}. \tag{6.2}$$

This will be done by localizing at a smaller scale N_1 and then using the induction hypothesis at that scale. We may without loss of generality assume that $\text{dist}(R, Q_0) \le 2N^{1+\varepsilon_1}$ (otherwise a much better inequality holds).

Let N_1 be a number with

$$\sqrt{N} \le N_1 \ll N; \tag{6.3}$$

we shall later see that the choice $N_1 = \sqrt{N}$ will be optimal for our proof. Set $\delta_1 = N_1^{-1}$ and let $\{\Delta\}$ be a tiling of $\mathbb{R}^d$ by N_1-cubes. For each such Δ let $\widetilde{\Delta}$ be a cube with same center as Δ but with sidelength equal to $5N_1^{1+\varepsilon_1}$.

Now since $\min_{x \in Q} \psi_Q(x) \ge c > 0$ with a universal constant c we have

$$\big|\{x \in Q_0 : |f(x)| > \lambda\}\big| \le \sum_{\Delta : \Delta \cap Q_0 \neq \emptyset} \big|\{x \in \Delta : |f\psi_\Delta(x)| > c\lambda\}\big| \tag{6.4}$$

for some constant $c > 0$. Given a fixed Δ, the function $f\psi_\Delta$ has Fourier transform supported in $\Sigma^{c\delta_1}$. Note that $f\psi_\Delta$ is in general not a packet. However, by Lemma 2.5, $f\psi_\Delta$ can be decomposed on Δ in terms of N_1-packets:

Lemma 6.1. *Let R and Q_0 be N-cubes as above, let f be an (N,R,E)-packet and let λ be as in (6.2). Then there exists $\lambda_1 > 0$ so that for every N_1-cube Δ which intersects Q_0 there is a plate family $\mathcal{P}_\Delta$, a set $E_\Delta \subset \mathcal{Z}(N_1^{1/2})$, and a stable $(N_1, \widetilde{\Delta}, E_\Delta)$-packet f_Δ so that*

$$\big|\{\, x \in Q_0 \ : \ |f(x)| > \lambda \,\}\big| \lessapprox \sum_{\Delta \cap Q_0 \neq \emptyset} \big|\{\, x \in \Delta \ : \ |f_\Delta(x)| \geq \lambda_1 \,\}\big| \tag{6.5}$$

and

$$\frac{\#\mathcal{P}_\Delta}{\#E_\Delta} \lessapprox \frac{\lambda_1^2}{\lambda^2} \frac{\|f\psi_\Delta\|_2^2}{N_1^{\frac{d+1}{2}}} \lessapprox \frac{\lambda_1^2}{\lambda^2} N_1^{\frac{d-1}{2}} . \tag{6.6}$$

Moreover, for $2 \leq p < \infty$,

$$\frac{\#\mathcal{P}_\Delta}{\#E_\Delta} \lessapprox \frac{\lambda_1^p}{\lambda^p} \frac{\|f\psi_\Delta\|_{p,2;\delta_1}^p}{N_1^{\frac{d+1}{2}}}. \tag{6.7}$$

Proof. Fix an N_1 cube Δ intersecting Q_0 and let $g \equiv g^\Delta = f\psi_\Delta$, which has Fourier transform supported in $\Sigma^{c\delta_1}$ and satisfies

$$\|g^\Delta\|_{\infty,2;c\delta_1} \lesssim (N/N_1)^{(d-1)/4} = A.$$

By Lemma 2.5 we can write

$$g^\Delta(x) = C \sum_{N_1^{-10d} \lesssim 2^j \lesssim N_1^d} 2^j \sum_{\ell=1}^{n_{j,\Delta}} g^\Delta_{[j,\ell]}(x) \, + \, h^\Delta(x), \quad x \in \Delta, \tag{6.8}$$

where

$$\sup_{x\in\Delta} |h_\Delta(x)| \leq C_{\varepsilon_1} N_1^{-8d} A, \tag{6.9}$$

$$n_{j,\Delta} \leq C_{\varepsilon_1} (\log N_1)^2; \tag{6.10}$$

moreover, for each (j,ℓ,Δ) there is a subset $E^\Delta_{j,\ell}$ of $\mathcal{Z}(N_1^{1/2})$ so that $g^\Delta_{(j,\ell)}$ is a stable $(N_1, \widetilde{\Delta}, E^\Delta_{j,\ell})$-packet, with associated plate family $\mathcal{P}^\Delta_{j,\ell}$, which contains only N_1-plates π with $\operatorname{dist}(\Delta,\pi) \lesssim N_1^{1+\varepsilon_1}$, and

$$2^{jp} N_1^{\frac{d+1}{2}} \#\mathcal{P}^\Delta_{j,\ell} \lesssim \|f\psi_\Delta\|^p_{p,2;\delta_1} \#E^\Delta_{j,\ell}, \qquad 2 \leq p < \infty. \tag{6.11}$$

As there are only $O(\log N)$ values of j and $O((\log N)^2)$ values of ℓ a simple pidgeonhole argument shows for λ in the range (6.2)

$$\begin{aligned} \Big|\big\{x \in \Delta \ : \ |g^\Delta| > c\lambda\big\}\Big| &\leq \Big|\big\{x \in \Delta \ : \ \big|\textstyle\sum_{N_1^{-10d} \lesssim 2^j \lesssim N_1^d} 2^j \sum_{\ell=1}^{n_{j,\Delta}} g^\Delta_{[j,\ell]}(x)\big| > \frac{c\lambda}{2}\big\}\Big| \\ &\leq \Big|\big\{x \in \Delta \ : \ |2^{j_\Delta} g^\Delta_{[j_\Delta,\ell_\Delta]}(x)\big| > \tfrac{\lambda}{C(\log N)^3}\big\}\Big| \end{aligned}$$

for some fixed j_Δ, ℓ_Δ.

Pigeonholing once again we can find, among the (j_Δ, ℓ_Δ)'s, a fixed $j_*, \ell_* \in \mathbb{Z}$ (independent of Δ) so that

$$\sum_\Delta \Big|\{x \in \Delta \ : \ |g^\Delta| > c\lambda\}\Big| \le C(\log N)^3 \sum_\Delta \Big|\{x \in \Delta \ : \ |2^{j_*} g^\Delta_{[j_*,\ell_*]}(x)| > \tfrac{\lambda}{C(\log N)^3}\}\Big|.$$

This means that (6.5) holds with $\lambda_1 = 2^{-j_*}\lambda/(C\log N)^3$, $f_\Delta = g^\Delta_{[j_*,\ell_*]}$, $E_\Delta = E^\Delta_{j_*,\ell_*}$ and $\mathcal{P}_\Delta = \mathcal{P}(g^\Delta_{[j_*,\ell_*]})$.

To prove (6.7) just observe that, by (6.11)

$$\frac{\#\mathcal{P}_\Delta}{\#E_\Delta} \lesssim 2^{-j_* p} N_1^{-(d+1)/2} \|f\psi_\Delta\|^p_{p,2;\delta_1} \approx (\log N)^{3p} \frac{\lambda_1^p}{\lambda^p N_1^{\frac{d+1}{2}}} \|f\psi_\Delta\|^p_{p,2;\delta_1}.$$

The first inequality in (6.6) follows from the case $p = 2$ of (6.7). For the second inequality in (6.6) we observe that if $f = \sum_k f_k$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$ then the Fourier transforms $\widehat{f_k\psi_\Delta}$ are supported in essentially disjoint $C\sqrt{\delta}$-cubes (here we use that $N_1 \ge \sqrt{N}$). Thus we have the crucial orthogonality estimate

$$\|f\psi_\Delta\|_2^2 \lesssim \sum_k \|f_k\psi_\Delta\|_2^2 \lesssim |\Delta| \sum_{k\in E} \|f_k\|^2_\infty \lesssim N_1^d \tag{6.12}$$

since f was assumed to be an (N,R,E)-packet. The second inequality in (6.6) follows. ☐

We wish to use the bound in (6.6) to argue that Proposition 4.4 can be applied to the pair (f_Δ, λ_1). The next lemma, shows how to conclude the theorem for (f, λ) in such case. Basically, one rescales the problem and uses one more time the induction hypothesis at scale N/N_1.

Lemma 6.2. *Let $p > 2$ and assume $\mathcal{H}^{str}(p, \gamma_0)$. Let f be a (N,R,E)-packet for some N-cube R, let Q_0 be an N-cube and let λ as in (6.2). Let $\omega > 0$ and suppose that for every N_1-cube Δ intersecting Q_0, the quadruplet $(f_\Delta, \mathcal{P}_\Delta, E_\Delta, \lambda_1)$ defined in Lemma 6.1 satisfies*

$$\big|\{\, x \in \Delta \ : \ |f_\Delta(x)| > \lambda_1 \,\}\big| \lessapprox \frac{N_1^{(\beta(p)+\omega)p}}{\lambda_1^p} N_1^{\frac{d+1}{2}} \frac{\#\mathcal{P}_\Delta}{\#E_\Delta}. \tag{6.13}$$

Then, we also have

$$\big|\{\, x \in Q_0 \ : \ |f(x)| > \lambda \,\}\big| \lessapprox \lambda^{-p} \frac{N^{(\beta(p)+\gamma_0)p}}{N_1^{(\gamma_0-\omega)p}} N^{\frac{d+1}{2}} \frac{\#\mathcal{P}(f)}{\#E}. \tag{6.14}$$

This is saying that if we have an improvement in (6.13) with an $\omega < \gamma_0$ then we also get an improvement in our main bound (6.14).

Proof of Theorem 3.4, given Lemma 6.2. We choose $N_1 = \sqrt{N}$. We need to verify that (6.13) holds with $\omega > \gamma(1 - \varepsilon_0/2)$. Then Lemma 6.2 tells us that (6.14) holds with $\beta > \gamma(1-\varepsilon_0/4)$ (where, say, ε_0 is chosen as in (2.1)). Proposition 4.4 says that (6.13) holds if the plate families $\mathcal{P}_\Delta$ satisfy $\#\mathcal{P}_\Delta \lesssim t_1^{10d}\lambda_1^2\#E_\Delta$ where $t_1 = \delta_1^{\varepsilon_0}$. By (6.6) and the lower bound on λ, $\lambda \gtrsim N^{\frac{d-1}{4}-\frac{1}{2(p-q)}}$ we have

$$\lambda_1^{-2}\frac{\#\mathcal{P}_\Delta}{\#E_\Delta} \lesssim N_1^{(d-1)/2}\lambda^{-2} = N^{(d-1)/4}\lambda^{-2} \lesssim N^{\frac{1}{p-q}-\frac{d-1}{4}},$$

and we are done if $N^{\frac{1}{p-q}-\frac{d-1}{4}} \lesssim t_1^{10d} = N^{-5d\varepsilon_0}$. This holds if $1/(p-q)-(d-1)/4 < -5d\varepsilon_0$ or equivalently $p > q+4/(d-1-20d\varepsilon_0)$. Note that this inequality is implied by (2.1) (and that the precise choice of ε_0 is not important in the argument).

Proof of Lemma 6.2. By (6.5) and (6.13) we have

$$\begin{aligned} \big|\{\,x\in Q_0 \;:\; |f|>\lambda\,\}\big| &\lessapprox \sum_\Delta \big|\{\,x\in\Delta \;:\; |f_\Delta|>\lambda_1\,\}\big| \\ &\lessapprox \sum_\Delta \lambda_1^{-p}\, N_1^{(\beta(p)+\omega)p}\, N_1^{\frac{d+1}{2}}\,\frac{\#\mathcal{P}_\Delta}{\#E_\Delta}. \end{aligned}$$

Thus, the result will be established if we can show

$$\sum_\Delta N_1^{\frac{d+1}{2}}\,\frac{\#\mathcal{P}_\Delta}{\#E_\Delta} \lesssim \frac{\lambda_1{}^p}{\lambda^p}\,(N/N_1)^{(\beta(p)+\gamma_0)p}\,N^{\frac{d+1}{2}}\,\frac{\#\mathcal{P}(f)}{\#E}. \tag{6.15}$$

Now consider functions Ξ_l so that their Fourier transforms $\widehat{\Xi}_l$ are bump functions associated to the $\delta_1^{1/2}\times\ldots\times\delta_1^{1/2}\times\delta_1$-plates $\Pi_l^{\delta_1}$. Then by (6.7) we have for each Δ,

$$\begin{aligned} &N_1^{\frac{d+1}{2}}\,\frac{\#\mathcal{P}_\Delta}{\#E_\Delta} \lessapprox \frac{\lambda_1{}^p}{\lambda^p}\,\|f\psi_\Delta\|^p_{p,2;\delta_1} \lesssim \frac{\lambda_1{}^p}{\lambda^p}\Big(\sum_l \big\|(f\psi_\Delta)*\Xi_l\big\|_p^2\Big)^{p/2} \\ &\lesssim \frac{\lambda_1{}^p}{\lambda^p}\Big(\sum_l \Big\|\big[\psi_\Delta\big(\sum_{k:\Pi_k^{(\delta)}\subset C\Pi_l^{(\delta_1)}} f_k\big)\big]*\Xi_l\Big\|_p^2\Big)^{p/2} \lesssim \frac{\lambda_1{}^p}{\lambda^p}\Big(\sum_l \Big\|\psi_\Delta\big(\sum_{\Pi_k^{(\delta)}\subset C\Pi_l^{(\delta_1)}} f_k\big)\Big\|_p^2\Big)^{p/2}. \end{aligned}$$

We sum in Δ and apply Minkowski's inequality to obtain

$$\begin{aligned} &\sum_\Delta N_1^{\frac{d+1}{2}}\,\frac{\#\mathcal{P}_\Delta}{\#E_\Delta} \lessapprox \frac{\lambda_1^p}{\lambda^p}\sum_\Delta\Big(\sum_l\Big\|\psi_\Delta\big(\sum_{\Pi_k^{(\delta)}\subset C\Pi_l^{(\delta_1)}} f_k\big)\Big\|_p^2\Big)^{p/2} \\ &\lesssim \frac{\lambda_1^p}{\lambda^p}\Big(\sum_l\Big[\sum_\Delta\Big\|\psi_\Delta\big(\sum_{\Pi_k^{(\delta)}\subset C\Pi_l^{(\delta_1)}} f_k\big)\Big\|_p^p\Big]^{2/p}\Big)^{p/2} \lesssim \frac{\lambda_1^p}{\lambda^p}\Big(\sum_l\Big\|\sum_{\Pi_k^{(\delta)}\subset C\Pi_l^{(\delta_1)}} f_k\Big\|_p^2\Big)^{p/2}. \end{aligned}$$

Now, we apply Hypothesis $\mathcal{H}^{str}(p,\gamma_0)$ in the rescaled version of Lemma 5.1 and bound for each l

$$\Big\|\sum_{k:\Pi_k^{(\delta)}\subset C\Pi_l^{(\delta_1)}} f_k\Big\|_p \lesssim (N/N_1)^{\beta(p)+\gamma_0}\Big(\sum_{k:\Pi_k^{(\delta)}\subset C\Pi_l^{(\delta_1)}}\|f_k\|_p^2\Big)^{1/2}.$$

This yields, using the convexity inequality (2.7) and $\|f\|_{\infty,2;\delta} \lesssim 1$,

$$\Big(\sum_l \Big\| \sum_{\Pi_k^{(\delta)} \subset c\Pi_l^{(\delta_1)}} f_k \Big\|_p^2\Big)^{p/2} \lessapprox (N/N_1)^{(\beta(p)+\gamma_0)p} \Big(\sum_l \sum_{\Pi_k^{(\delta)} \subset c\Pi_l^{(\delta_1)}} \big\|f_k\big\|_p^2\Big)^{p/2}$$
$$\lesssim (N/N_1)^{(\beta(p)+\gamma_0)p} \sum_k \big\|f_k\big\|_2^2 \Big(\sum_{k'} \big\|f_{k'}\big\|_\infty^2\Big)^{(p-2)/2} \lesssim (N/N_1)^{(\beta(p)+\gamma_0)p} N^{\frac{d+1}{2}} \frac{\#\mathcal{P}(f)}{\#E},$$

and thus we get the asserted (6.15). □

References

[1] D. Békollé, A. Bonami, G. Garrigós and F. Ricci. Littlewood-Paley decompositions related to symmetric cones and Bergman projections in tube domains. Proc. London Math. Soc. (3) **89** (2004), 317-360.

[2] J. Bourgain. Besicovitch type maximal operators and applications to Fourier analysis, Geometric and Functional analysis, 147-187.

[3] C. Fefferman. A note on spherical summation multipliers. Israel J. Math. 15 (1973), 44–52.

[4] G. Garrigós, W. Schlag, A. Seeger. Improvements in Wolff's inequality for decompositions of cone multipliers.

[5] G. Garrigós, A. Seeger, On plate decompositions for cone multipliers, to appear in Proc. Edinburgh Math. Soc.

[6] I. Łaba and M. Pramanik. Wolff's inequality for hypersurfaces. Proceedings of the 7th International Conference on Harmonic Analysis and Partial Differential Equations, El Escorial, 2004. Collect. Math. 2006, Vol. Extra, 293–326.

[7] I. Łaba and T. Wolff. A local smoothing estimate in higher dimensions. J. Anal. Math. **88** (2002), 149-171.

[8] S. Lee. Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators. Duke Math. J. 122 (2004), no. 1, 205–232.

[9] M. Pramanik and A. Seeger. L^p regularity of averages over curves and bounds for associated maximal operators. Amer. J. Math. 129 (2007), 61–103.

[10] E.M. Stein. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.

[11] T. Tao. A sharp bilinear restrictions estimate for paraboloids. Geom. Funct. Anal. 13 (2003), no. 6, 1359–1384.

[12] T. Tao, A. Vargas and L. Vega. A bilinear approach to the restriction and Kakeya conjectures. J. Amer. Math. Soc. 11 (1998), no. 4, 967–1000.

[13] T. Wolff. Local smoothing type estimates on L^p for large p. Geom. Funct. Anal. **10** (2000), 1237-1288.

G. Garrigós, Dep. Matemáticas C-XV, Universidad Autónoma de Madrid, 28049 Madrid, Spain

E-mail address: gustavo.garrigos@uam.es

A. Seeger, Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA

E-mail address: seeger@math.wisc.edu