A NOTE ON MAXIMAL OPERATORS ASSOCIATED WITH HANKEL MULTIPLIERS

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Abstract. Let m have compact support in $(0,\infty)$. For $1 < p < 2d/(d+1)$, we give a necessary and sufficient condition for the $L^p_{\mathrm{rad}}(\mathbb{R}^d)$-boundedness of the maximal operator associated with the radial multiplier $m(|\xi|)$. More generally we prove a similar result for maximal operators associated with multipliers of modified Hankel transforms. The result is obtained by modifying the proof of the characterization of Hankel multipliers given by the authors in [2].

1. Introduction and statement of results

Let $m \in L^\infty[0,\infty)$, and denote by K the associated radial convolution kernel in $\mathbb{R}^d$, whose (distributional) Fourier transform is given by the identity

$$\widehat{K}(\xi) = m(|\xi|), \quad \xi \in \mathbb{R}^d.$$

Consider the usual dilation $K_t = t^{-d}K(t^{-1}\cdot)$, so that $\widehat{K_t}(\xi) = m(t|\xi|)$, and define the convolution operator T_K and the associated maximal operator T^*_K by

$$T_K G = K * G \quad \text{and} \quad T^*_K G = \sup_{t>0} |T_{K_t} G|,$$

at least for functions $G \in \mathcal{S}(\mathbb{R}^d)$.

In [2] we gave necessary and sufficient conditions for the boundedness of T_K in the subspace of radial functions of $L^p(\mathbb{R}^d)$, denoted $L^p_{\mathrm{rad}}(\mathbb{R}^d)$, for values of p in the range $1 < p < \frac{2d}{d+1}$. In particular we showed that T_K extends to a bounded operator on $L^p_{\mathrm{rad}}(\mathbb{R}^d)$ if and only if

$$\sup_{t>0} \|K_t * \Phi\|_{L^p(\mathbb{R}^d)} < \infty \tag{1.1}$$

for some fixed radial (and not identically zero) $\Phi \in \mathcal{S}(\mathbb{R}^d)$.

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When the multiplier m is *compactly supported in* $(0,\infty)$, then the $L^p_{\rm rad}(\mathbb{R}^d)$-boundedness of T_K is equivalent to the simpler statement

$$\big\|K\big\|_{L^p_{\rm rad}(\mathbb{R}^d)} < \infty. \tag{1.2}$$

The range of p in this statement is best possible, since when $p \geq \frac{2d}{d+1}$ one can construct $K \in L^p_{\rm rad}(\mathbb{R}^d)$ for which $\widehat{K}$ is compactly supported away from the origin but is not even a bounded function.

Yuichi Kanjin, at the 2007 Miraflores Conference on Harmonic Analysis and Orthogonal Systems, posed the question whether one could also characterize the $L^p_{\rm rad}(\mathbb{R}^d)$-boundedness of the maximal operator T^*_K. We use the methods of [2] to prove such a result when the multiplier m is compactly supported in $(0,\infty)$.

Theorem 1.1. *Let m be an integrable function with compact support in $(0,\infty)$ and let $K = \mathcal{F}^{-1}_{\mathbb{R}^d}[m(|\cdot|)]$. If $1 < p < 2d/(d+1)$ then T^*_K extends to a bounded operator on $L^p_{\rm rad}(\mathbb{R}^d)$ if and only if*

$$\Big\|\sup_{t\in I} |K_t|\Big\|_{L^p(\mathbb{R}^d)} < \infty \tag{1.3}$$

for some fixed interval $I \Subset (0,\infty)$.

Remark. One could ask whether the conditions in (1.1) or (1.3) do actually characterize the boundedness of the operators T_K or T^*_K in the full space $L^p(\mathbb{R}^d)$. In [4] this has been shown to be true for T_K, at least for sufficiently high dimensions for each fixed $p \in (1,2)$. The situation is different for the maximal operators T^*_K: consider the (truncated) Bochner-Riesz multipliers defined by

$$m_\alpha(r) = \chi(r)\big(1-r^2\big)^\alpha_+, \tag{1.4}$$

with $\chi \in C^\infty_c(1/2,2)$ so that $\chi(1) = 1$, and let $K = \mathcal{F}^{-1}_{\mathbb{R}^d}[m_\alpha(|\cdot|)]$. Then $\sup_{t\in I}|K_t| \in L^p(\mathbb{R}^d)$ if $\alpha > d(1/p-1/2)-1/2$, for any $I \Subset (0,\infty)$. However, Tao [6] showed that T^*_K is not bounded on $L^p(\mathbb{R}^d)$ if $\alpha < d(1/p-1/2)-1/(2p)$. Thus Theorem 1.1 does not extend to general L^p-functions.

1.1. **A more general formulation.** As in [2], Theorem 1.1 can be formulated in the more general setting of multipliers of (modified) Hankel (or Fourier-Bessel) transforms. We use the notation

$$B_d(s) = s^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(s)\,,$$

where J_ν stands for the usual Bessel function. Observe that, with this normalization,

$$B_d(s) = O(1) \text{ as } s \to 0, \quad \text{and} \quad B_d(s) = O(s^{-\frac{d-1}{2}}) \text{ as } s \to \infty.$$

L^p spaces on $\mathbb{R}^+ = (0,\infty)$ will be defined with respect to the measure μ_d given by

$$d\mu_d(r) = r^{d-1}\,dr. \tag{1.5}$$

When $G(x) = f(|x|)$ is a radial function in $L^1(\mathbb{R}^d)$, we can write its Fourier transform as

$$\widehat{G}(\xi) \,=\, \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(|x|)\,dx \,=\, c_d \int_0^\infty B_d(r|\xi|)\,f(r)\,d\mu_d(r),$$

where c_d is a suitable positive constant. For every *real number* $d \geq 1$ we define a (modified) Hankel transform acting on functions in $L^1(\mu_d)$ by

$$(\mathcal{B}_d f)(\rho) = \int_0^\infty B_d(\rho r)\,f(r)\,d\mu_d(r).$$

and given $m \in L^1_{\rm loc}(0,\infty)$, the corresponding *Hankel multiplier operator* by

$$\mathcal{T}_m f = \mathcal{B}_d[m\mathcal{B}_d f].$$

We remark that, as long as $m \in L^1_{\rm loc}(0,\infty)$, the operator $\mathcal{T}_m$ is well defined acting on functions f in $\mathcal{B}_d(C_c^\infty(0,\infty))$ which is a dense subspace of $L^p(\mu_d)$ when $1 < p < \infty$; see *e.g.* [5, Thm 4.7]. We also define the corresponding maximal operator

$$\mathcal{T}_m^* f = \sup_{t>0} \big|\mathcal{B}_d[m(t\cdot)\mathcal{B}_d f]\big|.$$

Theorem 1.2. *Let $m \in L^1_c(0,\infty)$ (i.e., integrable and compactly supported in $(0,\infty)$). If $1 < p < \frac{2d}{d+1}$ then $\mathcal{T}_m^*$ extends as a bounded operator to $L^p(\mu_d)$ if and only if*

$$\Big\| \sup_{t\in I}\big|\mathcal{B}_d[m(t\cdot)]\,\big|\,\Big\|_{L^p(\mu_d)} < \infty \tag{1.6}$$

for some fixed interval $I \Subset (0,\infty)$. Moreover, (1.6) *is equivalent with*

$$\Big\| (1+|\cdot|)^{-\frac{d-1}{2}} \sup_{t\in I} |\kappa_t|\,\Big\|_{L^p(\mathbb{R},\widetilde{\mu}_d)} \,<\, \infty, \tag{1.7}$$

where $\kappa_t = \mathcal{F}_{\mathbb{R}}^{-1}[m(t\cdot)]$ and $\widetilde{\mu}_d(r) = (1+|r|)^{d-1}\,dr$.

When d is an integer and $G = f(|x|)$, then $T_K G(x) = c\,\mathcal{T}_m f(|x|)$. Thus Theorem 1.2 reduces easily to Theorem 1.1. We give the proof of the sufficiency in §3, and the necessary conditions and some preliminary kernel estimates in §2. Finally, section 4 contains some extensions and remarks.

2. Necessary conditions and kernel bounds

2.1. **The condition (1.6).** The necessity of the condition in (1.6) is easily verified. Given an interval $I \Subset (0,\infty)$, take a function $\chi \in C_c^\infty(0,\infty)$ such that $m(t\cdot) = \chi m(t\cdot)$, $\forall\, t \in I$. Then,

$$\sup_{t\in I}\big|\,\mathcal{B}_d[m(t\cdot)]\,\big| \,=\, \sup_{t\in I}\big|\,\mathcal{B}_d[\chi m(t\cdot)]\,\big| \,\leq\, \mathcal{T}^*(\mathcal{B}_d\chi).$$

Taking $L^p(\mu_d)$ norms we see that the boundedness of $\mathcal{T}^*$ implies (1.6).

2.2. **The condition (1.7).** The argument given in [2] can be easily modified to show that (1.6) implies (1.7). We state the result separately and sketch the proof below.

Lemma 2.1. *Let* $m \in L^1_c(0,\infty)$*,* $I \Subset (0,\infty)$*, and* $\kappa_t = \mathcal{F}^{-1}_{\mathbb{R}}[m(t\cdot)]$ *for* $t \in I$*. Then for all* $1 \le p \le 2$

$$\Big\| \frac{\sup_{t\in I} |\kappa_t|}{(1+|\cdot|)^{\frac{d-1}{2}}} \Big\|_{L^p(\mathbb{R},\widetilde{\mu}_d)} \le C \, \Big\| \sup_{t\in I} \big| \mathcal{B}_d[m(t\cdot)] \big| \Big\|_{L^p(\mu_d)}, \tag{2.1}$$

where

$$\widetilde{\mu}_d(r) = (1+|r|)^{d-1}\, dr, \tag{2.2}$$

and C *is a constant depending on* I *and supp* m*.*

Proof. Fix an integer N_0 so that supp $m(t\cdot) \subset [1/N_0, N_0]$ for all $t \in I$; the constants below may depend on this number N_0. Let $\chi \in C^\infty_c(1/(2N_0), 2N_0)$ with $\chi \equiv 1$ in $[1/N_0, N_0]$, so that $m(t\cdot) = \chi m(t\cdot)$. We first claim that, for all $1 \le p \le 2$,

$$\Big\| (1+\cdot)^{-\frac{d-1}{2}} \sup_{t\in I} \big| \mathcal{B}_1[\chi \mathcal{B}_d g_t] \big| \Big\|_{L^p(\widetilde{\mu}_d)} \le C \, \Big\| \sup_{t\in I} |g_t| \Big\|_{L^p(\mu_d)}, \tag{2.3}$$

where g_t, $t \in I$, are functions so that the right hand side is finite. This is a "vector-valued" version of the inequality in [2, (4.4)], which is obtained with exactly the same proof; namely, the function on the left hand side of (2.3) is estimated pointwise as in [2, Corollary 3.2], with the $\sup_{t\in I}$ taken inside the integrals (a valid operation since the kernels involved are positive), and from here the proof is the same as in [2, p.46].

Choosing $g_t = \mathcal{B}_d[m(t\cdot)]$ in (2.3) and using the fact that $\mathcal{B}_1$ is the cosine transform, one obtains (2.1) with $\kappa_t = \mathcal{F}^{-1}_{\mathbb{R}}[m(t\cdot)]$ replaced by $h_t = \mathcal{F}^{-1}_{\mathbb{R}}[m_{\text{even}}(t\cdot)]$, where m_{even} denotes the even extension of m. Since $\widehat{\kappa}_t = \widehat{h}_t \chi$, we have

$$\sup_{t\in I} |\kappa_t| \le \big(\sup_{t\in I} |h_t|\big) * |\check{\chi}|.$$

Hence, taking $L^p(\mathbb{R},\widetilde{\mu}_d)$ norms of the above quantities multiplied by $(1+|r|)^{-(d-1)/2}$, and using the elementary Lemma 2.2 in [2] to dispense with the convolution, one controls the left hand side of (2.1) by the same expression with h_t in place of κ_t, therefore establishing the result. $\square$

2.3. **Kernel bounds.** Let $m \in L^1_c(0,\infty)$. Using Fubini's theorem we can write

$$\mathcal{T}_m f(r) = \int_0^\infty \mathcal{K}(r,s)\, f(s)\, d\mu_d(s),$$

where $\mathcal{K} = \mathcal{K}_m$ is given by

$$\mathcal{K}(r,s) = \int_0^\infty B_d(ru)\, B_d(su)\, m(u)\, d\mu_d(u), \quad r,s>0. \tag{2.4}$$

The following estimate is shown in [2], being a crucial step in proving the sufficiency of (1.1). We denote by $\kappa = \mathcal{F}_{\mathbb{R}}^{-1}[m]$, where m is extended as zero outside $\mathbb{R}^+$.

Lemma 2.2. *Let $m \in L^1$ with supp $m \subset [1/2, 2]$. Then, for every $n \in \mathbb{N}$, there exists $C_n > 0$ such that*

$$|\mathcal{K}(r,s)| \le C_n \frac{|\kappa| * \omega_n(\pm r \pm s)}{(1+r)^{\frac{d-1}{2}}(1+s)^{\frac{d-1}{2}}}, \quad \forall\, r,s>0 \tag{2.5}$$

where $\omega_n(u) = (1+|u|)^{-n}$. The same estimate is valid with $|\mathcal{K}(r,s)|$ replaced by $\big|\partial_s[\mathcal{K}(r,s)]\big|$.

We refer to [2, Lemma 3.1] for details about the proof. We only mention that (2.5) is obtained from the asymptotic expansion of Bessel functions, namely

$$B_d(u) = u^{-\frac{d-1}{2}} \sum_{\ell=0}^{M} \frac{c_\ell e^{iu} + d_\ell e^{-iu}}{u^\ell} + u^{-M} E_M(u), \quad u \ge 1, \tag{2.6}$$

for suitable constants c_ℓ, d_ℓ, and where $E_M(u)$ is a bounded C^∞ function with all its derivatives bounded.

3. The proof of Theorem 1.2

The proof is very much analogous to the proof of the L^p Hankel multiplier result in [2]. Since m is fixed, we write $\mathcal{T}^*$ instead of $\mathcal{T}^*_m$ (and $\mathcal{T}$ instead of $\mathcal{T}_m$). In view of Lemma 2.1 we must show that $\mathcal{T}^*$ is bounded in $L^p(\mu_d)$ provided that

$$A(p) \equiv \left\| \frac{\sup_{t\in I} |\kappa_t(\cdot)|}{(1+|\cdot|)^{\frac{d-1}{2}}} \right\|_{L^p(\mathbb{R},\widetilde{\mu}_d)} < \infty,$$

where as before $\kappa_t = \mathcal{F}_{\mathbb{R}}^{-1}[m(t\cdot)]$. Dilating m if necessary, we may assume that $I = [1,\lambda]$ for some $\lambda > 1$. We also fix an integer N_0 so that supp $m(t\cdot) \subset [1/N_0, N_0]$, $\forall\, t \in I$, and a bump function $\phi \in C_c^\infty(1/(2N_0), 2N_0)$ with $\phi \equiv 1$ in $[1/N_0, N_0]$, so that $m(t\cdot) = \phi m(t\cdot)$. The constants below may depend on λ or N_0.

3.1. Decomposition of $\mathcal{T}^*$.

For $j \in \mathbb{Z}$, define

$$\mathcal{T}_j^t f = \mathcal{B}_d[m(\lambda^j t\cdot)\mathcal{B}_d f] \quad \text{and} \quad \mathcal{T}_j^* f = \sup_{t\in[1,\lambda]} \left|\mathcal{T}_j^t f\right|,$$

and set

$$L_j f = \mathcal{B}_d[\phi(\lambda^j\cdot)\mathcal{B}_d f].$$

Clearly

$$\mathcal{T}^* f \le \Big(\sum_{j\in\mathbb{Z}} \sup_{s\in[\lambda^j,\lambda^{j+1}]} \big|\mathcal{B}_d[m(s\cdot)\mathcal{B}_d f]\big|^2\Big)^{\frac12} = \Big(\sum_{j\in\mathbb{Z}} \big|\mathcal{T}_j^*[L_j f]\big|^2\Big)^{\frac12}.$$

As discussed in [2, p.37], the following Littlewood-Paley inequality holds

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|L_jf|^2\Big)^{\frac12}\Big\|_{L^p(\mu_d)}\le C_p\,\|f\|_{L^p(\mu_d)},\quad 1<p<\infty;$$

this is well-known when d is an integer, and in general follows by standard arguments from a Hörmander type multiplier theorem as in Gasper-Trebels [3]. Thus, to establish Theorem 1.2 it suffices to prove the inequality

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j^*f_j|^2\Big)^{\frac12}\Big\|_{L^p(\mu_d)}\lesssim A(p)\,\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{\frac12}\Big\|_{L^p(\mu_d)}\tag{3.1}$$

for all functions f_j in the dense class $\mathcal{B}_d(C_c^\infty)$.

As in [2, §5], we break each operator $\mathcal{T}_j^t$ into various parts. Call $I_m=[\lambda^m,\lambda^{m+1})$, and write

$$\mathcal{T}_j^tf=\sum_{m\in\mathbb{Z}}\mathcal{T}_j^t[f\chi_{I_m}]=\sum_{m\in\mathbb{Z}}\Big[H_{j,m}^tf+\mathcal{S}_{j,m}^tf+E_{j,m}^tf\Big],$$

where

$$\begin{aligned}H_{j,m}^tf&=\chi_{[\lambda^{m+2},\infty)}\,\mathcal{T}_j^t[f\chi_{I_m}]\\ \mathcal{S}_{j,m}^tf&=\chi_{[\lambda^{m-2},\lambda^{m+2}]}\,\mathcal{T}_j^t[f\chi_{I_m}]\\ E_{j,m}^tf&=\chi_{(0,\lambda^{m-2}]}\,\mathcal{T}_j^t[f\chi_{I_m}]\,.\end{aligned}$$

Respective operators $H_{j,m}^*,\mathcal{S}_{j,m}^*,E_{j,m}^*$ are defined taking the $\sup_{t\in[1,\lambda]}$ of the modulus of each of the above expressions. We shall deduce the theorem from the following three propositions.

Proposition 3.1. *For $j,m\in\mathbb{Z}$ and $1\le p<\frac{2d}{d+1}$ we have*

$$\big\|H_{j,m+j}^*f\big\|_{L^p(\mu_d)}\lesssim A(p)\,2^{-|m|\delta(p)}\,\big\|f\chi_{I_{m+j}}\big\|_{L^p(\mu_d)}\,,\tag{3.2}$$

where $\delta(p)=\frac{d}{p}-\frac{d+1}{2}$.

In the next proposition we write, for $\varepsilon\ge0$,

$$B(\varepsilon)=\big\|\sup_{t\in I}|\kappa_t|\,\big\|_{L^1(\mathbb{R},(1+|r|)^\varepsilon\,dr)}.$$

When $p<\frac{2d}{d+1}$, it follows easily from Hölder's inequality that there is $\varepsilon(p)>0$ such that $B(\varepsilon)\lesssim A(p)$ for $\varepsilon<\varepsilon(p)$.

Proposition 3.2. *For $j,m\in\mathbb{Z}$, $\varepsilon\ge0$ and $1\le p\le2$ we have*

$$\big\|E_{j,m+j}^*f\big\|_{L^p(\mu_d)}\lesssim B(\varepsilon)\,2^{-|m|\varepsilon}\,\big\|f\chi_{I_{m+j}}\big\|_{L^p(\mu_d)}\,.$$

Finally, we write $I_m^*=[\lambda^{m-2},\lambda^{m+2}]$.

Proposition 3.3. *For every $m \in \mathbb{Z}$, $\varepsilon > 0$ and $1 < p \le 2$ we have*

$$\Big\| \big(\sum_{j\in\mathbb{Z}} |\mathcal{S}^*_{j,m} f_j|^2\big)^{\frac12} \Big\|_{L^p(I^*_m,\mu_d)} \lesssim B(\varepsilon) \Big\| \big(\sum_{j\in\mathbb{Z}} |f_j|^2\big)^{\frac12} \Big\|_{L^p(I_m,\mu_d)}. \tag{3.3}$$

3.2. **Proof of Propositions 3.1 and 3.2.** For each $j \in \mathbb{Z}$ and $t \in I$, the kernel $\mathcal{K}^t_j(r,s)$ of the operator $\mathcal{T}^t_j$ satisfies the estimate

$$\begin{aligned} \big|\mathcal{K}^t_j(r,s)\big| &= \Big| \int_0^\infty B_d(r\rho)\, B_d(s\rho)\, m(\lambda^j t\rho)\, d\mu_d(\rho) \Big| \\ &\lesssim C_n \sum_{(\pm,\pm)} \lambda^{-jd} \frac{|\kappa_t| * \omega_n(\lambda^{-j}(\pm r \pm s))}{(1+\lambda^{-j} r)^{\frac{d-1}{2}} (1+\lambda^{-j}s)^{\frac{d-1}{2}}}, \quad r,s>0, \end{aligned} \tag{3.4}$$

where $\omega_n(u) = (1+|u|)^{-n}$ and n can be chosen as large as desired. This is obtained changing variables $\lambda^j \rho = u$ and using Lemma 2.2. Recall that supp $m(t\cdot) \subset [1/N_0, N_0]$, for some N_0, so that the application of the lemma produces constants depending on N_0, but independent of $t \in I$. Thus, if we denote $W^* \equiv [\sup_{t\in I} |\kappa_t|] * \omega_n$, we have

$$\sup_{t\in I} \lambda^{jd} \big|\mathcal{K}^t_j(\lambda^j r, \lambda^j s)\big| \lesssim \frac{W^*(\pm r \pm s)}{(1+r)^{\frac{d-1}{2}} (1+s)^{\frac{d-1}{2}}}. \tag{3.5}$$

Using this kernel estimate, Propositions 3.1 and 3.2 can be proved as in [2], with the function $W^*(x)$ in place of the function $W_j(x)$ which appears in [2, (6.2)]. For completeness, we briefly sketch the arguments.

Proof of Proposition 3.1. Using the kernel estimate in (3.5) and Minkowski's integral inequality, one easily controls the quantity $\big\|H^*_{j,m+j} f\big\|_{L^p(\mu_d)}$ by a constant multiple of

$$\int_{I_{m+j}} \Big\| \frac{W^*(\lambda^{-j}(\pm\cdot \pm s))}{(1+\lambda^{-j}\cdot)^{\frac{d-1}{2}}} \Big\|_{L^p([\lambda^{m+j+2},\infty), d\mu_d(r))} \frac{\lambda^{-jd}\,|f(s)|}{(1+\lambda^{-j}s)^{\frac{d-1}{2}}}\, d\mu_d(s).$$

Changing variables $\lambda^{-j}(\pm r \pm s) = u$ inside the norm, since $r \gg s$ we see that $\lambda^{-j} r \approx |u|$. Performing also the change $\lambda^{-j}s = v$, the above expression becomes bounded by

$$\int_{I_m} \lambda^{jd/p} \Big\| \frac{W^*}{(1+|\cdot|)^{\frac{d-1}{2}}} \Big\|_{L^p(\mathbb{R}, d\tilde\mu_d)} \frac{|f(\lambda^j v)|}{(1+v)^{\frac{d-1}{2}}}\, d\mu_d(v).$$

Now, $\|(1+|\cdot|)^{-\frac{d-1}{2}} W^*\|_{L^p(\mathbb{R},\tilde\mu_d)} \lesssim A(p)$, since the convolution with ω_n is a harmless operation by [2, Lemma 2.2]. The remaining integral is handled with Hölder's inequality, leading to

$$\big\|H^*_{j,m+j} f\big\|_{L^p(\mu_d)} \lesssim A(p)\, \lambda^{jd/p}\, \big\|f(\lambda^j\cdot)\chi_{I_m}\big\|_{L^p(\mu_d)} \Big(\int_{I_m} \frac{s^{d-1}}{(1+s)^{p'(d-1)/2}}\, ds \Big)^{1/p'},$$

from which the right hand side of (3.2) follows easily.

Proof of Proposition 3.2. Again, we use the kernel estimate in (3.5) to write

$$E^*_{j,m+j}f(r) \lesssim \sum_{(\pm,\pm)} \int_{\lambda^2 r}^{\infty} \frac{\lambda^{-jd}\, W^*(\lambda^{-j}(\pm r \pm s))}{[(1+\lambda^{-j}r)(1+\lambda^{-j}s)]^{(d-1)/2}}\, \chi_{I_{m+j}}(s)\, |f(s)|\, s^{d-1}\, ds.$$

The range of integration is justified by the fact that, by definition of $E^*_{j,j+m}$, we have $r \leq \lambda^{m+j-2}$ and $s \geq \lambda^{m+j}$. To control the $L^p(\mu_d)$-norm of this operator we may use Schur's lemma; that is, it suffices to find some $\alpha \in \mathbb{R}$ so that

$$\int_{\lambda^2 r}^{\infty} \frac{\lambda^{-jd}\, W^*(\lambda^{-j}(\pm r \pm s))}{[(1+\lambda^{-j}r)(1+\lambda^{-j}s)]^{\frac{d-1}{2}}}\, \chi_{I_{m+j}}(s)\, s^{-\alpha p'}\, s^{d-1}\, ds \lesssim B'(\varepsilon)\, 2^{-\varepsilon|m|}\, r^{-\alpha p'} \tag{3.6}$$

$$\int_{0}^{r/\lambda^2} \frac{\lambda^{-jd}\, W^*(\lambda^{-j}(\pm r \pm s))}{[(1+\lambda^{-j}r)(1+\lambda^{-j}s)]^{\frac{d-1}{2}}}\, \chi_{I_{m+j}}(s)\, r^{-\alpha p}\, r^{d-1}\, dr \lesssim B'(\varepsilon)\, 2^{-\varepsilon|m|}\, s^{-\alpha p}\,, \tag{3.7}$$

where $B'(\varepsilon) \equiv \|(1+|\cdot|)^{\varepsilon} W^*\|_{L^1(\mathbb{R})} \lesssim B(\varepsilon)$ (again by [2, Lemma 2.2]). The inequalities (3.6) and (3.7) follow from straightforward pointwise estimates. Changing variables $\rho = \lambda^j r$ and $s = \lambda^j \sigma$ we may assume that $j = 0$. For the first inequality it suffices to see that

$$\frac{s^{-\alpha p'}\, r^{\alpha p'}\, s^{d-1}}{[(1+r)(1+s)]^{\frac{d-1}{2}}\, (1+|r \pm s|)^{\varepsilon}} \lesssim 2^{-\varepsilon|m|}, \qquad \text{for } \lambda^2 r \leq s, \quad s \in I_m. \tag{3.8}$$

Since $|r \pm s| \approx s$, separating the cases $m \geq 0$ and $m \leq 0$, (3.8) is easily verified as long as $\alpha p' \geq (d-1)/2$. Arguing similarly, (3.7) is implied by the pointwise estimate

$$\frac{r^{-\alpha p}\, s^{\alpha p}\, r^{d-1}}{[(1+r)(1+s)]^{\frac{d-1}{2}}\, (1+|r \pm s|)^{\varepsilon}} \lesssim 2^{-\varepsilon|m|}, \qquad \text{for } \lambda^2 r \leq s, \quad s \in I_m, \tag{3.9}$$

which again, separating into two cases according to the sign of m, it is valid as long as $\alpha p \leq (d-1)/2$. Thus, it suffices to choose a number α so that

$$\frac{d-1}{2p'} \leq \alpha \leq \frac{d-1}{2p}\,,$$

which is always possible when $p \leq 2$.

3.3. **Proof of Proposition 3.3.** By definition of $\mathcal{S}^*_{j,m}$, (3.3) follows from

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |\mathcal{T}^*_j f_j|^2 \Big)^{\frac12} \Big\|_{L^p(I^*_m, \mu_d)} \lesssim B(\varepsilon) \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^2 \Big)^{\frac12} \Big\|_{L^p(\mu_d)}, \tag{3.10}$$

for all functions f_j supported in I_m. When $p = 2$, the same Schur's lemma argument in the proof of Proposition 3.2 (with $\varepsilon = 0$) gives, for each $j \in \mathbb{Z}$,

$$\big\| \mathcal{T}^*_j f_j \big\|_{L^2(I^*_m, \mu_d)} \lesssim B(0)\, \|f_j\|_{L^2(\mu_d)},$$

where $\operatorname{supp} f_j \subset I_m$. The fact that $r \approx s \in I_m$ does not affect the previous proof, since when $\varepsilon = 0$ we do not need to estimate the factors $(1+|r \pm s|)^{\varepsilon}$. Thus, (3.10) holds for $p = 2$.

By Marcinkiewicz interpolation theorem, it suffices to prove the weak type (1,1) version of (3.10), which in "vector-valued" notation can be written as follows: if $\vec{f} = \{f_j\}_{j\in\mathbb{Z}}$ belongs to the space $L^1_{\ell^2}(I_m, \mu_d)$, then we must show that

$$\mathbf{T}(\vec{f}) \;\equiv\; \big\{\mathcal{T}^t_j(f_j)\big\}_{j\in\mathbb{Z},t\in I} \in L^{1,\infty}_{\mathbb{B}}(I^*_m, \mu_d),$$

where $\mathbb{B} = \ell^2(\mathbb{Z}; L^\infty(I_m))$. At this point one can proceed exactly as in the proof of [2, Prop. 5.3], with the crucial step now being the kernel estimate

$$\int_{I^*_m\setminus(2J_\nu)} \big|\mathbf{T}(x,y) - \mathbf{T}(x,y_\nu)\big|_{\ell^1(\mathbb{Z},L^\infty(I_m))}\, d\mu_d(x) \;\lesssim\; B(\varepsilon), \quad y \in J_\nu. \tag{3.11}$$

For completeness we briefly describe the proof, and establish (3.11) at the end of the section. Given $\alpha > 0$, we consider a Calderón-Zygmund decomposition of $\vec{f}$ of height $\alpha/B(\varepsilon)$, that is $\vec{f} = \vec{g} + \vec{b}$ such that

(i) $|\vec{g}(x)|_{\ell^2} \lesssim \alpha/B(\varepsilon)$ for a.e. $x \in I_m$;

(ii) there are dyadic subintervals $J_\nu \subset I^*_m$, $\nu = 1, 2, \ldots$, with disjoint interiors so that

$$\sum_{\nu=1}^{\infty} \mu_d\big(J_\nu\big) \;\lesssim\; \tfrac{B(\varepsilon)}{\alpha}\, \|\vec{f}\|_{L^1_{\ell^2}(I_m,\mu_d)}\,;$$

(iii) $\vec{b} = \sum_\nu \vec{b}_\nu$, and for each ν we have

$$\text{supp}\ \vec{b}_\nu \subset J_\nu, \qquad \int \vec{b}_\nu\, d\mu_d = \vec{0} \quad \text{and} \quad \int |\vec{b}_\nu|_{\ell^2}\, d\mu_d \;\lesssim\; \tfrac{\alpha}{B(\varepsilon)}\, \mu_d(J_\nu).$$

Clearly, this implies

$$\|\vec{b}\|_{L^1_{\ell^2}(\mu_d)} \le \sum_\nu \|\vec{b}_\nu\|_{L^1_{\ell^2}(\mu_d)} \lesssim \|\vec{f}\|_{L^1_{\ell^2}(\mu_d)} \quad \text{and} \quad \|\vec{g}\|_{L^1_{\ell^2}(\mu_d)} \lesssim \|\vec{f}\|_{L^1_{\ell^2}(\mu_d)}. \tag{3.12}$$

From this construction and the inequality (3.10) for $p = 2$ we obtain

$$\mu_d\big\{x \in I^*_m \;:\; |\mathbf{T}\vec{g}|_{\mathbb{B}} > \tfrac{\alpha}{2}\big\} \;\lesssim\; \tfrac{B(\varepsilon)^2}{\alpha^2}\, \|\vec{g}\|^2_{L^2_{\ell^2}(\mu_d)} \;\lesssim\; \tfrac{B(\varepsilon)}{\alpha}\, \|\vec{g}\|_{L^1_{\ell^2}(\mu_d)} \;\lesssim\; \tfrac{B(\varepsilon)}{\alpha}\, \|\vec{f}\|_{L^1_{\ell^2}(\mu_d)}.$$

In view of (ii) it suffices to show that

$$\mu_d\big\{x \in I^*_m \setminus [\cup_\nu \widetilde{J}_\nu] \;:\; |\mathbf{T}\vec{b}|_{\mathbb{B}} > \tfrac{\alpha}{2}\big\} \;\lesssim\; \tfrac{B(\varepsilon)}{\alpha}\, \|\vec{f}\|_{L^1_{\ell^2}(\mu_d)}, \tag{3.13}$$

where $\widetilde{J}_\nu$ denotes the interval with same center as J_ν (call it y_ν), and twice its length. We write $\mathbf{T}(x,y) = \{\mathcal{K}^t_j(x,y)\}_{j,t}$ for the $\mathbb{B}$-valued kernel of $\mathbf{T}$. The left hand side of (3.13) is handled by a standard argument (using the cancellation of $\vec{b}_\nu$):

$$\begin{aligned}
LHS &\leq \frac{2}{\alpha}\int_{I_m^*\setminus[\cup_\nu \widetilde{J}_\nu]} |\mathbf{T}\vec{b}(x)|_{\mathbb{B}}\,d\mu_d(x)\\
&\leq \frac{2}{\alpha}\sum_\nu \int_{I_m^*\setminus\widetilde{J}_\nu} |\mathbf{T}\vec{b}_\nu(x)|_{\mathbb{B}}\,d\mu_d(x)\\
&= \frac{2}{\alpha}\sum_\nu \int_{I_m^*\setminus\widetilde{J}_\nu} \Big(\sum_j \big[\sup_{t\in I}\int_{J_\nu}[\mathcal{T}_j^t(x,y)-\mathcal{T}_j^t(x,y_\nu)](b_\nu)_j(y)\,d\mu_d(y)\big]^2\Big)^{\frac12}\,d\mu_d(x)\\
&\leq \frac{2}{\alpha}\sum_\nu \int_{J_\nu}\int_{I_m^*\setminus\widetilde{J}_\nu} \Big|\mathbf{T}(x,y)-\mathbf{T}(x,y_\nu)\Big|_{\ell^\infty(\mathbb{Z},L^\infty(I_m))}\,d\mu_d(x)\,\big|\vec{b}_\nu(y)\big|_{\ell^2}\,d\mu_d(y).
\end{aligned}$$

Assuming the validity of (3.11), and using (3.12) above, one easily obtains (3.13).

Thus, it only remains to justify (3.11). We shall use Lemma 2.2 to control, for each fixed $j\in\mathbb{Z}$, the individual term

$$\mathbb{I}_j(y)\equiv\int_{I_m^*\setminus\widetilde{J}_\nu}\sup_{t\in I}\big|\mathcal{K}_j^t(x,y)-\mathcal{K}_j^t(x,y_\nu)\big|\,d\mu_d(x),\quad y\in J_\nu. \tag{3.14}$$

Consider first integers $j\in\mathbb{Z}$ such that $\lambda^{-j}|J_\nu|\leq 1$. We use the pointwise estimate

$$\Big|\partial_y\mathcal{K}_j^t(x,y)\Big| \lesssim C_n\,\frac{\lambda^{-j}\lambda^{-jd}\,|\kappa_t|*\omega_n(\lambda^{-j}(\pm x\pm y))}{(1+\lambda^{-j}x)^{\frac{d-1}{2}}(1+\lambda^{-j}y)^{\frac{d-1}{2}}},$$

which is obtained as in (3.4) applying Lemma 2.2 to $\lambda^{-jd}\partial_y[\mathcal{K}_{m(t\cdot)}(\lambda^{-j}x,\lambda^{-j}y)]$. Since $|x|\approx|z|\approx\lambda^m$ when $z\in J_\nu$ and $x\in I_m^*$ we have

$$\begin{aligned}
\mathbb{I}_j(y) &\lesssim |y-y_\nu|\int_{I_m^*\setminus\widetilde{J}_\nu}\lambda^{-2j}\int_0^1 W^*(\lambda^{-j}(\pm x\pm[y_\nu+s(y-y_\nu)]))ds\,dx\\
&\lesssim \lambda^{-j}|J_\nu|\int_{|u|\gtrsim\lambda^{-j}|J_\nu|}W^*(u)\,du,
\end{aligned}$$

where in the second inequality we took out the s integral and changed variables $u=\lambda^{-j}(\pm x\pm[y_\nu+s(y-y_\nu)])$, so that $|u|\gtrsim\lambda^{-j}|J_\nu|$ (since $x\notin\widetilde{J}_\nu$ and $y_\nu+s(y-y_\nu)\in J_\nu$). Therefore, for all $y\in J_\nu$,

$$\sum_{j\in\mathbb{Z}\ :\ \lambda^{-j}|J_\nu|\leq 1}\mathbb{I}_j(y)\lesssim\int_{\mathbb{R}}|W^*(u)|\,du\lesssim B(0).$$

On the other hand, when $\lambda^{-j}|J_\nu|\geq 1$ we directly apply the estimate in (3.5) to $\mathcal{K}_j^t(x,y)$ and $\mathcal{K}_j^t(x,y_\nu)$, obtaining after a similar reasoning that for all $y\in J_\nu$

$$\begin{aligned}
\sum_{\lambda^j\leq|J_\nu|}\mathbb{I}_j(y) &\lesssim \sum_{\lambda^j\leq|J_\nu|}\int_{|u|\gtrsim\lambda^{-j}|J_\nu|}W^*(u)\,du\\
&\lesssim \sum_{\lambda^j\leq|J_\nu|}\big(\lambda^{-j}|J_\nu|\big)^{-\varepsilon}\int_{\mathbb{R}}W^*(u)\,(1+|u|)^\varepsilon\,du\lesssim B(\varepsilon),\quad\forall\,y\in J_\nu.
\end{aligned}$$

This justifies (3.11) and completes the proof of Proposition 3.3.

3.4. **Proof of** (3.1)**.** We argue as in [2, p. 50]. That is, for each $m \in \mathbb{Z}$ we control the $H^*_{j,m+j}$ terms as follows:

$$\Big\|\big(\sum_{j\in\mathbb{Z}} |H^*_{j,m+j}f_j|^2\big)^{\frac12}\Big\|_{L^p(\mu_d)} \le \Big(\sum_{j\in\mathbb{Z}} \|H^*_{j,m+j}f_j\|^p_{L^p(\mu_d)}\Big)^{\frac1p}$$

$$\lesssim A(p)\,2^{-|m|\delta(p)}\Big(\sum_{j\in\mathbb{Z}}\int_{I_{m+j}} |f_j|^p\,d\mu_d\Big)^{\frac1p} \le A(p)\,2^{-|m|\delta(p)}\Big\|\big(\sum_{\ell\in\mathbb{Z}}|f_\ell|^2\big)^{\frac12}\Big\|_{L^p(\mu_d)} \tag{3.15}$$

where in the first inequality we used the inclusion $\ell^p \hookrightarrow \ell^2$ (since $p<2$), in the second one Proposition 3.1, and in the last one we have majorized $|f_j|$ by the square function and then summed the integrals over disjoint intervals.

The same reasoning, using Proposition 3.2, gives

$$\Big\|\big(\sum_{j\in\mathbb{Z}} |E^*_{j,m+j}f_j|^2\big)^{\frac12}\Big\|_{L^p(\mu_d)} \lesssim B(\varepsilon)\,2^{-|m|\varepsilon}\Big\|\big(\sum_{j\in\mathbb{Z}}|f_j|^2\big)^{\frac12}\Big\|_{L^p(\mu_d)}. \tag{3.16}$$

Since for $1<p<\frac{2d}{d+1}$ we have $\delta(p)>0$ and $B(\varepsilon)\lesssim A(p)$ for some $\varepsilon>0$, we can sum in $m\in\mathbb{Z}$ the expressions in (3.15) and (3.16), obtaining the right hand side of (3.1). On the other hand, the remaining term is controlled with Proposition 3.3, since

$$\Big\|\big(\textstyle\sum_{j\in\mathbb{Z}}\big|\sum_{m\in\mathbb{Z}}\mathcal{S}^*_{j,m}f_j\big|^2\big)^{\frac12}\Big\|^p_{L^p(\mu_d)} = \textstyle\sum_{m\in\mathbb{Z}}\Big\|\big(\sum_{j\in\mathbb{Z}}\big|\sum_{i=-1}^{2}\mathcal{S}^*_{j,m+i}f_j\big|^2\big)^{\frac12}\Big\|^p_{L^p(I_m,\mu_d)}$$
$$\lesssim \textstyle\sum_{m\in\mathbb{Z}}\Big\|\big(\sum_{j\in\mathbb{Z}}|\mathcal{S}^*_{j,m}f_j|^2\big)^{\frac12}\Big\|^p_{L^p(I^*_m,\mu_d)} \lesssim B(\varepsilon)^p\sum_{m\in\mathbb{Z}}\Big\|\big(\sum_{j\in\mathbb{Z}}|f_j|^2\big)^{\frac12}\Big\|^p_{L^p(I_m,\mu_d)},$$

which yields the desired expression by disjointness of the I_m's.

4. Remarks and further comments

4.1. **Weak type boundedness.** One can also obtain the following characterization.

Theorem 4.1. *Let $m\in L^1_c(0,\infty)$. If $1<p<2d/(d+1)$ then $\mathcal{T}^*_m$ extends as a bounded operator from $L^p(\mu_d)$ into $L^{p,\infty}(\mu_d)$ if and only if*

$$\Big\|\sup_{t\in I}\big|\mathcal{B}_d[m(t\cdot)]\big|\Big\|_{L^{p,\infty}(\mu_d)}<\infty \tag{4.1}$$

for some fixed interval $I \Subset (0,\infty)$.

We sketch the modifications required for this case, following the strategy in [2]. Propositions 3.2 and 3.3 hold with L^p replaced by $L^{p,\sigma}$, for any $1\le\sigma\le\infty$ and $1<p<2$; this is a consequence of the Marcinkiewicz interpolation theorem and the present statements. The same reasoning applies to Lemma 2.1 (this time interpolating the inequality in (2.3)). On the other hand, the proof of Proposition 3.1 can be easily adapted to obtain the following generalization:

Proposition 4.2. *For $j, m \in \mathbb{Z}$, $1 < p < \frac{2d}{d+1}$ and $1 \le \sigma \le \infty$ we have*

$$\big\| H^*_{j,m+j} f \big\|_{L^{p,\sigma}(\mu_d)} \lesssim A(p,\sigma)\, 2^{-|m|\delta(p)} \,\big\| f \chi_{I_{m+j}} \big\|_{L^{p,\infty}(\mu_d)},$$

where $\delta(p) = \frac{d}{p} - \frac{d+1}{2}$ and

$$A(p,\sigma) \equiv \left\| \frac{\sup_{t\in I} |\kappa_t(r)|}{1 + |r|^{(d-1)/2}} \right\|_{L^{p,\sigma}(\mathbb{R},\widetilde{\mu}_d)}.$$

Using these facts, one can establish the inequality

$$\Big\| \big(\sum_{j\in\mathbb{Z}} |\mathcal{T}_j^* f_j|^2\big)^{\frac12} \Big\|_{L^{p,\infty}(\mu_d)} \lesssim A(p,\infty) \Big\| \big(\sum_{j\in\mathbb{Z}} |f_j|^2\big)^{\frac12} \Big\|_{L^p(\mu_d)} \tag{4.2}$$

(which implies Theorem 4.1) with minor modifications in the arguments presented in §3.4. Namely, the crucial term to bound is

$$\begin{aligned}
\Big\| \big(\sum_{j\in\mathbb{Z}} |H^*_{j,m+j}(f_j)|^2\big)^{\frac12} \Big\|_{L^{p,\infty}(\mu_d)} &\lesssim \Big(\sum_{j\in\mathbb{Z}} \|H^*_{j,m+j}(f_j)\|^p_{L^{p,\infty}(\mu_d)}\Big)^{\frac1p} \\
&\lesssim A(p,\infty)\, 2^{-|m|\delta(p)} \Big(\sum_{j\in\mathbb{Z}} \|f_j \chi_{I_{m+j}}\|^p_{L^{p,\infty}(\mu_d)}\Big)^{\frac1p} \\
&\le A(p,\infty)\, 2^{-|m|\delta(p)} \Big\| \big(\sum_{\ell\in\mathbb{Z}} |f_\ell|^2\big)^{\frac12} \Big\|_{L^p(\mu_d)},
\end{aligned}$$

where in the second inequality we have used Proposition 4.2 (with $\sigma = \infty$), and the first inequality can be justified from general facts about Lorentz norms (see [2, p.50] for details). Similar arguments for the operators $E^*_{j,m+j}$ and $\mathcal{S}^*_{j,m}$ give (4.2).

Remark. We do not know whether condition (4.1) may actually imply boundedness of $\mathcal{T}_m^*$ from $L^{p,\infty}(\mu_d)$ into $L^{p,\infty}(\mu_d)$. It is known that this is the case for the Bochner-Riesz multipliers m_α defined in (1.4) above (see [1]).

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