MAXIMAL FUNCTIONS ASSOCIATED WITH FAMILIES OF HOMOGENEOUS CURVES: L^p BOUNDS FOR $p \leq 2$

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ABSTRACT. Let $M^{(u)}$, $H^{(u)}$ be the maximal operator and Hilbert transform along the parabola (t, ut^2). For $U \subset (0,\infty)$ we consider L^p estimates for the maximal functions $\sup_{u\in U} |M^{(u)}f|$ and $\sup_{u\in U} |H^{(u)}f|$, when $1 < p \leq 2$. The parabolae can be replaced by more general non-flat homogeneous curves.

1. Introduction and statement of results

Let $b > 1$, $u > 0$, and $\gamma_b : \mathbb{R} \to \mathbb{R}$ homogeneous of degree b, i.e. $\gamma_b(st) = s^b\gamma_b(t)$ for $s > 0$. Also suppose $\gamma_b(\pm 1) \neq 0$. For a Schwartz function f on $\mathbb{R}^2$ we let

$$M^{(u)}f(x) = \sup_{R>0} \frac{1}{R}\int_0^R |f(x - (t, u\gamma_b(t)))|\, dt,$$
$$H^{(u)}f(x) = p.v. \int_{\mathbb{R}} f(x - (t, u\gamma_b(t)))\frac{dt}{t},$$

denote the maximal function and Hilbert transform of f along the curve $(t, u\gamma_b(t))$. For an arbitrary nonempty $U \subset (0,\infty)$ we consider the maximal functions

$$\mathcal{M}^U f(x) = \sup_{u\in U} M^{(u)}f(x), \quad \mathcal{H}^U f(x) = \sup_{u\in U} |H^{(u)}f(x)|. \tag{1.1}$$

For $2 < p < \infty$ the operators $\mathcal{M}^U$ are bounded on $L^p(\mathbb{R}^2)$ for all U; this was shown by Marletta and Ricci [8]. For the operators $\mathcal{H}^U$ a corresponding satisfactory theorem was proved in a previous paper [6] of the authors. To describe the result let

$$\mathfrak{N}(U) = 1 + \#\{n \in \mathbb{Z} : [2^n, 2^{n+1}] \cap U \neq \emptyset\}.$$

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Then, for $2 < p < \infty$, $\mathcal{H}^U$ is bounded on $L^p(\mathbb{R}^2)$ if and only if $\mathfrak{N}(U)$ is finite, and we have the equivalence

$$c_p \le \frac{\|\mathcal{H}^U\|_{L^p\to L^p}}{(\log \mathfrak{N}(U))^{1/2}} \le C_p, \quad 2 < p < \infty,$$

with nonzero constants c_p, C_p. Moreover, for all $p > 1$ we have the lower bound $\|\mathcal{H}^U\|_{L^p\to L^p} \gtrsim \sqrt{\log \mathfrak{N}(U)}$. The consideration of such results in [6] was motivated by results in [5], [4], and by the analogous questions for Hilbert transform along straight lines. See [7] for lower bounds, [3] for upper bounds, and the bibliography of [6] for a list of related works.

In this paper we seek to find efficient upper bounds for the operator norms of $\mathcal{M}^U$ and $\mathcal{H}^U$ in the case $1 < p \le 2$. As pointed out in [6], with reference to [10], L^p boundedness for $p \le 2$ fails, for both $\mathcal{M}^U$ and $\mathcal{H}^U$, when $U = [1, 2]$; therefore some additional sparseness condition needs to be imposed. To formulate such results let, for each $r > 0$

$$U^r = r^{-1}U \cap [1, 2] = \{\rho \in [1, 2] : r\rho \in U\}.$$

For $0 < \delta < 1$ we let $N(U^r, \delta)$ the δ–covering number of U^r, i.e. the minimal number of intervals of length δ needed to cover U^r. It is obvious that $\sup_{r>0} N(U^r, \delta) \lesssim \delta^{-1}$. Define

$$\mathcal{K}_p(U, \delta) = \delta^{1-\frac{1}{p}} \sup_{r>0} N(U^r, \delta)^{\frac{1}{p}}. \tag{1.2}$$

These definitions, and the results below are motivated by considerations for spherical maximal functions in [11] (see also [12], [10]). Define

$$p_{\rm cr}(U) = 1 + \limsup_{\delta\to 0+} \frac{\sup_{r>0} \log N(U^r, \delta)}{\log(\delta^{-1})} \tag{1.3}$$

Notice that $1 \le p_{\rm cr}(U) \le 2$, and that $p_{\rm cr}(U) = 1$ for lacunary U. We have $p_{\rm cr}(U) = 2$ if U contains any intervals. Moreover if $p_{\rm cr}(U) < p < 2$ there exists an $\varepsilon = \varepsilon(p, U) > 0$ such that $\sup_{0<\delta<1} \delta^{-\varepsilon}\mathcal{K}_p(U, \delta) < \infty$. If $1 < p < p_{\rm cr}(U)$ then there is $\varepsilon' = \varepsilon'(p, U) > 0$ and a sequence $\delta_n \to 0$ such that $\limsup_n \delta_n^{\varepsilon'}\mathcal{K}_p(U, \delta_n) > 0$.

Theorem 1.1. *Let $1 < p \le 2$ and $p_{\rm cr}(U)$ as in (1.3).*

(i) If $p_{\rm cr}(U) < p \le 2$ then $\mathcal{M}^U$ is bounded on $L^p(\mathbb{R}^2)$.

(ii) If $1 < p < p_{\rm cr}(U)$ then $\mathcal{M}^U$ is not bounded on $L^p(\mathbb{R}^2)$.

(iii) For every $\varepsilon > 0$ we have

$$c_p \sup_{\delta>0} \mathcal{K}_p(U, \delta) \le \|\mathcal{M}^U\|_{L^p\to L^p} \le C_{\varepsilon,p} \sup_{\delta>0} \delta^{-\varepsilon}\mathcal{K}_p(U, \delta)\,.$$

Here $c_p, C_{p,\varepsilon}$ are constants only depending on p or p, ε, respectively.

Theorem 1.2. *Let $1 < p \le 2$ and $p_{\rm cr}(U)$ as in (1.3).*

(i) If $p_{\rm cr}(U) < p \le 2$ then $\mathcal{H}^U$ is bounded on $L^p(\mathbb{R}^2)$ if and only if $\mathfrak{N}(U) < \infty$.

(ii) If $1 < p < p_{\rm cr}(U)$ then $\mathcal{H}^U$ is not bounded on $L^p(\mathbb{R}^2)$.

(iii) For every $\varepsilon > 0$ we have

$$\|\mathcal{H}^U\|_{L^p\to L^p} \le C_p\sqrt{\log(\mathfrak{N}(U))} + C_{\varepsilon,p}\sup_{\delta>0}\delta^{-\varepsilon}\mathcal{K}_p(U,\delta).$$

and

$$c_p\big(\sqrt{\log(\mathfrak{N}(U))} + \sup_{\delta>0}\mathcal{K}_p(U,\delta)\big) \le \|\mathcal{H}^U\|_{L^p\to L^p}.$$

Here $c_p, C_p, C_{p,\varepsilon}$ are constants only depending on p or p,ε, respectively.

We note that part (i), (ii) of the theorems follow immediately from part (iii) of the respective theorem. The term $C_{\varepsilon,p}\delta^{-\varepsilon}$ can be replaced by a logarithmic dependence, namely $C_p[\log(2/\delta)]^A$ for $A > 14/p - 6$. More precisely, we have the following

Theorem 1.3. *Let $1 < p \le 2$. Then there is C independent of p and U so that*

$$\|\mathcal{M}^U\|_{L^p\to L^p} \le C\sum_{\ell\ge 1}\vartheta_{p,\ell}\mathcal{K}_p(U,2^{-\ell}), \tag{1.4}$$

where $\vartheta_{p,\ell} = (p-1)^{3-\frac{10}{p}}\mathbb{1}_{\ell\le(p-1)^{-1}} + \ell^{7(\frac{2}{p}-1)}\mathbb{1}_{\ell>(p-1)^{-1}}$ and

$$\|\mathcal{H}^U\|_{L^p\to L^p} \le C(p-1)^{-7}\sqrt{\log(\mathfrak{N}(U))} + C(p-1)^{-2}\sum_{\ell\ge 1}\vartheta_{p,\ell}\mathcal{K}_p(U,2^{-\ell}). \tag{1.5}$$

Structure of the paper. In §2 we decompose the operators $\mathcal{M}^U$, $\mathcal{H}^U$ in the spirit of [6] in order to prepare for the proof of Theorem 1.3. The proof of Theorem 1.3 is then completed in §3 and §4. Finally, the lower bounds claimed in Theorem 1.1 and Theorem 1.2 are addressed in §5.

2. Basic reductions

We recall some notation and basic reductions from [6]. By the assumption of homogeneity and $\gamma_b(\pm 1) \ne 0$ there are $c_\pm \ne 0$ such that $\gamma_b(t) = c_+ t^b$ for $t > 0$, and $\gamma_b(t) = c_-(-t)^b$ for $t < 0$, and finally $\gamma_b(0) = 0$. We note that by scaling we may always assume that $c_- = 1$. Let $\chi_+ \in C_c^\infty$ be supported in $(1/2, 2)$ such that

$$\sum_{j\in\mathbb{Z}}\chi_+(2^j t) = 1 \text{ for } t > 0.$$

Let $\chi_-(t) = \chi_+(-t)$ and $\chi = \chi_+ + \chi_-$. We define measures τ_0, σ_0, $\sigma_\pm$ by

$$\langle\tau_0, f\rangle = \int f(t,\gamma_b(t))\chi_+(t)dt,$$
$$\langle\sigma_\pm, f\rangle = \int f(t,\gamma_b(t))\chi_\pm(t)\frac{dt}{t},$$
$$\sigma_0 = \sigma_+ + \sigma_-.$$

Let, for $j \in \mathbb{Z}$, the measures τ_j^u, σ_j^u be defined by

$$\langle \tau_j^u, f\rangle = \int f(t, u\gamma_b(t)) 2^j \chi_+(2^j t) dt,$$
$$\langle \sigma_j^u, f\rangle = \int f(t, u\gamma_b(t)) \chi(2^j t) \frac{dt}{t}.$$

By homogeneity of γ_b we have $\tau_j^u = 2^{j(1+b)} \tau_0^u(\delta_{2^j}^b \cdot)$ with $\delta_t^b x = (tx_1, t^b x_2)$, as well as the analogous relation between σ_j^u and σ_0^u. We note that the τ_j^u are positive measures and the σ_j^u have cancellation.

For Schwartz functions f the Hilbert transform along Γ_b^u can be written as

$$H^{(u)} f = \sum_{j\in\mathbb{Z}} \sigma_j^u * f.$$

For the maximal function it is easy to see that there is the pointwise estimate

$$M^{(u)} f(x) \le C \sup_{j\in\mathbb{Z}} \tau_j^u * |f|. \tag{2.1}$$

Following [6, §2] we further decompose σ_0 and τ_0. Choose Schwartz function η_0, supported in $\{|\xi| \le 100\}$ and equal with $\eta_0(\xi) = 1$ for $|\xi| \le 50$. Let $\varsigma_+ \in C_c^\infty(\mathbb{R})$ be supported in $(b(1/4)^{b-1}, b4^{b-1})$ and equal to 1 on $[b(2/7)^{b-1}, b(7/2)^{b-1}]$. Let $\varsigma_- \in C_c^\infty(\mathbb{R})$ be supported on $(-b4^{b-1}, -b(1/4)^{b-1})$ and equal to 1 on $[-b(7/2)^{b-1}, -b(2/7)^{b-1}]$.

One then decomposes

$$\sigma_0 = \phi_0 + \mu_{0,+} + \mu_{0,-}$$
$$\tau_0 = \varphi_0 + \rho_0$$

where ϕ_0, φ_0 are given by

$$\widehat{\phi_0}(\xi) = \eta_0(\xi)\widehat{\sigma}_0(\xi) + (1-\eta_0(\xi))\big(1 - \varsigma_-(\tfrac{\xi_1}{c_+\xi_2})\big)\widehat{\sigma}_+(\xi) + (1-\eta_0(\xi))\big(1 - \varsigma_+(\tfrac{\xi_1}{c_-\xi_2})\big)\widehat{\sigma}_-(\xi)$$

and

$$\widehat{\varphi_0}(\xi) = \eta_0(\xi)\widehat{\tau}_0(\xi) + (1-\eta_0(\xi))\big(1 - \varsigma_-(\tfrac{\xi_1}{c_+\xi_2})\big)\widehat{\tau}(\xi).$$

The measures and $\mu_{0,\pm}$ and ρ_0 are given via the Fourier transform by

$$\widehat{\mu}_{0,+}(\xi) = (1-\eta_0(\xi))\varsigma_-(\tfrac{\xi_1}{c_+\xi_2})\widehat{\sigma}_+(\xi),$$
$$\widehat{\mu}_{0,-}(\xi) = (1-\eta_0(\xi))\varsigma_+(\tfrac{\xi_1}{c_-\xi_2})\widehat{\sigma}_-(\xi)$$

and

$$\widehat{\rho}_0(\xi) = (1-\eta_0(\xi))\varsigma_-(\tfrac{\xi_1}{c_+\xi_2})\widehat{\tau}_0(\xi). \tag{2.2}$$

As in Lemma 2.1 of [6], the functions φ_0, ϕ_0 are Schwartz functions. In addition we have $\widehat{\phi}_0(0) = 0$.

Define, for $j \in \mathbb{Z}$, φ_j and ϕ_j by scaling via $\widehat{\varphi}_j(\xi) = \widehat{\varphi}_0(2^{-j}\xi_1, 2^{-jb}\xi_2)\widehat{f}(\xi)$ and $\widehat{\phi}_j(\xi) = \widehat{\phi}_0(2^{-j}\xi_1, 2^{-jb}\xi_2)\widehat{f}(\xi)$. Define $A^u_{j,0}f$ by

$$\widehat{A^u_{j,0}f}(\xi) = \widehat{\varphi}_j(\xi_1, u\xi_2)\widehat{f}(\xi)$$

and let $\mathcal{M}_0 f(x) = \sup_{j\in\mathbb{Z}} \sup_{u\in\mathbb{R}} |A^u_{j,0}f(x)|$. Let

$$\widehat{S^{(u)}f}(\xi) = \sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi_1, u\xi_2)\widehat{f}(\xi).$$

Let $M^{\mathrm{str}}f$ denote the strong maximal function of f. For $p \in (1,2]$ we have

$$\|M^{\mathrm{str}}\|_{L^p\to L^p} \le C(p-1)^{-2}. \tag{2.3}$$

Lemma 2.1. *There exists a constant C such that for all $p \in (1,2]$,*
(i)

$$\|\mathcal{M}_0 f\|_p \le C(p-1)^{-2}\|f\|_p.$$

(ii)

$$\|\sup_{u\in U} |S^{(u)}f|\|_p \le C(p-1)^{-7}\sqrt{\log \mathfrak{N}(U)}\|f\|_p.$$

Proof. Part (i) follows from the estimate

$$|A^u_{j,0}f(x)| \le CM^{\mathrm{str}}f(x). \tag{2.4}$$

Part (ii) is more substantial and relies on the Chang–Wilson–Wolff bounds for martingales, [2]. This is the subject of Theorem 2.2 in [6]. The dependence on p was not specified there, but can be obtained by a literal reading of the proof provided in [6, §4]. We remark that the exponent 7 can likely be improved, but it is satisfactory for our purposes here. □

We also decompose $\widehat{\rho}_0$ and $\widehat{\mu_{0,\pm}}$ further by making an isotropic decomposition for large frequencies. Let $\zeta_0 \in C^\infty_c(\mathbb{R}^2)$ supported in $\{\xi : |\xi| < 2\}$ and such that $\zeta_0(\xi) = 1$ for $|\xi| \le 5/4$. For $\ell = 1, 2, 3, \dots$ let

$$\zeta_\ell(\xi) = \zeta_0(2^{-\ell}\xi) - \zeta_0(2^{1-\ell}\xi).$$

Then for $\ell > 0$, ζ_ℓ is supported in the annulus $\{\xi : 2^{\ell-1} < |\xi| < 2^{\ell+1}\}$ and we have $1 = \sum_{\ell>0}\zeta_\ell(\xi)$ for ξ in the support of $\widehat{\rho}_0, \widehat{\mu_{0,\pm}}$.

Define operators $A^u_{j,\ell}$ and $T^u_{j,\ell,\pm}$ by

$$\widehat{A^u_{j,\ell}f}(\xi) = \zeta_\ell(2^{-j}\xi_1, 2^{-jb}u\xi_2)\widehat{\rho}_0(2^{-j}\xi_1, 2^{-jb}u\xi_2)\widehat{f}(\xi), \tag{2.5}$$

$$\widehat{T^u_{j,\ell,\pm}f}(\xi) = \zeta_\ell(2^{-j}\xi_1, 2^{-jb}u\xi_2)\widehat{\mu_{0,\pm}}(2^{-j}\xi_1, 2^{-jb}u\xi_2)\widehat{f}(\xi). \tag{2.6}$$

We shall show

Proposition 2.2. *There is $C > 0$ such that for each $\ell > 0$, $p \in (1,2]$ we have*

$$\big\| \sup_{u\in U}\sup_{j\in\mathbb{Z}} |A^u_{j,\ell}f| \big\|_p \le C\vartheta_{p,\ell}\mathcal{K}_p(U, 2^{-\ell})\|f\|_p, \tag{2.7}$$

where $\vartheta_{p,\ell} = (p-1)^{3-\frac{10}{p}}\mathbb{1}_{\ell\le(p-1)^{-1}} + \ell^{7(\frac{2}{p}-1)}\mathbb{1}_{\ell>(p-1)^{-1}}$ *and*

$$\Big\| \sup_{u\in U} \Big| \sum_{j\in\mathbb{Z}} T^u_{j,\ell,\pm} f \Big| \Big\|_p \le C(p-1)^{-2}\vartheta_{p,\ell}\mathcal{K}_p(U,2^{-\ell})\|f\|_p. \tag{2.8}$$

We claim that Proposition 2.2 implies Theorem 1.3. Indeed, we have for non–negative f,

$$\mathcal{M}^U f \lesssim \mathcal{M}_0 f + \sum_{\ell>0} \sup_{u\in U}\sup_{j\in\mathbb{Z}} |A^u_{j,\ell} f|$$

and thus (1.4) follows from part (i) of Lemma 2.1 and (2.7). It remains to show (1.5). But in view of the decomposition,

$$H^{(u)} = S^{(u)} + \sum_{\pm}\sum_{\ell>0}\sum_{j\in\mathbb{Z}} T^u_{j,\ell,\pm},$$

this follows from part (ii) of Lemma 2.1 and (2.8). This finishes the proof of Theorem 1.3.

We conclude this section with some estimates that will be used in the proof of Proposition 2.2. We will harvest the required decay in ℓ from the following simple estimate. For $p\in[1,2]$, $\ell>0$, $j\in\mathbb{Z}$, $u\in(0,\infty)$ we have

$$\|A^u_{j,\ell} f\|_p \le C2^{-\ell(1-1/p)}\|f\|_p. \tag{2.9}$$

Indeed, the endpoint $p=2$ is a consequence of Plancherel's theorem and van der Corput's lemma, while $p=1$ follows because the convolution kernel of $A^u_{j,\ell} f$ is L^1–normalized. Another key ingredient will be the following pointwise estimate. From the definition of $A^u_{j,\ell}$ in (2.5) we have for $\ell>0$, $j\in\mathbb{Z}$, $u\in(0,\infty)$ that

$$|A^u_{j,\ell} f| \le CM^{\mathrm{str}}(\tau^u_j * |f|). \tag{2.10}$$

This follows because we have

$$A^u_{j,\ell} f = (f*\tau^u_j) * \kappa^u_{j,\ell},$$

with $\kappa^u_{j,\ell}$ certain Schwartz functions that can be read off from the definitions (2.2), (2.5) and satisfy $|f*\kappa^u_{j,\ell}| \le CM^{\mathrm{str}} f$ with $C>0$ not depending on j,ℓ,u.

We also need to introduce appropriate Littlewood–Paley decompositions. Let $\chi^{(1)}$ be an even C^∞ function supported on

$$\{\xi_1 : |c_+|b2^{-3b-1} \le |\xi_1| \le |c_+|b2^{3b+1}\}$$

and equal to 1 for $|c_+|b2^{-3b} \le |\xi_1| \le |c_+|b2^{3b}$. Let $\chi^{(2)}$ be an even C^∞ function supported on

$$\{\xi_2 : 2^{-2b-1} \le |\xi_2| \le 2^{2b+1}\}$$

and equal to 1 for $2^{-2b} \le |\xi_2| \le 2^{2b}$. Define $P^{(1)}_{k_1,\ell}$, $P^{(2)}_{k_2,\ell,b}$ by

$$\widehat{P^{(1)}_{k_1,\ell} f}(\xi) = \chi^{(1)}(2^{-k_1-\ell}\xi_1)\widehat{f}(\xi)$$
$$\widehat{P^{(2)}_{k_2,\ell,b} f}(\xi) = \chi^{(2)}(2^{-k_2 b-\ell}\xi_2)\widehat{f}(\xi)$$

Then for $s \in [1, 2^b]$,

$$A^{2^{bn}s}_{j,\ell} = A^{2^{bn}s}_{j,\ell} P^{(2)}_{j-n,\ell,b} P^{(1)}_{j,\ell} = P^{(1)}_{j,\ell} P^{(2)}_{j-n,\ell,b} A^{2^{bn}s}_{j,\ell}. \tag{2.11}$$

For $p \in (1, 2]$ we have the Littlewood–Paley inequalities

$$\Big\| \Big(\sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} |P^{(1)}_{k_1,\ell} P^{(2)}_{k_2,\ell,b} f|^2 \Big)^{1/2} \Big\|_p \le C(p-1)^{-2} \|f\|_p \tag{2.12}$$

and

$$\Big\| \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} P^{(1)}_{k_1,\ell} P^{(2)}_{k_2,\ell,b} f_{k_1,k_2} \Big\|_p \le C(p-1)^{-2} \Big\| \Big(\sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} |f_{k_1,k_2}|^2 \Big)^{1/2} \Big\|_p, \tag{2.13}$$

which also hold for Hilbert space valued functions.

3. A positive bilinear operator

In this section we are given for every $n \in \mathbb{Z}$ an at most countable set

$$\mathfrak{S}(n) = \{ s_n(i) : i = 1, 2, \dots \} \subset [1, 2^b].$$

Proposition 3.1. *There is a constant C independent of the choice of the sets $\mathfrak{S}(n) = \{s_n(i)\}$, $n \in \mathbb{N}$, such that for $1 < p \le 2$ and $\ell > 0$,*

$$\Big\| \Big(\sum_{j,n\in\mathbb{Z}} \sum_{i\in\mathbb{N}} |w_n(i)\, \mathcal{A}^{2^{bn}s_n(i)}_{j,\ell} f|^2 \Big)^{1/2} \Big\|_p \le C(p-1)^{3-\frac{10}{p}} 2^{-\ell(p-1)/2} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \|f\|_p$$

for all functions f and $w_n : \mathbb{N} \to \mathbb{C}$. This holds for $\mathcal{A}^{2^{bn}s_n(i)}_{j,\ell}$ being any one of the following:

$$A^{2^{bn}s_n(i)}_{j,\ell},\ 2^{-\ell}\frac{d}{ds} A^{2^{bn}s}_{j,\ell}|_{s=s_n(i)},\ T^{2^{bn}s_n(i)}_{j,\ell,\pm},\ 2^{-\ell}\frac{d}{ds} T^{2^{bn}s}_{j,\ell,\pm}\big|_{s=s_n(i)}.$$

We will only detail the proof in the case $\mathcal{A}^{2^{bn}s_n(i)}_{j,\ell} = A^{2^{bn}s_n(i)}_{j,\ell}$. The other cases follow *mutatis mutandis*. To this end note that the corresponding variants of the main ingredients (2.9), (2.10), (2.11) also hold for each of the other cases, the underlying reasoning being identical in each case.

In the proof of the proposition we use a bootstrapping argument by Nagel, Stein and Wainger [9] in a simplified and improved form given in unpublished work by Christ (see [1] for an exposition).

We first introduce an auxiliary maximal operator. For $R \in \mathbb{N}$ let

$$\mathfrak{M}_R[f,w](x) = \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} \big|w_n(i)\, \tau_j^{2^{bn}s_n(i)} * f(x)\big|.$$

We let $B_p(R)$ be the best constant C in the inequality

$$\|\mathfrak{M}_R[f,w]\|_p \le C \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \|f\|_p,$$

that is,

$$B_p(R) = \sup\{\|\mathfrak{M}_R[f,w]\|_p :\ \|f\|_p \le 1,\ \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \le 1\}. \tag{3.1}$$

The positive number $B_p(R)$ is finite, as from the uniform L^p-boundedness of the operator $f \mapsto \tau_j^u * f$ we have $B_p(R) \le C(2R+1)^{2/p}$. It is our objective to show that $B_p(R)$ is independent of R. More precisely, we claim that there is a constant C independent of the choice of the sets $\mathfrak{S}(n)$, such that for $1 < p \le 2$,

$$B_p(R) \le C(p-1)^{2-10/p}. \tag{3.2}$$

We begin with an estimate for a vector–valued operator.

Lemma 3.2. *Let $1 < p \le 2$, $p \le q \le \infty$. Then*

$$\begin{aligned} &\Big\| \Big(\sum_{-R\le j,n\le R} \sum_{i\in\mathbb{N}} |w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} g_{j,n}|^q \Big)^{1/q} \Big\|_p \\ &\le C(p-1)^{-2(1-\frac{p}{q})} B_p(R)^{1-\frac{p}{q}} 2^{-\ell(1-\frac{1}{p})\frac{p}{q}} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \Big\| \Big(\sum_{j,n\in\mathbb{Z}} |g_{j,n}|^q \Big)^{1/q} \Big\|_p \end{aligned} \tag{3.3}$$

Proof. The case $q = p$ of (3.3) follows from (2.9). For $q = \infty$ we use (2.10) to estimate

$$\begin{aligned} &\Big\| \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} g_{j,n}| \Big\|_p \\ &\le C \Big\| \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)|\, M^{\mathrm{str}}[\tau_j^{2^{bn}s_n(i)} * |g_{j,n}|] \Big\|_p \\ &\le C \Big\| M^{\mathrm{str}}\Big[\sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)|\, \tau_j^{2^{bn}s_n(i)} * \big(\sup_{j',n'\in\mathbb{Z}} |g_{j',n'}| \big) \Big] \Big\|_p \end{aligned}$$

where we have used the positivity of the operators $f \mapsto \tau_j^u * f$. By (2.3) we can dominate the last displayed expression by

$$\begin{aligned} &C'(p-1)^{-2} \Big\| \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)|\, \tau_j^{2^{bn}s_n(i)} * \big[\sup_{j',n'\in\mathbb{Z}} |g_{j',n'}| \big] \Big\|_p \\ &\lesssim (p-1)^{-2} B_p(R) \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \Big\| \sup_{j',n'\in\mathbb{Z}} |g_{j',n'}| \Big\|_p \end{aligned}$$

which establishes the case $q = \infty$. The case $p < q < \infty$ follows by interpolation. □

Proof of Proposition 3.1. We use the decomposition $\tau_j^u * f = \sum_{\ell=0}^{\infty} A_{j,\ell}^u f$. By (2.4) we get

$$\Big\| \sup_{j,n\in\mathbb{Z}} \sup_{i\in\mathbb{N}} \big|w_n(i)\, A_{j,0}^{2^{bn}s_n(i)} f\big| \Big\|_p \lesssim (p-1)^{-2} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^\infty} \|f\|_p.$$

For $\ell > 0$ we have,

$$\Big\| \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} \big|w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} f\big| \Big\|_p \le \Big\| \Big(\sum_{-R\le j,n\le R} \sum_{i\in\mathbb{N}} \big|w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} f\big|^2 \Big)^{1/2} \Big\|_p$$

and, by (2.11) and Lemma 3.2 for $q = 2$, and (2.12),

(3.4)

$$\begin{aligned}
&\Big\| \Big(\sum_{-R\le j,n\le R} \sum_{i\in\mathbb{N}} \big|w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} f\big|^2 \Big)^{1/2} \Big\|_p \\
&\lesssim (p-1)^{-2(1-\frac{p}{2})} B_p(R)^{1-\frac{p}{2}} 2^{-\ell(1-\frac{1}{p})\frac{p}{2}} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \Big\| \Big(\sum_{j,n\in\mathbb{Z}} \big|P_{j-n,\ell,b}^{(2)} P_{j,\ell}^{(1)} f\big|^2 \Big)^{1/2} \Big\|_p \\
&\lesssim (p-1)^{p-4} 2^{-\ell(p-1)/2} B_p(R)^{1-p/2} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \|f\|_p.
\end{aligned}$$

This implies, for $1 < p \le 2$

$$\begin{aligned}
B_p(R) &\lesssim \Big[(p-1)^{-2} + \sum_{\ell>0} (p-1)^{p-4} 2^{-\ell(p-1)/2} B_p(R)^{1-p/2} \Big] \\
&\lesssim (p-1)^{-2} + (p-1)^{p-5} B_p(R)^{1-p/2}
\end{aligned}$$

which leads to

$$B_p(R) \lesssim (p-1)^{2-10/p}.$$

If we use this inequality in (3.4) and observe

$$p - 4 + (2 - 10/p)(1 - p/2) = 3 - 10/p,$$

then the claimed inequality in Proposition 3.1 follows by the monotone convergence theorem. □

4. Proof of Proposition 2.2

For $n \in \mathbb{Z}$ let $U_n \subset [1, 2^b]$ be defined by

$$U_n = \{2^{-bn} u : u \in [2^{bn}, 2^{b(n+1)}] \cap U\}$$

and let

$$\mathcal{N}_{n,\ell}(U) = \#\big\{k : [2^{-\ell}k, 2^{-\ell}(k+1)) \cap U_n \neq \emptyset\big\}.$$

Then we have

$$2^{-\ell(1-\frac{1}{p})} \sup_{n\in\mathbb{Z}} \mathcal{N}_{n,\ell}(U) \approx \mathcal{K}_p(U, 2^{-\ell})$$

We cover each set U_n with dyadic intervals of the form

$$I_{k,\ell} = [k2^{-\ell}, (k+1)2^{-\ell})$$

where $k \in \mathbb{N}$. Denote by $\mathfrak{S}_{n,\ell}$ the left endpoints of these intervals and note $\mathcal{N}_{n,\ell}(U) = \#\mathfrak{S}_{n,\ell}$. We label the set of points in $\mathfrak{S}_{n,\ell}$, by $\{s_{n,\ell}(i)\}_{i=1}^{\mathcal{N}_{n,\ell}(U)}$ and write

$$\begin{aligned}\sup_{j\in\mathbb{Z}}\sup_{u\in U}|A^u_{j,\ell}f(x)| &= \sup_{j\in\mathbb{Z}}\sup_{n\in\mathbb{Z}}\sup_{s\in U_n}|A^{2^{nb}s}_{j,\ell}f(x)|\\ &\le \sup_{j,n\in\mathbb{Z}}\sup_{i=1,\dots\mathcal{N}_{n,\ell}(U)}|A^{2^{nb}s_{n,\ell}(i)}_{j,\ell}f(x)|\\ &\quad+\sup_{j,n\in\mathbb{Z}}\sup_{i=1,\dots\mathcal{N}_{n,\ell}(U)}\int_0^{2^{-\ell}}\Big|\frac{d}{d\alpha}A^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell}f(x)\Big|d\alpha.\end{aligned}$$

Hence

$$\begin{aligned}\Big\|\sup_{j\in\mathbb{Z}}\sup_{u\in U}|A^u_{j,\ell}f|\Big\|_p &\le \Big\|\Big(\sum_{j,n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}|A^{2^{nb}s_{n,\ell}(i)}_{j,\ell}f|^2\Big)^{1/2}\Big\|_p\\ &\quad+\int_0^{2^{-\ell}}\Big\|\Big(\sum_{j,n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}\Big|\frac{d}{d\alpha}A^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell}f|^2\Big)^{1/2}\Big\|_p d\alpha\end{aligned}$$

and by part (ii) of Proposition 3.1 both expressions on the right hand side can be estimated by

$$C(p-1)^{3-10/p}2^{-\ell(p-1)/2}\sup_{n\in\mathbb{Z}}\mathcal{N}_{n,\ell}(U)^{1/p}\|f\|_p. \tag{4.1}$$

This estimate is efficient for $1 < p < 1+\ell^{-1}$. Note that in this range $2^{-C\ell(1-1/p)} \approx 1$ and $\mathcal{N}_{n,\ell}(U)^{1/p} \approx \mathcal{K}_p(U, 2^{-\ell})$. For $p = 2$ we have the inequality

$$\begin{aligned}&\Big\|\Big(\sum_{j,n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}|A^{2^{nb}s_{n,\ell}(i)}_{j,\ell}f|^2\Big)^{1/2}\Big\|_2\\ &\qquad+\int_0^{2^{-\ell}}\Big\|\Big(\sum_{j,n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}\Big|\frac{d}{d\alpha}A^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell}f|^2\Big)^{1/2}\Big\|_2 d\alpha\\ &\lesssim 2^{-\ell/2}\sup_{n\in\mathbb{Z}}\mathcal{N}_{n,\ell}(U)^{1/2}\|f\|_2.\end{aligned} \tag{4.2}$$

For $p_\ell := 1+\ell^{-1} < p < 2$ we use the Riesz–Thorin interpolation theorem (together with the fact that $(p_\ell-1)^{C/\ell} \approx_C 1$ and $(p_\ell-1)^{-A} = \ell^A$). We then

obtain for $p_\ell < p < 2$

$$\Big\|\Big(\sum_{j,n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}|A_{j,\ell}^{2^{nb}s_{n,\ell}(i)}f|^2\Big)^{1/2}\Big\|_p + \int_0^{2^{-\ell}}\Big\|\Big(\sum_{j,n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}\Big|\frac{d}{d\alpha}A_{j,\ell}^{2^{nb}(s_{n,\ell}(i)+\alpha)}f|^2\Big)^{1/2}\Big\|_p d\alpha$$

$$\lesssim 2^{-\ell(1-\frac{1}{p})}\sup_{n\in\mathbb{Z}}\mathcal{N}_{n,\ell}(U)^{1/p}\ell^{7(\frac{2}{p}-1)}\|f\|_p. \tag{4.3}$$

Thus we have established (2.7). The proof of (2.8) is similar but the reduction to a square–function estimate requires one more use of a Littlewood–Paley estimate. We have, using the analogue of (2.11) for $T_{j,\ell,+}^{2^{bn}s}$

$$\Big\|\sup_{n\in\mathbb{Z}}\sup_{u\in U\cap[2^{nb},2^{(n+1)b}]}\Big|\sum_{j\in\mathbb{Z}}T_{j,\ell,+}^{u}f\Big|\Big\|_p$$

$$\le\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}}\Big|\sum_{j\in\mathbb{Z}}P_{j,\ell}^{(1)}P_{j-n,\ell,b}^{(2)}T_{j,\ell,+}^{2^{nb}s_{n,\ell}(i)}f\Big|^2\Big)^{1/2}\Big\|_p + \int_0^{2^{-\ell}}\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}\Big|\sum_{j\in\mathbb{Z}}P_{j,\ell}^{(1)}P_{j-n,\ell,b}^{(2)}\frac{d}{d\alpha}T_{j,\ell,+}^{2^{nb}(s_{n,\ell}(i)+\alpha)}f\Big|^2\Big)^{1/2}\Big\|_p d\alpha$$

which by (2.13) is bounded by

$$C(p-1)^{-2}\Big[\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}\sum_{j\in\mathbb{Z}}|T_{j,\ell,+}^{2^{nb}s_{n,\ell}(i)}f|^2\Big)^{1/2}\Big\|_p + \int_0^{2^{-\ell}}\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mathcal{N}_{n,\ell}(U)}\sum_{j\in\mathbb{Z}}|\frac{d}{d\alpha}T_{j,\ell,+}^{2^{nb}(s_{n,\ell}(i)+\alpha)}f|^2\Big)^{1/2}\Big\|_p d\alpha\Big].$$

From here on the estimation is exactly analogous to the previous square function – just replace $A_{j,\ell}^u$ with $T_{j,\ell,+}^u$. The arguments for the corresponding terms with $T_{j,\ell,-}^u$ are similar (or could be reduced to the previous case by a change of variable, and curve). This concludes the proof of Theorem 2.2.

5. Lower bounds for $p \le 2$

As mentioned before the lower bound $(\log\mathfrak{N}(U))^{1/2}$ for $\|\mathcal{H}^U\|_{L^p\to L^p}$, based on ideas of Karagulyan [7], was established in [6]. We now show the easier lower bound in terms of the quantity $\sup_{\delta>0}\mathcal{K}_p(U,\delta)$ (where we only have to consider the cases $\delta<1$). The same calculation gives the same type of lower bound for $\|\mathcal{M}^U\|_{L^p\to L^p}$.

By rescaling in the second variable and reflection we may assume that $c_+ = 1$. For $u \in U$ and $\delta \in (0,1)$ we define

$$V_\delta(u) = \{(x_1, x_2) : 1 \le x_1 \le 2, |x_2 - ux_1^b| \le \delta/4\}$$

and let f_δ be the characteristic function of the ball of radius δ centered at the origin. Observe that for $1 \le x_1, u \le 2$, $\varepsilon < 1$ and $x_1 \le t \le x_1 + \varepsilon\delta$ we have $u(t^b - x_1^b) \le 2b\cdot 3^{b-1}\varepsilon\delta$. Thus for $\varepsilon_b = (8b\cdot 3^{b-1})^{-1}$ we get $f_\delta(x_1 - t, x_2 - ut^b) = 1$ and thus

$$H^{(u)} f_\delta(x) \ge \frac{1}{3}\int_{x_1}^{x_1+\varepsilon_b\delta} f_\delta(x_1 - t, x_2 - ut^b)dt \ge \frac{\varepsilon_b}{3}\delta, \quad x \in V_\delta(u).$$

By rescaling in the second variable we have for every $r > 0$ that

$$\|\mathcal{H}^U\|_{L^p\to L^p} \ge \|\mathcal{H}^{U^r}\|_{L^p\to L^p},$$

where $U^r = r^{-1}U \cap [1,2]$. Let $U^r(\delta)$ be a maximal $2^b\delta$–separated subset of U^r, then $\#U^r(\delta) \gtrsim N(U^r, \delta)$. This implies

$$\mathcal{H}^{U^r(\delta)} f_\delta(x) \gtrsim \delta \text{ for } x \in V_{r,\delta} := \bigcup_{u\in U^r(\delta)} V_\delta(u).$$

For different $u_1, u_2 \in U^r(\delta)$ the sets $V_\delta(u_1)$ and $V_\delta(u_2)$ are disjoint and therefore we have $\mathrm{meas}(V_{r,\delta}) \gtrsim \delta\#(U_r(\delta))$. Hence we get

$$\|\mathcal{H}^{U^r(\delta)} f_\delta\|_p \ge c\delta^{1+1/p}\#(U_r(\delta))^{1/p}.$$

Since also $\|f_\delta\|_p \lesssim \delta^{2/p}$ we obtain

$$\|\mathcal{H}^U\|_{L^p\to L^p} \ge \|\mathcal{H}^{U^r(\delta)}\|_{L^p\to L^p} \gtrsim \delta^{1-\frac{1}{p}}\#(U^r(\delta))^{\frac{1}{p}} \gtrsim \delta^{1-\frac{1}{p}} N(U^r,\delta)^{\frac{1}{p}}$$

which gives the uniform lower bound

$$\|\mathcal{H}^U\|_{L^p\to L^p} \gtrsim \mathcal{K}_p(U,\delta) \tag{5.1}$$

for sufficiently small δ.

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