RADIAL FOURIER MULTIPLIERS IN HIGH DIMENSIONS

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Abstract. Given a fixed $p \neq 2$ we prove a simple and effective characterization of all radial multipliers of $\mathcal{F}L^p(\mathbb{R}^d)$ provided that the dimension d is sufficiently large. The method also yields new L^q space-time regularity results for solutions of the wave equation in high dimensions.

Introduction

In this paper we study convolution operators with radial kernels acting on functions defined in $\mathbb{R}^d$. These can also be described as Fourier multiplier transformations T_m defined by

$$\widehat{T_m f} = m\widehat{f},$$

with radial m. The main question we will be interested in is when the operator T_m is bounded in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$. By duality, the boundedness of T_m in L^p is equivalent to its boundedness in $L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$, so we may restrict ourselves to the range $1 \le p \le 2$.

A simple characterization of convolution operators bounded in L^p (whether radial or not) is known only in two cases: $p = 1$ and $p = 2$; namely, boundedness in L^1 holds if and only if the convolution kernel is a bounded Borel measure and boundedness in L^2 holds if and only if the multiplier is an essentially bounded function (see [12]). It is currently widely believed that for $1 < p < 2$, a full characterization of all L^p-multipliers in reasonable terms is impossible. For the class of *radial* multipliers we deal with in this paper, numerous sufficient conditions for boundedness in L^p have been obtained in the literature. Many of them are in some or another sense close to being necessary (*cf.* [3], [1], [14], [2], [26], [16], and references in those papers) but no nice necessary and sufficient conditions have been known. However, recently, Garrigós and the second author [9] obtained a perhaps surprising characterization of the radial multiplier transformations that are bounded on the invariant subspace L^p_{rad} of radial L^p functions in the range

Date: This version of the paper (2008) will remain unpublished. The results in a slightly improved range of exponents, will be published in a joint paper with Yaryong Heo.

1991 *Mathematics Subject Classification.* 42B15.

Supported in part by NSF grants.

$1 < p < \frac{2d}{d+1}$ (which is optimal for their result). This raised the question whether the necessary and sufficient conditions in [9] actually give a characterization of the radial multiplier transformations bounded on the entire space $L^p(\mathbb{R}^d)$. The main result of the present paper is to show that this is indeed the case in dimensions $d \ge 5$ for $1 < p < p_d$ where $p_d \to 2$ as $d \to \infty$.

1. Statement of results

Theorem 1.1. *Let $d \ge 5$, $1 < p < p_d := \frac{2(d^2-2d-3)}{d^2-5}$, and let m be radial. Fix an arbitrary Schwartz function η that is not identically* 0*. Then*

$$\big\|T_m\big\|_{L^p\to L^p} \asymp \sup_{t>0} t^{d/p}\big\|T_m[\eta(t\cdot)]\big\|_{L^p}\,. \tag{1.1}$$

The finiteness of the right hand side is, obviously, necessary for the L^p boundedness, and the main result here is that it is also sufficient. The constants implicit in this characterization depend (of course) on the choice of η. The condition in (1.1) is equivalent to $\sup_{t>0}\|\mathcal{F}^{-1}[m(t\cdot)\widehat{\eta}]\|_p < \infty$. If one chooses η to be radial and such that $\widehat{\eta}$ is compactly supported away from the origin, then one recovers one of the characterizations for L^p_{rad}-boundedness in [9]. Consequently, in the given range L^p-boundedness is equivalent to L^p_{rad}-boundedness. For other equivalent formulations, we refer the reader to [9].

One special situation is worth mentioning here. Namely when m is compactly supported away from the origin, the convolution operator is bounded in L^p if and only if the (radial) convolution kernel belongs to L^p.

We have no reason to believe that the range for p in Theorem 1.1 is even close to the optimal one. It is conceivable that the characterization holds in low dimensions or even in the optimal range $p < \frac{2d}{d+1}$, but proving that will certainly require new ideas. We also emphasize that the theorem gives no improvements for the Bochner-Riesz multiplier problem that is by now understood in the range $p < \frac{2d+4}{d+4}$, $d \ge 2$ ([3], [14]). Our result just goes in a different direction: it applies to all, however irregular, radial kernels and it is to be expected that, using some additional structural or regularity conditions, one may get some better range of p for each particular case. Nevertheless, our technique does yield some improvements upon the existing results in the so-called local smoothing problem for the wave equation in high dimensions. This concerns inequalities of the form

$$\Big(\int_I \|e^{it\sqrt{-\Delta}}f\|_q^q\,dt\Big)^{1/q} \le C_I\|f\|_{L^q_\alpha}\,, \tag{1.2}$$

for $q > 2$; here I is compact interval and $L^q_\alpha(\mathbb{R}^d)$ denotes the usual Sobolev (or potential) space where q is the Lebesgue exponent and α is the number of derivatives. Sharp L^q-Sobolev inequalities for fixed time were obtained by Miyachi [15] and Peral [18]; they showed that the operator $e^{it\sqrt{-\Delta}}$ maps $L^q_\beta(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ provided that $\beta \geq (d-1)|1/2 - 1/q|$, $1 < q < \infty$. In [21] Sogge raised the question whether the averaged inequality (1.2) could hold with a gain of almost $1/q$ derivatives compared to the fixed time estimate, *i.e.*, with $\alpha > \alpha(q) = d(1/2 - 1/q) - 1/2$, in the best possible range $q > 2d/(d-1)$ for such an estimate. This conjecture is at the top of a tree of other conjectures in harmonic analysis (including the cone multiplier, Bochner-Riesz, Fourier-restriction and Kakeya conjectures) and the relation between the different questions is discussed for example in [23]. The current techniques seem to be insufficient to settle this problem, as well as many of its consequences, in the full range of q's. Some evidence for the smoothing conjecture can be found in [16] where the analogous question for the $L^q_{\text{rad}}(L^2_{\text{sph}})$ scale of spaces is settled. For the L^q spaces even partial results proved to be rather hard and the first result was obtained by Wolff [26]; he established, in a deep and fundamental paper, the validity of Sogge's conjecture in two dimension for the range $q > 74$. Versions of this result for the higher dimensional cases were obtained by Łaba and Wolff [13] and further improvements on the range of q's are in [8], [10]; it is now known that Wolff's main $\ell^q(L^q) \to L^q$ inequality for plate decompositions of cone multipliers, which implies (1.2) for $\alpha > \alpha(q)$, holds with $q > 20$ if $d = 2$ and $q > 2 + \frac{8}{d-2}\frac{2d+1}{2d+2}$ if $d \geq 3$, *cf.* [10].

We improve the current results on the smoothing problem in two ways. First we widen the range in dimensions $d \geq 6$ to $q > q_d$ where $q_d = p_d' = 2 + 4d^{-1} + O(d^{-2})$ as $d \to \infty$. Secondly, we strengthen Sogge's conjecture to obtain the endpoint result in (1.2), in dimensions $d \geq 5$, for $q > q_d$.

Theorem 1.2. *Suppose $d \geq 5$ and $q > q_d = \frac{2(d^2-2d-3)}{d^2-4d-1}$. Then there is a constant C_q so that for all $L > 0$*

$$\frac{1}{2L}\int_{-L}^{L} \big\|e^{it\sqrt{-\Delta}} f\big\|_q^q \, dt \leq C_q^q \big\|(I - L^2\Delta)^{\alpha/2} f\big\|_q^q \tag{1.3}$$

holds for $\alpha = \alpha(q) = d(1/2 - 1/q) - 1/2$.

We remark that this result can be strengthened further by using suitable Triebel-Lizorkin spaces, see §9. A similar phenomenon occurs for solutions of Schrödinger type equations, see [19].

A downside of our method is of course that it currently does not yield results in low dimensions. However when it does apply it is somewhat simpler than the induction on scales methods introduced by Wolff. We also remark that we do not improve on the current range of the abovementioned Wolff inequality for plate decompositions which has other applications and is interesting in its own right.

Structure of the paper. In §2 we explain the basic idea in the paper, which is that weak orthogonality properties may be combined with support size estimates to prove satisfactory L^p bounds. Here we also state a basic interpolation lemma which is related to the Marcinkiewicz theorem and will be used throughout the paper. The main section is §3 where we outline the proof of a discretized version of Theorem 1.1 for a fixed scale. A crucial L^2 estimate needed for this proof is done in §4. The characterization of L^p boundedness for radial multipliers that are compactly supported away from the origin is proved in §5. In §6 we give an important refinement of the earlier estimates, which is crucial for putting scales together. This is completed in §7 where the relevant atomic decomposition techniques are introduced and applied. The proof of Theorem 1.1 is concluded in §8. The last section §9 contains the proof of (a somewhat strengthened version of) Theorem 1.2.

Notation. For two quantities A and B we shall write $A \lesssim B$ if $A \le CB$ for some absolute positive constant C (depending on the dimension). We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$. The cardinality of a finite set $\mathcal{E}$ is denoted by $\#E$. The Lebesgue measure on $\mathbb{R}^d$ of a subset E will be denoted by $\text{meas}(E)$ or by $|E|$.

Acknowledgement. The authors would like to thank Gustavo Garrigós and Keith Rogers for useful comments on preliminary versions of this paper.

2. L^2 bounds versus support: A simple model case

Since we do not know how to exploit cancellations in L^p directly, we use the strategy of controlling the L^2 norm and the size of the support simultaneously to get our L^p bounds. We describe a simple model case for which we have some limited orthogonality, but not enough to prove a favorable L^2 bound.

Lemma 2.1. *Suppose we are given finite number of complex-valued L^2-functions $\{f_z\}$ indexed by $z \in \mathbb{Z}^d$ so that each function f_z is supported on a cube Q_z of sidelength 1. Suppose also that the family $\{f_z\}$ satisfies*

$$|\langle f_z, f_{z'}\rangle| \le (1 + |z - z'|)^{-\beta}, \tag{2.1}$$

for some $\beta \in (0, d)$. Then

$$\Big\| \sum_z a_z f_z \Big\|_p \le C_{\beta,p} \Big(\sum_z |a_z|^p \Big)^{1/p}, \quad p < \frac{2d}{2d - \beta}. \tag{2.2}$$

We remark that if (2.1) were assumed for some $\beta > d$, then inequality (2.2) would be true for $p = 2$ and the result would follow for $1 \le p \le 2$ by interpolation with the trivial $\ell^1 \to L^1$ bound. The assumption (2.1) for

$\beta < d$ is too weak to yield the $\ell^2 \to L^2$ bound. Instead we have to use some improved support properties when several of the intervals I_k overlap.

Proof of Lemma 2.1. We shall first prove a weaker (so-called restricted strong type) inequality that includes the endpoint; namely

$$\Big\| \sum_{z\in E} a_z f_z \Big\|_p \le C_\beta (\#E)^{1/p} \sup_z |a_z|, \quad p \le \frac{2d}{2d-\beta}. \tag{2.3}$$

We may assume that $\sup_z |a_z| = 1$. Let $x_z \in \mathbb{R}^d$ be the center of the cube Q_z of sidelength 1 supporting f_z. Split $\mathbb{R}^d$ into nonoverlapping cubes J of sidelength 1, put $E_J = \{z \in E : x_z \in J\}$ and define $u_J = \#E_J$ so that $\#E = \sum_J u_J$. Define $F_J = \sum_{z\in E_J} a_z f_z$ so that we have to bound the L^p norm of $\sum_J F_J$.

Now observe that at each point $x \in \mathbb{R}$, at most 3^d of the functions F_J can be non-zero simultaneously. Therefore

$$\Big\|\sum_J F_J\Big\|_p^p \le 3^{dp} \sum_J \|F_J\|_p^p .$$

Now, according to our weak orthogonality assumption about the functions f_z, we have

$$\begin{aligned}\|F_J\|_2^2 &\le \sum_{z\in E_J} \sum_{z'\in E_J} (1+|z-z'|)^{-\beta} \\ &\le \sum_{z\in E_J} \sum_{z':|z-z'|\le\sqrt{d}u_J^{1/d}} (1+|z-z'|)^{-\beta} \lesssim u_J^{2-\frac{\beta}{d}}.\end{aligned}$$

The measure of the support of F_J is at most 2^d and therefore, by Hölder's inequality, $\|F_J\|_p \lesssim \|F_J\|_2$. Hence

$$\Big\|\sum_J F_J\Big\|_p \lesssim \Big(\sum_J \|F_J\|_2^p\Big)^{1/p} \lesssim \Big(\sum_J u_J^{(2-\beta/d)p/2}\Big)^{1/p}$$

and if $(2-\beta/d)p/2 \le 1$ then the last expression is bounded by $(\sum_J u_J)^{1/p} \le (\#E)^{1/p}$. This yields (2.3).

The improved bound (2.2) can be deduced by using interpolation theorems for Lorentz spaces (see [22], ch. V): Consider the operator on sequences $\mathfrak{a} = \{a_z\}_{z\in\mathbb{Z}^d}$, given by $T[\mathfrak{a}] = \sum_z a_z f_z$. Then (2.3) states that T maps the Lorentz space $\ell^{p,1}$ to L^p, for $p \le 2d/(2d-\beta)$ and, by interpolation, one deduces the inequality (2.2) in the open range $p < 2d/(2d-\beta)$ □

We wish to give a direct proof of the last interpolation result based on a dyadic interpolation lemma, which will be frequently used in this paper. For closely related considerations see also the expository note [24] by Tao.

Lemma 2.2. *Let $0 < p_0 < p_1 < \infty$. Let $\{F_j\}$ be a sequence of measurable functions on a measure space $\{\Omega, \mu\}$, and let $\{s_j\}$ be a sequence of nonnegative numbers. Assume that, for all j, the inequality*

$$\|F_j\|_{p_\nu}^{p_\nu} \le 2^{jp_\nu} M^{p_\nu} s_j \tag{2.4}$$

holds for $\nu = 0$ and $\nu = 1$. Then for all $p \in (p_0, p_1)$ there is a constant $C = C(p_0, p_1, p)$ so that

$$\Big\| \sum_j F_j \Big\|_p^p \le C^p M^p \sum_j 2^{jp} s_j \tag{2.5}$$

There is an analogous statement for the case $p_0 = 0$ where the assumption (2.4) for $\nu = 0$ is replaced with $\operatorname{meas}(\{x : |F_j(x)| \ne 0\}) \le s_j$, and the conclusion (2.5) holds for $0 < p < p_1$.

To see how this is used to derive (2.2) from (2.3) we consider the sets of indices $E_j = \{z \in \mathbb{Z} : 2^{j-1} < |a_z| \le 2^j\}$ and define $F_j = \sum_{z \in E_j} a_z f_z$. Then $\|F_j\|_{L^p}^p \lesssim 2^{jp} \# E_j$ for all $p \in (0, 2d/(2d-\beta)]$, by (2.3). Thus Lemma 2.2 immediately yields $\| \sum_k a_z f_z\|_p^p = \| \sum_j F_j\|_p^p \lesssim_p \sum_j 2^{pj} \# E_j \lesssim \sum_j |c_k|^p$ for all $p < 2d/(2d-\beta)$.

Proof of Lemma 2.2. First, replacing F_j by $M^{-1}F_j$, we can reduce the statement to the case $M = 1$. Now let, for $n \in \mathbb{Z}$, denote by $E_{j,n}$ the set where $2^{j+n} \le |F_j| < 2^{j+n+1}$ and put $F_{j,n} = \chi_{E_{j,n}} F_j$. Then $F_j = \sum_{n \in \mathbb{Z}} F_{j,n}$. Observe that if a_j is any numerical sequence such that for every j, the absolute value of a_j either is 0 or belongs to $[2^j, 2^{j+1})$, then $|\sum_j a_j|^p \lesssim \sum_j |a_j|^p$. Applying this observation to $2^{-n} \sum_j F_{j,n}$, we see that for fixed n and x

$$\Big| \sum_j F_{j,n}(x) \Big| \lesssim \Big(\sum_j |F_{j,n}(x)|^p \Big)^{1/p}$$

and therefore

$$\Big\| \sum_j F_{j,n} \Big\|_p^p \lesssim \sum_j \|F_{j,n}\|_p^p \lesssim \sum_j 2^{(j+n)p} \operatorname{meas}\big(\{x : |F_j| \ge 2^{j+n}\}\big).$$

By Chebyshev's inequality,

$$\operatorname{meas}\big(\{x : |F_j| \ge 2^{j+n}\}\big) \le \min\{2^{-p_0 n}, 2^{-p_1 n}\} s_j\,.$$

Thus,

$$\Big\| \sum_j F_{j,n} \Big\|_p \lesssim 2^{-\sigma|n|/p} \Big(\sum_j 2^{jp} s_j \Big)^{1/p}$$

where $\sigma = \min\{p_1 - p, p - p_0\}$. We sum in n to get the statement of the lemma for the case $p_0 > 0$. The case $p_0 = 0$ is very similar and left to the reader. ∎

3. The main inequality

In this section we shall prove the main inequality of this paper, which turns out to be the key estimate for the case that our multiplier has compact support away from the origin; this application is discussed at the end of this section.

In what follows, we denote by σ_r the surface measure on the $(d-1)$-dimensional sphere of radius r centered at the origin. We shall denote by $\psi_\circ$ a fixed radial C^∞ function that is compactly supported in a ball of radius $(2d)^{-1}$ centered at the origin, and whose Fourier transform $\widehat{\psi_\circ}$ vanishes to high order (say $20d$) at the origin. We set $\psi = \psi_\circ * \psi_\circ$.

Consider a 1-separated set $\mathcal{Y}$ of points in $\mathbb{R}^d$ and a 1-separated set $\mathcal{R}$ of radii ≥ 1. Also set

$$\mathcal{R}_k = \mathcal{R} \cap [2^k, 2^{k+1}), \qquad k \geq 0.$$

For $y \in \mathcal{Y}$ and $r \in \mathcal{R}$ define

$$F_{y,r} = \sigma_r * \psi(\cdot - y). \tag{3.1}$$

Proposition 3.1. *Let $\mathcal{E}$ be a finite subset of $\mathcal{Y}\times\mathcal{R}$ and let $\mathcal{E}_k = \mathcal{E}\cap(\mathcal{Y}\times\mathcal{R}_k)$. Let $c : \mathcal{E} \to \mathbb{C}$ be a function satisfying $|c(y,r)| \leq 1$ for all $(y,r) \in \mathcal{E}$. Then for $p < p_d$*

$$\Big\| \sum_{(y,r)\in\mathcal{E}} c(y,r) F_{y,r} \Big\|_p^p \lesssim \sum_k 2^{k(d-1)}\, \#\mathcal{E}_k; \tag{3.2}$$

here the implied constant depends only on p, d and ψ.

Proposition 3.1 implies a stronger estimate, namely

Corollary 3.2. *For $F_{y,r}$ as in* (3.1)*, $(y,r) \in \mathcal{Y}\times\mathcal{R}$, and $p < p_d$,*

$$\Big\| \sum_{(y,r)} \gamma(y,r) F_{y,r} \Big\|_p \lesssim \Big(\sum_{y,r} |\gamma(y,r)|^p r^{d-1} \Big)^{1/p}. \tag{3.3}$$

Indeed let, for $j \in \mathbb{Z}$, denote by $\mathcal{E}^j$ the set of all $(y,r) \in \mathcal{Y}\times\mathcal{R}$ for which $2^{j-1} < |\gamma(y,r)| \leq 2^j$. By Proposition 3.1 we see that $\| \sum_{(y,r)\in\mathcal{E}^j} \gamma(y,r) F_{y,r} \|_p^p$ is dominated by $C_p^p 2^{jp} \sum_{(y,r)\in\mathcal{E}^j} r^{d-1}$ for all $p < p_d$, and the assertion follows from the proposition by the dyadic interpolation Lemma 2.2.

If γ has a tensor product structure, namely $\gamma(y,r) = \alpha(y)\beta(r)$ then the expression $\sum_{(y,r)\in\mathcal{E}} c(y,r) F_{y,r}$ can be interpreted as the convolution operator with kernel $\sum_r \beta_r \sigma_r * \psi$ acting on $f = \sum_y \alpha(y)\delta(\cdot - y)$. Here δ is the

Dirac measure at the origin. In §5 we shall show how by a simple averaging argument this model case implies the version of our theorem for radial multipliers compactly supported away from the origin.

We shall now outline the proof of Proposition 3.1 (leaving one part to the next section).

Estimates for scalar products. We are aiming at a good L^2 estimate for $\sum c_{y,r}F_{y,r}$ and make use of some (albeit weak) orthogonality property of the summands. This property is expressed by

Lemma 3.3. *For any choice of $r, r' > 1$ and $y, y' \in \mathbb{R}^d$*

$$\big|\langle F_{y,r}, F_{y',r'}\rangle\big| \lesssim \frac{(rr')^{\frac{d-1}{2}}}{(1+|y-y'|+|r-r'|)^{\frac{d-1}{2}}}. \tag{3.4}$$

Proof. Note that $\sigma_r = r^{-1}\sigma_1(r^{-1}\cdot)$, in the sense of measures, and that $\widehat{\sigma}_r(\xi) = r^{d-1}\widehat{\sigma}_1(r\xi)$. Next $\widehat{\sigma}_1(\xi) = B_d(|\xi|)$ where $B_d(s) = cs^{-(d-2)/2}J_{(d-2)/2}(s)$ (the usual Bessel function). Thus $|B_d(s)| \lesssim (1+|s|)^{-(d-1)/2}$ (see [22], ch. IV). Now $\widehat{\psi}$ is radial and we can write $\widehat{\psi}(\xi) = a(|\xi|)$ where a is rapidly decaying and a vanishes to high order at the origin. By Plancherel's theorem the scalar product $\langle F_{y,r}, F_{y',r'}\rangle$ is equal to a constant times

$$\begin{aligned}&\int \widehat{\sigma}_r(\xi)\widehat{\sigma}_{r'}(\xi)|\psi(\xi)|^2 e^{i\langle y'-y,\xi\rangle}d\xi \\ &= c\,(rr')^{d-1}\int B_d(r\rho)B_d(r'\rho)B_d(|y-y'|\rho)|a(\rho)|^2\rho^{d-1}d\rho\end{aligned}$$

The decay properties of B_d and the behavior of a imply that

$$\big|\langle F_{y,r}, F_{y',r'}\rangle\big| \lesssim \frac{(rr')^{\frac{d-1}{2}}}{(1+|y-y'|)^{\frac{d-1}{2}}}$$

which gives the claimed bound for the range $|r-r'| \le C(1+|y-y'|)$. But if $|r-r'| \gg (1+|y-y'|)$ then $F_{y,r}$ and $F_{y',r'}$ have disjoint supports. Thus in this case $\langle F_{y,r}, F_{y',r'}\rangle = 0$. The lemma is proved. $\square$

Remark 3.4. Taking into account the oscillation of the Bessel functions one can obtain the improved bound

$$|\langle F_{y,r}, F_{y',r'}\rangle| \le C_N (rr')^{\frac{d-1}{2}}(1+|y-y'|)^{-\frac{d-1}{2}}\sum_{\pm,\pm}\big(1+\big|r\pm r'\pm|y-y'|\big|\big)^{-N}.$$

We shall not use it in our proof.

The exponent $(d-1)/2$ in the denominator in (3.4) is too small to use orthogonality in a straightforward way; this is analogous to the weak orthogonality assumption in Lemma 2.1. However if we impose a suitable density assumption on the sets $\mathcal{E}_k$, then we can prove a satisfactory L^2 bound. To quantify this we give a definition.

Definition 3.5. Fix a parameter $h \in (1, d)$, and fix $R \geq 1$ and $u \geq 1$. Let E be a finite 1-separated subset of $\mathbb{R}^d \times [R, 2R)$. We say that E is of *h-density type* (u, R) if

$$\#(B \cap E) \leq 2(u + \rho^h)$$

for any ball B of radius $\rho \leq 2^5 R$.

We shall prove in section §4 the following L^2 inequality involving sets of h-density type with $h < (d-1)/2$. The proof will be based on Lemma 3.3.

Lemma 3.6. *Suppose $h < \frac{d-1}{2}$ and let $u \geq 1$. For each $k \geq 0$, let $\mathcal{E}_k \subset \mathcal{Y} \times \mathcal{R}_k$ be a set of h-density type $(u, 2^k)$. Assume $|c(y,r)| \leq 1$ for $(y,r) \in \mathcal{Y} \times \mathcal{R}$. Then*

$$\Big\| \sum_k \sum_{(y,r)\in\mathcal{E}_k} c(y,r) F_{y,r} \Big\|_2^2 \lesssim_h u^{\frac{2}{d+1}} \log(2+u) \sum_k 2^{k(d-1)} \#\mathcal{E}_k. \tag{3.5}$$

In order to use this estimate efficiently we shall need to decompose the sets $\mathcal{E}_k$ into subsets of h-density type $(u, 2^k)$ for various values of u.

Density decompositions of sets. Fix $h < d$. Assume that $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$ is a 1-separated set and let $\mathcal{E}_k = \mathcal{E} \cap (\mathcal{Y} \times \mathcal{R}_k)$ (*i.e.* only radii in $[2^k, 2^{k+1})$ are involved). We consider $u \in \mathcal{U} = \{2^\nu, \nu = 0, 1, 2, \dots\}$. For $u \in \mathcal{U}$, let

$$R_{k,u} = \min\{2^{k+5}, u^{1/h}\}$$

and let $B_{k,u}(y,r)$ be the ball (in $\mathbb{R}^{d+1}$) of radius $2R_{k,u}$ centered at $(y,r) \in \mathbb{R}^{d+1}$.

For $k \in \mathbb{N}$ and $u \in \mathcal{U}$, consider a maximal $R_{k,u}$-separated set $\Lambda_k(u) \subset \mathbb{R}^d \times [2^k, 2^{k+1})$. Let

$$\Lambda_k(u, \mathcal{E}) = \{(y,r) \in \Lambda_k(u) : \#\big(\mathcal{E}_k \cap B_{k,u}(y,r)\big) \geq u\}.$$

Let

$$\widehat{\mathcal{E}}_k(u) = \mathcal{E}_k \cap \bigcup_{(y,r)\in\Lambda_k(u,\mathcal{E})} B_{k,u}(y,r),$$

and

$$\mathcal{E}_k(u) = \widehat{\mathcal{E}}_k(u) \setminus \bigcup_{\substack{u'\in\mathcal{U}\\ u'>u}} \widehat{\mathcal{E}}_k(u').$$

Finally set $\mathcal{E}(u) = \bigcup_k \mathcal{E}_k(u)$.

Lemma 3.7. *The sets $\mathcal{E}(u)$ have the following properties.*

(i) $\mathcal{E} = \bigcup_{u\in\mathcal{U}} \mathcal{E}(u) = \bigcup_{u\in\mathcal{U}} \bigcup_{k\geq 0} \mathcal{E}_k(u)$ *and the unions are disjoint.*

(ii) $\mathcal{E}_k(u)$ *can be covered by* $\lesssim_d u^{-1}(\#\mathcal{E}_k)$ *balls of radius* $2R_{k,u}$ *each.*

(iii) If B *is any ball of radius* $R_{k,u}$ *containing at least* u *points of* $\mathcal{E}_k$*, then*

$$B \cap \mathcal{E}_k \subset \bigcup_{\substack{u\in\mathcal{U}\\ u'\geq u}} \mathcal{E}_k(u) = \widehat{\mathcal{E}}_k(u).$$

(iv) $\mathcal{E}_k(u)$ *is a set of* h*-density type* $(u, 2^k)$ *(i.e., for every* $(y, r) \in \mathbb{R}^d \times [2^k, 2^{k+1})$ *and every* $\rho \leq 2^{k+5}$ *the ball of radius* ρ *centered at* (y, r) *contains no more than* $2(u + \rho^h)$ *points in* $\mathcal{E}_k(u)$*.)*

Proof. In order to prove (i), it suffices to observe that $\widehat{\mathcal{E}}_k(2^0) = \mathcal{E}_k$ and $\widehat{\mathcal{E}}_k(u) = \varnothing$ when u is sufficiently large.

To prove (ii), first note that $\mathcal{E}_k(u)$ is empty if $u > \#\mathcal{E}_k$. Observe that the family of balls $B_{k,u}(y, r)$ (with $(y, r) \in \Lambda_k(u)$) has the covering number $\lesssim_d 1$. This implies that

$$\#\Lambda_k(u, \mathcal{E}) \lesssim_d u^{-1}\#\mathcal{E}_k,$$

so we may just use the balls $B_{k,u}(y, r)$ with $(y, r) \in \Lambda_k(u, \mathcal{E})$.

(iii) immediately follows from the observation that every ball of radius $R_{k,u}$ is contained in one of the balls $B_{k,u}(y, r)$ (of radius $2R_{k,u}$) with $(y, r) \in \Lambda_k(u)$.

It remains to prove (iv). Let B be a ball of radius ρ centered at (y, r). If $\#(B \cap \mathcal{E}) \leq 2u + 2\rho^h$, there is nothing to prove. Suppose that the opposite inequality holds. Let $\widetilde{u}$ be the smallest number in $\mathcal{U}$ that exceeds $u + \rho^h$. Then $\rho \leq \widetilde{u}^{1/h}$ and the ball B whose radius is $\rho \leq R_{k,\widetilde{u}} = \min\{2^{k+5}, \widetilde{u}^{1/h}\}$ contains at least $\widetilde{u}$ points of $\mathcal{E}$. Therefore, by (iii), $B \cap \mathcal{E}_k \subset \widehat{\mathcal{E}}_k(\widetilde{u})$ and (since $\widetilde{u} > u$) we conclude that in this case $B \cap \mathcal{E}_k(u) = \varnothing$. □

We now set

$$G_{u,k} = \sum_{(y,r)\in\mathcal{E}_k(u)} c(y, r)F_{y,r} \quad \text{and} \quad G_u = \sum_k G_{u,k}. \tag{3.6}$$

From the support properties of $\sigma_r * \psi$ it follows immediately that $G_{u,k}$ is supported in a set of measure $\lesssim 2^{k(d-1)}\#\mathcal{E}_{k,u}$, hence of measure $\lesssim 2^{k(d-1)}\#\mathcal{E}_k$. By the properties of $\mathcal{E}_{k,u}$ we get the following improved bound.

Lemma 3.8. *For $u \in \mathcal{U}$,*

$$\mathrm{meas}(\mathrm{supp}(G_{u,k})) \lesssim u^{-\frac{h-1}{h}} 2^{k(d-1)} \# \mathcal{E}_k.$$

Proof. Recall that $\mathcal{E}_{k,u}$ can be covered by $\lesssim u^{-1} \# \mathcal{E}_k$ balls of radius $2R_{k,u} \le 2 \cdot u^{1/h}$. Let $(y_\circ, r_\circ)$ be the center of one such ball. Then, for every pair (y, r) contained in this ball, the support of $c(y,r)\sigma_r * \psi(\cdot - y)$ is contained in the annulus of width not exceeding $4R_{k,u} + 1$ built on the sphere centered at $y_\circ$ of radius $r_\circ$. Note also that we can assume that $r_\circ \le 2^{k+7}$ because otherwise the ball centered at $(y_\circ, r_\circ)$ of radius $2R_{k,u}$ does not intersect $\mathcal{E}_k$ and we can just remove this ball from the covering. Also, note that the estimate for the width of the annulus does not exceed the estimate for the radius of the sphere it is built upon, so we can conclude that the volume of this annulus is $\lesssim_d 2^{k(d-1)} u^{1/h}$. Consequently the measure of the support of $G_{u,k}$ does not exceed $C u^{-1} u^{1/h} 2^{k(d-1)} \# \mathcal{E}_k$. □

We now combine the L^2 bound of Lemma 3.6 and the support bound of Lemma 3.8 to get an L^p bound; for later reference in §6 this is formally stated as

Lemma 3.9. *Suppose $d \ge 4$. Let G_u be as in (3.6) where the sets $\mathcal{E}_{k,u}$ are defined using the density decomposition of $\mathcal{E}_k$, with a parameter $h \in (1, \frac{d-1}{2})$. Let*

$$p_{h,d} := \frac{2(h-1)(d+1)}{(h-1)(d+1) + 2h}. \tag{3.7}$$

Let $p < p_{h,d}$ and let $\delta < \frac{h-1}{h}(\frac{1}{p} - \frac{1}{p_{h,d}})$. Then

$$\|G_u\|_p^p \le C_\delta u^{-\delta p} \sum_k 2^{k(d-1)} \# \mathcal{E}_k.$$

Proof. Since we assume $h < (d-1)/2$ we have by Lemma 3.6 that $\|G_u\|_2^2 \lesssim_p \log(2+u) u^{2/(d+1)} \sum_k 2^{k(d-1)} \# \mathcal{E}_k$. Combining this with the support bound of Lemma 3.8 we obtain

$$\begin{aligned} \|G_u\|_p^p &\le \big(\mathrm{meas}(\mathrm{supp}(G_u))\big)^{1-p/2} \|G_u\|_2^p \\ &\le \Big(\sum_k \mathrm{meas}(\mathrm{supp}(G_{u,k}))\Big)^{1-p/2} \|G_u\|_2^p \\ &\lesssim u^{-\frac{h-1}{h}(1-\frac{p}{2})} \big(\log(2+u) u^{\frac{2}{d+1}}\big)^{\frac{p}{2}} \sum_k 2^{k(d-1)} \# \mathcal{E}_k. \end{aligned} \tag{3.8}$$

This implies the conclusion of the lemma. □

In order to finish the proof of Proposition 3.1 we fix $p < \frac{2(d^2-2d-3)}{d^2-5} = p_d$. If we choose $h < \frac{d-1}{2}$ sufficiently close to $\frac{d-1}{2}$ then $p > p_{h,d}$ and thus the conclusion of Lemma 3.9 holds with a positive δ. Consequently we can sum the bounds for each $\|G_u\|_p$ in $u \in \mathcal{U}$ and the proof of Proposition 3.1 is complete.

4. Proof of Lemma 3.6

We are working with sets $\mathcal{E}_k \subset \mathcal{Y} \times \mathcal{R}_k$, which have the property that every ball of radius $\rho \le 2^{k+5}$ contains $\lesssim u + \rho^h$ points in $\mathcal{E}_k$. Let

$$G_k = \sum_{(y,r)\in\mathcal{E}_k} c(y,r) F_{y,r}$$

with $\|c\|_\infty \le 1$, and it is our task to estimate the L^2 norm of $\sum_k G_k$. We may break up this sum into ten separate sums, each with the property that k ranges over a 10-separated set of natural numbers. We shall assume this separation property in all sums involving a k-summation.

It will be convenient to avoid scalar products of expressions of G_k involving $k \lesssim \log(2+u)$. We therefore let $N(u)$ be the smallest integer larger than $10\log_2(2+u)$ and split the sum as $\sum_{k\le N(u)} + \sum_{k>N(u)} G_k$, and then apply the Cauchy-Schwarz inequality with respect to the first sum. We thus obtain

$$\Big\|\sum G_k\Big\|_2^2 \lesssim \log(2+u)\left[\sum_{k\le N(u)} \|G_k\|_2^2 + \Big\|\sum_{k>N(u)} G_k\Big\|_2^2\right]$$

$$\lesssim \log(2+u)\left[\sum_k \|G_k\|_2^2 + 2\sum_{k'>k>N(u)} \big|\langle G_{k'}, G_k\rangle\big|\right]. \tag{4.1}$$

We begin with estimating the double sum $\sum_{k'>k>N(u)} \big|\langle G_{k'}, G_k\rangle\big|$. In this sum we have various scalar products of $F_{y,r}$ with $F_{Y,R}$ where $r \le R2^{-5}$. Let us fix the pair (Y,R) and examine the sum of the absolute values of such scalar products when (y,r) runs over $\mathcal{E}_k$ with $2^k < R/4$. The scalar product $\langle F_{y,r}, F_{Y,R}\rangle$ can be different from 0 only if y lies in the annulus of width $2^{k+1}+2$ built upon the sphere of radius R centered at Y, moreover $2^k \le r < 2^{k+1}$. Now the set of all pairs $(y,r) \in \mathcal{Y}\times\mathcal{R}$ satisfying these conditions can be covered by $\lesssim R^{d-1}2^{-k(d-1)}$ balls (in $\mathbb{R}^{d+1}$) of radius 2^k. Each such ball can contain only $O(u + 2^{kh})$ pairs $(y,r) \in \mathcal{E}_k$ by our assumption on $\mathcal{E}_k$. For each such (y,r), the scalar product $\langle F_{y,r}, F_{Y,R}\rangle$ is $O(2^{k(d-1)/2})$ by

Lemma 3.3. Consequently, for fixed (Y, R),

$$\sum_{(y,r)\in\mathcal{E}_k} \big|\langle F_{y,r}, F_{Y,R}\rangle\big| \lesssim R^{d-1} 2^{-k(d-1)/2}(u + 2^{kh}).$$

and therefore (as $N(u) = 10\log_2(2+u)$)

$$\sum_{2^{N(u)}<2^k<R/4}\ \sum_{(y,r)\in\mathcal{E}_k} \big|\langle F_{y,r}, F_{Y,R}\rangle\big| \lesssim R^{d-1} \sum_{k>N(u)} 2^{-k(d-1)/2}(u + 2^{kh}) \lesssim R^{d-1};$$

here we used $h < (d-1)/2$ and summed a decaying geometric progression whose maximal term corresponds to $k = N(u) + 10$. Since $(d-1)/2 \geq 1/2$, we see that the geometric series cancels the large term u in the last displayed formula. Now it remains to sum this estimate over pairs (Y, R) to get the bound $\sum_k 2^{k(d-1)} \#\mathcal{E}_k$ for the sum of scalar products in (4.1).

Now that we have dealt with the interaction of incomparable radii, we can concentrate on estimating $\|G_k\|_2^2$ for each k separately. It is convenient to arrange the radii in intervals of length u^a, for some $a \in (0, 1/h)$, and then apply the estimates of Lemma 3.3 to scalar products arising from different intervals; we shall see later that that the choice of $a = 2/(d+1)$ is optimal.

Now let $I_{k,\mu} = [2^k + (\mu-1)u^a, 2^k + \mu u^a$ for $\mu = 1, 2, \dots$, and let $\mathcal{E}_{k,\mu}$ be the set of all $(y, r) \in \mathcal{Y} \times I_{k,\mu}$ that belong to $\mathcal{E}_k$. Set

$$G_{k,\mu} = \sum_{(y,r)\in\mathcal{E}_{k,\mu}} c(y, r)F_{y,r}.$$

We need to estimate the L^2 norm of $\sum_\mu G_{k,\mu}$. By splitting the μ sum into ten different sums we may assume that μ ranges over a 10-separated set and bound

$$\Big\|\sum_\mu G_{k,\mu}\|_2^2 \lesssim \sum_\mu \big\|G_{k,\mu}\big\|_2^2 + 2\sum_{\mu'>\mu} \big|\langle G_{k,\mu'}, G_{k,\mu}\rangle\big|.$$

Again, we shall first estimate the sum of the various scalar products, using strongly the assumption that the sets $\mathcal{E}_k$ are of h density type $(u, 2^k)$. We claim that

$$\sum_{\mu'>\mu} \big|\langle G_{k,\mu'}, G_{k,\mu}\rangle\big| \lesssim u^{1-a\frac{d-1}{2}} 2^{k(d-1)} \#\mathcal{E}_k. \tag{4.2}$$

To see this we pick again some pair $(Y, R) \in \mathcal{E}_{k,\mu'}$ and examine how it interacts with pairs in $\mathcal{E}_{k,\mu}$ where $\mu \leq \mu'-10$. Note that if (y, r) is such a pair for which the scalar product is non-zero, then we must have $|y - Y| \leq 2^{k+3}$ and, since $|r - R| \leq 2^{k+1}$, we conclude that $|(y, r) - (Y, R)| \leq 2^{k+4}$ in $\mathbb{R}^{d+1}$. Moreover $|r - R| \geq u^a$ and thus the sum of the scalar products in which the pair (Y, R) participates is

$$\lesssim 2^{k(d-1)} \sum_{\substack{(y,r)\in\mathcal{E}_k: \\ u^a\leq|(y,r)-(Y,R)|\leq 2^{k+5}}} |(y, r) - (Y, R)|^{-(d-1)/2}\,.$$

We split this sum into two parts, one involving the terms for which the distance between (y, r) and (Y, R) is $\geq u^{1/h}$ and one involving the terms for which this distance is between u^a and $u^{1/h}$. Note that if $u^{1/h} \leq T \leq 2^{k+5}$ then every ball of radius T centered at (Y, R) contains only $O(T^h)$ points. Since $h < (d-1)/2$ we obtain the uniform bound

$$\sum_{\substack{(y,r)\in\mathcal{E}_k:\\ u^{1/h}\leq|(y,r)-(Y,R)|\leq 2^{k+5}}} |(y,r)-(Y,R)|^{-(d-1)/2} \lesssim_h 1.$$

We also know that the ball of radius $u^{1/h}$ centered at (Y, R) contains at most $O(u)$ points in $\mathcal{E}_k$. This implies that

$$\sum_{\substack{(y,r)\in\mathcal{E}_k:\\ u^{a}\leq|(y,r)-(Y,R)|\leq u^{1/h}}} |(y,r)-(Y,R)|^{-(d-1)/2} \lesssim u\cdot u^{-a\frac{d-1}{2}}.$$

We combine these two estimates, add over all $(Y, R) \in \mathcal{E}_{k,\mu'}$ and then add over all μ'. Then the left hand side of (4.2) is

$$\lesssim u^{1-a\frac{d-1}{2}} 2^{k(d-1)} \sum_{\mu} \#\mathcal{E}_{k,\mu};$$

thus (4.2) follows.

We now estimate the L^2 norm of each $G_{k,\mu}$. For each $r \in \mathcal{R}_{k,\mu} := I_{k,\mu} \cap \mathcal{R}$ let

$$G_{k,\mu,r} = \sum_{(y,r)\in\mathcal{E}_k} c(y,r)F_{y,r}.$$

The conclusion of Lemma 3.3 is now too weak to give satisfactory results; instead we apply the Cauchy-Schwarz inequality with respect to r and use that the cardinality of $\mathcal{R}_{k,\mu}$ is $\lesssim u^a$. Thus

$$\|G_{k,\mu}\|_2^2 \lesssim u^a \sum_{r\in\mathcal{R}_{k,\mu}} \|G_{k,\mu,r}\|_2^2.$$

Now $G_{k,\mu,r}$ is the convolution of $\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}} c(y,r)\psi_\circ(\cdot - y)$ with $\sigma_r * \psi_\circ$. By the standard decay estimate for the Fourier transform of the surface measure on the unit sphere we have

$$|\widehat{\sigma}_r(\xi)| \leq r^{d-1}(1+r|\xi|)^{-\frac{d-1}{2}}$$

and since $\widehat{\psi}_\circ$ vanishes to high order at the origin we also have, for $r \geq 1$,

$$\|\widehat{\sigma}_r\widehat{\psi}_\circ\|_\infty \lesssim r^{(d-1)/2}.$$

Since $\mathcal{Y}$ is 1-separated and the support of ψ is contained in a ball of radius 1/2, we conclude that

$$\|G_{k,\mu,r}\|_2^2 \lesssim r^{d-1}\#\{y\in\mathcal{Y} : (y,r)\in\mathcal{E}_{k,\mu}\}$$

and thus

$$\sum_{\mu} \|G_{k,\mu}\|_2^2 \lesssim u^a \sum_{\mu} \sum_{r\in\mathcal{R}_{k,\mu}} \|G_{k,\mu,r}\|_2^2 \lesssim u^a 2^{k(d-1)} \#\mathcal{E}_k.$$

Combining this bound with (4.2) yields

$$\big\|G_k\big\|_2^2 \lesssim \big(u^a + u^{1-a\frac{d-1}{2}}\big) 2^{k(d-1)} \#\mathcal{E}_k.$$

The two terms balance if $a = 2/(d+1)$ and with this choice the previous bound becomes

$$\big\|G_k\big\|_2^2 \lesssim u^{\frac{2}{d+1}} 2^{k(d-1)} \#\mathcal{E}_k.$$

Now we use this to estimate the first term in (4.1) and combine with the earlier bound for the mixed terms in (4.1) to complete the proof of the lemma. □

5. Application to compactly supported multipliers

Now let m be a radial Fourier multiplier supported in $\{1/2 < |\xi| < 2\}$ and let $K = \mathcal{F}^{-1}[m]$; since K is radial we can also write $K = \kappa(|\cdot|)$ for suitable κ. We shall prove the estimate

$$\|K * f\|_p \lesssim \|K\|_p \|f\|_p, \quad 1 \le p < p_d. \tag{5.1}$$

Let $\eta_\circ$ be a radial Schwartz function whose Fourier transform is supported in $\{1/4 < |\xi| < 3\}$ so that $\widehat{\eta}_\circ(\xi) = 1$ on the support of m. Let $\psi_\circ$ be a radial C^∞ function with compact support in $\{|x| \le 10^{-1}\}$, with the property that $\widehat{\psi}_\circ$ and all its derivatives up to order $20d$ vanish at the origin, but $\widehat{\psi}_\circ(\xi) > 0$ on $\{1/4 \le |\xi| \le 4\}$. This is easy to achieve (take a radial function $\chi \in C_0^\infty$ so that $\widehat{\chi}(0) = 1$, then define $\psi_\circ = \lambda^d \Delta^{10d}[\chi(\lambda\cdot)]$, for a sufficiently large λ; here Δ denotes the Laplacian in $\mathbb{R}^d$.

Let $\eta = \mathcal{F}^{-1}[\widehat{\eta}_\circ(\widehat{\psi}_\circ)^{-2}]$. Then $K * f = \psi_\circ * K * \psi_\circ * g$ where $g = \eta * f$ and clearly $\|g\|_p \lesssim \|f\|_p$. We split $K = K_0 + K_\infty$ where $K(x) = K_0(x)$ if $|x| \le 1$ and $K(x) = K_\infty(x)$ if $|x| > 1$. Since K_0 is a bounded compactly supported function the operator of convolution with K_0 is clearly bounded on all L^p, $1 \le p \le \infty$, and therefore it suffices to show that the L^p norm of $\psi_\circ * K_\infty * \psi_\circ * g$ is controlled by $C\|K\|_p\|g\|_p$. We now write

$$K_\infty = \sum_{n=1}^{\infty} \int_0^1 \kappa(n+\tau)\sigma_{n+\tau} d\tau,$$

and set $\psi = \psi_\circ * \psi_\circ$. Let $Q_\circ = [0,1)^d$. Then $\|\psi_\circ * K_\infty * \psi_\circ * g\|_p$ is dominated by

$$\iint_{\substack{(\tau,w)\in\\ [0,1)\times Q_0}} \Big\| \sum_{n=1}^{\infty} \sum_{z\in\mathbb{Z}^d} g(z+w)\kappa(n+\tau)\sigma_{n+\tau} * \psi(\cdot - z - w)\Big\|_p d\tau dw$$

We set $\gamma(y,r)=g(y)\kappa(r)$, with $y\in\mathcal{Y}:=w+\mathbb{Z}^d$, $r\in\mathcal{R}:=\tau+\mathbb{N}$, and apply Corollary 3.2. It follows that the last displayed expression is bounded by

$$C\int_0^1\Big(\sum_{n=1}^\infty|\kappa(n+\tau)|^p(n+\tau)^{d-1}\Big)^{1/p}d\tau\int_{Q_\circ}\Big(\sum_z|g(z+w)|^p\Big)^{1/p}dw,$$

which after applying Hölder's inequality with respect to τ and w is bounded by $C\|K_\infty\|_p\|g\|_p\lesssim\|K\|_p\|f\|_p$. This establishes (5.1). □

6. A variant of Corollary 3.2 involving large radii

The following estimate for convolution operators with radial kernels will be used in conjunction with atomic decompositions to extend the one scale situation of §5 to the general case. The crucial feature is an exponential gain for large radii, which will be useful when putting different scales together. For $\nu\in\mathbb{Z}$, let $\mathcal{W}^\nu$ be the tiling of $\mathbb{R}^d$ with dyadic cubes of sidelength 2^ν, *i.e.*, the set of cubes of the form

$$[z_1 2^\nu,(z_1+1)2^\nu)\times\cdots\times[z_d 2^\nu,(z_d+1)2^\nu),\quad z=(z_1,\dots,z_d)\in\mathbb{Z}^d.$$

Proposition 6.1. *For $1<p<p_d$ there is $\varepsilon=\varepsilon(p)>0$ so that the following holds. Let K be a radial convolution kernel supported in $\{x:|x|>2^\ell\}$. For $s\in\mathbb{Z}$ let $K_s=2^{sd}K(2^s\cdot)$, $\psi_s=2^{sd}\psi(2^s\cdot)$. Let $\ell\geq 0$. Then*

$$\big\|\psi_s*K_s*g\big\|_p\lesssim\|K\|_p 2^{-\ell\varepsilon}\Big(\sum_{W\in\mathcal{W}^{\ell-s}}\mathrm{meas}(W)\,\|g\chi_W\|_\infty^p\Big)^{1/p}\tag{6.1}$$

We shall base the proof on the arguments in §3 and first prove a discretized version for the functions $F_{y,r}$ in (3.1); here (y,r) is taken from $\mathcal{Y}\times\mathcal{R}$ where $\mathcal{Y}$ is a 1-separated set of $\mathbb{R}^d$ and $\mathcal{R}$ is a 1-separated set of $\mathbb{R}^+$. We prove a variant of Corollary 3.2, which involves only radii $r\geq 2^\ell$. This corresponds to the case $s=0$ of the proposition.

Lemma 6.2. *For $p<p_d$, there is $\varepsilon=\varepsilon(p)>0$ so that for $\ell\geq 0$,*

$$\Big\|\sum_{\substack{(y,r)\in\mathcal{Y}\times\mathcal{R}\\ r\geq 2^\ell}}\gamma(y,r)F_{y,r}\Big\|_p\lesssim 2^{-\ell\varepsilon}2^{\ell d/p}\Big(\sum_r\sum_{W\in\mathcal{W}^\ell}\sup_{y\in\mathcal{Y}\cap W}|\gamma(y,r)|^p r^{d-1}\Big)^{1/p}.$$

Proof. Let $p_{h,d}$ be as in (3.7). Since $p_{h,d}\to p_d$ as $h\to\frac{d-1}{2}$ it suffices to prove the inequality for $p<p_{h,d}$, for a fixed $h\in(1,\frac{d-1}{2})$. We choose $\delta>0$ so that $\delta<\frac{h-1}{h}(\frac1p-\frac{1}{p_{h,d}})$.

For $j\in\mathbb{Z}$, $r\in\mathcal{R}$, let $\mathcal{W}^\ell(j,r)$ be the set of all $W\in\mathcal{W}^\ell$ for which $2^j\leq\sup_{x\in W}|\gamma(x,r)|<2^{j+1}$. For each $y\in\mathcal{Y}$ let $W(y)$ be the unique cube in $\mathcal{W}^\ell$ that contains y, and for each $j \in \mathbb{Z}$, let $\mathcal{E}_k(j)$ be the set of all $(y,r) \in \mathcal{Y} \times \mathcal{R}_k$ with the property that $W(y) \in \mathcal{W}^\ell(j,r)$.

We now apply the density decomposition of Lemma 3.7 to the sets $\mathcal{E}_k(j)$ and write $\mathcal{E}_k(j) = \sum_{u \in \mathcal{U}} \mathcal{E}_k(j,u)$ as in this lemma. Lemma 3.9 applied to the set $\cup_{k \geq \ell} \mathcal{E}_k(j,u)$ yields

$$\Big\| \sum_{\substack{(y,r)\in \\ \cup_{k\geq \ell}\mathcal{E}_k(j,u)}} \gamma(y,r) F_{y,r} \Big\|_p^p \lesssim_p u^{-\delta p} 2^{jp} \sum_{k \geq \ell} \sum_{(y,r)\in \mathcal{E}_k(j,u)} r^{d-1}. \tag{6.2}$$

Now we use that $\mathcal{E}_k(j,u)$ is of h-density type $(u, 2^k)$. Since $k \geq \ell$ this implies that for every $u \in \mathcal{U}$, every j, every $W \in \mathcal{W}^\ell$ and every $r \in [2^k, 2^{k+1})$, the slice $\mathcal{E}_k(j,u,W,r) := \{y \in \mathcal{Y} \cap W : (y,r) \in \mathcal{E}_k(j,u)\}$ contains no more than $2u + 2 \cdot 2^{\ell h}$ points. Also, since $\mathcal{Y}$ is 1-separated, the cardinality of each slice is $\lesssim 2^{\ell d}$. Therefore the right hand side of (6.2) is controlled by

$$\begin{aligned} &2^{jp} u^{-\delta p} \sum_{k\geq \ell} \sum_{r \in \mathcal{R}_k} r^{d-1} \sum_{W \in \mathcal{W}^\ell} \#\mathcal{E}_k(j,u,r,W) \\ &\lesssim 2^{jp} C(\ell,u) \sum_{k \geq \ell} \sum_{r\in\mathcal{R}_k} r^{d-1} \#\mathcal{W}^\ell(j,r), \end{aligned}$$

with $C(\ell,u) := u^{-\delta p} \min\{u + 2^{\ell h},\, 2^{\ell d}\}$. By interpolation (Lemma 2.2),

$$\begin{aligned} &\Big\| \sum_j \sum_{(y,r)\in \cup_{k\geq\ell}\mathcal{E}_k(j,u)} \gamma(y,r) F_{y,r} \Big\|_p^p \\ &\qquad \lesssim_p C(\ell,u) \sum_j 2^{jp} \sum_{k\geq \ell} \sum_{r\in\mathcal{R}_k} r^{d-1} \#\mathcal{W}^\ell(j,r) \\ &\qquad \lesssim_p C(\ell,u) \sum_{W\in\mathcal{W}^\ell} \sum_{r\in\mathcal{R}} r^{d-1} \sup_{y\in W} |\gamma(y,r)|^p. \end{aligned}$$

We sum geometric progressions to get $\sum_{u\in\mathcal{U}} C(\ell,u)^{1/p} \lesssim_p 2^{-\ell\varepsilon} 2^{\ell d/p}$, with $\varepsilon = \min\{(d-h)/p, \delta\}$. Hence

$$\Big\| \sum_j \sum_{\substack{(y,r)\in \\ \cup_{k\geq\ell}\mathcal{E}_k(j)}} \gamma(y,r) F_{y,r} \Big\|_p^p \lesssim_p 2^{-\ell\varepsilon p} \sum_{r\in\mathcal{R}} r^{d-1} \sum_{W\in\mathcal{W}^\ell} |W| \sup_{y\in W} |\gamma(y,r)|^p.$$

This proves the lemma. □

Proof of Proposition (6.1). By rescaling we can immediately reduce to the situation where $s = 0$. Then $|W| = 2^{\ell d}$ for all W involved. We can write

$$K(x) = \sum_{n \geq 2^\ell} \int_n^{n+1} \kappa(r) \sigma_r dr$$

where $\int |\kappa(r)|^p r^{d-1} dr = c\|K\|_p^p$. As in §3 we have

$$\psi * K * f = \int_{w\in Q_0} \int_{\tau=0}^{1} \sum_{z,n} \kappa(n+\tau) f(z+w) \psi * \sigma_{n+\tau}(\cdot - z - w)\, d\tau\, dw\,.$$

We fix w, τ and apply Lemma 6.2 to bound

$$\|\psi * K * f\|_p \lesssim 2^{-\ell\varepsilon} 2^{\ell d/p} \times$$
$$\iint_{Q_0\times[0,1]} \Big(\sum_{n\ge 2^\ell} \sum_{W\in\mathcal{W}^\ell} \sup_{z\in W\cap\mathbb{Z}^d} |\kappa(n+\tau) f(z+w)|^p (n+\tau)^{d-1} \Big)^{1/p} dw\, d\tau.$$

The double integral is equal to the product of

$$\int_0^1 \Big(\sum_{n\ge 2^\ell} |\kappa(n+\tau)|^p (n+\tau)^{d-1} \Big)^{1/p} d\tau$$

and

$$\int_{Q_0} \Big(\sum_{W\in\mathcal{W}^\ell} \sup_{z\in W\cap\mathbb{Z}^d} |f(z+w)|^p \Big)^{1/p} dw.$$

Now (6.1) for $s = 0$ follows quickly from applications of Hölder's inequality. □

7. Atomic decompositions and the proof of Theorem 1.1

The purpose of this chapter is to prove Theorem 1.1 for one particular Schwartz-function η whose Fourier transform is compactly supported away from the origin (for the extension to more general η see §8). We follow the presentation in §3.1 and introduce a radial Schwartz function $\eta_\circ$ such that $\widehat{\eta_\circ}$ is supported in $\{\xi : 1/2 < |\xi| < 2\}$ and satisfies

$$\sum_{s\in\mathbb{Z}} [\widehat{\eta_\circ}(2^{-s}\xi)]^2 = 1 \tag{7.1}$$

for all $\xi \neq 0$. Let $\psi_\circ$ be a C^∞ function compactly supported in $\{x : |x| \le 1/10\}$ such that $\widehat{\psi_\circ}$ does not vanish in $\{\xi : 1/4 \le |\xi| \le 4\}$ and such that $\widehat{\psi_\circ}$ does vanish to order $10d$ at the origin. Let $\psi = \psi_\circ * \psi_\circ$ and

$$\eta = \mathcal{F}^{-1}[\widehat{\eta_\circ}/\widehat{\psi}]. \tag{7.2}$$

We shall use this particular η in the assumption of our theorem; in other words, we shall assume that $\sup_{t>0} \|T_m[t^{d/p}\eta(t\cdot)]\|_p \le B < \infty$. For $s \in \mathbb{Z}$, let

$$H_s = \mathcal{F}^{-1}[\widehat{\eta}(\cdot) m(2^s \cdot)].$$

By our assumption,

$$\sup_{s\in\mathbb{Z}} \|H_s\|_p \le B. \tag{7.3}$$

Now let $K_s = 2^{sd} H_s(2^s \cdot)$, $\psi_s = 2^{sd}\psi(2^s \cdot) = 2^{sd}\psi_\circ * \psi_\circ(2^s \cdot)$, $\eta_s = 2^{sd}\eta(2^s \cdot)$. By (7.1), and our definitions, we have the decomposition

$$T_m f = \sum_s \psi_s * \psi_s * K_s * f_s$$

where

$$f_s = \eta_s * f. \tag{7.4}$$

We may assume that f_s is a Schwartz function whose Fourier transform is compactly supported away from the origin; this class is dense in $L^p(\mathbb{R}^d)$, $1 < p < \infty$. For those functions the sum in s is finite.

We shall work with atomic decompositions constructed from Peetre's maximal square-function (*cf.* [17], [25] and [20]), using ideas from work by Chang and Fefferman [4]. The nontangential version of Peetre's expression is

$$Sf(x) = \Big(\sum_s \sup_{|y| \le 10d \cdot 2^{-s}} |f_s(x+y)|^2 \Big)^{1/2}.$$

Then the L^p norm of Sf is controlled by $\|f\|_p$ if $1 < p < \infty$, and by the Hardy space (quasi-)norm $\|f\|_{H^p}$ if $p \le 1$. These statements follow for example from the Fefferman-Stein inequalities for the vector-valued Hardy-Littlewood maximal operator ([5]).

Put $\Psi_s = \psi_s * \psi_s$. The proof of L^p boundedness of T_m reduces to the inequality

$$\Big\| \sum_s \Psi_s * K_s * f_s \Big\|_p \lesssim B \|Sf\|_p, \quad 1 < p < p_d; \tag{7.5}$$

here we may assume that the sum in s is over a finite set of integers. In what follows, we will make several decompositions of the Schwartz functions f_s (involving even rough cutoffs) and the a priori convergence of various sums can be justified by using the rapid decay of the functions.

The cancellation of the functions ψ_s is crucial for the estimation of the left hand side in (7.5) and various similar expressions. A simple tool is the inequality

$$\Big\| \sum_s \psi_s * h_s \Big\|_p \le C \Big(\sum_s \|h_s\|_p^p \Big)^{1/p}, \quad 1 \le p \le 2, \tag{7.6}$$

with a constant C depending only on ψ. This is immediate from Plancherel's theorem for $p = 2$, trivial for $p = 1$ and true by interpolation for $1 < p < 2$. Inequality (7.6) is not enough to put the estimates for the various scales together, and in addition we have to use an "atomic decomposition" of each f_s which we now describe.

For fixed s, we tile $\mathbb{R}^d$ by the dyadic cubes of sidelength 2^{-s}; this family of cubes is denoted by $\mathcal{Q}^s$, and we write $L(Q) = -s$ if we want to indicate that the sidelength of a dyadic cube is 2^{-s}. For each integer j, we introduce the set $\Omega_j = \{x : Sf(x) > 2^j\}$. Let $\mathcal{Q}_j^s$ be the set of all dyadic cubes in $\mathcal{Q}^s$ with the property that $|Q \cap \Omega_j| \ge |Q|/2$ but $|Q \cap \Omega_{j+1}| < |Q|/2$. We also set

$$\Omega_j^* = \{x : M\chi_{\Omega_j}(x) > 100^{-d}\}$$

where M is the Hardy-Littlewood maximal operator. Ω_j^* is an open set containing Ω_j and $|\Omega_j^*| \lesssim |\Omega_j|$. We work with a Whitney decomposition $\mathcal{W}_j$ of Ω_j^* into dyadic cubes W. Specifically $\mathcal{W}_j$ is the set of all dyadic cubes W for which the 20-fold dilate of W is contained in Ω_j^* and W is maximal with respect to this property. We note that each $Q \in \mathcal{Q}_j^s$ is contained in a unique $W \in \mathcal{W}_j$. This is verified by showing that the 20-fold dilate Q^* of Q belongs to Ω_j^*. Indeed, $|Q^* \cap \Omega|/|Q^*| \ge 20^{-d}|Q \cap \Omega|/|Q| \ge 40^{-d}$; hence $Q^* \subset \Omega_j^*$.

We now define some building blocks that are analogous to the usual atoms; however they are not normalized, and, since we are mainly interested in L^p bounds for $p > 1$, we do not insist on cancellation. For each $W \in \mathcal{W}_j$, set

$$A_{s,W,j} = \sum_{\substack{Q \in \mathcal{Q}_j^s \\ Q \subset W}} f_s \chi_Q;$$

note that only terms with $L(W) + s \ge 0$ occur. We also need to consider "cumulative atoms", as any dyadic cube W can be a Whitney cube for several Ω_j^*. We set

$$A_{s,W} = \sum_{j : W \in \mathcal{W}_j} A_{s,W,j}.$$

Note that

$$f_s = \sum_W A_{s,W} = \sum_j \sum_{W \in \mathcal{W}_j} A_{s,W,j}.$$

The following observations about atomic decomposition are standard (see *e.g.* [4]), but included here for completeness.

Lemma 7.1. *For each $j \in \mathbb{Z}$ the following inequalities hold.*

(i)

$$\sum_{W \in \mathcal{W}_j} \sum_s \|A_{s,W,j}\|_2^2 \lesssim 2^{2j} \mathrm{meas}(\Omega_j).$$

(ii) There is a constant C_d so that for every assignment $W \mapsto s(W)$ defined on $\mathcal{W}_j$, and $0 \le p \le 2$,

$$\sum_{W \in \mathcal{W}_j} \mathrm{meas}(W) \|A_{s(W),W,j}\|_\infty^p \le C_d 2^{pj} \mathrm{meas}(\Omega_j).$$

Proof. Using the definitions of the atoms part (i) follows from the inequality

$$\sum_s \sum_{Q\in\mathcal{Q}_j^s} \|f_s\chi_Q\|_2^2 \lesssim 2^{2j}\text{meas}(\Omega_j).$$

To see this observe that $\text{meas}(Q\setminus\Omega_{j+1}) \geq \text{meas}(Q)/2$ for each $Q\in\mathcal{Q}_j^s$, and we also have $Q\subset\Omega_j^*$. We use this together with Fubini's theorem and see that the left hand side of the first inequality is bounded by

$$\sum_s \sum_{Q\in\mathcal{Q}_j^s} \text{meas}(Q)\|f_s\chi_Q\|_\infty^2 \leq \sum_s \sum_{Q\in\mathcal{Q}_j^s} 2\,\text{meas}(Q\setminus\Omega_{j+1})\,\|f_s\chi_Q\|_\infty^2$$
$$\leq 2\int_{\Omega_j^*\setminus\Omega_{j+1}} \sum_s \sup_{|y|\leq\sqrt{d}2^{-s}} |f_s(x+y)|^2 dx \leq 2\cdot 2^{2(j+1)}\text{meas}(\Omega_j^*)$$

which is $\lesssim 2^{2j}\text{meas}(\Omega_j)$.

Part (ii) of the lemma follows since

$$\|A_{s,W,j}\|_\infty \lesssim \sup_{\substack{Q\in\mathcal{Q}_j^s\\ Q\subset W}} \big|f_s\chi_Q\big| \leq \sup_{x\in\Omega^*\setminus\Omega_{j+1}} |Sf(x)| \leq 2^{j+1}$$

and $\sum_{W\in\mathcal{W}_j}|W| \leq |\Omega_j^*| \lesssim |\Omega_j|$. □

To establish (7.5) we need to verify the inequality

$$\Big\|\sum_s\sum_j\sum_{\ell\geq 0}\sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-s+\ell}} \Psi_s * K_s * A_{s,W,j}\Big\|_p \lesssim B\|Sf\|_p. \tag{7.7}$$

For each integer ℓ in this sum we split the convolution operator K_s into a short range and a long-range piece, $K_{s,\ell}^{\text{sh}}$ and $K_{s,\ell}^{\text{lg}}$. To define them we first look at the rescaled kernels H_s and set $H_{s,\ell}^{\text{sh}}(x) = H_s(x)$ if $|x|\leq 2^\ell$ and $H_{s,\ell}^{\text{sh}}(x) = 0$ if $|x| > 2^\ell$. Also $H_{s,\ell}^{\text{lg}}(x) = H_s(x) - H_{s,\ell}^{\text{sh}}$. Now set $K_{s,\ell}^{\text{sh}} = 2^{sd}H_{s,\ell}^{\text{sh}}(2^s\cdot)$ and $K_{s,\ell}^{\text{lg}} = 2^{sd}H_{s,\ell}^{\text{lg}}(2^s\cdot)$. Finally, we split the sum in (7.7) into two parts, replacing K_s by $K_{s,\ell}^{\text{sh}}$ and $K_{s,\ell}^{\text{lg}}$, respectively.

Now consider W with $L(W) = -s+\ell$ and note that the short range convolution $\psi_s * K_{s,\ell}^{\text{sh}} * A_{s,W,j}$ is supported in the quadruple dilate W^* of W; thus for fixed j, all these terms are supported in Ω_j^*. In order to prove the short range inequality

$$\Big\|\sum_s\sum_j\sum_{\ell}\sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-s+\ell}} \Psi_s * K_{s,\ell}^{\text{sh}} * A_{s,W,j}\Big\|_p \lesssim B\|Sf\|_p \tag{7.8}$$

for $p<2$, it suffices to show that for fixed j, and for $q\le 2$,

$$\Big\|\sum_s\sum_\ell\sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-s+\ell}}\Psi_s*K^{\rm sh}_{s,\ell}*A_{s,W,j}\Big\|_q^q\lesssim B^q2^{jq}\text{meas}(\Omega_j). \tag{7.9}$$

Indeed, by Lemma 2.2, inequality (7.9) implies that the left hand side of (7.8) is controlled for $p<2$ by $B^p\sum_j 2^{jp}\text{meas}(\Omega_j)\lesssim B^p\|Sf\|_p^p$.

Inequality (7.9) for $q<2$ follows from (7.9) for $q=2$ by Hölder's inequality. Here we use that the relevant expressions are supported in Ω_j^* and $|\Omega_j^*|\lesssim|\Omega_j|$. To prove (7.9) for $q=2$ we use a standard estimate for the Fourier transform of radial kernels $K=\int_0^\infty\kappa(r)\sigma_r dr$, namely,

$$\|\widehat K\widehat\psi\|_\infty\le C_p\|K\|_p=c\Big(\int_0^\infty|\kappa(r)|^pr^{d-1}dr\Big)^{1/p},\qquad p<\frac{2d}{d+1}. \tag{7.10}$$

Indeed using Bessel functions as in the proof of Lemma 3.3 one can estimate by Hölder's inequality

$$\begin{aligned}|\widehat K(\xi)|&=c'\int_0^\infty\kappa(r)r^{d-1}B_d(r|\xi|)dr\\ &\lesssim\Big(\int_0^\infty|\kappa(r)|^pr^{d-1}dr\Big)^{1/p}\Big(\int_0^\infty r^{d-1}(1+r|\xi|)^{-\frac{d-1}{2}p'}dr\Big)^{1/p'}\end{aligned}$$

and it is easy to see that the last $L^{p'}$ norm is $O(|\xi|^{-d/p'})$ provided that $p<2d/(d+1)$. The bound (7.10) follows since $\widehat\psi$ is a Schwartz function that vanishes to high order at 0.

We return to (7.9) for $q=2$. As $\Psi_s*K^{\rm sh}_{s,\ell}*A_{s,W,j}$ is supported in W^* and the W^* have bounded overlap, we can dominate the left hand side of the inequality by

$$\begin{aligned}&\sum_W\Big\|\sum_s\psi_s*\psi_s*K^{\rm sh}_{s,L(W)+s}*A_{s,W,j}\Big\|_2^2\\ &\lesssim\sum_W\sum_s\big\|\psi_s*K^{\rm sh}_{s,L(W)+s}*A_{s,W,j}\big\|_2^2\\ &\lesssim\sup_{s,\nu}\|\widehat{\psi_s}\widehat{K^{\rm sh}_{s,\nu}}\|_\infty^2\sum_{W\in\mathcal{W}_j}\sum_s\|A_{s,W,j}\|_2^2.\end{aligned}$$

Here we used (7.6) for $p=2$. Now by (7.10) the Fourier transform of $\psi_s*K^{\rm sh}_{s,\nu}$ has L^∞ norm $\lesssim\|H^{\rm sh}_{s,\nu}\|_p\lesssim\|H_s\|_p\le B$. Thus, by Lemma 7.1, (i), the last displayed quantity is $\lesssim B^22^{2j}|\Omega|$. This finishes the proof of (7.9).

We now turn to the long range estimate, that is

$$\Big\|\sum_s\sum_\ell\sum_j\sum_{\substack{W\in\mathcal{W}_j\\ L(W)=-s+\ell}}\psi_s*\psi_s*K^{\rm lg}_{s,\ell}*A_{s,W,j}\Big\|_p\lesssim B\|Sf\|_p. \tag{7.11}$$

We use the j-sum to combine the atoms into the cumulative atoms $A_{s,W}$, take out the ℓ-sum by Minkowski's inequality and use (7.6). Thus the left hand side of the previous inequality is dominated by a constant times

$$\sum_{\ell\ge 0}\Big(\sum_s \Big\|\psi_s * K^{\mathrm{lg}}_{s,\ell} * \sum_{W:L(W)=-s+\ell} A_{s,W}\Big\|_p^p\Big)^{1/p}. \tag{7.12}$$

Now $\|H^{\mathrm{lg}}_{s,\ell}\|_p \le \|H_s\|_p \le B$ and therefore Proposition 6.1 implies that, for fixed ℓ,

$$\Big\|\psi_s * K^{\mathrm{lg}}_{s,\ell} * \sum_{W:L(W)=-s+\ell} A_{s,W}\Big\|_p$$
$$b \lesssim 2^{-\ell\varepsilon} B\Big(\sum_{W:L(W)=-s+\ell} \mathrm{meas}(W)\,\|A_{s,W}\|^p_{L^\infty(W)}\Big)^{1/p}$$

for $p < p_d$, with some $\varepsilon = \varepsilon(p) > 0$. Now also use that for fixed s, W the functions $A_{s,W,j}$ live on disjoint sets (since the dyadic cubes of sidelength 2^{-s} are disjoint and each is in exactly one family $\mathcal{Q}^s_j$). Thus it follows that the expression (7.12) is

$$\lesssim B\sum_\ell 2^{-\ell\varepsilon}\Big(\sum_j\sum_{W\in\mathcal{W}_j}\mathrm{meas}(W)\|A_{\ell-L(W),W,j}\|^p_{L^\infty(W)}\Big)^{1/p}$$
$$\lesssim B\sum_\ell 2^{-\ell\varepsilon}\Big(\sum_j \mathrm{meas}(\Omega_j)\,2^{jp}\Big)^{1/p} \lesssim B\|Sf\|_p,$$

by part (ii) of Lemma 7.1. Thus we obtain (7.11). Finally, (7.7) follows from (7.8) and (7.11). This concludes the proof of the L^p boundedness of T_m, under the assumption (7.3). □

8. Conclusion of the proof

We still have to show the equivalence (1.1) for arbitrary choices of η. To this end we fix the radial multiplier m and consider the family Θ of all C^∞ functions compactly supported away from the origin such that the condition

$$\big\|\mathcal{F}^{-1}[\varphi m(t\cdot)]\big\|_p < \infty \tag{8.1}$$

holds. Note that if $\varphi \in \Theta$, then $\varphi(\lambda\cdot) \in \Theta$ for every $\lambda > 0$, moreover $\varphi(R\cdot) \in \Theta$ for every rotation R of $\mathbb{R}^d$ (here we use the fact that m is radial). Also if χ is any compactly supported C^∞ function then $\chi\varphi \in \Theta$, simply because χ is an $\mathcal{F}L^p$ multiplier. Finally if $\varphi_1, \varphi_2 \in \Theta$, then $\varphi_1 + \varphi_2 \in \Theta$.

Now assume that there exists at least one not identically zero function $\varphi_0 \in \Theta$. Let V be a non-empty open subset of $\mathbb{R}^{d+1}$ such that $|\varphi_0| > 0$ on V. Let φ be any other C^∞ function compactly supported away from the origin. For every $\xi \in \mathbb{R}^d \setminus \{0\}$, one can find a rotation R_ξ and a number $\lambda_\xi > 0$ such that $\lambda_\xi R_\xi \xi \in V$ or, equivalently, $\xi \in \lambda_\xi^{-1} R_\xi^{-1} V$. Then the open sets $\lambda_\xi^{-1} R_\xi^{-1} V$, $\xi \in \operatorname{supp}\varphi$, form a cover of $\operatorname{supp}\varphi$. Choose a finite subcover $\lambda_{\xi_j}^{-1} R_{\xi_j}^{-1} V$, $j = 1, \dots, n$, and put

$$\zeta = \sum_{j=1}^{n} \overline{\varphi_0(\lambda_{\xi_j} R_{\xi_j} \cdot)}\varphi_0(\lambda_{\xi_j} R_{\xi_j} \cdot)\,.$$

Note that $\zeta \in \Theta$ and $\zeta > 0$ on $\bigcup_{j=1}^{n} \lambda_{\xi_j}^{-1} R_{\xi_j}^{-1} V \supset \operatorname{supp}\varphi$. Hence, the function χ defined as φ/ζ on $\operatorname{supp}\varphi$ and 0 on $\mathbb{R}^d \setminus \operatorname{supp}\varphi$ is a C^∞ function with compact support, so $\varphi = \chi\zeta \in \Theta$.

Proof of Theorem 1.1, concluded. Let g be an arbitrary Schwartz function, then the condition $\sup_{t>0} \|T_m[t^{d/p} g(t\cdot)]\|_p < \infty$ is clearly necessary for L^p boundedness. Conversely, suppose that this condition is satisfied; it is equivalent to $\sup_{t>0} \|\mathcal{F}^{-1}[m(t\cdot)\widehat{g}]\|_p < \infty$. We may pick $\chi \in C^\infty$ with compact support in $\mathbb{R}^d \setminus \{0\}$ so that $\chi\widehat{g}$ is not identically 0. Since χ is a Fourier multiplier we see that $\chi\widehat{g} \in \Theta$. By the above consideration we also have $\widehat{\eta} \in \Theta$ where η is as in (7.2). But for this η the characterization is already proved and the L^p boundedness of T_m follows. □

9. The regularity result for the wave equation

In this section we shall prove Theorem 1.2. We first note that by a standard scaling argument it suffices to prove the inequality

$$\Big(\int_1^2 \big\|e^{it\sqrt{-\Delta}} f\big\|_q^q\, dt\Big)^{1/q} \lesssim \|(I-\Delta)^{\alpha/2} f\|_q. \tag{9.1}$$

Indeed let us first show how (1.3) follows assuming (9.1) (here $q < \infty$). We may assume by symmetry that in (1.3) we integrate over $[0, L]$. We then write

$$\Big(L^{-1}\int_0^L \|e^{it\sqrt{-\Delta}} f\|_q^q\, dt\Big)^{1/q} \le \sum_{n=1}^{\infty} \Big(L^{-1}\int_{2^{-n}L}^{2^{-n+1}L} \|e^{it\sqrt{-\Delta}} f\|_q^q\, dt\Big)^{1/q}$$
$$= \sum_{n=1}^{\infty} 2^{-n/q} \Big(\int_1^2 \|e^{iL2^{-n}s\sqrt{-\Delta}} f\|_q^q\, ds\Big)^{1/q} = \sum_{n=1}^{\infty} 2^{-n/q} (*)_n$$

where

$$(*)_n = \Big(\int_1^2 \int_{\mathbb{R}^d} \big|e^{is\sqrt{-\Delta}} f_{L,n}(L^{-1}2^n x)\big|^q dx\, ds\Big)^{1/q} \text{ and } f_{L,n}(y) = f(L2^{-n}y).$$

We change variables in x, apply (9.1), and then change variables again to see that

$$(*)_n \lesssim L^{d/q} 2^{-nd/q} \|(I-\Delta)^{\alpha/2} f_{L,n}\|_q = \|(I - 2^{-2n} L^2 \Delta)^{\alpha/2} f\|_q.$$

Now, by standard L^1 Fourier multiplier results, we have for $\alpha \ge 0$, $n \ge 0$,

$$\|(I - 2^{-2n} L^2 \Delta)^{\alpha/2} f\|_q \le C \|(I - L^2\Delta)^{\alpha/2} f\|_q$$

where C does not depend on L and n. Thus $(*)_n$ is bounded by the right hand side of (1.3), uniformly in $n \ge 1$, and therefore, as $q < \infty$, we can sum $\sum_{n=1}^{\infty} 2^{-n/q} (*)_n$.

Now we shall actually show an improvement of (9.1) where the Sobolev space on the right hand side is replaced by the larger (Besov-Triebel-Lizorkin) space $B^q_{\alpha,q} = F^q_{\alpha,q}$ and the spatial L^q space on the left hand side is replaced by the Triebel-Lizorkin space $F^q_{0,1}$ which is imbedded in L^q. We use arguments that are very similar to those in the proof of Theorem 1.1.

Theorem 9.1. *Suppose $d \ge 5$ and $q > \frac{2(d^2-2d-3)}{d^2-4d-1}$. Then*

$$\Big(\int_1^2 \|e^{it\sqrt{-\Delta}} f\|^q_{F^q_{0,1}} \, dt \Big)^{1/q} \lesssim \|f\|_{F^q_{\alpha,q}}, \quad \alpha = d(\frac12 - \frac1q) - \frac12.$$

If $\eta_\circ$ is as in (7.1) and if P_k is, say, defined by $\widehat{P_k f} = (\widehat{\eta}_\circ(2^{-k}\xi))^2 \widehat{f}$ for $k > 0$ and $P_0 = I - \sum_{k \ge 1} P_k$ then the inequality of the theorem can be expressed as

$$\Big(\int_1^2 \Big\| \sum_{k \ge 0} |P_k e^{it\sqrt{-\Delta}} f| \Big\|_q^q \, dt \Big)^{1/q} \lesssim \Big(\sum_{k \ge 0} 2^{k\alpha q} \|P_k f\|_q^q \Big)^{1/q}, \tag{9.2}$$

with $\alpha = d(\frac12 - \frac1q) - \frac12$. We remark that we have chosen k as our index for the dyadic frequency pieces instead of s, first, to distinguish it from the homogeneous expression ($s \in \mathbb{Z}$) used earlier and secondly to match it with the notation in §3; the term for large frequencies $\approx 2^k$ corresponds, after a rescaling, to the situation of Corollary 3.2 when the radii are taken in $[2^k, 2^{k+1}]$.

It will be convenient to dispose of the terms for $k = 0, 1$. Let χ_0 be a radial $C_0^\infty(\mathbb{R}^d)$ function so that $\chi_0(\xi) = 1$ for $|\xi| \le 1$ and $\chi_0(\xi) = 0$ for $|\xi| \ge 3/2$. One easily checks that $\chi_0(\xi/\lambda) e^{i|\xi|}$ is the Fourier transform of an L^1 function for any λ (with L^1 norm growing in λ for $\lambda \to \infty$). The contribution of the multiplier near the origin is handled by considering $m_\kappa(\xi) = (\chi_0(2^\kappa \xi) - \chi_0(2^{\kappa+1}\xi))(e^{i|\xi|} - 1)$. One bounds the derivatives of $m_\kappa(2^{-\kappa}\xi)$ for $\kappa > 0$ to see that the L^1 norm of $\mathcal{F}^{-1}[m_\kappa]$ is $O(2^{-\kappa})$.

Next, we describe a further reduction to an inequality involving spherical means (*cf.* (9.7), (9.6) below). This can be done in various ways. One way is to apply the method of stationary phase in conjunction with multiplier theorems. We give a more direct approach based on the principle that every radial function can be written as a superposition of spherical measures. As before we let σ_ρ denote the surface measure of the sphere of radius ρ.

Let ϑ be a C^∞-function on the real line supported in $(1/8, 8)$ so that $\vartheta(s) = 1$ on $(1/4, 4)$. For $k \ge 1$ let the convolution kernel K_k be defined by

$$\widehat{K_k}(\xi) = e^{i|\xi|}\vartheta(2^{-k}|\xi|).$$

Lemma 9.2. *For $k \ge 1$,*

$$K_k = 2^{k(d-1)/2} \int_{1/2}^{2} w_k(\rho)\sigma_\rho d\rho \,+\, E_k \tag{9.3}$$

where

$$\sup_k \int |w_k(\rho)| d\rho < \infty, \tag{9.4}$$

and, for any N,

$$\|E_k\|_1 \le C_N 2^{-kN}.$$

Proof. Straightforward integration by parts arguments show that K_k is rapidly decreasing away from the unit sphere; in fact

$$|K_k(x)| \le c_N 2^{-kN}(1 + |x|)^{-N} \qquad \text{if } |x| < 1/2 \text{ or } |x| > 2.$$

Thus if with $\mathcal{A} = \{x : 1/2 \le |x| \le 2\}$ we set $E_k := K_k(1 - \chi_{\mathcal{A}})$, it is clear that the error term E_k has an L^1 norm that is rapidly decreasing as $k \to \infty$. We now examine $K_k(x)$ for $1/2 \le |x| \le 2$. We use polar coordinates and then write an integral over the sphere S^{d-1} in terms of integrals over $d-2$ dimensional spheres perpendicular to x. We get

$$\begin{aligned}
(2\pi)^d K_k(x) &= \int_{\mathbb{R}^d} \vartheta(2^{-k}|\xi|) e^{i|\xi|} e^{i\langle \xi, x\rangle} d\xi \\
&= 2^{k(d-1)} \int_0^\infty \vartheta(2^{-k}s)(2^{-k}s)^{d-1} e^{is} \int_{S^{d-1}} e^{is|x|\langle \frac{x}{|x|}, \theta\rangle} d\sigma(\theta)\, ds \\
&= c_{d-2} 2^{k(d-1)} \int_{-1}^{1} 2^k \Theta(2^k(1 + \tau|x|))(1-\tau^2)^{\frac{d-3}{2}} d\tau,
\end{aligned}$$

where c_{d-2} is the surface measure of the unit sphere S^{d-2} and

$$\Theta(\sigma) = \int_0^\infty \vartheta(s) s^{d-1} e^{is\sigma} ds.$$

Clearly $\Theta \in \mathcal{S}(\mathbb{R})$. Since ϑ and therefore $\widehat{\Theta}$ is supported in $(1/8, 8)$ the function Θ is the derivative of order M of a Schwartz function Θ_M.

From the above formula it is clear that (9.3) holds with

$$w_k(\rho) = c_{d-2}(2\pi)^{-d}2^{k(d-1)/2}\int_{-1}^{1} 2^k\Theta(2^k(1+\tau\rho))(1-\tau^2)^{\frac{d-3}{2}}d\tau.$$

We now prove for $\beta \geq 1$, $1/2 \leq \rho \leq 2$,

$$\int_{-1}^{1}\beta\Theta(\beta(1+\tau\rho))(1-\tau^2)^{\frac{d-3}{2}}d\tau \leq C_N\beta^{-\frac{d-1}{2}}\frac{\beta}{(1+\beta|1-\rho|)^N} \tag{9.5}$$

for any $N > 1$. This immediately yields the uniform bound (9.4).

For the proof of (9.5) one first considers the case where $|1-\rho| \leq \beta^{-1}$. In this case the asserted bound $O(\beta^{-(d-3)/2})$ follows from a straightforward estimation using the rapid decay of Θ.

Now assume $|1-\rho| > \beta^{-1}$. We make a dyadic decomposition of the integral in terms of the distance to the boundary. Let $v \in C_0^\infty(\mathbb{R})$ so that $v(s) = 1$ if $|s| \leq 1/2$ and $v(s) = 0$ for $|s| > 3/4$ and let $v_{\beta,0}(s) = v(\beta(1-\tau^2))$ so that v is supported in a β^{-1} neighborhood of the boundary of $[-1,1]$. For $\ell \geq 1$ set $v_{\beta,\ell}(s) = v_{\beta,0}(2^{-\ell}(1-\tau^2)) - v_{\beta,0}(2^{1-\ell}(1-\tau^2))$ and split the integral as $\sum_{\ell=0}^\infty I_\ell^\beta(\rho)$ where

$$I_\ell^\beta(\rho) = \int_{-1}^{1}\beta\Theta(\beta(1+\tau\rho))v_{\beta,\ell}(\tau)(1-\tau^2)^{\frac{d-3}{2}}d\tau.$$

We note that $v_{\beta,\ell}$ is supported where $|\tau| \leq 1$ and $\mathrm{dist}(\tau,\pm 1) \approx 2^\ell/\beta$; in particular for $2^\ell \gg \beta$ the term $I_\ell^\beta(\rho)$ is identically zero. To estimate $I_0^\beta(\rho)$ we only use the bound $|(1-\tau^2)^{\frac{d-3}{2}}| \lesssim \beta^{-(d-3)/2}$ in the support of $v_{\beta,0}$, and the rapid decay of Θ. It is then easy to see that $I_0^\beta(\rho)$ is dominated by a constant times the right hand side of (9.5). Next, we consider the terms for $\ell > 0$, and we can integrate by parts to obtain

$$I_\ell^\beta(\rho) = (-1)^M\int_{-1}^{1}(\beta\rho)^{-M}\beta\Theta_M(\beta(1+\tau\rho))\big[v_{\beta,\ell}(\tau)(1-\tau^2)^{\frac{d-3}{2}}\big]^{(M)}d\tau.$$

The ρ^{-M} term is irrelevant as $\rho \approx 1$. We gain powers of $2^{-\ell}$ in this integration by parts but we also have to take into account the larger support of the integrand. A straightforward computation shows that $I_\ell^\beta(\rho)$ is bounded by $C_{M,N}2^{-\ell(M-(d-1)/2)}$ times the right hand side of (9.5). Choosing M large we may now sum in ℓ. □

We continue with the proof of (9.2). Let $K_{k,t} = t^{-d}K_k(t^{-1}\cdot)$ with K_k as in the lemma and observe that

$$P_k[e^{it\sqrt{-\Delta}}f] = P_k[K_{k,t} * f], \quad 1/2 \leq t \leq 2.$$

Now define

$$\mu_{k,t} = \int_{1/2}^2 w_k(\rho)\sigma_{\rho t} d\rho, \tag{9.6}$$

with w_k satisfying (9.4). In view of Lemma 9.2 it suffices to prove, for $q > q_d$, the estimate

$$\Big(\int_1^2 \Big\| \sum_{k=2}^\infty 2^{k\frac{d-1}{2}} \big|\mu_{k,t} * \psi_k * f_k\big|\Big\|_q^q dt\Big)^{1/q} \lesssim \Big(\sum_k \|f_k\|_q^q 2^{kq(d(\frac12-\frac1q)-\frac12)}\Big)^{1/q};$$

for all $\{f_k\}_{k=2}^\infty$ with $\widehat{f_k}$ supported in $\mathcal{A}_k := \{\xi : 2^{k-1} < |\xi| < 2^{k+1}\}$. Here ψ_k are suitably chosen so that $\psi_k = 2^{kd}\psi(2^k\cdot)$, $\psi = \psi_\circ * \psi_\circ$, $\psi_\circ$ supported in $\{|x| \le 10^{-1}\}$ with $10d$ vanishing moments (*cf.* the discussion leading to (7.2)).

It suffices to prove this inequality for families $\{f_k\}$ for which all but finitely many of the f_k are zero, with constant independent of the number of summands. By duality the desired bound then follows from

$$\Big(\sum_{k=2}^\infty 2^{k\frac{d}{p'}p}\Big\| \int_1^2 \mu_{k,t} * \psi_k * g_k(\cdot,t)\, dt\Big\|_p^p\Big)^{1/p} \lesssim \Big(\int_1^2 \big\|\sup_k |g_k(\cdot,t)|\big\|_p^p\, dt\Big)^{1/p}, \quad p < p_d, \tag{9.7}$$

for all $\{g_k\}_{k=2}^\infty$, with the property that the (spatial) Fourier transform of $g_k(\cdot,t)$ is supported in $\mathcal{A}_k$.

To prove (9.7) we need the following inequality for fixed k (which will be a straightforward consequence of Lemma 6.2). Let $\mathcal{W}^{\ell-k}$ denote the set of dyadic cubes of sidelength $2^{\ell-k}$.

Proposition 9.3. *For $1 \le p < p_d$ there is $\varepsilon = \varepsilon(p) > 0$ so that*

$$\Big\| \int_1^2 \psi_k * \mu_{k,t} * g(\cdot,t)\, dt\Big\|_p \lesssim 2^{-kd/p'} 2^{-\ell\varepsilon}\Big(\sum_{W\in\mathcal{W}^{\ell-k}} |W| \int_1^2 \sup_{y\in W} |g(y,t)|^p dt\Big)^{1/p}. \tag{9.8}$$

Proof. We first prove the inequality

$$\Big\| \int_1^2 \psi_k * \sigma_t * g(\cdot,t)\, dt\Big\|_p \lesssim 2^{-kd/p'} 2^{-\ell\varepsilon}\Big(\sum_{W\in\mathcal{W}^{\ell-k}} |W| \int_1^2 \sup_{y\in W} |g(y,t)|^p dt\Big)^{1/p}. \tag{9.9}$$

We apply a rescaling and averaging argument to deduce it from Lemma 6.2. Define $H_{k,t}$ by $\widehat{H_{k,t}}(\xi) = \widehat{\psi}(\xi)\widehat{\sigma}_1(2^k t\xi)$. The expression on the left hand side of (9.9) can be written as

$$\Big\| \int_1^2 2^{kd} H_{k,t}(2^k \cdot) * g(\cdot,t)\, t^{d-1} dt \Big\|_p = 2^{-kd/p} \Big\| \int_1^2 H_{k,t} * g(2^{-k}\cdot,t)\, t^{d-1} dt \Big\|_p$$
$$= 2^{-kd/p} \Big\| \int_{2^k}^{2^{k+1}} \psi * \sigma_r * 2^{-kd} g(2^{-k}\cdot, 2^{-k} r) dr \Big\|_p .$$

We write out the convolution and discretize as in the proof of Proposition 6.1. Then the last expression is dominated by

$$2^{-kd/p} \iint_{\substack{v\in Q_\circ\\ \tau\in[0,1]}} \Big\| \sum_{n=2^k}^{2^{k+1}-1} \psi*\sigma_{n+\tau}(x-z-v) 2^{-kd} g(2^{-k}(z+v), 2^{-k}(n+\tau)) \Big\|_p dv\, d\tau$$

and by Lemma 6.2 this is bounded by a constant times

$$2^{-kd/p} 2^{-\ell\varepsilon} 2^{\ell d/p} \times$$
$$\int_{\tau=0}^1 \Big(\sum_{W'\in\mathcal{W}^\ell} \sum_{n=2^k}^{2^{k+1}-1} \sup_{y'\in W'} |2^{-kd} g(2^{-k}y', 2^{-k}(n+\tau))|^p (n+\tau)^{d-1} \Big)^{1/p} d\tau,$$

which is dominated by

$$2^{-\ell\varepsilon} \Big(\sum_{W\in\mathcal{W}^{\ell-k}} |W| \sum_{n=2^k}^{2^{k+1}-1} \int_0^1 \sup_{y\in W} |g(y, 2^{-k}(n+\tau))|^p 2^{k(d-1-dp)} d\tau \Big)^{1/p}$$
$$\lesssim 2^{-\ell\varepsilon} 2^{-kd/p'} \Big(\sum_{W\in\mathcal{W}^{\ell-k}} |W| \int_1^2 \sup_{y\in W} |g(y,t)|^p dt \Big)^{1/p}.$$

It remains to show how (9.9) implies the assertion of the proposition. Since $\int |w_k(\rho)| d\rho$ is uniformly bounded it suffices, by averaging, to show the uniform bound

$$\Big\| \int_1^2 \psi_k * \sigma_{\rho t} * g(\cdot,t)\, dt \Big\|_p \lesssim 2^{-kd/p'} 2^{-\ell\varepsilon} \Big(\sum_{W\in\mathcal{W}^{\ell-k}} |W| \int_1^2 \sup_{y\in W} |g(y,t)|^p dt \Big)^{1/p}, \qquad \frac{1}{2} \le \rho \le 2. \tag{9.10}$$

This is a consequence of (9.9), by scaling. For the details assume $\rho \in (1,2]$ and after a change of variables we have to estimate the L^p norm of

$$\int_\rho^2 + \int_2^{2\rho} \big[\psi_k * \sigma_1 * g(\cdot, \rho^{-1} t) \big](x) \frac{dt}{\rho}.$$

We apply (9.9) with the function $g(\cdot,\rho^{-1}t)\chi_{[\rho,1]}(t)$ to bound the first integral. The second integral is equal to

$$\frac{2}{\rho}\int_1^\rho \big[\psi_k * \sigma_{2s} * g(\cdot, \tfrac{2s}{\rho})\big](x)ds = \frac{2^d}{\rho}\int_1^\rho \big[\psi_{k+1} * \sigma_s * g(2\cdot, \tfrac{2s}{\rho})\big]\big(\frac{x}{2}\big)ds$$

and after conjugation with a dilation operator we may apply (9.9) (with ψ_k replaced by ψ_{k+1}). Note that replacing $\mathcal{W}^{\ell-k}$ with $\mathcal{W}^{\ell-k-1}$ on the right hand side of (9.10) yields an equivalent norm. The argument for $\rho\in[1/2,1)$ is similar. $\square$

We now use the arguments of §7 based on atomic decompositions for the functions $g_k(\cdot,t)$, *for any fixed* $t\in[1,2]$. We work with the ℓ^∞ variant of Peetre's operator, namely

$$\mathcal{M}G(x,t) = \sup_{k>0}\sup_{|y|\le 10d\cdot 2^{-k}} |g_k(x+y,t)|,$$

where it will always be understood that $G=\{g_k\}_{k=1}^\infty$ and $g_k(\cdot,t)$ has spectrum in the annulus $\mathcal{A}_k$. Then with this specification Peetre's inequality says that

$$\big\|\mathcal{M}G(\cdot,t)\big\|_{L^p(\mathbb{R}^d)} \lesssim_p \|\sup_k |g_k(\cdot,t)|\|_p, \quad 0<p\le\infty. \tag{9.11}$$

As before, denote by $\mathcal{Q}^k$ the family of dyadic cubes in $\mathbb{R}^d$ of sidelength 2^{-k}. Now for each $t\in[1,2]$ let $\Omega_j(t)=\{x\in\mathbb{R}^d : \mathcal{M}f(x,t)>2^j\}$ and let $\mathcal{Q}_j^k(t)$ be the set of all dyadic cubes in $\mathcal{Q}^k$ with the property that $|Q\cap\Omega_j(t)|\ge |Q|/2$ but $|Q\cap\Omega_{j+1}(t)|<|Q|/2$. Moreover let $\Omega_j^*(t)=\{x : M\chi_{\Omega_j(t)}(x) > 10^{-d}\}$ where M is the Hardy-Littlewood maximal operator. We work with Whitney-cubes of $\Omega_j^*(t)$ and the set of these Whitney-cubes is denoted by $\mathcal{W}_j(t)$. For each W, j, t define

$$A_{k,W,j}(x,t) = \sum_{\substack{Q\in\mathcal{Q}_j^k(t)\\ Q\subset W}} g_k(x,t)\chi_Q(x);$$

and for each dyadic cube W we combine those atoms for which the appropriate Whitney cube is W; *i.e.*, we set

$$A_{k,W}(x,t) = \sum_{j: W\in\mathcal{W}_j(t)} A_{k,W,j}(x,t).$$

By Lemma 7.1, (ii), we have, for all $t\in[1,2]$,

$$\sum_{W\in\mathcal{W}_j(t)} |W|\|A_{k(W),W,j}(\cdot,t)\|_\infty^p \lesssim 2^{pj}|\Omega_j(t)|,$$

for any assignment $W\mapsto k(W)$. We can then decompose

$$g_k(x,t) = \sum_{\ell\ge 0}\sum_{W\in\mathcal{W}^{\ell-k}}\sum_{j: W\in\mathcal{W}_j(t)} A_{k,W,j}(x,t).$$

Using this decomposition and Minkowski's inequality we estimate the left hand side of (9.7) by

$$\sum_{\ell\ge 0}\Big(\sum_k 2^{k\frac{d}{p'}p}\Big\|\int_1^2 \mu_{k,t}*\psi_k*\sum_{W\in\mathcal{W}^{\ell-k}}\sum_{j:W\in\mathcal{W}_j(t)} A_{k,W,j}(\cdot,t)\,dt\Big\|_p^p\Big)^{1/p}$$

and by Proposition 9.3 the term corresponding to a fixed ℓ is

$$\lesssim 2^{-\ell\varepsilon}\Big(\sum_k\sum_{W\in\mathcal{W}^{\ell-k}}|W|\int_1^2\sup_{y\in W}\Big|\sum_{j:W\in\mathcal{W}_j(t)}A_{k,W,j}(y,t)\Big|^p dt\Big)^{1/p}.$$

For each fixed k, W, t the functions $y\mapsto A_{k,W,j}(y,t)$, $j\in\mathbb{Z}$, live on disjoint sets and therefore the last expression is

$$\lesssim 2^{-\ell\varepsilon}\Big(\int_1^2\sum_j\sum_{W\in\mathcal{W}_j(t)}|W|\|A_{\ell-L(W),W,j}(\cdot,t)\chi_W\|_\infty^p\,dt\Big)^{1/p}$$

$$\lesssim 2^{-\ell\varepsilon}\Big(\int_1^2\sum_j 2^{jp}|\Omega_j(t)|dt\Big)^{1/p}\lesssim 2^{-\ell\varepsilon}\Big(\int_1^2\big\|\sup_k\mathcal{M}_k g_k(\cdot,t)\big\|_p^p dt\Big)^{1/p}.$$

We sum in ℓ and use (9.11) to conclude the proof of (9.7). □

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