ON MAXIMAL FUNCTIONS FOR MIKHLIN-HÖRMANDER MULTIPLIERS

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Abstract. Given Mikhlin-Hörmander multipliers m_i, $i = 1, \dots, N$, with uniform estimates we prove an optimal $\sqrt{\log(N+1)}$ bound in L^p for the maximal function $\sup_i |\mathcal{F}^{-1}[m_i \widehat{f}]|$ and related bounds for maximal functions generated by dilations. These improve results in [7].

1. Introduction

Given a symbol m satisfying

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^{-\alpha} \tag{1.1}$$

for all multiindices α, then by classical Calderón-Zygmund theory the operator $f \mapsto \mathcal{F}^{-1}[m\widehat{f}]$ defines an L^p bounded operator. We study two types of maximal operators associated to such symbols.

First we consider N multipliers $m_1, \dots, m_N$ satisfying uniformly the conditions (1.1) and ask for bounds

$$\Big\| \sup_{1\le i\le N} |\mathcal{F}^{-1}[m_i \widehat{f}]| \Big\|_p \le A(N) \|f\|_p, \tag{1.2}$$

for all $f \in \mathcal{S}$.

Secondly we form two maximal functions generated by dilations of a single multiplier,

$$\mathcal{M}_m^{\mathrm{dyad}} f(x) = \sup_{k\in\mathbb{Z}} |\mathcal{F}^{-1}[m(2^k\cdot)\widehat{f}]| \tag{1.3}$$

$$\mathcal{M}_m f(x) = \sup_{t>0} |\mathcal{F}^{-1}[m(t\cdot)\widehat{f}]| \tag{1.4}$$

and ask under what additional conditions on m these define bounded operators on L^p.

Concerning (1.3), (1.4) a counterexample in [7] shows that in general additional conditions on m are needed for the maximal inequality to hold; moreover positive results were shown using rather weak decay assumptions on m. The counterexample also shows that the optimal uniform bound in (1.2) satisfies

$$A(N) \ge c\sqrt{\log(N+1)}. \tag{1.5}$$

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The extrapolation argument in [7] only gives the upper bound $A(N) = O(\log(N+1))$ and the main purpose of this paper is to close this gap and to show that the upper bound is indeed $O(\sqrt{\log(N+1)})$.

We will formulate our theorems with minimal smoothness assumptions that will be described now.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be supported in $\{\xi : 1/2 < |\xi| < 2\}$ so that

$$\sum_{k\in\mathbb{Z}} \phi(2^{-k}\xi) = 1$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$. Let $\eta_0 \in C_c^\infty(\mathbb{R}^d)$ so that η_0 is even, $\eta_0(x) = 1$ for $|x| \le 1/2$ and η_0 is supported where $|x| \le 1$. For $\ell > 0$ let $\eta_\ell(x) = \eta_0(2^{-\ell}(x)) - \eta_0(2^{-\ell+1}x)$ and define

$$H_{k,\ell}[m](x) = \eta_\ell(x)\mathcal{F}^{-1}[\phi m(2^k\cdot)](x).$$

In what follows we set

$$\|m\|_{Y(q,\alpha)} := \sup_{k\in\mathbb{Z}} \sum_{\ell\ge 0} 2^{\ell\alpha}\|H_{k,\ell}[m]\|_{L^q}.$$

Using the Hausdorff-Young inequality one gets

$$\|m\|_{Y(r',\alpha)} \lesssim \sup_{k\in\mathbb{Z}} \|\phi m(2^k\cdot)\|_{B^r_{\alpha,1}}, \qquad \text{if } 1 \le r \le 2 \tag{1.6}$$

where $B^r_{\alpha,1}$ is the usual Besov space; this is well known, for a proof see Lemma 3.3 below. Thus if m belongs to $Y(2, d/2)$, then it is a Fourier multiplier on $L^p(\mathbb{R}^d)$, for $1 < p < \infty$ (this follows from a slight modification of Stein's approach in [16], ch. IV.3, see also [15] for a related endpoint bound).

Theorem 1.1. *Suppose that $1 \le r < 2$ and suppose that the multipliers m_i, $i = 1, \dots, N$ satisfy the condition*

$$\sup_i \|m_i\|_{Y(r',d/r)} \le B < \infty. \tag{1.7}$$

Then for $r < p < \infty$

$$\Big\| \sup_{i=1,\dots,N} \big|\mathcal{F}^{-1}[m_i\widehat{f}]\big| \Big\|_p \le C_{p,r} B\sqrt{\log(N+1)}\|f\|_p.$$

In particular, the conclusion of Theorem 1.1 holds if the multipliers m_i satisfy estimates (1.1) uniformly in i. By (1.6) we immediately get

Corollary 1.2. *Suppose that $1 < r < 2$, and*

$$\sup_{1\le i\le N} \sup_{t>0} \|\phi m_i(t\cdot)\|_{B^r_{d/r,1}} \le A. \tag{1.8}$$

Then for $r < p < \infty$

$$\Big\| \sup_{i=1,\dots,N} \big|\mathcal{F}^{-1}[m_i\widehat{f}]\big| \Big\|_p \le C_{p,r} A\sqrt{\log(N+1)}\|f\|_p.$$

Remark. If one uses $Y(\infty, d+\varepsilon)$ in (1.7) or $B^1_{d+\varepsilon,1}$ in (1.8) one can use Calderón-Zygmund theory (see [8], [7]) to prove the $H^1 - L^1$ boundedness and the weak type $(1,1)$ inequality, both with constant $O(\sqrt{\log(N+1)})$.

Our second result is concerned with the operators $\mathcal{M}_m^{\text{dyad}}$, $\mathcal{M}_m$ generated by dilations.

Theorem 1.3. *Suppose* $1 < p < \infty$, $q = \min\{p, 2\}$.

(i) Suppose that

$$\|\phi m(2^k \cdot)\|_{L^q_\alpha} \le \omega(k), \quad k \in \mathbb{Z}, \tag{1.9}$$

holds for $\alpha > d/q$ *and suppose that the nonincreasing rearrangement* ω^* *satisfies*

$$\omega^*(0) + \sum_{l=2}^{\infty} \frac{\omega^*(l)}{l\sqrt{\log l}} < \infty. \tag{1.10}$$

Then $\mathcal{M}_m^{\text{dyad}}$ *is bounded on* $L^p(\mathbb{R}^d)$.

(ii) Suppose that (1.10) *holds and* (1.9) *holds for* $\alpha > d/p + 1/p'$ *if* $1 < p \le 2$ *or for* $\alpha > d/2 + 1/p$ *if* $p > 2$. *Then* $\mathcal{M}_m$ *is bounded on* $L^p(\mathbb{R}^d)$.

If (1.9), (1.10) *are satisfied with* $q = 1$, $\alpha > d$ *then* $\mathcal{M}_m$ *is of weak type* $(1,1)$, *and* $\mathcal{M}_m$ *maps* H^1 *to* L^1.

This improves the earlier result in [7] where the conclusion is obtained under the assumption $\sum_{l=2}^{\infty} \omega^*(l)/l < \infty$, however somewhat weaker smoothness assumptions were made in [7].

In §2 we shall discuss model cases for Rademacher expansions. In §3 we shall give the outline of the proof of Theorem 1.1 which is based on the $\exp(L^2)$ estimate by Chang-Wilson-Wolff [5], for functions with bounded Littlewood-Paley square-function. The proof of a critical pointwise inequality is given in §4. The proof of Theorem 1.3 is sketched in §5. Some open problems are mentioned in §6.

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2. Dyadic model cases for Rademacher expansions

Before we discuss the proof of Theorem 1.1 we give a simple result on expansions for Rademacher functions r_j on $[0,1]$ which motivated the proof.

Proposition 2.1. *Let* $a^i \in \ell^2$. *and let*

$$F_i(s) = \sum_j a^i_j r_j(s), \quad s \in [0,1].$$

Then

$$\big\| \sup_{i<N} |F_i| \big\|_{L^2[0,1]} \lesssim \sup \|a^i\|_{\ell^2} \sqrt{\log(N+1)}.$$

Proof. We use the well known estimate for the distribution function of the Rademacher expansions ([16], p. 277),

$$\text{meas}\big(\{s \in [0,1] : |F_i(s)| > \lambda\}\big) \le 2\exp\big(-\tfrac{\lambda^2}{4\|a^i\|_{\ell^2}^2}\big) \tag{2.1}$$

Set $u_N = (4\log(N+1))^{1/2}\sup_{1\le i\le N}\|a^i\|_{\ell^2}$. Then

$$\begin{aligned}\big\|\sup_{i=1,\dots,N}|F_i|\big\|_2^2 &\le u_N^2 + 2\sum_{i=1}^N\int_{u_N}^\infty \lambda\,\text{meas}\big(\{s : |F_i(s)| > \lambda\}\big)d\lambda \\ &\le u_N^2 + 4\sum_{i=1}^N\int_{u_N}^\infty \lambda e^{-\lambda^2/(4\|a^i\|_{\ell^2}^2)}d\lambda \le u_N^2 + 4\sup_{i=1,\dots,N}\|a^i\|_{\ell^2}^2 N e^{-u_N^2/4}\end{aligned}$$

which is bounded by $(1+4\log(N+1))\sup_i\|a^i\|_{\ell^2}^2$. The claim follows. □

There is a multiplier interpretation to this inequality. One can work with a single function $f = \sum a_j r_j$ and a family of bounded sequences (or multipliers) $\{b^i\}$ and one forms $F_i(s) = \sum_j b_j^i a_j r_j(s)$. The norm then grows as a square root of the logarithm of the number of multipliers; i.e. we have

Corollary 2.2.

$$\Big\|\sup_{i=1,\dots,N}\big|\sum_j b_j^i a_j r_j\big|\Big\|_{L^2([0,1])} \lesssim \sup_i\|b^i\|_\infty\sqrt{\log(N+1)}\Big\|\sum_j a_j r_j\Big\|_{L^2([0,1])}.$$

We shall now consider a dyadic model case for the maximal operators generated by dilations.

Proposition 2.3. *Consider a sequence $b = \{b_i\}_{i\in\mathbb{Z}}$ which satisfies*

$$b^*(l) \le \frac{A}{(\log(l+2))^{1/2}}.$$

Then for any sequence $a = \{a_n\}_{n=1}^\infty$ we have

$$\Big\|\sup_{k\in\mathbb{Z}}\big|\sum_{j=0}^\infty b_{j-k}a_j r_j\big|\Big\|_2 \le CA\|a\|_2.$$

Proof. We may assume that both a and b are real valued sequences. Let

$$H_k(s) = \sum_{j=1}^\infty b_{j-k}a_j r_j(s).$$

Then by orthogonality of the Rademacher functions

$$\|H_k\|_2^2 = \sum_{j=1}^\infty [b_{j-k}a_j]^2.$$

We shall use a result of Calderón [4] which states that if some linear operator is bounded on $L^1(\mu)$ and on $L^\infty(\mu)$ on a space with σ-finite measure μ, then it is bounded on all rearrangement invariant function spaces on that space.

In our case the intermediate space is the Orlicz space $\exp \ell$, which coincides with the space of all sequences $\gamma = \{\gamma_j\}_{j\in\mathbb{Z}}$ that satisfy the condition

$$\gamma^*(l) \le \frac{C}{\log(l+2)}, \quad l \ge 0, \tag{2.2}$$

and the best constant in 2.2 is equivalent to the norm in $\exp(\ell)$. We apply Calderón's result to the operator T defined by

$$[T\gamma]_k = \sum_{j=1}^{\infty} \gamma_{j-k} a_j^2$$

and get

$$\sup_{l\ge 0} \log(l+2)(T\gamma)^*(l) \le C \big\| \{a_n^2\} \big\|_{\ell^1} \sup_{l\ge 0} \log(l+2)\gamma^*(l).$$

Let $c_k = \|H_k\|_2 \equiv ([T(b^2)]_k)^{1/2}$ where b^2 stands for the sequence $\{b_j^2\}$; then by our bound for $T\gamma$ and the assumption on b it follows that

$$c^*(l) \le C_1 A \|a\|_{\ell^2} \big(\log(2+l)\big)^{-1/2}. \tag{2.3}$$

We can proceed with the proof as in Proposition 2.1, using again (2.1), *i.e.*

$$\text{meas}(\{s \in [0,1] : |H_k(s)| > \alpha\}) \le 2e^{-\alpha^2/4c_k^2}.$$

Then we obtain for $u > 0$

$$\begin{aligned} \big\| \sup_k |H_k| \big\|_2 &\le u^2 + 4 \sum_k \int_u^{\infty} \alpha e^{-\alpha^2/4c_k^2} \\ &\le u^2 + 8 \sum_k c_k^2 e^{-u^2/(4c_k^2)} \\ &= u^2 + 8 \sum_{l\ge 0} (c^*(l))^2 e^{-u^2/4(c^*(l))^2}. \end{aligned}$$

We set the cutoff level to be $u = 10 C_1 A \|a\|_2$ and obtain

$$\| \sup_k |H_k| \|_2^2 \le u^2 + C_1^2 A^2 \sum_{l\ge 0} (2+l)^{-5/2} \lesssim A^2 \|a\|_2^2$$

which is what we wanted to prove. □

Remark: Since the L^p norm of $\sum a_j r_j$ is equivalent to the ℓ^2 norm of $\{a_j\}$ one can also prove L^p analogues of the two propositions, for $0 < p < \infty$.

3. Proof of Theorem 1.1

To prove (1.2) we may assume that $\widehat{f}$ is compactly supported in $\mathbb{R}^d \setminus \{0\}$ and thus we may assume that the multipliers m_i are compactly supported on a finite union of dyadic annuli. In view of the scale invariance of the assumptions we may assume without loss of generality that

$$m_i(\xi) = 0, \quad |\xi| \le 2^N, \quad i = 1, \dots, N. \tag{3.1}$$

In the case of Fourier multipliers the inequality (2.1) will be replaced by a "good-λ inequality" involving square-functions for martingales as proved by Chang, Wilson and Wolff [5]. To fix notation let, for any $k \geq 0$, $\mathfrak{Q}_k$ denote the family of dyadic cubes of sidelength 2^{-k}; each Q is of the form $\prod_{i=1}^d [n_i 2^{-k}, (n_i+1)2^{-k})$. Denote by $\mathbb{E}_k$ the conditional expectation,

$$\mathbb{E}_k f(x) = \sum_{Q \in \mathfrak{Q}_k} \chi_Q(x) \frac{1}{|Q|} \int_Q f(y) dy$$

and by $\mathbb{D}_k$ the martingale differences,

$$\mathbb{D}_k f(x) = \mathbb{E}_{k+1} f(x) - \mathbb{E}_k f(x).$$

The square function for the dyadic martingale is defined by

$$S(f) = \Big(\sum_{k \geq 0} |\mathbb{D}_k f(x)|^2 \Big)^{1/2};$$

one has the inequality $\|S(f)\|_p \leq C_p \|f\|_p$ for $1 < p < \infty$ (see [3], [2] for the general martingale case, and for our special case *cf.* also Lemma 3.1 below).

The result from [5] says that there is a constant $c_d > 0$ so that for all $\lambda > 0$, $0 < \varepsilon < 1$, one has

$$\text{(3.2)} \quad \begin{aligned} &\text{meas}\big(\big\{x : \sup_{k \geq 0} |\mathbb{E}_k g(x) - \mathbb{E}_0 g(x)| > 2\lambda,\, S(g) < \epsilon\lambda\big\}\big) \\ &\qquad \leq C \exp(-\frac{c_d}{\epsilon^2}) \text{meas}\big(\big\{x : \sup_{k \geq 0} |\mathbb{E}_k g(x)| > \varepsilon\lambda\big\}\big); \end{aligned}$$

see [5] (Corollary 3.1 and a remark on page 236). To use (3.2) we need a pointwise inequality for square functions applied to convolution operators.

Choose a radial Schwartz function ψ which equals 1 on the support of ϕ (defined in the introduction) and is compactly supported in $\mathbb{R}^d \setminus \{0\}$, and define the Littlewood-Paley operator L_k by

$$\text{(3.3)} \qquad \widehat{L_k f}(\xi) = \psi(2^{-k}\xi) \widehat{f}(\xi)$$

Let M be the Hardy-Littlewood maximal operator and define the operator M_r by

$$M_r = (M(|f|^r))^{1/r}.$$

Denote by $\mathfrak{M} = M \circ M \circ M$ the three-fold iteration of the maximal operator. Now define

$$\text{(3.4)} \qquad G_r(f) = \Big(\sum_{k \in \mathbb{Z}} \big(\mathfrak{M}[|L_k f|^r]\big)^{2/r} \Big)^{1/2}.$$

From the Fefferman-Stein inequality for vector-valued maximal functions [9],

$$\text{(3.5)} \qquad \|G_r(f)\|_p \leq C_{p,r} \|f\|_p, \quad 1 < r < 2,\ r < p < \infty.$$

Lemma 3.1. *Let* $Tf = \mathcal{F}^{-1}[m\widehat{f}]$ *and let* $1 < r \le \infty$. *Then for* $x \in \mathbb{R}^d$,

$$S(Tf)(x) \le A_r \|m\|_{Y(r',d/r)} G_r(f)(x). \tag{3.6}$$

The proof will be given in §4.

We shall also need

Lemma 3.2. *Let* $Tf = \mathcal{F}^{-1}[m\widehat{f}]$ *and suppose that* $m(\xi) = 0$ *for* $|\xi| \le 2^N$. *Then*

$$|\mathbb{E}_0 Tf(x)| \le C 2^{-N/r} C_r \|m\|_{Y(r',d/r)} (\mathfrak{M}(|f|^r))^{1/r}. \tag{3.7}$$

We now give the proof of Theorem 1.1. Let $T_i f = \mathcal{F}^{-1}[m_i \widehat{f}]$. We need to estimate

$$\big\| \sup_{1\le i\le N} |T_i f| \big\|_p = \Big(p4^p \int_0^\infty \lambda^{p-1} \text{meas}(\{x : \sup_i |T_i f(x)| > 4\lambda\}) d\lambda \Big)^{1/p}.$$

Now by Lemma 3.1 one gets the pointwise bound

$$S(T_i f) \le A_r B\, G_r(f). \tag{3.8}$$

We note that

$$\{x : \sup_{1\le i\le N} |T_i f(x)| > 4\lambda\} \subset E_{\lambda,1} \cup E_{\lambda,2} \cup E_{\lambda,3}$$

where with

$$\varepsilon_N := \Big(\frac{c_d}{10\log(N+1)}\Big)^{1/2} \tag{3.9}$$

we have set

$$\begin{aligned} E_{\lambda,1} &= \{x : \sup_{1\le i\le N} |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda, G_r(f)(x) \le \frac{\varepsilon_N \lambda}{A_r B}\}, \\ E_{\lambda,2} &= \{x : G_r(f)(x) > \frac{\varepsilon_N \lambda}{A_r B}\}, \\ E_{\lambda,3} &= \{x : \sup_{1\le i\le N} |\mathbb{E}_0 T_i f(x)| > 2\lambda\}. \end{aligned}$$

By (3.8),

$$E_{\lambda,1} \subset \bigcup_{i=1}^N \{x : |T_i f(x)| > 2\lambda, S(T_i f) \le \varepsilon_N \lambda\}, \tag{3.10}$$

and thus using the good-λ inequality (3.2) we obtain

$$\begin{aligned} \text{meas}(E_{\lambda,1}) &\le \sum_{i=1}^N \text{meas}\big(\{x : |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda, S(T_i f) \le \varepsilon_N \lambda\}\big) \\ &\le \sum_{i=1}^N C \exp(-\tfrac{c_d}{\varepsilon_N^2}) \text{meas}(\{x : \sup_k |\mathbb{E}_k(T_i f)| > \lambda\}). \end{aligned}$$

Hence

$$\begin{aligned}
&\Big(p\int_0^\infty \lambda^{p-1}\mathrm{meas}(E_{\lambda,1})d\lambda\Big)^{1/p}\\
&\lesssim \Big(\sum_{i=1}^N \exp(-\tfrac{c_d}{\varepsilon_N^2})\big\|\sup_k|\mathbb{E}_k(T_if)|\big\|_p^p\Big)^{1/p}\\
&\lesssim \Big(\sum_{i=1}^N \exp(-\tfrac{c_d}{\varepsilon_N^2})\big\|T_if\big\|_p^p\Big)^{1/p}\\
&\lesssim B\big(N\exp(-\tfrac{c_d}{\varepsilon_N^2})\big)^{1/p}\|f\|_p \lesssim B\|f\|_p \qquad (3.11)
\end{aligned}$$

uniformly in N (by our choice of ε_N in (3.9)).

Next, by a change of variable,

$$\begin{aligned}
\Big(p\int_0^\infty \lambda^{p-1}\mathrm{meas}(E_{\lambda,2})d\lambda\Big)^{1/p} &= \frac{A_rB}{\varepsilon_N}\big\|G_r(f)\big\|_p\\
&\lesssim B\sqrt{\log(N+1)}\|f\|_p \qquad (3.12)
\end{aligned}$$

Finally, from Lemma 3.2 and the Fefferman-Stein inequality

$$\mathrm{meas}(E_{\lambda,3}) \le \sum_{i=1}^N \mathrm{meas}\big(\{x : |\mathbb{E}_0T_if(x)| > 2\lambda\}\big)$$

and thus

$$\begin{aligned}
\Big(p\int_0^\infty \lambda^{p-1}\mathrm{meas}(E_{\lambda,3})d\lambda\Big)^{1/p} &= 2\big\|\sup_{i=1,\dots,N}|\mathbb{E}_0(T_if)|\big\|_p\\
\le 2\Big(\sum_{i=1}^N \big\|\mathbb{E}_0(T_if)\big\|_p^p\Big)^{1/p} &\lesssim BN^{1/p}2^{-N/r}\|f\|_p \lesssim B\|f\|_p. \qquad (3.13)
\end{aligned}$$

The asserted inequality follows from (3.11), (3.12), and (3.13). □

For completeness we mention the well known relation of the $Y(r',\alpha)$ conditions with Besov and Sobolev norms.

Lemma 3.3. *Let* $1 \le r \le 2$ *and* $\alpha > d/r$. *Then*

$$\begin{aligned}
\|m\|_{Y(r',d/r)} &\lesssim \sup_k \|\phi m(2^k\cdot)\|_{B^r_{d/r,1}}\\
&\lesssim \sup_k \|\phi m(2^k\cdot)\|_{L^r_\alpha} \lesssim \sup_k \|\phi m(2^k\cdot)\|_{L^2_\alpha}
\end{aligned}$$

Proof. By the Hausdorff-Young inequality and the definition of the Besov space we have

$$\sum_{\ell=0}^\infty 2^{\ell d/r}\|H_{k,\ell}\|_{r'} \lesssim \sum_{\ell=0}^\infty 2^{\ell d/r}\|[\phi m(2^k\cdot)] * \widehat{\eta}_\ell\|_r \lesssim \|\phi m(2^k\cdot)\|_{B^r_{d/r,1}}.$$

By elementary imbedding properties $\|g\|_{B^r_{d/r,1}} \lesssim \|g\|_{L^r_\gamma}$ if $\gamma > d/r$. Finally $\|\phi m(2^k\cdot)\|_{L^r_\gamma} \lesssim C'_r \|\phi m(2^k\cdot)\|_{L^2_\gamma}$, if $1 < r \le 2$. In this last inequality we used that for $\chi \in C_c^\infty$ we have $\|\chi g\|_{L^{r_0}_\gamma} \lesssim \|g\|_{L^{r_1}_\gamma}$ for $r_0 \le r_1$, $\gamma \ge 0$; this is trivial for integers γ from Hölder's inequality and follows for all $\gamma \ge 0$ by interpolation. □

4. Proofs of Lemma 3.1 and Lemma 3.2

Choose a radial Schwartz function β with the property that $\widehat{\beta}$ is supported in $\{x : |x| \le 1/4\}$ so that $\beta(\xi) \ne 0$ in $\{\xi : 1/4 \le |\xi| \le 4\}$ and $\beta(0) = 0$. Now choose a function $\widetilde{\psi} \in C_c^\infty$ so that $\widetilde{\psi}(\xi)(\beta(\xi))^2 = 1$ for all $\xi \in \operatorname{supp} \phi$, here ϕ is as in the formulation of the theorem. Define operators T_k, B_k, $\widetilde{L}_k$ by

$$\widehat{T_k f}(\xi) = \phi(2^{-k}\xi) m(\xi) \widehat{f}(\xi)$$
$$\widehat{B_k f}(\xi) = \beta(2^{-k}\xi) \widehat{f}(\xi)$$
$$\widehat{\widetilde{L}_k f}(\xi) = \widetilde{\psi}(2^{-k}\xi) \widehat{f}(\xi).$$

Then $T = \sum_k T_k = \sum_k B_k^2 \widetilde{L}_k T_k L_k$ and we write

$$\mathbb{D}_k T f = \sum_{n\in\mathbb{Z}} (\mathbb{D}_k B_{k+n})(B_{k+n}\widetilde{L}_{k+n}) T_{k+n} L_{k+n} f. \tag{4.1}$$

Sublemma 4.1.

$$|B_k \widetilde{L}_k f(x)| \lesssim M f(x). \tag{4.2}$$

Proof. Immediate. □

Sublemma 4.2. *For $s \ge 0$,*

$$|\mathbb{E}_{k+1} B_{k+s} f(x)| + |\mathbb{E}_k B_{k+s} f(x)| \lesssim 2^{-s/q'} M_q f(x) \tag{4.3}$$

and

$$|\mathbb{D}_k B_{k-s} f(x)| \lesssim 2^{-s} M f(x). \tag{4.4}$$

Proof. We give the proof although the estimates are rather standard (for similar calculations in other contexts see for example [6], [12], [10], [13]).

For (4.3) first note this inequality is trivial if s is small and assume, say, $s \ge 10$. For $Q \in \mathfrak{Q}_k$, $s > 0$ let $b_s(Q)$ be the set of all $x \in Q$ for which the ℓ^∞ distance to the boundary of Q is $\le 2^{-k-s+1}$.

Fix a cube $Q_0 \in \mathfrak{Q}_{k+1}$. If Q' is a dyadic subcube of sidelength 2^{-k-s+1} subcube which is not contained in $b_s(Q)$ then $B_{k+s}[f\chi_{Q'}]$ is supported in Q_0 and using the cancellation of $\mathcal{F}^{-1}[\beta]$ we see that $\mathbb{E}_{k+1} B_{k+s}[\chi_{Q'} g] = 0$ for all g. Let $\mathcal{V}_s(Q_0)$ be the union over all dyadic cubes of sidelength 2^{-k-s+1} whose closures intersect the boundary of Q_0. Then

$$\mathbb{E}_{k+1} B_{k+s}[\chi_{Q_0} g] = \mathbb{E}_{k+1} B_{k+s}[g \chi_{\mathcal{V}_s(Q_0)}]$$

for all g. In view of the support properties of $\widehat{\beta}$ we note that $B_{k+s}[g\chi_{\mathcal{V}_s(Q_0)}]$ is also supported in $\mathcal{V}_{s-1}(Q_0)$. Observe that this set has measure $O(2^{-kd}2^{-s})$.

It follows that for $x \in Q_0$

$$\begin{aligned}|\mathbb{E}_{k+1}B_{k+s}f(x)| &\le 2^d|Q_0|^{-1}\int_{\mathcal{V}_{s-1}(Q_0)}|B_{k+s}[\chi_{\mathcal{V}_s(Q_0)}f](y)|dy\\ &\lesssim |Q_0|^{-1}\Big(\int_{Q_0}|f(y)|^q dy\Big)^{1/q}2^{-(kd+s)/q'}\\ &\lesssim 2^{-s/q'}\big(M(|f|^q)\big)^{1/q}\end{aligned}$$

By the same argument one obtains this bound also for $|\mathbb{E}_kB_{k+s}f|$ and thus (4.3) follows.

The inequality (4.4) $\mathbb{D}_kB_{k-s}f$ is a simple consequence of the smoothness of the convolution kernel of B_{k-s} and the cancellation properties of the operator $\mathbb{D}_k = \mathbb{E}_{k+1} - \mathbb{E}_k$. □

Sublemma 4.3. *Let $1 < r < \infty$. We have*

$$|T_kf(x)| \le C\|m\|_{Y(r',d/r)}M_rf(x). \tag{4.5}$$

Proof. We may decompose T_k using the kernels $H_{k,l}$ and obtain

$$\begin{aligned}|T_kf(x)| &= \Big|\sum_{\ell=0}^{\infty}\int 2^{kd}H_{k,\ell}(2^ky)f(x-y)dy\Big|\\ &\le \sum_{\ell=0}^{\infty}\Big(2^{kd}\int|H_{k,\ell}(2^ky)|^{r'}dy\Big)^{1/r'}\Big(2^{kd}\int_{|y|\le 2^{-k+\ell}}|f(x-y)|^rdy\Big)^{1/r}\\ &\le \sum_{\ell=0}^{\infty}2^{\ell d/r}\|H_{k,\ell}\|_{r'}\big(M(|f|^r)(x)\big)^{1/r}. \quad \square\end{aligned}$$

Proof of Lemma 3.1. To estimate the terms in (4.1) we use Sublemma 4.1 to bound $B_{k+n}\widetilde{L}_{k+n}$, Sublemma 4.2 to bound $\mathbb{D}_kB_{k+n}$ and Sublemma 4.3 to bound T_{k+n}. This yields that

$$\begin{aligned}|\mathbb{D}_kB^2_{k+n}\widetilde{L}_{k+n}T_{k+n}L_{k+n}f(x)| &\lesssim \|m\|_{Y(r',d/r)}\\ &\quad\times\begin{cases}2^{-n/q'}M_q\circ M\circ M_r(L_{k+n}f)(x) & \text{if } n\ge 0\\ 2^nM\circ M\circ M_r(L_{k+n}f)(x) & \text{if } n<0,\end{cases}\end{aligned}$$

and straightforward estimates imply the asserted bound. □

Proof of Lemma 3.2. We split $\mathbb{E}_0Tf = \sum_{k\ge N-2}\mathbb{E}_0B_k^2\widetilde{L}_kT_k$, and by the sublemmas we get

$$|\mathbb{E}_0B_k^2\widetilde{L}_kT_kf(x)| \lesssim 2^{-k/r}\|m\|_{Y(r',d/r)}M_r\circ M\circ M_r(f)(x)$$

which implies the assertion. □

5. Maximal functions generated by dilations

For the proof of Theorem 1.3 we use arguments in [7] and applications of Theorem 1.1. Let us first consider the dyadic maximal operator $\mathcal{M}_m^{\mathrm{dyad}}$.

Let

$$\mathcal{I}_j = \{k \in \mathbb{Z} : \omega^*(2^{2^j}) < |\omega(k)| \le \omega^*(2^{2^{j-1}})\}.$$

We split $m = \sum_j m_j$ where m_j is supported in the union of dyadic annuli $\cup_{k\in\mathcal{I}_j}\{\xi : 2^{k-1} < |\xi| < 2^{k+1}\}$.

By Lemma 3.1 in [7] we can find a sequence of integers $B = \{i\}$ so that for each j the sets $b_i + \mathcal{I}_j$ are pairwise disjoint, and $\mathbb{Z} = \cup_{n=-4^{2^j+1}}^{4^{2^j+1}}(n+B)$.

Let $T_k^j f = \mathcal{F}^{-1}[m_j(2^k\cdot)\widehat{f}]$. We write

$$\sup_k |T_k f| = \sup_{|n|\le 4^{2^j+1}} \sup_{i\in\mathbb{Z}} |T_{b_i+n} f| \tag{5.1}$$

and split the sup in i according to whether $i > 0$, $i = 0$, $i < 0$. We use the standard equivalence of the L^p norm of expansions of Rademacher functions $\{r_i\}_{i=1}^\infty$ with the ℓ^2 norm of the sequence of coefficients (see [16], p. 276).

Then

$$\begin{aligned}
\Big\| \sup_{|n|\le 4^{2^j+1}} \sup_{i>0} |T^j_{b_i+n} f| \Big\|_p &\le \Big\| \sup_{|n|\le 4^{2^j+1}} \Big(\sum_{i>0} |T^j_{b_i+n} f|^2 \Big)^{1/2} \Big\|_p \\
&\le C_p \Big\| \sup_{|n|\le 4^{2^j+1}} \Big(\int_0^1 \Big| \sum_{i=1}^\infty r_i(s) T^j_{b_i+n} f \Big|^p ds \Big)^{1/p} \Big\|_p \\
&\le C_p \Big\| \Big(\int_0^1 \sup_{|n|\le 4^{2^j+1}} \Big| \sum_{i=1}^\infty r_i(s) T^j_{b_i+n} f \Big|^p ds \Big)^{1/p} \Big\|_p \\
&= C_p \Big(\int_0^1 \Big\| \sup_{|n|\le 4^{2^j}} \Big| \sum_{i=1}^\infty r_i(s) T^j_{b_i+n} f \Big| \Big\|_p^p ds \Big)^{1/p}
\end{aligned}$$

which reduce matters for the dyadic maximal function to an application of Theorem 1.1 (of course the terms above with $i \le 0$ are handled similarly). Thus we obtain the estimate

$$\|M^{\mathrm{dyad}}_{m_j}\|_{L^p\to L^p} \lesssim 2^{j/2}\omega^*(2^{2^{j-1}}).$$

For the full maximal operator we use standard decompositions by smoothing out the rescaled dyadic pieces. We just sketch the argument. Assume that $p \ge 2$ and that the assumption of Theorem 1.3, (ii), with $\alpha > d/2 + 1/p$ holds. Then one can decompose $m_j = \sum_{l\ge 0} m_{j,l}$ where $m_{j,l}$ has essentially the same support property as m_j (with slightly extended dyadic annuli) and where

$$\|\phi m_{j,l}(2^k\cdot)\|_{L^2_{\alpha-1/p}} + 2^{-l}\|\phi\,\langle \xi, \nabla\rangle [m_{j,l}(2^k\cdot)]\|_{L^2_{\alpha-1/p}} \lesssim \omega^*(2^{2^{j-1}})2^{-l/p}.$$

One then uses a standard argument (see *e.g.* [17], p. 499) to see that

$$\sup_{t>0} |\mathcal{F}^{-1}[m_{j,l}(t\cdot)\widehat{f}]| \le C \sup_{k>0} |\mathcal{F}^{-1}[m_{j,l}(2^k\cdot)\widehat{f}]| +$$
$$C\Big(\int_1^2 |\mathcal{F}^{-1}[m_{j,l}(2^k u\cdot)\widehat{f}]|^p du\Big)^{\frac{1}{p'p}} \Big(\int_1^2 \big|(\partial/\partial u)\mathcal{F}^{-1}[m_{j,l}(2^k u\cdot)\widehat{f}]|^p du\Big)^{\frac{1}{p^2}}$$

and straightforward estimates reduce matters to the dyadic case treated above. For the weak-type estimate (or the $H^1 \to L^1$ estimate) one has to combine this argument with Calderón-Zygmund theory and the L^p estimates for $1 < p < 2$ follow then by an analytic interpolation. Similar arguments appear in [8] and [7]; we omit the details. □

6. Open problems

Concerning Theorem 1.1 one can ask about L^p boundedness for $p > 2$ under merely the assumption $m_i \in Y(p', \alpha)$, $\alpha > d/p$. Combining our present result with those in [7] one can show that if for some $2 < r < \infty$

$$\sup_i \|m_i\|_{Y(r',\alpha)} \le A, \quad \alpha > d/r \tag{6.1}$$

then for $r \le p < \infty$

$$\Big\| \sup_{i=1,\dots,N} \big|\mathcal{F}^{-1}[m_i\widehat{f}]\big| \Big\|_p \le C_{p,r,\alpha} A (\log(N+1))^{1/r'} \|f\|_p. \tag{6.2}$$

Indeed one can imbed the multipliers in analytic families so that for $L^\infty \to BMO$ boundedness one has $Y(1+\varepsilon_1, \varepsilon_2)$ conditions and the $O(\log(N+1))$ result of [7] applies. For $p = 2$ on has the usual $Y(2, d/2+\varepsilon)$ conditions and Theorem 1.1 applies giving an $O((\log(N+1))^{1/2})$ bound.

Problem 1: Does (6.2) hold with an $O(\sqrt{\log(N+1)})$ bound if we assuming (6.1) with $r > 2$?

Problem 2: To which extent can one relax the smoothness conditions in Theorems 1.1 and 1.3 to obtain L^2 bounds? In particular what happens in Theorem 1.3 if one imposes localized L^2_α conditions for $\alpha < d/2$, assuming again minimal decay assumptions on ω^*.

Finally we discuss possible optimal decay estimates for the maximal operators generated by dilations. The hypothesis in Theorem 1.3 is equivalent with the assumption

$$\{2^{j/2}\omega^*(2^{2^j})\} \in \ell^1.$$

The counterexamples in [7] leave open the possibility that the conclusion of Theorem 1.1 might hold under the weaker assumption $\{2^{j/2}\omega^*(2^{2^j})\} \in \ell^\infty$, *i.e.*

$$\omega^*(l) \le C\big(\log(2+l)\big)^{-1/2}; \tag{6.3}$$

this is in fact suggested by the dyadic model case in Proposition 2.3. The latter condition would be optimal and leads us to formulate

Problem 3. Suppose m is a symbol satisfying (1.9) for sufficiently large α. Does L^p boundedness hold merely under the assumption (6.3)?

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