ON PLATE DECOMPOSITIONS OF CONE MULTIPLIERS

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Abstract. An important inequality due to Wolff on plate decompositions of cone multipliers is known to have consequences for sharp L^p results on cone multipliers, local smoothing for the wave equation, convolutions with radial kernels, Bergman projections in tubes over cones, averages over finite type curves in $\mathbb{R}^3$ and associated maximal functions. We observe that the range of p in Wolff's inequality, for the conic and the spherical versions, can be improved by using bilinear restriction results. We also use this inequality to give some improved estimates on square functions associated to decompositions of cone multipliers in low dimensions. This gives a new L^4 bound for the cone multiplier operator in $\mathbb{R}^3$.

1. Introduction

Let $\Gamma = \{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d \ : \ \tau = |\xi|\,\}$ denote the forward light-cone in $\mathbb{R}^{d+1}$, $d \geq 2$. For fixed $c > 0$ and small $\delta > 0$, we consider δ-neighborhoods of the truncated cone

$$\Gamma_\delta(c) = \{(\tau,\xi) \in \mathbb{R}^{d+1} \ : \ 1 \leq \tau \leq 2 \quad \text{and} \quad \big|\tau - |\xi|\big| \leq c\delta\},$$

with the usual decomposition into plates subordinated to a $\sqrt{\delta}$-*separated* sequence in the sphere $\{\omega_k\} \subset S^{d-1}$:

$$\Pi_k^{(\delta)} = \Big\{(\tau,\xi) \in \Gamma_\delta(c) \ : \ \big|\frac{\xi}{|\xi|} - \omega_k\big| \leq c'\sqrt{\delta}\,\Big\}; \qquad \text{dist}(\omega_k,\omega_{k'}) \geq \sqrt{\delta} \quad \text{if } k \neq k'. \tag{1.1}$$

Let

$$\alpha(p) := d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, \tag{1.2}$$

the standard Bochner-Riesz critical index in d dimensions. Then Wolff's inequality asserts that for all $\varepsilon > 0$

$$\Big\|\sum_k f_k\Big\|_p \leq C_\varepsilon\, \delta^{-\alpha(p)-\varepsilon}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p} \tag{1.3}$$

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provided that

$$\operatorname{supp}\widehat{f}_k \subset \Pi_k^{(\delta)}. \tag{1.4}$$

The power $\alpha(p)$ is optimal for each p (except perhaps for $\varepsilon > 0$), and the inequality is conjectured to hold for all $p > 2 + \frac{4}{d-1}$. In his fundamental work [22], Wolff developed a method to prove such inequalities for large values of p, and obtained a positive answer for $d = 2$ and $p > 74$. Subsequently the method has been extended in the paper by Łaba and Wolff [11] to higher dimensions. It is shown there that (1.3) holds for $p > 2 + \frac{32}{3d-7}$ when $d \geq 3$ and $p > 2 + \frac{8}{d-3}$ when $d \geq 4$. In this paper we modify the weakest part of their proof to obtain a better range of exponents in all dimensions (see Table 1 below). The improvement relies on certain square-function bounds which follow from Wolff's bilinear Fourier extension theorem, [23].

Dimension	[22], [11]	Improvements	Conjecture
$d = 2$	$p > 74$	$p > p_2 := 63 + 1/3$	$p > 6$
$d = 3$	$p > 18$	$p > p_3 := 15$	$p > 4$
$d = 4$	$p > 8.4$	$p > p_4 := 7.28$	$p > \frac{10}{3}$
$d \geq 5$	$p > 2 + \frac{8}{d-3}$	$p > p_d := 2 + \frac{8}{d-3}(1 - \frac{1}{d+1})$	$p > 2 + \frac{4}{d-1}$

TABLE 1. Range of exponents for the validity of (1.3) for light-cones.

Theorem 1.1. *Let $d \geq 2$ and p_d as in Table 1. Then, under the assumption* (1.4) *the inequality (1.3) holds for all $\varepsilon > 0$ and all $p \geq p_d$.*

Remark: Various further more technical improvements on the range of Theorem 1.1 (and by implication on the results of Corollaries 1.3 and 1.4 below) have been obtained by the authors, and also by Wilhelm Schlag (personal communication). We intend to take up these matters in a joint paper (*cf.* [9]).

A similar result can be proved for decompositions of spheres in $\mathbb{R}^d$. We now let

$$\mathcal{U}_\delta(c) = \{\xi \in \mathbb{R}^d : \big||\xi| - 1\big| \leq c\delta\},$$

and consider the decomposition into plates subordinated to a $\sqrt{\delta}$-separated sequence in the sphere $\{\omega_k\} \subset S^{d-1}$,

$$B_k^{(\delta)} = \Big\{\xi \in \mathcal{U}_\delta(c) : \big|\xi/|\xi| - \omega_k\big| \leq c'\sqrt{\delta}\Big\}.$$

Theorem 1.2. *The analogue of Wolff's inequality for the sphere,*

$$\Big\|\sum_k f_k\Big\|_p \leq C_\varepsilon\, \delta^{-\alpha(p)-\varepsilon}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}, \quad \textit{supp}\ \widehat{f}_k \subset B_k^{(\delta)}, \tag{1.5}$$

holds for $p \geq 2 + \frac{8}{d-1} - \frac{4}{(d-1)d}$ and all $\varepsilon > 0$.

Again (1.5) is conjectured to hold for the optimal range $p > 2 + 4/(d-1)$. It has been known to hold for $p > 2 + 8/(d-1)$; this follows from a modification of the argument in [11], see also [10]. Note that in two dimensions the range is improved from previously $p > 10$ to $p > 8$.

Remark: Theorem 1.2 may be extended to convex surfaces with nonvanishing Gaussian curvature and similarly Theorem 1.1 may be extended to cones with $d-1$ positive principal curvatures. This can be achieved by using scaling and induction on scales arguments such as in §2 of [16], see also the article by Łaba and Pramanik [10] for related results.

We proceed to list some of the known implications of Theorem 1.1.

Corollary 1.3. *Let $d \geq 2$ and p_d as in Table 1. Then*

(i) For all $p > p_d$, $\alpha > \frac{d-1}{2} - \frac{d}{p}$, we have

$$\Big(\int_1^2 \big\|e^{it\sqrt{-\Delta}} f\big\|_{L^p(\mathbb{R}^d)}^p dt\Big)^{1/p} \lesssim \|f\|_{L^p_\alpha(\mathbb{R}^d)}. \tag{1.6}$$

(ii) For all $p \in (p_d, \infty)$, $\alpha > \frac{d-1}{2} - \frac{d}{p}$ the Fourier multiplier

$$m_\alpha(\tau, \xi) = (1 - |\xi|^2/\tau^2)_+^\alpha \tag{1.7}$$

defines a bounded operator in $L^p(\mathbb{R}^{d+1})$.

(iii) Let $K \in \mathcal{S}'(\mathbb{R}^d)$ be radial, let $\varphi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ so that φ is radial and not identically zero, and let $\varepsilon > 0$. Let $K_t = \mathcal{F}^{-1}[\varphi \widehat{K}(t\cdot)]$. Then for all Schwartz functions f and $1 < p < \frac{p_d}{p_d - 1}$

$$\|K * f\|_p \leq C_\varepsilon \sup_{t>0} \|K_t\|_{p+\varepsilon} \|f\|_p.$$

(iv) Let $\chi \in C_0^\infty(\mathbb{R})$ and $s \mapsto \gamma(s) \in \mathbb{R}^3$ a smooth curve satisfying $\sum_{j=1}^n |\langle \theta, \gamma^{(j)}(s)\rangle| \neq 0$ for every unit vector θ and every $s \in supp\ \chi$. For $t > 0$ define the convolution operator A_t by

$$A_t f(x) = \int f(x - t\gamma(s))\chi(s)ds.$$

Suppose that $\max\{n, 32 + 2/3\} < p < \infty$. Then A_t maps $L^p(\mathbb{R}^3)$ into the L^p-Sobolev space $L^p_{1/p}(\mathbb{R}^3)$. Moreover the maximal function $Mf = \sup_t |A_t f|$ defines a bounded operator on $L^p(\mathbb{R}^3)$.

Parts (i), (ii), (iii) are standard consequences of Theorem 1.1; see [22] for (i) and the local version of (ii). The global version follows by results on dyadic decompositions of multipliers and L^p Calderón-Zygmund theory (see [6] or [17]). The proof of Theorem 1.6 in [15] together with these arguments can be used to deduce (iii) from Theorem 1.1. For (iv) see [16].

Besides the connection to cone multipliers a major motivation for this paper is the relevance of inequalities for plate decompositions for the boundedness properties of the Bergman projection in tube domains over full light cones, see [1], [2]. Denote by $\Delta(Y) = y_0^2 - |y'|^2$ the Lorentz form and consider the forward light cone on which Δ is positive;

$$\Lambda^{d+1} = \{Y = (y_0, y') \in \mathbb{R} \times \mathbb{R}^d : y_0^2 - |y'|^2 > 0, y_0 > 0\}.$$

Let $\mathcal{T}^{d+1} \subset \mathbb{C}^{d+1}$ be the tube domain over Λ^{d+1}, i.e.

$$\mathcal{T}^{d+1} = \mathbb{R}^{d+1} + i\Lambda^{d+1}.$$

Let $w_\gamma(Y) = \Delta(Y)^\gamma$ and consider the weighted space $L^p(\mathcal{T}^{d+1}, w_\gamma)$ with norm

$$\|F\|_{p,\gamma} = \Big(\iint_{\mathcal{T}^{d+1}} |F(X+iY)|^p \Delta^\gamma(Y)\, dY dX \Big)^{1/p}.$$

Let $\mathcal{P}_\gamma$ be the orthogonal projection mapping the weighted space $L^2(\mathcal{T}^{d+1}, w_\gamma)$ to its subspace $\mathcal{A}^p_\gamma$ consisting of the holomorphic functions. Only the case $\gamma > -1$ is interesting since $\mathcal{A}^p_\gamma = \{0\}$ for $\gamma \le -1$. We are interested in the L^p boundedness properties of $\mathcal{P}_\gamma$. For $\gamma > -1$ the operator $\mathcal{P}_\gamma$ can only be bounded on $L^p(\mathcal{T}^{d+1}, w_\gamma)$ in the range

$$1 + \frac{d-1}{2(\gamma+d+1)} < p < 1 + \frac{2(\gamma+d+1)}{d-1}, \tag{1.8}$$

see e.g. Theorem 4.3 in [3], and (1.8) is indeed the conjectured range for L^p boundedness.

Corollary 1.4. *Let $d \ge 2$ and p_d as in Table 1. Then for all $\gamma \ge \frac{d-1}{2}(p_d - \frac{2(d+1)}{d-1})$, the Bergman projection $\mathcal{P}_\gamma$ is a bounded operator in $L^p(\mathcal{T}^{d+1}, w_\gamma)$ in the sharp range* (1.8).

Beyond Corollary 1.4 both Theorem 1.1 and Theorem 4.4 below have implications for the range of boundedness of the Bergman projector $\mathcal{P}_\gamma$ in natural weighted mixed norm spaces. For the derivation of Corollary 1.4 and further discussion of mixed norm estimates we refer to [2] (*cf.* in particular Proposition 5.5 and Corollaries 5.12 and 5.17).

Our approach to Theorem 1.1 is based on bilinear methods, for which we consider a closely related inequality:

$$\Big\| \sum_k f_k \Big\|_p \le C_\alpha\, \delta^{-\alpha} \Big(\sum_k \|f_k\|_p^2 \Big)^{1/2}. \tag{1.9}$$

One can conjecture the validity of (1.9) for all $\alpha > 0$ and all $2 < p < 2 + \frac{4}{d-1}$, but for the moment no positive result for any such p seems to be known. The limiting point $p = \frac{2(d+1)}{d-1}$ should be the hardest case, since by interpolation and Hölder's inequality it implies both (1.9) and (1.3) in all the conjectured ranges. This kind of inequalities arises naturally in the study of weighted mixed norm inequalities for the Bergman projection operator $\mathcal{P}_\gamma$, see [2].

We shall deduce Theorem 1.1 by using a stronger version of (1.9) for $p = 2(d+3)/(d+1)$, but with a power of $1/\delta$ which is (probably) not optimal. Namely under the assumption (1.4) we have

$$\Big\| \sum_k f_k \Big\|_{\frac{2(d+3)}{d+1}} \le C_\varepsilon\, \delta^{-\frac{d-1}{4(d+3)} - \varepsilon} \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_{\frac{2(d+3)}{d+1}}, \tag{1.10}$$

for all $\varepsilon > 0$. We prove this inequality in §2 using the bilinear approach of Tao and Vargas [20, §5] and the optimal bilinear cone extension inequality of T. Wolff [23], see Proposition 2.3 below. By Minkowski's inequality and interpolation (1.10) trivially implies non optimal estimates for the inequality (1.9) for all $p \in (2, \infty)$ (see Corollary 2.4 below). In §3 we use these to refine a part of Wolff's proof of (1.3) and obtain the new sharp estimates for large p announced in Table 1. In §4 we improve on some of the square-function results in low dimensions; these yield in particular the following estimate for the cone multiplier in $\mathbb{R}^{2+1}$.

Theorem 1.5. *Suppose $\alpha > \frac{5}{44}\big(\frac{p_2-4}{p_2-\frac{41}{11}}\big)$. Then the cone Fourier multiplier m_α defines a bounded operator on $L^4(\mathbb{R}^3)$ and the local smoothing result* (1.6) *holds in two dimensions.*

This is a small improvement over the known range $\alpha > 5/44$ which follows from a combination of [20] and [23].

Notation. We shall use the notation $A \lesssim B$ if there is a constant (which may depend on d) so that $A \le CB$. For families (A_δ, B_δ), $\delta \le 1$ we use $A_\delta \lessapprox B_\delta$ if for every $\varepsilon \in (0,1)$ there is a constant C_ε so that $A_\delta \le C_\varepsilon \delta^{-\varepsilon} B_\delta$ for $\delta < 1$.

2. The bilinear estimate

Following the approach by Tao and Vargas, we first establish an equivalence between linear and bilinear versions of (1.10), which is a higher dimensional analogue of Lemma 5.2 in [20].

Lemma 2.1. *Let $d \ge 2$, and suppose that for some $p \in [2,\infty)$ and $\alpha > \max\{0, (d-1)(1/4 - 1/p)\}$*

$$\Big\| \Big(\sum_{\omega_k \in \Omega} f_k \Big) \Big(\sum_{\omega_{k'} \in \Omega'} f_{k'} \Big) \Big\|_{p/2} \le C\, \delta^{-2\alpha} \Big\| \Big(\sum_{\omega_k \in \Omega} |f_k|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_{\omega_{k'} \in \Omega'} |f_{k'}|^2 \Big)^{1/2} \Big\|_p, \tag{2.1}$$

holds for all $f_k \in \mathcal{S}(\mathbb{R}^{d+1})$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$, all pairs of 1-separated subsets $\Omega, \Omega' \subset S^{d-1}$ and all $\delta \ll 1$. Then, it must also hold

$$\Big\| \sum_k f_k \Big\|_p \le C'\, \delta^{-\alpha} \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p, \quad \textit{supp } \widehat{f_k} \subset \Pi_k^{(\delta)}. \tag{2.2}$$

We remark that the restriction on α for $p > 4$ is never severe. To see this we note that the condition $(d-1)(1/4 - 1/p) \le \alpha(p)/2$ holds iff $d \ge 2$ and that (2.2) cannot hold with $\alpha < \alpha(p)/2$; this can be proved using Knapp examples.

Proof of Lemma 2.1. Let $\Phi : Q \equiv [0,1]^{d-1} \to S^{d-1}$ be a smooth parametrization of (a compact subset of) the sphere and let $\mathcal{D}$ denote the set of all dyadic intervals $I \subset Q$ with $|I| \ge \delta^{\frac{d-1}{2}}$. As in [21, p. 971], we may consider a Whitney decomposition $Q \times Q = \biguplus_{I \sim J} I \times J$, where $I \sim J$ means:

(i) $I, J \in \mathcal{D}$ and $|I| = |J|$;

(ii) If $|I| > \delta^{\frac{d-1}{2}}$, then I and J are not adjacent but their parents are.

(iii) If $|I| = \delta^{\frac{d-1}{2}}$, then I, J have adjacent or equal parents.

For simplicity, we assume (by splitting the sphere into finitely many pieces) that all $\omega_k \in \Phi(Q)$ and let $y_k = \Phi^{-1}(\omega_k) \in Q$. We also denote $\mathcal{D}_j = \{I \in \mathcal{D} \;:\; |I| = 2^{-j(d-1)}\}$. Then

$$\Big(\sum_k f_k \Big)^2 = \sum_{y_k, y_{k'} \in Q} f_k\, f_{k'} = \sum_{\sqrt{\delta} \le 2^{-j} \le 1} \sum_{\substack{I,J \in \mathcal{D}_j \\ I \sim J}} \big(\sum_{y_k \in I} f_k \big) \big(\sum_{y_{k'} \in J} f_{k'} \big).$$

To establish (2.2) we take $L^{p/2}$-norms in the above expression and use Minkowski's inequality in j, so that we reduce the problem to show, for each j

$$\Big\| \sum_{\substack{I,J \in \mathcal{D}_j \\ I \sim J}} \Big(\sum_{y_k \in I} f_k \Big) \Big(\sum_{y_{k'} \in J} f_{k'} \Big) \Big\|_{p/2} \lesssim (2^{2j}\delta)^{-2\alpha} \max\{1, 2^{j(d-1)(1-4/p)}\} \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p^2. \tag{2.3}$$

Inequality (2.3) is trivial when $2^{-j} \approx \sqrt{\delta}$ since by assumption the number of y_k's in each I is approximately constant. We consider the general case $\sqrt{\delta} < 2^{-j} \le 1$. By construction we must have

$$\sum_{I\in\mathcal{D}_j}\sum_{J\sim I}\chi_{I+J} \lesssim 1. \tag{2.4}$$

Indeed, if c_I denotes the center of I, then

$$I+J \subset \big(c_I + B_{c2^{-j}}\big) + \big(c_J + B_{c2^{-j}}\big) \subset 2c_I + B_{c'2^{-j}}.$$

Since for each I there are at most $O(1)$ cubes J with $J \sim I$, and since the centers c_I are 2^{-j} separated, (2.4) follows easily.

From (2.4) it follows that the functions $F_{I,J} = \big(\sum_{y_k\in I} f_k\big)\big(\sum_{y_{k'}\in J} f_{k'}\big)$ have pairwise (almost) disjoint spectra when $I \sim J \in \mathcal{D}_j$. We may conclude by orthogonality and standard interpolation arguments

$$\Big\|\sum_{I\sim J\in\mathcal{D}_j} F_{I,J}\Big\|_{p/2} \lesssim \max\{1, 2^{j(d-1)(1-4/p)}\}\Big(\sum_{I\sim J\in\mathcal{D}_j}\|F_{I,J}\|_{p/2}^{p/2}\Big)^{2/p}. \tag{2.5}$$

(the case $p/2 = 2$ follows by orthogonality and the cases $p/2 = 1$ and $p/2 = \infty$ are trivial; see e.g. Lemma 7.1 in [20]). Next, we wish to use the bilinear assumption (2.1) to estimate $\|F_{I,J}\|_{p/2}$. This can only be used directly when $2^j \approx 1$, since $\operatorname{dist}(I,J) \sim 1$. For other j's we must use Lorentz transformations to rescale the problem. To do this, let $\{\eta_1, \dots, \eta_d\}$ be an orthonormal basis of $\mathbb{R}^d$ with η_1 being the center of $\Phi(I)$. Then we define $L \in SO(1,d)$ acting on a basis of $\mathbb{R}^{d+1}$ by

$$L(1,\eta_1) = (1,\eta_1), \quad L(-1,\eta_1) = \tfrac{\sigma}{\delta}(-1,\eta_1) \quad \text{and} \quad L(0,\eta_\ell) = \sqrt{\tfrac{\sigma}{\delta}}\,(0,\eta_\ell),\ \ell = 2, ..., d,$$

where we choose $\sigma = 2^{2j}\delta$ (so that $\delta < \sigma < 1$). The functions $f_k \circ L$ have now spectrum in (perhaps a multiple) of the plates $\Pi_k^{(\sigma)}$ corresponding to the $\sqrt{\sigma}$-separated centers $\{L(1,\omega_k)\}$. Moreover, by the choice of σ, the plates corresponding to $y_k \in I$ and $y_{k'} \in J$ are c-separated, and therefore after a change of variables we can apply (2.1) at scale σ to obtain

$$\begin{aligned}\|F_{I,J}\|_{p/2} &= \Big\|\Big(\sum_{y_k\in I} f_k\Big)\Big(\sum_{y_{k'}\in J} f_{k'}\Big)\Big\|_{p/2}\\ &\lesssim (2^{2j}\delta)^{-2\alpha}\Big\|\Big(\sum_{y_k\in I}|f_k|^2\Big)^{1/2}\Big\|_p\Big\|\Big(\sum_{y_{k'}\in J}|f_{k'}|^2\Big)^{1/2}\Big\|_p,\end{aligned}$$

and then also

$$\begin{aligned}\Big(\sum_{I\sim J\in\mathcal{D}_j}\|F_{I,J}\|_{p/2}^{p/2}\Big)^{2/p} &\lesssim (2^{2j}\delta)^{-2\alpha}\Big[\sum_{I\sim J\in\mathcal{D}_j}\Big\|\Big(\sum_{y_k\in I}|f_k|^2\Big)^{1/2}\Big\|_p^{\frac{p}{2}}\Big\|\Big(\sum_{y_{k'}\in J}|f_{k'}|^2\Big)^{1/2}\Big\|_p^{\frac{p}{2}}\Big]^{\frac{2}{p}}\\ &\lesssim (2^{2j}\delta)^{-2\alpha}\Big[\int\Big(\sum_I\sum_{y_k\in I}|f_k|^2\Big)^{p/2}\Big]^{2/p} \le (2^{2j}\delta)^{-2\alpha}\Big\|\Big(\sum_k|f_k|^2\Big)^{1/2}\Big\|_p^2,\end{aligned}$$

where in the second inequality we have used $2ab \le a^2 + b^2$ followed by the imbedding $\ell^1 \hookrightarrow \ell^{\frac{p}{2}}$. Combining this with (2.5) we obtain

$$\Big\|\sum_{I\sim J\in\mathcal{D}_j} F_{I,J}\Big\|_{p/2} \lesssim (2^{2j}\delta)^{-2\alpha}\max\{1, 2^{j(d-1)(1-4/p)}\}\Big\|\Big(\sum_k|f_k|^2\Big)^{1/2}\Big\|_p^2, \tag{2.6}$$

This proves (2.3). By our assumption on α we may sum in j and the lemma follows. $\square$

We turn to the proof of (a generalization of) the square function estimate (1.10). We shall use the following statement of Wolff's Fourier extension theorem.

Wolff's bilinear estimate. [23, p. 680]. *Let $p \geq \frac{d+3}{d+1}$, $\varepsilon > 0$ and let E, E' be 1-separated subsets of $\Gamma_{1/N}$. Then, for all smooth f and g supported in E and E', and all N-cubes Q, we have*

$$\big\|\widehat{f}\,\widehat{g}\big\|_{L^p(Q)} \leq C_\varepsilon\, N^{-1+\varepsilon}\, \|f\|_2\, \|g\|_2. \tag{2.7}$$

Denote by $\mathcal{Q} \equiv \mathcal{Q}(\delta^{-1/2})$ a tiling of $\mathbb{R}^{d+1}$ with cubes Q of disjoint interior and sidelength $\delta^{-1/2}$, with centers c_Q in $\delta^{-\frac{1}{2}}\mathbb{Z}^{d+1}$.

Proposition 2.2. *Let $d \geq 2$, and suppose that supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$, supp $\widehat{g_k} \subset \Pi_k^{(\delta)}$ and let $\Omega, \Omega' \subset S^{d-1}$ be 1-separated subsets. Suppose $\frac{2(d+3)}{d+1} \leq q \leq p \leq \infty$ and let*

$$\mu(p) = \frac{d}{4} - \frac{d+1}{2p}. \tag{2.8}$$

Then, for all $\varepsilon > 0$

$$\Big(\sum_{Q\in\mathcal{Q}(\delta^{-1/2})} \Big\| \big(\sum_{\omega_k\in\Omega} f_k\big)\big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\Big\|_{L^{q/2}(Q)}^{p/2}\Big)^{2/p} \lesssim \delta^{-2\mu(p)-\varepsilon}\Big\|\Big(\sum_{\omega_k\in\Omega}|f_k|^2\Big)^{1/2}\Big\|_p \Big\|\Big(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\Big)^{1/2}\Big\|_p. \tag{2.9}$$

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$ be so that supp $\widehat{\psi} \subset B_{1/10}$ and $\psi(x) > 1$ if $|x_i| \leq 2$, $i = 1, \ldots, d+1$; then $\sum_{n\in\mathbb{Z}^{d+1}} \psi(\cdot + n)^2 \approx 1$. Let $\psi_Q = \psi(\sqrt{\delta}(\cdot - c_Q))$, so that $\sum_Q \psi_Q^2 \approx 1$. We write

$$F^Q = \big(\sum_{\omega_k\in\Omega} f_k\big)\,\psi_Q \quad \text{and} \quad G^Q = \big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\,\psi_Q,$$

so that the supports of $\widehat{F^Q}$ and $\widehat{G^Q}$ are 1-separated sets in $\Gamma_{\sqrt{\delta}}$. Thus, we can use Wolff's estimate (2.7) with $N = \delta^{-1/2}$ to obtain

$$\Big\|\Big(\sum_{\omega_k\in\Omega} f_k\Big)\Big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\Big)\Big\|_{L^{q/2}(Q)} \lesssim \|F^Q\,G^Q\|_{L^{q/2}(Q)} \lessapprox \delta^{1/2}\,\|\widehat{F^Q}\|_2\,\|\widehat{G^Q}\|_2. \tag{2.10}$$

Now, by almost orthogonality we can write

$$\|\widehat{F^Q}\|_2^2 \approx \sum_k \|\widehat{f}_k * \widehat{\psi_Q}\|_2^2 = \Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\psi_Q\Big\|_2^2,$$

and similarly for G^Q. We write $S_\Omega = (\sum_{\omega_k\in\Omega}|f_k|^2)^{1/2}$, $\widetilde{S}_{\Omega'} = (\sum_{\omega_k\in\Omega'}|g_k|^2)^{1/2}$, raise (2.10) to the power $p/2$ and sum in Q. Thus

$$\Big(\sum_Q\Big\|\big(\sum_{\omega_k\in\Omega} f_k\big)\big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\Big\|_{L^{q/2}(Q)}^{p/2}\Big)^{2/p} \lessapprox \sqrt{\delta}\,\Big(\sum_Q \|S_\Omega\psi_Q\|_2^{p/2}\|\widetilde{S}_{\Omega'}\psi_Q\|_2^{p/2}\Big)^{2/p}$$

and by the Cauchy-Schwarz and Hölder inequalities the right hand side is

$$\lesssim \sqrt{\delta}\Big(\sum_Q \big\|S_\Omega\psi_Q\big\|_2^p\Big)^{1/p}\Big(\sum_Q \big\|\widetilde{S}_{\Omega'}\psi_Q\big\|_2^p\Big)^{1/p}$$
$$\lesssim \sqrt{\delta}\Big(\sum_Q \big\|S_\Omega\psi_Q\big\|_p^p|Q|^{-1+p/2}\Big)^{1/p}\Big(\sum_Q \big\|\widetilde{S}_{\Omega'}\psi_Q\big\|_p^p|Q|^{-1+p/2}\Big)^{1/p}$$
$$\lesssim \delta^{\frac12-(d+1)(\frac12-\frac1p)}\,\big\|S_\Omega\big\|_p\,\big\|\widetilde{S}_{\Omega'}\big\|_p$$

which yields the assertion. □

We combine Proposition 2.2 for $q = p$ and Lemma 2.1 to obtain

Proposition 2.3. *Let $d \ge 2$, let $\mu(p)$ be as in* (2.8) *and suppose that $p > \frac{2(d+3)}{d+1}$. Then, for all $\varepsilon > 0$*

$$\Big\|\sum_k f_k\Big\|_p \le C_\varepsilon\,\delta^{-\mu(p)-\varepsilon}\,\Big\|\big(\sum_k |f_k|^2\big)^{1/2}\Big\|_p \quad \text{if supp } \widehat{f_k} \subset \Pi_k^{(\delta)}. \tag{2.11}$$

We may apply Minkowski's inequality on the right hand side of (2.11) and obtain (1.9) for the limiting case $p = 2(d+3)/(d+1)$. It turns out this is all what is needed to obtain the claimed improvements in Theorem 1.1. This inequality can also be interpolated with the trivial estimates for L^2 and L^∞ to give:

Corollary 2.4. *The inequality (1.9) holds for all $\alpha > \frac{d-1}{4}(\frac12 - \frac1p)$, when $2 \le p \le \frac{2(d+3)}{d+1}$ and for all $\alpha > \frac{d-1}{4}(1 - \frac{2(d+2)}{p(d+1)})$ when $\frac{2(d+3)}{d+1} \le p \le \infty$.*

3. An improvement of Wolff's estimate

We turn to Theorem 1.1. The proof in [22, 11] for inequality (1.3) is based on a subtle localization procedure, induction on scales and certain combinatorial arguments. Here we only discuss the modifications leading to the claimed improvements based on Proposition 2.3. A more selfcontained exposition with further improvements will be in [9].

For simplicity, when δ is fixed (and small) we use the notation $A \lessapprox B$ to indicate the inequality $A \le C_\varepsilon\,\delta^{-\varepsilon}B$ for all $\varepsilon > 0$. Recall that the number of plates $\Pi_k^{(\delta)}$ covering Γ_δ is approximately $\delta^{-\frac{d-1}{2}}$. Also, throughout this section we fix $q(d) = 2(d+3)/(d+1)$.

Due to various reductions (see [11, §3]), it is enough to show that, for all f_k with supp $\hat f_k \subset \Pi_k^{(\delta)}$ and $\|f_k\|_\infty \le 1$, and for all $\lambda > 0$ we have

$$\big|\{|\textstyle\sum_k f_k| > \lambda\}\big| \lessapprox \lambda^{-p}\,\delta^{d-\frac{(d-1)p}{2}}\,\|f\|_2^2 \tag{3.1}$$

where $f = \sum_k f_k$. In [22, 11] it is observed that, by Chebyshev's inequality, this property trivially holds for small enough λ; namely for all $\lambda \le \delta^{-\frac{d-1}{2}+\frac{1}{p-2}}$. We use (1.10) to enlarge this range of λ.

Lemma 3.1. *Let $q = q(d) = 2(d+3)/(d+1)$. Then, inequality (3.1) holds for all*

$$\lambda \le \delta^{-\frac{d-1}{2}+\frac{q}{4(p-q)}}. \tag{3.2}$$

Proof. Let $\beta = \frac{d-1}{4(d+3)}$. By Chebyshev's inequality and (1.10), we have

$$\big|\{|f| > \lambda\}\big| \leq \lambda^{-q} \, \|f\|_q^q \ \lessapprox \ \delta^{-q\beta} \, \lambda^{-q} \, \big(\textstyle\sum_k \|f_k\|_q^2\big)^{q/2}$$

and estimate

$$\big(\textstyle\sum_k \|f_k\|_q^2\big)^{q/2} \lesssim \delta^{-\frac{d-1}{2}\frac{q/2}{(q/2)'}} \textstyle\sum_k \|f_k\|_q^q \lesssim \delta^{-\frac{d-1}{2}(\frac{q}{2}-1)} \textstyle\sum_k \|f_k\|_2^2 \, \sup_k \|f_k\|_\infty^{q-2}.$$

Since by assumption $\|f_k\|_\infty \leq 1$ and by almost orthogonality $\sum_k \|f_k\|_2^2 \approx \|f\|_2^2$, it suffices to show that in the desired range of λ we have $\delta^{-q\beta-\frac{d-1}{2}(\frac{q}{2}-1)}\lambda^{-q} \leq \delta^{-\frac{(d-1)p}{2}-d}\lambda^{-p}$ which is equivalent to (3.2). □

At this point one can proceed exactly as in the proof of Proposition 3.2 of [11] (or p. 1277 in [22], when $d = 2$). The desired gain comes from using $\lambda \geq \delta^{-\frac{d-1}{2}+\frac{q(d)}{4(p-q(d))}}$ (rather than $\lambda \leq \delta^{-\frac{d-1}{2}+\frac{1}{p-2}}$) in step (54) of [11] (or (68) of [22]).

For completeness, we shall briefly sketch this procedure here, referring always to the notation in [11]. Localizing with $\sqrt{N}$-cubes Δ as in Lemma 6.1 of [11], one can find a collection of functions $\{f_\Delta\}$ with spectrum in $\Gamma_{\sqrt{\delta}}$ and a number

$$\lambda_* \in (\lambda\delta^{\frac{d-1}{4}+\varepsilon}, c\delta^{-\frac{d-1}{4}}) \tag{3.3}$$

so that

$$\big|\{|f| > \lambda\}\big| \ \lessapprox \ \sum_\Delta \big|\{|f_\Delta| > \lambda_*\}\big|$$

and

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \lessapprox \lambda_*^2 \lambda^{-2} \, \delta^{-\frac{3d-1}{4}}. \tag{3.4}$$

Here $\mathcal{P}(f_\Delta)$ refers to the set of plates in the wave-packet decomposition of f_Δ. When the cardinality of this set is "small", a further localization argument and induction on scales allows to conclude the theorem (see Lemmas 6.2 and 6.3 in [11]).

In [11, 22], the size of card$(\mathcal{P}(f_\Delta))$ which ensures the validity of these arguments is controlled in three different ways, each depending on a different combinatorial estimate

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \leq c_\varepsilon \delta^\varepsilon \lambda_*^2, \tag{3.5}$$

or

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \leq c_\varepsilon \delta^{\frac{3d-3}{8}+\varepsilon} \lambda_*^4, \tag{3.6}$$

or, in three dimensions (i.e. $d = 2$) only,

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \leq c_\varepsilon \delta^{\frac{11}{8}+\varepsilon} \lambda_*^9. \tag{3.7}$$

the last estimate being by far the most difficult (see Lemmas 5.2 and 5.3 in [11] and Lemma 3.2 in [22]).

Given the lower bound for λ_* in (3.3) and

$$\lambda \geq \delta^{-\frac{d-1}{2}+\frac{q}{4(p-q)}} \tag{3.8}$$

and given (3.4) it remains to verify the estimates (3.5) in the claimed range $p > p_d$, $d \geq 5$, (3.6) for $p > p_d$, $d = 3, 4$ and (3.7) for $p > p_2$.

This is straightforward. By (3.4) and (3.8) we have

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \lessapprox \delta^{-\varepsilon}\lambda_*^2 \delta^{d-1-\frac{q(d)}{2(p-q(d))}}\delta^{-\frac{3d-1}{4}}$$

which gives in the case $d \geq 4$ the assertion (3.5) if $d - 1 - \frac{q(d)}{2(p-q(d))} - \frac{3d-1}{4} > 0$ or, after a short computation $p > q(1 + \frac{2}{d-3}) = 2 + \frac{8}{d-3}\frac{d}{d+1}$. This is the asserted range if $d \geq 5$.

Next we examine the validity of the inequality (3.6) under condition (3.8). We now have

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \;\leq C_\varepsilon \frac{\lambda_*^4 \delta^{-\frac{3d-1}{4}} - \varepsilon}{\lambda_*^2 \lambda^2} \;\leq\; \frac{\lambda_*^4 \delta^{-\frac{3d-1}{4}} - \varepsilon}{\lambda^4\, \delta^{\frac{d-1}{2}+2\varepsilon}} \;\leq\; \frac{\delta^{-\frac{5d-3}{4}-3\varepsilon}}{\delta^{-2(d-1)+\frac{q(d)}{p-q(d)}}}\,\lambda_*^4.$$

This quantity is $\lesssim \delta^\varepsilon \delta^{\frac{3d-3}{8}}\,\lambda_*^4$ if and only if $\frac{5d-3}{4} - 2(d-1) + \frac{q(d)}{p-q(d)} + 4\varepsilon < -\frac{3d-3}{8}$, which yields the range gives $p > q(d)(1 + \frac{8}{3d-7})$. Notice that this inequality amounts to $p > 7.28$ if $d = 4$ and $p > 15$ if $d = 3$ which is the assertion in those cases.

Finally we consider the case $d = 2$ when $q(2) = 10/3$. By (3.4) we need to have $\lambda_*^2\lambda^{-2}\delta^{-5/4-\varepsilon} \leq c_\varepsilon \delta^{11/8}\lambda_*^9$, i.e. $\lambda^{-2}\delta^{-21/8-\varepsilon} \leq c_\varepsilon \lambda_*^7$ provided that $\lambda_* > \lambda\delta^{1/4+\varepsilon}$. Thus taking the smallest possible λ^* yields $\delta^{-35/8-10\varepsilon} \leq \lambda^9$ and this has to hold for all λ satisfying (3.8), i.e. $\lambda \geq \delta^{-\frac{1}{2}+\frac{q(2)}{4(p-q(2))}}$. Taking the minimal λ this is achieved if $35/8 - 10\varepsilon < 9/2 - 9q/(4p - 4q)$ with $q = q(2) = 10/3$. Solving in p and letting $\varepsilon \to 0$ yields the range $p > 19q(2) = 63 + 1/3$. □

On the proof of Theorem 1.5. The proof is similar to the proof of Theorem 1.1. Instead of (1.10) we use a square function inequality for the sphere

$$\Big\|\sum_k f_k\Big\|_q \leq C_\varepsilon\, \delta^{-\alpha(q)/2-\varepsilon} \Big\| \big(\sum_k |f_k|^2\big)^{\frac12} \Big\|_q, \quad \operatorname{supp} \widehat{f_k} \subset B_k^{(\delta)}, \tag{3.9}$$

with $\alpha(q) = d(1/2 - 1/q) - 1/2$, and $q = 2(d+2)/d$. In two dimensions this is an old observation by C. Fefferman ([8]), and holds for $q = 4$ with $\varepsilon = 0$. In general the proof of (3.9) is rather analogous to the proof of Proposition 2.3; one uses Tao's bilinear Fourier extension inequality [19] (see also [12] for related results). Unlike (2.11) in the conic case the inequality (3.9) is essentially optimal for the given range $q \geq 2(d+2)/d$. We omit further details. □

4. More on square functions

We shall now discuss some improvements of the square function estimate in Proposition 2.3 in low dimensions; thus we seek for estimates of the form

$$\Big\|\sum_k f_k\Big\|_p \leq C_\varepsilon\, \delta^{-\beta-\varepsilon} \Big\| \Big(\sum_k |f_k|^2\Big)^{1/2} \Big\|_p, \quad \operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)}. \tag{4.1}$$

for some $\beta < \mu(p) = d/4-(d+1)/2p$. We shall assume throughout this chapter the Wolff assumption,

Hypothesis $\boldsymbol{\mathcal{W}(w,d)}$. *For all $\delta \in (0,1)$ and all families $\{h_k\}$ of functions satisfying supp $\widehat{h}_k \subset \Pi_k^{(\delta)}$,*

$$\Big\|\sum_k h_k\Big\|_w \leq C_\varepsilon \delta^{-\alpha(w)-\varepsilon}\Big(\sum_k \|h_k\|_w^w\Big)^{1/w}, \tag{4.2}$$

where $\alpha(w) = d(1/2 - 1/w) - 1/2$. *Cf.* Table 1.

We note that in view of the embedding $L^p(\ell^2) \subset L^p(\ell^p)$ the inequality (4.2) trivially implies (4.1) with $\beta = \alpha(p)$, for $w \le p < \infty$. Another trivial observation is that (4.1) holds with $\beta \ge (d-1)/4$ in view of the Cauchy-Schwarz inequality, as $\sum_k |f_k(x)| \lesssim \delta^{-(d-1)/4} (\sum_k |f_k(x)|^2)^{1/2}$ for every x.

The method for our improvement over the exponent $\min\{\mu(p), \frac{d-1}{4}\}$ will be limited to the case where

$$\alpha(p) < \min\{\mu(p), \tfrac{d-1}{4}\} \tag{4.3}$$

which holds if and only if $p < \min\{\frac{2(d-1)}{d-2}, \frac{4d}{d-1}\}$. We have the additional restriction $p > \frac{2(d+3)}{d+1}$ in Proposition 2.3. Summarizing we shall get an improvement which is limited to $d = 2, 3, 4$ and to the ranges

$$\begin{cases} d = 2, & 10/3 < p < \min\{8, w\}, \\ d = 3, & 3 < p < 4, \\ d = 4, & 14/5 < p < 3. \end{cases} \tag{4.4}$$

We emphasize that square-function estimates such as (4.1) cannot a priori be interpolated when subject to the Fourier support condition (1.4). We shall however start with a preliminary result which is proved using an interpolation.

We let φ_k be a bump function adapted to the plate $\Pi_k^{(\delta)}$ satisfying the natural estimates, so that φ_k equals 1 on the plate, and is supported on the "double plate". Define the operator P_k by

$$\widehat{P_k f} = \varphi_k \widehat{f}. \tag{4.5}$$

Each P_k is bounded on $L^p(\mathbb{R}^{d+1})$, $1 \le p \le \infty$, with uniform bounds.

Lemma 4.1. *Let $d = 2$, suppose that hypothesis $\mathcal{W}(w, 2)$ holds. Let*

$$\beta = \beta_*(p, w) = \frac{3w - 13}{6w - 20} - \frac{9w - 40}{(6w - 20)p} \tag{4.6}$$

and let $r = r(p, w)$ be defined by

$$\frac{1}{r(w,p)} = \frac{1}{2} - \frac{w-2}{6w-20}\Big(3 - \frac{10}{p}\Big) \tag{4.7}$$

Then, for $10/3 \le p \le w$,

$$\Big\| \sum_k P_k g_k \Big\|_p \le C_\varepsilon \delta^{-\beta-\varepsilon} \Big\| \Big(\sum_k |g_k|^r \Big)^{1/r} \Big\|_p,$$

for all families $\{g_k\}$ with $g_k \in \mathcal{S}(\mathbb{R}^{d+1})$.

Proof. By $\mathcal{W}(2, w)$ and the embedding $L^p(\ell^2) \subset \ell^p(L^p)$ we have the inequality

$$\begin{aligned} \Big\| \sum_k P_k g_k \Big\|_w &\le C_\varepsilon \delta^{-(\alpha(w)+\varepsilon)} \Big(\sum_k \|P_k g_k\|_w^w \Big)^{1/w} \\ &\lesssim C_\varepsilon \delta^{-(\alpha(w)+\varepsilon)} \Big\| \Big(\sum_k |g_k|^w \Big)^{1/w} \Big\|_w. \end{aligned} \tag{4.8}$$

We also observe that for $2 \le p \le 4$

$$\Big\| \Big(\sum_k |P_k g_k|^2 \Big)^{1/2} \Big\|_p \le C(1+\log \delta^{-1})^{1/2-1/p} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_p \tag{4.9}$$

Indeed the left hand side is estimated by

$$\sup_{\omega \in L^{(p/2)'}} \Big(\sum_k \int |P_k g_k|^2 \omega dx \Big)^{1/2} \lesssim \sup_{\omega \in L^{(p/2)'}} \Big(\sum_k \int |g_k|^2 M_\delta \omega dx \Big)^{1/2}$$

where M_δ is a Besicovitch-type maximal operator associated to the light cone which is bounded on L^2 with norm $O(\sqrt{\log(2+\delta^{-1})})$ if $\delta < 1/2$, see [7], [14]. Thus Hölder's inequality implies (4.9).

Now we can combine Proposition 2.3 with respect to the double plates, and $f_k = P_k g_k$, and (4.9) to obtain

$$\Big\| \sum_k P_k g_k \Big\|_{10/3} \le C_\varepsilon \delta^{-\frac{1}{20}-\varepsilon} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_{10/3}. \tag{4.10}$$

After a little arithmetic the claimed bound follows by interpolation between (4.8) and (4.10). $\square$

Since $r(p,w) \ge 2$ in Lemma 4.1 we immediately get

Corollary 4.2. *Let $d=2$, suppose that hypothesis $\mathcal{W}(w,2)$ holds. Then for all family of functions $\{f_k\}$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$ the estimate (4.1) holds for $10/3 \le p \le w$ with $\beta = \beta_*(p,w)$.*

In particular note that $\beta_*(4,w) = \frac{3w-12}{24w-80}$ so that $\beta_*(4,6) = 3/32$. If we use the exponent obtained in Theorem 1.1, *i.e.* $w = p_2 = 190/3$ we get only $\beta_*(4,p_2) = 89/720$ which is worse than $5/44$ exponent that is already known from [20], [23].

For large values of w one can improve on the result of Corollary 4.2. Our approach will be similar to the one by Tao and Vargas [20] in $2+1$ dimensions. By using $\mathcal{W}(w,2)$ in that approach one can slightly improve on the previously known exponents.

Theorem 4.3. *Let $2 \le d \le 4$, and p as in (4.4). If hypothesis $\mathcal{W}(w,d)$ holds then for all family of Schwartz functions $\{f_k\}$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$ the estimate (4.1) holds with*

$$\beta = \mu(p) - \frac{d-1}{2} \Big(\frac{\frac{d+1}{2(d+3)} - \frac{1}{p}}{\frac{d+1}{2(d+3)} + \frac{1}{p} - \frac{2(p-1)}{(w-1)p}} \Big) \Big(\frac{1}{p} - \frac{d-2}{2(d-1)} \Big) \tag{4.11}$$

The proof will be given in §5.

In $2+1$ dimensions Theorem 4.3 yields inequality (4.1) for the range $10/3 \le p \le w$ with β equal to

$$\beta_{**}(p,w) = \frac{1}{2p} \cdot \frac{(3p^2-2p-20)w - 23p^2 + 82p - 40}{(10+3p)w - 23p + 10}; \tag{4.12}$$

in particular we have $\beta_{**}(4,w) = \frac{5w-20}{44w-164}$ which (with $p_2 \equiv w$) occurs in Theorem 1.5. We compare with (4.6). Notice that $3/32 = \beta_*(4,6) < \beta_{**}(4,6) = 1/10$. A straightforward computation shows the inequality $\beta_{**}(p,w) < \beta_*(p,w)$ holds if and only if $(9p-30)w^2 + (-9p^2-39p+230)w + 23p(3p-10) > 0$ and after factoring we see that for $10/3 < p < w$ we have $\beta_{**}(p,w) < \beta_*(p,w)$ if and only if $(p-\frac{10}{3})(w-\frac{23}{3})(w-p) > 0$. Thus for any $p \in (10/3, w)$ we have

$$\beta_{**}(p,w) < \beta_*(p,w) \quad \iff w > \frac{23}{3} \tag{4.13}$$

so that the L^p result in Theorem 4.3 is better than the result of Corollary 4.2 in the range $w > 23/3$. We obtain the following corollary which yields Theorem 1.5.

Corollary 4.4. *Let $d = 2$ and suppose that $\mathcal{W}(w, 2)$ holds for some $w > 6$. Let $10/3 < p \le 4$ and let $\alpha > \min\{\beta_*(p, w), \beta_{**}(p, w)\}$ (i.e. $\alpha > \beta_{**}(p, w)\}$ if $w > 23/3$).*
Then

(i) the smoothing inequality (1.6) *holds true and*

(ii) the Fourier multiplier m_α in (1.7) *defines a bounded operator on $L^p(\mathbb{R}^3)$.*

We also observe that by interpolation we obtain the analogous boundedness results for the range $4 \le p \le w$ under the assumption that $\alpha > \frac{1}{2} - \frac{2}{p} + \frac{4(w-p)}{p(w-4)} \min\{\beta_*(4, w)), \beta_{**}(4, w)\}$.

If we use the result of Theorem 1.1 in $2 + 1$ dimensions (*i.e.* hypothesis $\mathcal{W}(w, 2)$ with $w = p_2 = 190/3$) we obtain this result for $\alpha > \beta_{**}(p, \frac{190}{3}) = \frac{501p^2-134p-3920}{2p(501p+1930)}$. which equals $445/3934$ if $p = 4$. This represents a slight improvement over the Tao-Vargas result [20] which yields the L^4 boundedness for $\alpha > \frac{5}{44} = 0.11\overline{363}$; note that $\frac{445}{3934} \approx 0.11311642....$ We also see from Corollary 4.2 that the validity of (1.3) for the optimal (conjectured) range $p \ge 6$ implies the L^4 boundedness for $\alpha > 3/32 = 0.09375$; however it has been conjectured that it should hold for all $\alpha > 0$.

Proof of Corollary 4.4. It remains to estimate the L^p norm of the square-function. But this is done exactly as in [13], [14] using the optimal $L^{(p/2)'}(\mathbb{R}^3)$ bounds of the appropriated Besicovitch maximal operators (note that $(p/2)' > 2$ by the assumption $p \le 4$). □

5. Proof of Theorem 4.3

We work with the operators P_k in (4.5) which localize in Fourier space to the doubles of the plates $\Pi_k^{(\delta)}$. We shall use the following consequence of $\mathcal{W}(w, d)$.

Lemma 5.1. *Suppose that $\mathcal{W}(w, d)$ holds. Let $w \le r \le \infty$ and let $p = \frac{r}{r-w+1}$ (i.e. $r = p'(w-1)$). Then, for every $\varepsilon > 0$,*

$$\Big\| \sum_k P_k h_k \Big\|_r \le C_\varepsilon \delta^{-\frac{w'}{p'}\alpha(w)-\varepsilon} \Big(\sum_k \|h_k\|_r^p \Big)^{1/p}. \tag{5.1}$$

Proof. This follows by interpolation between the Wolff inequality (i.e. (4.8) in d dimensions) and the trivial bound $\| \sum_k P_k h_k \|_\infty \lesssim \sum_k \|h_k\|_\infty$. □

To establish Theorem 4.3 we shall work with the following
Hypothesis $\mathcal{H}(\gamma,p)$. *For all $\delta < 1$, $\varepsilon > 0$*

$$\Big\| \sum_k h_k \Big\|_p \le C_\varepsilon \delta^{-\gamma-\varepsilon} \Big\| \Big(\sum_k |h_k|^2 \Big)^{1/2} \Big\|_p, \tag{5.2}$$

provided that supp $\widehat{h_k} \subset \Pi_k^{(\delta)}$.

By Proposition 2.3 we know already that for $p > 2(d+3)/(d+1)$ this inequality holds true with the exponent $\gamma = \mu(p) = d/4 - (d+1)/2p$ and we seek an improvement in the ranges (4.4).

We use Lemma 5.1 to prove the following proposition which amounts to an improved version of Proposition 5.4 in [20] (who considered the case $w = \infty$ in the $2+1$ dimensional situation). As in §2 we work with a covering $\mathcal{Q}(\delta^{-1/2})$ of $\sqrt{1/\delta}$ cubes.

Proposition 5.2. *Suppose that $2 < p < w$ and suppose that hypotheses $\mathcal{H}(\gamma, p)$ and $\mathcal{W}(w, d)$ hold. Let $r = p'(w-1)$. Then*

$$\Big(\sum_{Q\in\mathcal{Q}(\delta^{-1/2})} \Big\|\sum_k g_k\Big\|_{L^r(Q)}^p\Big)^{1/p} \le C_\varepsilon \delta^{-\frac{\gamma+\alpha(p)}{2}-\varepsilon}\Big\|\Big(\sum_k |g_k|^2\Big)^{1/2}\Big\|_p. \tag{5.3}$$

provided that supp $\widehat{g_k} \in \Pi_k^{(\delta)}$.

Proof. We group the indices k (and therefore the corresponding plates $\Pi_k^{(\delta)}$) into $O(\delta^{-(d-1)/4})$ disjoint families S_l so that $\mathrm{dist}(\omega_k, \omega_{k'}) \lesssim \delta^{1/4}$ for $k, k' \in S_l$. Define

$$G_l = \sum_{k\in S_l} g_k.$$

As in the proof of Proposition 2.2 we also work with the functions ψ_Q adapted to the cubes $Q \in \mathcal{Q}(\delta^{-1/2})$. By the support propery of $\widehat{\psi_Q}$ the Fourier transform of $\psi_Q G_l$ is supported in a $C\sqrt{\delta}$ plate and these plates form an essentially disjoint plate family. Therefore

$$\begin{aligned}\Big\|\sum_l G_l\Big\|_{L^r(Q)} &\lesssim \Big\|\psi_Q \sum_l G_l\Big\|_r \\ &\lessapprox \delta^{-\frac{w'}{p'}\frac{\alpha(w)}{2}}\Big(\sum_l \|\psi_Q G_l\|_r^p\Big)^{1/p}, \end{aligned}\tag{5.4}$$

by Lemma 5.1 with δ replaced by $\sqrt{\delta}$. By the support property of $\widehat{\psi_Q G_l}$ and Young's inequality

$$\|\psi_Q G_l\|_r \lesssim \delta^{\frac{d+1}{4}(\frac1p-\frac1r)}\|\psi_Q G_l\|_p \tag{5.5}$$

and therefore

$$\Big(\sum_Q \|\sum_l G_l\|_{L^r(Q)}^p\Big)^{1/p} \lessapprox \delta^{-\frac{w'}{p'}\frac{\alpha(w)}{2}+\frac{d+1}{4}(\frac1p-\frac1r)}\Big(\sum_{Q,l}\|\psi_Q G_l\|_p^p\Big)^{1/p}.$$

A little algebra shows

$$-\frac{w'}{p'}\frac{\alpha(w)}{2}+\frac{d+1}{4}(\frac1p-\frac1r) = -\frac{\alpha(p)}{2}$$

with $r = p'(w-1)$. Using some straightforward estimation using the decay of the ψ_Q we also get

$$\Big(\sum_Q \|\sum_l G_l\|_{L^r(Q)}^p\Big)^{1/p} \lessapprox \delta^{-\alpha(p)/2}\Big(\sum_l \|G_l\|_p^p\Big)^{1/p} \tag{5.6}$$

As $\widehat{G}_l$ is supported in a $C\sqrt{\delta}$ plate we may use rescaling arguments as in the proof of Lemma 2.1 to deduce from the hypothesis $\mathcal{H}(\gamma, p)$ applied with parameter $\sqrt{\delta}$ that

$$\|G_l\|_p \lessapprox \delta^{-\gamma/2}\Big\|\Big(\sum_{k\in S_l}|g_k|^2\Big)^{1/2}\Big\|_p$$

and hence

$$\Big(\sum_Q \|\sum_l G_l\|^p_{L^r(Q)}\Big)^{1/p} \le C_\varepsilon \delta^{-\frac{\alpha(p)+\gamma}{2}-\varepsilon}\Big(\sum_l \Big\|\Big(\sum_{k\in S_l}|g_k|^2\Big)^{1/2}\Big\|_p^p\Big)^{1/p}$$
$$\le C_\varepsilon \delta^{-\frac{\alpha(p)+\gamma}{2}-\varepsilon}\Big\|\Big(\sum_k |g_k|^2\Big)^{1/2}\Big\|_p$$

which is the assertion. □

Proof of Theorem 4.3, cont. We first note that Hypothesis $\mathcal{H}(p,\mu(p))$ holds by Proposition 2.3.

Assuming that $\mathcal{H}(p,\gamma)$ holds for some $\gamma \le \mu(p)$ the following estimate for bilinear expressions is an immediate consequence of Proposition 5.2; here $r = p'(w-1)$.

$$(5.7)\quad \Big(\sum_{Q\in\mathcal{Q}}\Big\|\big(\sum_{\omega_k\in\Omega} f_k\big)\big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\Big\|^{p/2}_{L^{r/2}(Q)}\Big)^{2/p} \lessapprox \delta^{-\alpha(p)-\gamma}\Big\|\big(\sum_{\omega_k\in\Omega}|f_k|^2\big)^{\frac12}\Big\|_p\Big\|\big(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\big)^{\frac12}\Big\|_p.$$

We now assume that Ω and Ω' are separated as in Proposition 2.2 and interpolate the inequalities (5.7) and (2.9) with $q = 2(d+3)/(d+1)$. As a result we obtain

$$\Big(\sum_{Q\in\mathcal{Q}}\Big\|\big(\sum_{\omega_k\in\Omega} f_k\big)\big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\Big\|^{p/2}_{L^{p/2}(Q)}\Big)^{2/p} \lessapprox \delta^{-2\Gamma(p,\gamma)}\Big\|\big(\sum_{\omega_k\in\Omega}|f_k|^2\big)^{\frac12}\Big\|_p\Big\|\big(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\big)^{\frac12}\Big\|_p.$$

where

$$\Gamma(p,\gamma) = (1-\vartheta)\mu(p) + \vartheta\frac{\alpha(p)+\gamma}{2} \qquad \text{with } \vartheta = \Big(\frac1q-\frac1p\Big)\Big/\Big(\frac1q-\frac1r\Big).$$

By Lemma 2.1 we also obtain

$$(5.8)\qquad \Big\|\sum_k f_k\Big\|_p \lessapprox \delta^{-\Gamma(p,\gamma)}\Big\|\big(\sum_{\omega_k\in\Omega}|f_k|^2\big)^{1/2}\Big\|_p.$$

Notice that $\alpha(p) < \Gamma(p,\gamma) \le \mu(p)$ provided that $\alpha(p) < \gamma \le \mu(p)$. Moreover $\gamma = \Gamma(p,\gamma)$ if and only if γ equals

$$\gamma_* = \frac{1}{1-\vartheta/2}\big((1-\vartheta)\mu(p) + \vartheta\frac{\alpha(p)}{2}\big) \;=\; \mu(p) - \frac{\vartheta}{2-\vartheta}\big(\mu(p)-\alpha(p)\big).$$

The fixed point is contained in the interval $(\alpha(p),\mu(p))$ and one observes that $\Gamma(p,\gamma) < \gamma$ for $\gamma^* < \gamma < \mu(p)$. Thus, if we define a sequence γ_n by setting $\gamma_0 = \mu(p)$ and $\gamma_{n+1} = \Gamma(p,\gamma_n)$ for $n \ge 0$, then γ_n is decreasing and bounded below and converges to γ^*. We compute that $\vartheta/(2-\vartheta) = (1/q-1/p)/(1/q+1/p-2/r)$ and $\alpha(p)-\mu(p) = (d-2)/4-(d-1)/2p$ and see that γ_* is equal to the right hand side of (4.11). Thus (5.8) and an iteration yields the assertion of the theorem. □

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