# ON POINTWISE CONVERGENCE OF SCHRÖDINGER MEANS

EVANGELOS DIMOU AND ANDREAS SEEGER

ABSTRACT. For functions in the Sobolev space $H^s$ and decreasing sequences $t_n \to 0$ we examine convergence almost everywhere of the generalized Schrödinger means on the real line, given by

$$S^a f(x, t_n) = \exp(it_n(-\partial_{xx})^{a/2})f(x);$$

here $a > 0$, $a \neq 1$. For decreasing convex sequences we obtain a simple characterization of convergence a.e. for all functions in $H^s$ when $0 < s < \min\{a/4, 1/4\}$ and $a \neq 1$. We prove sharp quantitative local and global estimates for the associated maximal functions. We also obtain sharp results for the case $a = 1$.

## 1. INTRODUCTION

For Schwartz functions $f$ defined on the real line consider the initial value problem

$$i\partial_t u(x, t) + (-\partial_{xx})^{a/2}u(x, t) = 0, \quad u(x, 0) = f(x);$$

so that for $a = 2$ we recover the Schrödinger equation. The solutions are given by

$$S^a f(x, t) = \int_{\mathbb{R}} e^{i(x\xi + t|\xi|^a)} \widehat{f}(\xi) \, \frac{d\xi}{2\pi},$$

and, for fixed time, the solution operator extends to all $f \in H^s$, where $H^s$ is the Sobolev space of all distributions $f$ with $\|f\|_{H_s} := (\int (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi)^{1/2} < \infty$.

One refers to the operators $f \mapsto S^a f(\cdot, t)$ as generalized Schrödinger means. For Schwartz functions $f$ it is clear that $\lim_{t \to 0} S^a f(x, t) = f(x)$ and that the convergence is uniform in $x$. One is interested in almost everywhere convergence for functions in $H^s$ for suitable $s > 0$. Following the fundamental result by Carleson [2], many authors have considered this question. It was shown in [2], [15] that

$$\lim_{t \to 0} S^a f(x, t) = f(x) \text{ a.e.}, \quad f \in H^{1/4},$$

when $a > 1$ and this result fails for some $f \in H^s$, if $s < 1/4$ ([5], [15]). If $0 < a < 1$, pointwise convergence for $f \in H^s$ holds when $s > a/4$ and may fail for $f \in H^s$ when $s < a/4$, see [21]. We remark that the problem in higher dimensions is much harder and not considered here. For the Schrödinger equation in higher dimensions a complete solution up to endpoints has been recently found in [8], [9] and relies on sophisticated methods from Fourier restriction theory. We refer to these papers for more references and a historical prospective.

In this paper, we consider, in one spatial dimension, the question of the solution converging to the initial data when the limit is taken over a decreasing sequence $\{t_n\}_{n=1}^{\infty}$, converging to zero. Here we always use the term 'decreasing' as synonymous with 'nonincreasing'. Given such a sequence we seek to find the precise range of $s$ such that $\lim_{n\to\infty} S^a f(x, t_n) = f(x)$ a.e. holds for every $f \in H^s$. This is partially motivated by the work [3] on approach regions for pointwise convergence for solutions of the Schrödinger equation, and also by the work [13] on the pointwise convergence of spherical means of $L^p$ functions (although the mathematical issues and expected outcomes for the latter problem are different).

For the class of convex decreasing sequences and any $s \in (0, \min\{a/4, 1/4\})$ we obtain a complete characterization of when pointwise convergence holds for all $f \in H^s$. This characterization involves the Lorentz space $\ell^{r,\infty}(\mathbb{N})$. By definition, for $0 < r < \infty$,

$$\{t_n\} \in \ell^{r,\infty} \iff \sup_{b>0} b^r \#\{n \in \mathbb{N} : |t_n| > b\} < \infty.$$

Note that $\ell^{r_1,\infty}(\mathbb{N}) \subset \ell^{r_2}(\mathbb{N}) \subset \ell^{r_2,\infty}(\mathbb{N}) \subset \ell^{\infty}(\mathbb{N})$ if $r_1 < r_2 < \infty$ and all inclusions are strict. A model example is given by $t_n = n^{-\gamma}$ which belongs to $\ell^{r,\infty}$ if and only if $r \geq 1/\gamma$. Another example is $\{n^{-\gamma} \log n\}$ which belongs to $\ell^{r,\infty}$ if and only if $r > 1/\gamma$.

**Theorem 1.1.** *Let $a > 0$, $a \neq 1$, and assume $0 < s < \min\{a/4, 1/4\}$. Let $\{t_n\}_{n=1}^{\infty}$ be a decreasing sequence with $\lim_{n\to\infty} t_n = 0$ and assume that $t_n - t_{n+1}$ is also decreasing. Then the following four statements are equivalent.*

*(a) The sequence $\{t_n\}$ belongs to $\ell^{r(s),\infty}(\mathbb{N})$, where $r(s) = \frac{2s}{a-4s}$.*

*(b) There is a constant $C_1$ such that for all $f \in H^s$ and for all sets $B$ of diameter at most 1 we have*

$$\| \sup_{n\in\mathbb{N}} |S^a f(x, t_n)| \|_{L^2(B)} \leq C_1 \|f\|_{H^s}.$$

*(c) There is a constant $C_2$ such that for all $f \in H^s$, for all sets $B$ of diameter at most 1, and for all $\alpha > 0$,*

$$\text{meas}(\{x \in B : \sup_{n\in\mathbb{N}} |S^a f(x, t_n)| > \alpha\}) \leq C_2\, \alpha^{-2} \|f\|_{H^s}^2.$$

*(d) For every $f \in H^s$ we have*

$$\lim_{n\to\infty} S^a f(x, t_n) = f(x) \quad a.e.$$

Here and in what follows we write meas$(A)$ for the Lebesgue measure of $A \subset \mathbb{R}$. The equivalence of (b) and (c) seems nontrivial, and we do not have a direct proof for it, without going through condition (a). In Theorem 1.1 the convexity assumption can be dropped for the sufficiency, i.e. statements (b), (c), (d) hold whenever $t_n$ is decreasing and belongs to $\ell^{\frac{2s}{a-4s},\infty}(\mathbb{N})$, see Proposition 2.3 below.

Regarding the maximal function inequalities we also have a global version:

**Theorem 1.2.** *Let $a > 0$, $a \neq 1$, and assume $0 < s < a/4$. Let $\{t_n\}_{n=1}^{\infty}$ be a decreasing sequence with $\lim_{n\to\infty} t_n = 0$, and assume that $t_n - t_{n+1}$ is also decreasing. Then the following statements (a), (b), (c) are equivalent.*

*(a) The sequence $\{t_n\}$ belongs to $\ell^{\frac{2s}{a-4s},\infty}(\mathbb{N})$.*

(b) There is a constant $C_1$ such that for all $f \in H^s$ we have

$$\|\sup_{n \in \mathbb{N}} |S^a f(x, t_n)|\|_{L^2(\mathbb{R})} \leq C_1 \|f\|_{H^s}.$$

(c) There is a constant $C_2$ such that for all $f \in H^s$ and all $\alpha > 0$,

$$\text{meas}(\{x \in \mathbb{R} : \sup_{n \in \mathbb{N}} |S^a f(x, t_n)| > \alpha\}) \leq C_2 \, \alpha^{-2} \|f\|_{H^s}^2.$$

We contrast the above results with the exceptional case $a = 1$ which covers solutions of the wave equation. Now the critical $r(s) = \frac{2s}{a-4s}$ in Theorem 1.1 has to be replaced with the smaller $\frac{2s}{1-2s}$, for all $s < 1/2$. Notice that $S^1$ corresponds to a family of translation operators, when acting on functions with spectrum in $[0, \infty)$ or $(-\infty, 0]$. The analysis is somewhat similar to the one for spherical means in [13], see also [12]. For $a = 1$ we have

**Theorem 1.3.** *Let* $0 < s < 1/2$ *and let* $\{t_n\}_{n=1}^\infty$ *be a decreasing sequence with* $\lim_{n \to \infty} t_n = 0$ *such that* $t_n - t_{n+1}$ *is also decreasing. Then the following four statements are equivalent.*

(a) *The sequence* $\{t_n\}$ *belongs to* $\ell^{\rho(s), \infty}(\mathbb{N})$, *where* $\rho(s) = \frac{2s}{1-2s}$.

(b) *There is a constant* $C_1$ *such that for all* $f \in H^s$ *we have*

$$\|\sup_{n \in \mathbb{N}} |S^1 f(x, t_n)|\|_{L^2(\mathbb{R})} \leq C_1 \|f\|_{H^s}.$$

(c) *There is a constant* $C_2$ *such that for all* $f \in H^s$, *for all sets* $B$ *of diameter at most 1, and for all* $\alpha > 0$,

$$\text{meas}(\{x \in B : \sup_{n \in \mathbb{N}} |S^1 f(x, t_n)| > \alpha\}) \leq C_2 \, \alpha^{-2} \|f\|_{H^s}^2.$$

(d) *For every* $f \in H^s$ *we have*

$$\lim_{n \to \infty} S^1 f(x, t_n) = f(x) \quad a.e.$$

The convexity condition is satisfied for the model case $t_n = n^{-\gamma}$ with $\gamma > 0$ and thus Theorems 1.1, 1.3 and the known results for $s = 1/4$, when $a > 1$, yield

**Corollary 1.4.** *Let* $0 < \gamma < \infty$.
(i) *If* $a > 1$, *then* $\lim_{n \to \infty} S^a f(x, n^{-\gamma}) = f(x)$ *a.e. holds for every* $f \in H^s(\mathbb{R})$ *if and only if* $s \geq \min\{\frac{a}{2\gamma+4}, \frac{1}{4}\}$.
(ii) *If* $0 < a < 1$, *then* $\lim_{n \to \infty} S^a f(x, n^{-\gamma}) = f(x)$ *a.e. holds for every* $f \in H^s(\mathbb{R})$ *if and only if* $s \geq \frac{a}{2\gamma+4}$.
(iii) *If* $a = 1$, *then* $\lim_{n \to \infty} S^1 f(x, n^{-\gamma}) = f(x)$ *a.e. holds for every* $f \in H^s(\mathbb{R})$ *if and only if* $s \geq \frac{1}{2\gamma+2}$.

This answer for the sequence $\{n^{-\gamma}\}$ reveals a perhaps surprising phenomenon for the case $a > 1$, namely that there is a gain over the general pointwise convergence result when $\gamma > 2(a - 1)$, but not when $0 < \gamma \leq 2(a - 1)$. When $0 < a \leq 1$, we have for all $\gamma \in (0, \infty)$ a gain over the general convergence result. The same remarks apply to the local $H^s \to L^2(B)$ maximal inequality. In contrast we get for the global maximal operator and $a \neq 1$:

**Corollary 1.5.** *Let $0 < \gamma < \infty$ and $a \in (0, \infty) \setminus \{1\}$. Then the global maximal function inequality*

$$\| \sup_n |S^a f(x, n^{-\gamma})| \|_{L^2(\mathbb{R})} \leq C \|f\|_{H^s}$$

*holds for some $C > 0$ and all $f \in H^s$ if and only if $s \geq \frac{a}{2\gamma + 4}$.*

*Remarks.* (i) Our results for the special case $\{n^{-\gamma}\}$, $a \neq 1$ as stated in Corollaries 1.4, 1.5 were already incorporated in the 2016 thesis [7] of the first named author. Moreover sufficiency in Theorem 1.1, merely for decreasing sequences but under the more restrictive assumption $\{t_n\} \in \ell^r$ for $r < \frac{2s}{a - 4s}$, follows already from Proposition 1.6 in [7].

(ii) The problem of convergence of Schrödinger means $S^a(f, t_n)$ for a decreasing sequence $\{t_n\}$ was independently considered in recent papers by Sjölin [17] and by Sjölin and Strömberg [18]. Their conditions are more restrictive, but apply in all dimensions. In [17] it is proved for $a > 1$ that the condition $\{t_n\} \in \ell^{2s/a}$ is sufficient for pointwise convergence. This is improved in [18] where for $s \leq 1/2$, $a > 2s$, the condition $\{t_n\} \in \ell^r$ for $r < \frac{2s}{a - 2s}$ is shown to be sufficient for pointwise convergence. Proposition 2.3 yields an improvement of these results and Theorem 1.1 gives the optimal result for decreasing convex sequences.

(iii) For $a \neq 1$ there are natural analogous open questions of necessary and sufficient conditions in higher dimensions, given the recent groundbreaking results for the full local Schrödinger maximal operator in [8], [9] which are sharp up to endpoints.

(iv) For $0 < a < 1$ there is still the open problem whether $S^a f(x, t) \to f(x)$ a.e. holds for all $f \in H^{a/4}(\mathbb{R})$. Likewise there is the problem of a global bound for the maximal function if $s = a/4$, and $a > 1$. One can show using a variant of the arguments in [21], [16] that a.e. convergence holds in the Besov space $B_{2,1}^{a/4}(\mathbb{R})$ which is properly contained in $B_{2,2}^{a/4} \equiv H^{a/4}$, see Proposition 2.4. For the case $a = 1$ we have pointwise convergence in $B_{2,1}^{1/2}(\mathbb{R})$, but pointwise convergence fails for some functions in $H^{1/2}(\mathbb{R})$, see Proposition 4.3.

*This paper.* In §2 we show for decreasing sequences that the $\ell^{r(s), \infty}$ condition is sufficient for pointwise convergence and the appropriate boundedness properties of the maximal operators. The necessity for decreasing convex sequences (converging to 0) is proved in §3. The case $a = 1$ is separately considered in §4. In §5 we include a short appendix regarding the relevant application of Stein-Nikishin theory.

## 2. Upper bounds for maximal functions

In the present section we prove maximal function results which imply the positive results of the theorems stated in the introduction. We already know the local estimate

$$(2.1) \qquad \left\| \sup_{t \in [0,1]} |S^a f(\cdot, t)| \right\|_{L^2(B)} \leq C \|f\|_{H^{1/4}},$$

which was established by Kenig and Ruiz [11] when $a = 2$ and Sjölin [15] for general $a > 1$. In view of (2.1) it now suffices to give the proof of the $L^2(\mathbb{R})$ bound in part (b) of Theorem 1.2, under the assumption of $\{t_n\} \in \ell^{\frac{2s}{a-4s}}$, whenever $s < a/4$.

Throughout this section we assume that $\{t_n\}$ is decreasing but we drop the convexity assumption in the introduction. Without loss of generality (dropping a finite number of terms in the sequence) we can assume that $t_n \in (0, 1)$ for all $n \in \mathbb{N}$. We first restrict our attention to the frequency localized operator

$$S_\lambda^a f(x, t) = \int_{\mathbb{R}} e^{i(x\xi + t|\xi|^a)} \widehat{f}(\xi) \chi(\xi/\lambda) \, \tfrac{d\xi}{2\pi},$$

where $\chi \in C^\infty$ is a real-valued, smooth function, supported in $\{1/2 \le |\xi| \le 1\}$. The following result is a variant of the inequality given in [11]:

**Proposition 2.1.** *If $J \subseteq [0, 1]$ is an interval and $0 < a \ne 1$, then*

$$\big\| \sup_{t \in J} |S_\lambda^a f(\cdot, t)| \big\|_{L^2(\mathbb{R})} \le C(1 + |J|^{1/4} \lambda^{a/4}) \|f\|_2.$$

*Proof.* We use the Kolmogorov-Seliverstov-Plessner method, by linearizing the maximal operator: let $x \mapsto t(x)$ be a measurable function, with values in $J$. It will then suffice to prove

$$\left( \int_{\mathbb{R}} |S_\lambda^a(x, t(x))|^2 \, dx \right)^{1/2} \le C(1 + |J|^{1/4} \lambda^{a/4}) \|f\|_2,$$

where the constant $C$ is independent of $t(\cdot)$ and $f$. Notice that

$$S_\lambda^a f(x, t(x)) = \int e^{i(x\xi + t(x)|\xi|^a)} \widehat{f}(\xi) \chi(\xi/\lambda) \, \tfrac{d\xi}{2\pi} = \lambda \, T_\lambda^a [\widehat{f}(\lambda \cdot)](x)$$

where

$$T_\lambda^a g(x) = \int e^{i(\lambda x\xi + \lambda^a t(x)|\xi|^a)} \chi(\xi) g(\xi) \, \tfrac{d\xi}{2\pi}.$$

Since $\|\widehat{f}(\lambda \cdot)\|_2 = c\lambda^{-1/2} \|f\|_2$ we need to show that

$$\|T_\lambda^a\|_{L^2 \to L^2} \lesssim \lambda^{(a-2)/4} |J|^{1/4} + \lambda^{-1/2},$$

which in turn follows from

(2.2) $$\|T_\lambda^a (T_\lambda^a)^*\|_{L^2 \to L^2} \lesssim \lambda^{(a-2)/2} |J|^{1/2} + \lambda^{-1}.$$

The kernel of $T_\lambda^a (T_\lambda^a)^*$ is

$$K_\lambda^a(x, y) = \int e^{i[\lambda(x-y)\xi + \lambda^a(t(x) - t(y))|\xi|^a]} \chi^2(\xi) \, \tfrac{d\xi}{2\pi}.$$

and the derivative of the phase $\Phi_\lambda^a(\xi) = \lambda(x - y)\xi + \lambda^a(t(x) - t(y))|\xi|^a$ is equal to

$$(\Phi_\lambda^a)'(\xi) = \lambda(x - y) + a\lambda^a(t(x) - t(y)) \, (\text{sign } \xi) \, |\xi|^{a-1}.$$

Therefore, if $|x - y| \gg \lambda^{a-1} |t(x) - t(y)|$, we have that $|(\Phi_\lambda^a)'(\xi)| \gtrsim \lambda|x - y|$ and integration by parts gives

$$|K_\lambda^a(x, y)| \lesssim_N (\lambda|x - y|)^{-N}.$$

In the case where $|x - y| \lesssim \lambda^{a-1} |t(x) - t(y)|$ we use van der Corput's lemma. The second derivative of the phase is $(\Phi_\lambda^a)''(\xi) = c_a \lambda^a(t(x) - t(y))|\xi|^{a-2}$, hence

$$|K_\lambda^a(x, y)| \lesssim \lambda^{-a/2} |t(x) - t(y)|^{-1/2} \lesssim (\lambda|x - y|)^{-1/2}.$$

Thus $\int_{\mathbb{R}} |K_\lambda^a(x,y)|\, dy$ can be estimated by

$$\int_{\substack{|x-y|\lesssim \\ \lambda^{a-1}|t(x)-t(y)|}} \lambda^{-1/2}|x-y|^{-1/2}dy + \int_{\substack{|x-y|\gg \\ \lambda^{a-1}|t(x)-t(y)|}} (1+\lambda|x-y|)^{-N}dy$$

$$\leq \int_{|x-y|\lesssim\lambda^{a-1}|J|} \lambda^{-1/2}|x-y|^{-1/2}dy + \int_{\mathbb{R}} (1+\lambda|x-y|)^{-N}dy$$

$$\lesssim \lambda^{(a-2)/2}|J|^{1/2} + \lambda^{-1}.$$

Therefore $\sup_{x\in\mathbb{R}} \int |K_\lambda^a(x,y)|\, dy \lesssim \lambda^{(a-2)/2}|J|^{1/2} + \lambda^{-1}$ and by symmetry we get the same bound for $\sup_{y\in\mathbb{R}} \int |K_\lambda^a(x,y)|\, dx$. Hence Schur's test gives the required bound (2.2). $\qquad\square$

We now use Proposition 2.1 to prove a sharp result for the frequency-localized operaors $S_\lambda^a$.

**Lemma 2.2.** *Let $0 < a \neq 1$, $0 < r < \infty$ and let $\{t_n\}$ be a sequence in $[0,1]$ which belongs to $\ell^{r,\infty}$. Then for $\lambda > 1$*

$$\left\| \sup_n |S_\lambda^a f(\cdot, t_n)| \right\|_{L^2(\mathbb{R})} \leq C\lambda^{\frac{ar}{2+4r}} \|f\|_{L^2(\mathbb{R})}.$$

*Proof.* We start by writing

$$\left\| \sup_n |S_\lambda^a f(\cdot, t_n)| \right\|_2 \leq \left\| \sup_{n:\, t_n\leq b} |S_\lambda^a f(\cdot, t_n)| \right\|_2 + \left\| \sup_{n:\, t_n>b} |S_\lambda^a f(\cdot, t_n)| \right\|_2.$$

By Proposition 2.1 we can bound the first term by $b^{1/4}\lambda^{a/4}\|f\|_2$. On the other hand, by using Plancherel's theorem and our assumption, we get

$$\left\| \sup_{n:\, t_n>b} |S_\lambda^a f(\cdot, t_n)| \right\|_{L^2(\mathbb{R})} \leq \left( \sum_{n:\, t_n>b} \|S_\lambda^a f(\cdot, t_n)\|_2^2 \right)^{1/2}$$

$$\leq \#(\{n : t_n > b\})^{1/2}\|f\|_2 \lesssim b^{-r/2}\|f\|_2.$$

We therefore have

$$\left\| \sup_n |S_\lambda^a f(\cdot, t_n)| \right\|_2 \lesssim (b^{1/4}\lambda^{a/4} + b^{-r/2})\|f\|_2$$

and choosing $b$ such that $b^{1/4}\lambda^{a/4} = b^{-r/2}$, namely $b = \lambda^{-\frac{a}{1+2r}}$, finishes the proof. $\qquad\square$

We wish to apply the Lemma 2.2 for $\lambda = 2^k$, $k > 1$. A more refined argument is needed to combine the dyadic scales.

**Proposition 2.3.** *Let $0 < a \neq 1$, and assume that $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$ is decreasing. Then*

(2.3) $$\left\| \sup_n |S^a f(\cdot, t_n)| \right\|_{L^2(\mathbb{R})} \leq C\|f\|_{H^s}, \quad s = \frac{ar}{2+4r}.$$

*Moreover $S^a f(x, t_n) \to f(x)$ a.e. whenever $f \in H^\sigma$ for $\sigma \geq \min\{\frac{1}{4}, \frac{ar}{2+4r}\}$.*

*Proof.* Define projection operators $P_k$ by

$$\widehat{P_0 f}(\xi) = \mathbb{1}_{[-1/2, 1/2]}(\xi)\widehat{f}(\xi)$$
$$\widehat{P_k f}(\xi) = (\mathbb{1}_{[2^{k-1}, 2^k]} + \mathbb{1}_{[-2^k, -2^{k-1}]})\widehat{f}(\xi), \quad k \geq 1$$

Clearly $P_k P_k = P_k$ and $\sum_{k \geq 0} P_k f = f$.

Next, for each integer $l \geq 0$ we set

$$\mathfrak{N}_l = \left\{ n \in \mathbb{N} : 2^{-(l+1)\frac{a}{1+2r}} < t_n \leq 2^{-l\frac{a}{1+2r}} \right\}.$$

By assumption $\{t_n\} \in \ell^{r,\infty}$ there is $C > 0$ so that

(2.4) $$\#(\mathfrak{N}_l) \leq C 2^{l\frac{ar}{1+2r}} = C 2^{2ls}.$$

We can then write

$$\sup_n |S^a f(x, t_n)| \leq \mathcal{E}_1(x) + \mathcal{E}_2(x) + \mathcal{E}_3(x)$$

where

$$\mathcal{E}_1(x) = \sup_l \sup_{n \in \mathfrak{N}_l} \Big| \sum_{k < \frac{l}{1+2r}} S^a P_k f(x, t_n) \Big|$$

$$\mathcal{E}_2(x) = \sup_l \sup_{n \in \mathfrak{N}_l} \Big| \sum_{\frac{l}{1+2r} \leq k < l} S^a P_k f(x, t_n) \Big|$$

$$\mathcal{E}_3(x) = \sup_l \sup_{n \in \mathfrak{N}_l} \Big| \sum_{k \geq l} S^a P_k f(x, t_n) \Big|$$

We first give the estimate for $\|\mathcal{E}_3\|_2$. We make the change of variable $k = l + m$ and and get

$$\mathcal{E}_3(x) \leq \sum_{m \geq 0} \Big( \sum_{l \geq 0} \sup_{n \in \mathfrak{N}_l} |S^a P_{l+m} f(x, t_n)|^2 \Big)^{1/2}.$$

From this,

$$\|\mathcal{E}_3\|_2 \leq \sum_{m \geq 0} \Big( \sum_{l \geq 0} \Big\| \sup_{n \in \mathfrak{N}_l} |S^a P_{l+m} f(\cdot, t_n)| \Big\|_2^2 \Big)^{1/2}$$

$$\leq \sum_{m \geq 0} \Big( \sum_{l \geq 0} \sum_{n \in \mathfrak{N}_l} \big\| S^a P_{l+m} f(\cdot, t_n) \big\|_2^2 \Big)^{1/2}$$

$$\leq \sum_{m \geq 0} \Big( \sum_{l \geq 0} \#(\mathfrak{N}_l) \big\| P_{l+m} f \big\|_2^2 \Big)^{1/2}$$

and using (2.4) this is further estimated by

$$\sum_{m \geq 0} \Big( \sum_{l \geq 0} 2^{2sl} \|P_{l+m} f\|_2^2 \Big)^{1/2} = \sum_{m \geq 0} 2^{-ms} \Big( \sum_{l \geq 0} 2^{2s(l+m)} \|P_{l+m} f\|_2^2 \Big)^{1/2} \lesssim \|f\|_{H^s}.$$

In order to deal with the first and second terms we use that by definition of $\mathfrak{N}_l$ the $t_n$ with $n \in \mathfrak{N}_l$ lie in the interval

$$J_l = [0, 2^{-l\frac{a}{1+2r}}].$$

For the term $\mathcal{E}_2$ we make the change of variables $k = l - j$ and estimate

$$\mathcal{E}_2(x) \leq \sup_l \sup_{n \in \mathfrak{N}_l} \Big| \sum_{0 < j \leq \frac{2r}{1+2r} l} S^a P_{l-j} f(x, t_n) \Big|$$

$$\leq \Big( \sum_{l \geq 0} \big( \sum_{0 < j \leq \frac{2r}{1+2r} l} \sup_{n \in \mathfrak{N}_l} |S^a P_{l-j} f(x, t_n)| \big)^2 \Big)^{1/2}$$

$$\leq \sum_{j > 0} \big( \sum_{l \geq j \frac{1+2r}{2r}} \sup_{n \in \mathfrak{N}_l} |S^a P_{l-j} f(x, t_n)|^2 \big)^{1/2}.$$

We can now use Proposition 2.1, with $J = J_l$ amd $\lambda = 2^k = 2^{l-j}$. Note that $l \geq j \frac{1+2r}{r}$ implies that $|J_l|^{\frac{1}{4}} 2^{\frac{a}{4}(l-j)} = 2^{-l \frac{a}{4(1+2r)}} 2^{\frac{a}{4}(l-j)} \geq 1$. Using $P_k P_k = P_k$ we then get

$$\|\mathcal{E}_2\|_2 \leq \sum_{j \geq 0} \Big( \sum_{l \geq j \frac{1+2r}{r}} \big\| \sup_{n \in \mathfrak{N}_l} |S^a P_{l-j} f(\cdot, t_n)| \big\|_{L^2(\mathbb{R})}^2 \Big)^{1/2}$$

$$\lesssim \sum_{j \geq 0} \Big( \sum_{l \geq j \frac{1+2r}{2r}} \big[ (1 + 2^{\frac{a}{4}(l-j)} 2^{-l \frac{a}{4(1+2r)}}) \|P_{l-j} f\|_2 \big]^2 \Big)^{1/2}$$

$$\lesssim \sum_{j \geq 0} \Big( \sum_{l \geq j \frac{1+2r}{2r}} \big[ 2^{(l-j) \frac{a}{4}(1 - \frac{1}{1+2r})} 2^{-j \frac{a}{4(1+2r)}} \|P_{l-j} f\|_2 \big]^2 \Big)^{1/2}$$

$$= \sum_{j \geq 0} 2^{-j \frac{a}{4(1+2r)}} \Big( \sum_{l \geq j \frac{1+2r}{2r}} \big[ 2^{s(l-j)} \|P_{l-j} f\|_2 \big]^2 \Big)^{1/2} \lesssim \|f\|_{H^s}.$$

Finally we consider the term $\mathcal{E}_1$ and estimate

$$\|\mathcal{E}_1\|_2 \leq \big\| \sup_l \sup_{n \in \mathfrak{N}_l} \sum_{k < \frac{l}{1+2r}} |S^a P_k f(x, t_n)| \big\|_2$$

$$\leq \sum_{k \geq 0} \big\| \sup_{l > k(1+2r)} \sup_{n \in \mathfrak{N}_l} |S^a P_k f(\cdot, t_n)| \big\|_2$$

The $t_n$ with $n \in \cup_{l > k(1+2r)} \mathfrak{N}_l$ lie in an interval of length $O(2^{-ka})$, namely in $J_{l(k)}$ with $l(k) = \lfloor k(1 + 2r) \rfloor$. We use again Proposition 2.1, with $J = J_{l(k)}$ and $\lambda = 2^k$, and observe that now $2^{ka/4} |J_{l(k)}|^{1/4} \lesssim 1$. Thus we can bound for each $k > 0$

$$\big\| \sup_{l > k(1+2r)} \sup_{n \in \mathfrak{N}_l} |S^a P_k f(\cdot, t_n)| \big\|_2 \lesssim (1 + 2^{ka/4} |J_{l(k)}|^{1/4}) \|P_k f\|_2 \lesssim \|P_k f\|_2.$$

We sum in $k$ and deduce that

$$\|\mathcal{E}_1\|_2 \lesssim \sum_{k \geq 0} \|P_k f\|_2 \leq C(s) \|f\|_{H^s}, \quad s > 0.$$

We combine the estimates for $\|\mathcal{E}_i\|$, $i = 1, 2, 3$ to finish the proof of the maximal inequality (2.3). Since $\lim_{t \to 0} S^a f(x, t) = f(x)$ for all $x \in \mathbb{R}$ whenever $f$ is a Schwartz function, and since Schwartz functions are dense in $H^\sigma$ the stated pointwise convergence result follows from (2.1), (2.3) by a standard argument (see e.g. [15] or [18]). $\qquad \square$

Finally we mention an endpoint result involving the Besov space $B_{2,1}^s(\mathbb{R})$ when $s = a/4$. We do not know whether $B_{2,1}^{a/4}$ can be replaced with $H^{a/4}$ in the following proposition.

**Proposition 2.4.** *Let $a > 0$, $a \neq 1$. Then, for all $f \in B_{2,1}^{a/4}(\mathbb{R})$,*

$$\big\| \sup_{t \in [0,1]} |S^a f(\cdot, t)| \big\|_{L^2(\mathbb{R})} \leq C \|f\|_{B_{2,1}^{a/4}}.$$

*Proof.* Write $f = \sum_{k \geq 0} S^a P_k P_k f$ as in the proof of Proposition 2.3. By Proposition 2.1 we have

$$\big\| \sup_{t \in [0,1]} S^a f \big\|_2 \leq \sum_{k \geq 0} \big\| \sup_{t \in [0,1]} |S^a P_k P_k f| \big\|_2 \lesssim \sum_{k \geq 0} 2^{ka/4} \|P_k f\|_2$$

and using Plancherel's theorem and the definition of Besov spaces via dyadic frequency decompositions we see that the last expression is dominated by $C \|f\|_{B_{2,1}^{a/4}}$. $\square$

## 3. Necessary conditions

In order to prove necessity in Theorem 1.1 we use arguments from Nikishin-Stein theory. We include the standard argument for the proof of the following proposition in Appendix §5.

**Proposition 3.1.** *Assume that for every $f \in H^s$, the limit $\lim_{n \to \infty} S^a f(x, t_n)$ exists for almost every $x \in \mathbb{R}$. Then for any compact set $K \subset \mathbb{R}$, there is a constant $C_K$, such that for all $\alpha > 0$,*

$$\mathrm{meas}(\{x \in K : \sup_n |S^a f(x, t_n)| > \alpha\}) \leq C_K \Big( \frac{\|f\|_{H^s}}{\alpha} \Big)^2.$$

We also need the following elementary lemma.

**Lemma 3.2.** *Let $\{t_n\}$ be a sequence of positive numbers in $[0,1]$, let $0 < r < \infty$ and assume that $\sup_{b>0} b^r \#(\{n : b < t_n \leq 2b\}) \leq A$. Then $\{t_n\} \in \ell^{r,\infty}$.*

*Proof.* For every $\beta > 0$,

$$\beta^r \#(\{n : t_n > \beta\}) = \beta^r \sum_{k \geq 0} \#(\{n : 2^k \beta < t_n \leq 2^{k+1} \beta\}) \leq \sum_{k \geq 0} A 2^{-kr} \lesssim A. \quad \square$$

We now turn to the proof of the necessity of the $\ell^{r,\infty}$-condition in Theorems 1.1 and Theorem 1.2.

**Proposition 3.3.** *Assume that $\{t_n\}$ is a decreasing sequence such that $t_n - t_{n+1}$ is also decreasing and $\lim_{n \to \infty} t_n = 0$. For $0 < s < a/4$, let*

$$r(s) = \frac{2s}{a - 4s}.$$

*(i) If $s < \min\{a/4, 1/4\}$ and if*

(3.1) $$\mathrm{meas}(\{x \in [0,1] : \sup_n |S^a f(x, t_n)| > 1/2\}) \leq C_\circ \|f\|_{H_s}^2$$

*holds for all $f \in H^s$, then $\{t_n\} \in \ell^{r(s),\infty}$.*

*(ii) If $s < a/4$ and if the global weak type inequality*

(3.2) $$\mathrm{meas}(\{x \in \mathbb{R} : \sup_n |S^a f(x, t_n)| > 1/2\}) \leq C_\circ \|f\|_{H_s}^2$$

*holds for all $f \in H^s$, then $\{t_n\} \in \ell^{r(s),\infty}$.*

*Proof.* We argue by contradiction and assume that $\{t_n\} \notin \ell^{r(s),\infty}$ while (3.1) holds if $s < \min\{a/4, 1/4\}$ or (3.2) holds in the case $a > 1$ and $1/4 \leq s < a/4$. By Lemma 3.2, this means

$$\sup_{0 < b < 1/2} b^{r(s)} \#(\{n : b < t_n \leq 2b\}) = \infty.$$

Hence there exists an increasing sequence $\{R_j\}$ with $\lim_{j \to \infty} R_j = \infty$ and a sequence of positive numbers $b_j$ with $\lim_{j \to \infty} b_j = 0$ so that

$$(3.3) \qquad\qquad \#(\{n : b_j < t_n \leq 2b_j\}) \geq R_j b_j^{-r(s)}.$$

We take another sequence

$$M_j \leq R_j \text{ with } \lim_{j \to \infty} M_j = \infty$$

such that in the case where $s < 1/4$

$$(3.4) \qquad\qquad a M_j^{\frac{2(a-1)}{a}} b_j^{\frac{1-4s}{a-4s}} \leq 1.$$

In the case $1/4 \leq s < a/4$ we simply take $M_j = R_j$.
    We now show that

$$(3.5) \qquad\qquad t_n - t_{n+1} \leq 2 M_j^{-1} b_j^{\frac{a-2s}{a-4s}}, \quad \text{if } t_n \leq b_j.$$

Indeed since $n \mapsto t_n - t_{n+1}$ is decreasing we get, for $t_n \leq b_j$,

$$t_n - t_{n+1} \leq \min\{t_m - t_{m+1} : t_m > b_j\} \leq \frac{2b_j}{\#(\{n : b_j < t_n \leq 2b_j\})}$$

$$\leq \frac{2b_j}{R_j b_j^{-r(s)}} \leq \frac{2b_j}{M_j b_j^{-r(s)}},$$

by (3.3), and (3.5) follows since $r(s) + 1 = \frac{a-2s}{a-4s}$.
    For our construction of a counterexample we rely on the idea originally proposed by Dahlberg and Kenig [5]. We introduce a family of Schwartz functions which is used to test (3.1). Choose a $C^\infty$ function $g$ with compact support in $[-1/2, 1/2]$ such that $g(\xi) \geq 0$ and $\int g(\xi) \, d\xi = 1$ and consider a family of functions $f_{\lambda,\rho}$, with large $\lambda$ and $\rho \ll \lambda$, defined via the Fourier transform by

$$\widehat{f}_{\lambda,\rho}(\eta) = \rho^{-1} g((\eta + \lambda)/\rho).$$

Thus $\widehat{f}_{\lambda,\rho}$ is supported in an interval of length $\rho \ll \lambda$ contained in $[-2\lambda, -\lambda/2]$. The assumption $\rho \ll \lambda$ clearly implies

$$(3.6) \qquad\qquad \|f_{\lambda,\rho}\|_{H^s} \lesssim \lambda^s \rho^{-1/2}.$$

We now examine the action of $S^a$ on $f_{\lambda,\rho}$. We have

$$|S^a f_{\lambda,\rho}(x, t_n)| = \left| \int e^{i(x\eta + t_n |\eta|^a)} \rho^{-1} g((\eta + \lambda)/\rho) \frac{d\eta}{2\pi} \right| = \left| \int e^{i\Phi_{\lambda,\rho}(\xi;x,t_n)} g(\xi) \frac{d\xi}{2\pi} \right|$$

where

$$\Phi_{\lambda,\rho}(\xi; x, t_n) = x(\rho\xi - \lambda) + t_n(\lambda - \rho\xi)^a.$$

We shall use, for $x$ in a suitable interval $I_j \subset I$, and for suitable choices of $\lambda_j, \rho_j$ and $n(x, j)$, the estimate

$$|S^a f_{\lambda_j, \rho_j}(x, t_{n(x,j)})| \geq \int g(\xi)\, d\xi - \int |e^{i\Phi_{\lambda_j, \rho_j}(\xi; x, t_{n(x,j)})} - 1|\, g(\xi)\, \tfrac{d\xi}{2\pi}$$

(3.7)
$$\geq 1 - \max_{|\xi| \leq 1/2} |e^{i\Phi_{\lambda_j, \rho_j}(\xi; x, t_{n(x,j)})} - 1|$$

and we will have to show that the subtracted term is small for our choices of $x$, $n(x, j)$ and $(\lambda_j, \rho_j)$.

By a standard Taylor expansion, we see that

$$(1 - \rho\xi/\lambda)^a = 1 - a\rho\xi/\lambda + \tfrac{a(a-1)}{2}(\rho\xi/\lambda)^2 + E_3(\rho\xi/\lambda)$$

where $E_3(t) = -\tfrac{1}{2}a(a-1)(a-2)(\int_0^1 (1 - st)^{a-3}(1 - s)^2 ds)\, t^3$. Hence

(3.8)  $\Phi_{\lambda, \rho}(\xi; x, t_n) =$

$$(x - a\lambda^{a-1}t_n)\rho\xi + \tfrac{a(a-1)}{2}\rho^2\lambda^{a-2}t_n\xi^2 + \lambda^a t_n E_3(\rho\xi/\lambda) + \lambda^a t_n - \lambda x.$$

Since terms that are independent of $\xi$ do not affect the absolute value of our integral, we only need to show an upper bound of the first three terms. We consider $t_n$ with $t_n \leq b_j/2$ and let $\varepsilon$ be such that $\varepsilon < 10^{-1}(a + 2)^{-1}$. We chose $(\lambda, \rho) = (\lambda_j, \rho_j)$ as

(3.9)  $\lambda_j = M_j^{2/a} b_j^{-\frac{1}{a-4s}}, \qquad \rho_j = \varepsilon b_j^{-1/2}\lambda_j^{1-a/2} = \varepsilon M_j^{\frac{2-a}{a}} b_j^{-\frac{1-2s}{a-4s}}$

and we consider these choices for large $j$ when $b_j \ll 1$ and $M_j \gg 1$. We then get

$$\rho_j/\lambda_j = \varepsilon M_j^{-1} b_j^{\frac{2s}{a-4s}} \leq \varepsilon;$$

hence for $|\xi| \leq 1/2$

(3.10a)  $|\tfrac{a(a-1)}{2}\rho_j^2\lambda_j^{a-2}t_n\xi^2| \leq \tfrac{(a+1)^2}{2}\rho_j^2\lambda_j^{a-2}b_j\xi^2$

$$\leq (a+1)^2\varepsilon^2 M_j^{\frac{4-2a}{a}} b_j^{-\frac{2-4s}{a-4s}} M_j^{\frac{2(a-2)}{a}} b_j^{-\frac{a-2}{a-4s}} b_j = (a+1)^2\varepsilon^2$$

and similarly

(3.10b)  $|\lambda_j^a t_n E_3(\rho_j\xi/\lambda)| \leq (a+2)^3\lambda_j^a b_j \left(\tfrac{\rho_j}{2\lambda_j}\right)^3 \leq (a+2)^3\varepsilon\lambda_j^{a-2}\rho_j^2 b_j \leq (a+2)^3\varepsilon^3.$

Next we consider $x$ in the interval

$$I_j := [0, a\lambda_j^{a-1}b_j/2].$$

Notice that in the case $s < 1/4$,

$$a\lambda_j^{a-1}b_j/2 = \frac{a}{2}M_j^{\frac{2(a-1)}{a}} b_j^{\frac{1-4s}{a-4s}} \leq 1/2,$$

by (3.4) and hence $I_j \subset [0, 1/2]$ in this case. If $a > 1$ and $1/4 \leq s < a/4$, no restriction on $I_j$ is required (as we are trying to disprove the global inequality (3.2) in this case). Each $x \in I_j$ is contained in an interval $(a\lambda_j^{a-1}t_{n+1}, a\lambda_j^{a-1}t_n]$ for a unique $n$, which we label $n(x, j)$. By (3.5) we have that

$$0 \leq t_{n(x,j)} - t_{n(x,j)+1} \leq 2M_j^{-1}b_j^{\frac{a-2s}{a-4s}}.$$

Hence

$$|(x - a\lambda_j^{a-1}t_{n(x,j)})\rho_j\xi| \leq a\lambda_j^{a-1}\rho_j(t_{n(x,j)} - t_{n(x,j)+1})$$

(3.10c)
$$\leq aM_j^{\frac{2(a-1)}{a}} b_j^{-\frac{a-1}{a-4s}} \varepsilon M_j^{\frac{2-a}{a}} b_j^{-\frac{1-2s}{a-4s}} 2M_j^{-1}b_j^{\frac{a-2s}{a-4s}} = 2a\varepsilon.$$

As $\varepsilon \leq 10^{-1}(a+2)^{-1}$ we obtain from (3.10a), (3.10b) and (3.10c)

$$\max_{|\xi|\leq 1/2} |e^{i\Phi_{\lambda_j,\rho_j}(\xi;x,t_{n(x,j)})} - 1| \leq 1/2$$

and thus from (3.7)

(3.11)
$$\sup_n |S^a f_{\lambda_j,\rho_j}(x,t_n)| \geq |S^a f_{\lambda_j,\rho_j}(x,t_{n(x,j)})| \geq \tfrac{1}{2}, \text{ for } x \in I_j = [0, a\lambda_j^{a-1}b_j/2],$$

and, as noted before, $I_j \subset [0,1]$ if $s < 1/4$. The assumption of (3.1) (in the case $s < \min\{a/4, 1/4\}$) or the assumption of (3.2), both yield

(3.12)
$$\mathrm{meas}(I_j) \leq 4C_\circ \|f_{\lambda_j,\rho_j}\|_{H^s}^2 = \widetilde{C}\lambda_j^{2s}\rho_j^{-1}.$$

This leads to

$$aM_j^{\frac{2(a-1)}{a}} b_j^{\frac{1-4s}{a-4s}} \leq \widetilde{C}M_j^{\frac{4s}{a}} b_j^{-\frac{2s}{a-4s}}\varepsilon^{-1}M_j^{\frac{a-2}{a}} b_j^{\frac{1-2s}{a-4s}}$$

and hence to

$$a\varepsilon\widetilde{C}^{-1} \leq M_j^{-\frac{a-4s}{a}}.$$

Since $\lim_{j\to\infty} M_j = \infty$ the right hand side converges to 0 as $j \to \infty$ and we obtain a contradiction. This means that if $\{t_n\} \notin \ell^{\frac{2s}{a-4s},\infty}$ then (3.1) (and therefore (3.2)) cannot hold with $s < \min\{a/4, 1/4\}$ and (3.2) cannot hold with $1/4 \leq s < a/4$. Thus both parts of the proposition are proved. $\qquad\square$

We are now able to combine previous results to give a proof of the theorems in the introduction.

*Proof of Theorem 1.1.* The implications (a) $\implies$ (b) and (a) $\implies$ (d) follow from Proposition 2.3. The implication (b) $\implies$ (c) is immediate by Tshebyshev's inequality. The implication (c) $\implies$ (a) follows from part (i) of Proposition 3.3. Finally, the implication (d) $\implies$ (c) follows from Proposition 3.1. $\qquad\square$

*Proof of Theorem 1.2.* The implication (a) $\implies$ (b) follows from Proposition 2.3. The implication (b) $\implies$ (c) is again immediate by Tshebyshev's inequality. The implication (c) $\implies$ (a) follows from part (ii) of Proposition 3.3. $\qquad\square$

## 4. THE CASE $a = 1$

We now give the sketch of the proof of Theorem 1.3. We start with an analog to Lemma 2.2, for the frequency-localized operator $S_\lambda^1$.

**Lemma 4.1.** *(i) Let $b > \lambda^{-1}$. Then*

$$\big\| \sup_{0\leq t\leq b} |S_\lambda^1 f(\cdot,t)| \big\|_2 \lesssim (\lambda b)^{1/2}\|f\|_2.$$

*(ii) Let $0 < r < \infty$ and let $\{t_n\}$ be a sequence in $[0,1]$ which belongs to $\ell^{r,\infty}$. Then for $\lambda > 1$*

$$\Big\| \sup_n |S_\lambda^1 f(\cdot,t_n)| \Big\|_{L^2(\mathbb{R})} \leq C\lambda^{\frac{r}{2+2r}}\|f\|_{L^2(\mathbb{R})}.$$

*Proof.* We use the elementary inequality

$$(4.1) \qquad \big\| \sup_{c \le t \le c+\lambda^{-1}} |S_\lambda^1 f(x,t)| \big\|_2 \lesssim \|f\|_2$$

which just follows from $L^2$ estimates for $S_\lambda f(\cdot,t)$ and $\partial_t S_\lambda f(\cdot,t)$. Now

$$\big\| \sup_{0 < t \le b} |S_\lambda^1 f(\cdot,t)| \big\|_{L^2(\mathbb{R})} \le \Big( \sum_{\substack{m \ge 0: \\ 0 \le m\lambda^{-1} \le b}} \big\| \sup_{m\lambda^{-1} \le t \le (m+1)\lambda^{-1}} |S_\lambda^1 f(\cdot,t)| \big\|_2^2 \Big)^{1/2}$$

which is bounded by a constant times $(\lambda b)^{1/2}\|f\|_2$.

To prove part (ii) we write as in the proof of Lemma 2.2, for $b > \lambda^{-1}$ to be determined,

$$\big\| \sup_n |S_\lambda^1 f(\cdot,t_n)| \big\|_2 \le \big\| \sup_{n:\, t_n \le b} |S_\lambda^1 f(\cdot,t_n)| \big\|_2 + \big\| \sup_{n:\, t_n > b} |S_\lambda^1 f(\cdot,t_n)| \big\|_2.$$

For the first term we have

$$\big\| \sup_{n:\, t_n \le b} |S_\lambda^1 f(\cdot,t_n)| \big\|_{L^2(\mathbb{R})} \lesssim (\lambda b)^{1/2}\|f\|_2,$$

by part (i). For the second term we may estimate as in Lemma 2.2

$$\big\| \sup_{n:\, t_n > b} |S_\lambda^1 f(\cdot,t_n)| \big\|_{L^2(\mathbb{R})} \lesssim b^{-r/2}\|f\|_2.$$

Choosing $b$ such that $(\lambda b)^{1/2} = b^{-r}$ yields the claimed result. $\qquad \square$

**Proposition 4.2.** *Let $0 < r < \infty$ and assume that $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$ is decreasing. Then*

$$(4.2) \qquad \big\| \sup_n |S^1 f(\cdot,t_n)| \big\|_{L^2(\mathbb{R})} \le C\|f\|_{H^s}, \quad s = \frac{r}{2+2r}.$$

*Proof.* We set for $l \ge 0$ we set

$$\mathfrak{N}(l) = \big\{ n \in \mathbb{N} : 2^{-(l+1)\frac{1}{1+r}} < t_n \le 2^{-l\frac{1}{1+r}} \big\}.$$

By assumption $\{t_n\} \in \ell^{r,\infty}$ there is $C > 0$ so that

$$(4.3) \qquad \#(\mathfrak{N}(l)) \le C 2^{l\frac{r}{1+r}} = C 2^{2ls}.$$

Arguing as in the proof of Proposition 4.2 we can estimate $\|\sup_n |S^a f(\cdot,t_n)|\|_2 \le \|\mathcal{E}_1\|_2 + \|\mathcal{E}_2\|_2$ where

$$\mathcal{E}_1(x) = \sum_{j \ge 0} \Big( \sum_{l \ge j} \sup_{n \in \mathfrak{N}(l)} |S^1 P_{l-j} f(x,t_n)|^2 \Big)^{1/2}$$

$$\mathcal{E}_2(x) = \sum_{m \ge 0} \Big( \sum_{l \ge 0} \sup_{n \in \mathfrak{N}(l)} |S^1 P_{l+m} f(x,t_n)|^2 \Big)^{1/2}.$$

Again as in the proof of Proposition 2.3

$$\|\mathcal{E}_2\|_2 \le \sum_{m \ge 0} \Big( \sum_{l \ge 0} \#(\mathfrak{N}(l)) \|P_{l+m} f\|_2^2 \Big)^{1/2} \lesssim \sum_{m \ge 0} \Big( \sum_{l \ge 0} 2^{2sl} \|P_{l+m} f\|_2^2 \Big)^{1/2}$$

$$= \sum_{m \ge 0} 2^{-ms} \Big( \sum_{l \ge 0} 2^{2s(l+m)} \|P_{l+m} f\|_2^2 \Big)^{1/2} \lesssim \|f\|_{H^s}.$$

In order to deal with the first sum, we use that $\mathfrak{N}(l) \subset [0, b_l]$ with $b_l = 2^{-l/(1+r)}$. Hence by Lemma 4.1

$$\|\mathcal{E}_1\|_2 \leq \sum_{j\geq 0} \Big( \sum_{l\geq j} \Big\| \sup_{n\in\mathfrak{N}(l)} |S^1 P_{l-j} f(\cdot, t_n)| \Big\|^2_{L^2(\mathbb{R})} \Big)^{1/2}$$

$$\lesssim \sum_{j\geq 0} \Big( \sum_{l\geq j} \big[ 2^{\frac{l-j}{2}} 2^{-l\frac{1}{2+2r}} \|P_{l-j} f\|_2 \big]^2 \Big)^{1/2}$$

$$\lesssim \sum_{j\geq 0} 2^{-j\frac{1}{2+2r}} \Big( \sum_{l\geq j} \big[ 2^{(l-j)\frac{r}{2+2r}} \|P_{l-j} f\|_2 \big]^2 \Big)^{1/2} \lesssim \|f\|_{H^s}$$

with $s = \frac{r}{2r+2}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For completeness we state the case $s = 1/2$, $a = 1$ analog of Proposition 2.4 which is sharp in this case.

**Proposition 4.3.** *For all $f \in B^{1/2}_{2,1}(\mathbb{R})$,*

$$\big\| \sup_{t\in[0,1]} |S^1 f(\cdot, t)| \big\|_{L^2(\mathbb{R})} \leq C \|f\|_{B^{1/2}_{2,1}}.$$

*The space $B^{1/2}_{2,1}$ cannot be replaced by $B^{1/2}_{2,\nu}$ for any $\nu > 1$.*

*Proof.* The first part is immediate from Lemma 4.1. For the second part one recalls that there are unbounded functions in $B^{1/2}_{2,\nu}$ whose Fourier transform is supported in $(-\infty, 0]$, *cf.* [1]. For such functions $S^1 f(x, t) = f(x - t)$ and thus, for $\nu > 1$ one can easily find $f \in B^{1/2}_{2,\nu}$ such that $\sup_{t\in[0,1]} |S^1 f(x, t)| = \infty$ on a set $A \subset [0, 1]$ with $\text{meas}(A) > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Proposition 4.4.** *Assume that $\{t_n\}$ is a decreasing sequence such that $t_n - t_{n+1}$ is also decreasing and $\lim_{n\to\infty} t_n = 0$. For $s < 1/2$ let*

$$\rho(s) = \frac{2s}{1 - 2s}.$$

*Then the validity of the inequality*

$$(4.4) \qquad \text{meas}(\{x \in [0, 1] : \sup_n |S^1 f(x, t_n)| > 1/2\}) \leq C_\circ \|f\|^2_{H_s}.$$

*for all $f \in H^s$, implies that $\{t_n\} \in \ell^{\rho(s),\infty}$.*

*Proof.* Assume that $\{t_n\} \notin \ell^{\rho(s),\infty}$. Arguing as in the proof of Proposition 3.3 we find an increasing sequence $M_j$ with $\lim M_j = \infty$ and a sequence of positive numbers $b_j$ with $\lim_{j\to\infty} b_j = 0$ so that

$$\#(\{n : b_j < t_n \leq 2b_j\}) \geq M_j b_j^{-\rho(s)}.$$

As in the previous proof we also have

$$(4.5) \qquad t_n - t_{n+1} \leq 2M_j^{-1} b_j^{\rho(s)+1}, \ \ \text{if} \ \ t_n \leq b_j.$$

Let $g \in C^\infty$ be nonnegative, supported in $(-1/2, 1/2)$, such that $\int g(\xi)\, d\xi = 1$. Define $f_\lambda$, for large $\lambda$, by

$$\widehat{f_\lambda}(\xi) = 10\lambda^{-1} g(10\lambda^{-1}(\xi + \lambda)).$$

Then $\|f_\lambda\|_{H^s} \leq C_\circ \lambda^{s-1/2}$. We write

$$|S^1 f_\lambda(x,t)| = \left| \int e^{i10^{-1}\lambda(x-t)\xi} g(\xi) \frac{d\xi}{2\pi} \right| \geq 1 - \left| \int (e^{i10^{-1}\lambda(x-t)\xi} - 1) g(\xi) \frac{d\xi}{2\pi} \right|$$

and see that

(4.6) $$|S^1 f_\lambda(x,t)| \geq 1/2, \quad \text{if } |x-t| \leq \lambda^{-1}.$$

We now set $\lambda_j = M_j b_j^{-\frac{1}{1-2s}}$. Note that $\rho(s) + 1 = (1-2s)^{-1}$, and thus we have $t_n - t_{n+1} \leq M_j \lambda_j^{-1}$ for $t_n \leq b_j$, by (4.5). Hence by (4.6) we see that

$$\sup_n |S^1 f_{\lambda_j}(x)| \geq 1/2, \quad \text{for } 0 < x < b_j/2.$$

Therefore the asserted weak type inequality implies

$$b_j/2 \leq 4\|f_{\lambda_j}\|_{H^s}^2 \leq 4C_\circ \lambda_j^{2s-1} = 4C_\circ M_j^{2s-1} b_j$$

and thus $8C_\circ M_j^{2s-1}$ is bounded below as $j \to \infty$. This yields a contradiction as we have $\lim_{j \to 0} M_j^{2s-1} = 0$ for $s < 1/2$. □

*Proof of Theorem 1.3.* The implication (a) $\implies$ (b) follows from Proposition 4.2. The implication (b) $\implies$ (c) follows from Tshebyshev's inequality. The implication (c) $\implies$ (a) follows from Proposition 4.4. The implication (c) $\implies$ (d) follows by a standard argument using the weak type inequality and the density of Schwartz functions. The implication (d) $\implies$ (c) follows from Proposition 3.1. □

## 5. APPENDIX: PROOF OF PROPOSITION 3.1

We need to use a theorem by Nikishin, whose proof can be found, for example, in [6] (Chapter VI, Corollary 2.7), see also [19]. Nikishin's theorem asserts that if $M : L^2(Y,\mu) \to L^0(\mathbb{R}^d, |\cdot|)$ is a continuous sublinear operator (with $(Y,\mu)$ an arbitrary measure space), then there exists a measurable function $w$ with $w(x) > 0$ a.e. such that

$$\int_{\{x: |Mf(x)|>\alpha\}} w(x)\, dx \leq \alpha^{-2}\|f\|_{L^2(\mu)}^2.$$

To prove Proposition 3.1, let $M^a f(x) = \sup_n |S^a f(x,t_n)|$ and consider $T_n^a g(x) = (2\pi)^{-1} \int e^{i(x\xi + t_n|\xi|^a)} g(\xi)\, d\xi$, so that $T_n^a \hat{f}(x) = S^a f(x,t_n)$. Then $T_n^a$ acts on functions in the weighted $L^2$ space $L^2(\mu_s)$, where $d\mu_s(\xi) = (1+|\xi|^2)^s d\xi$. Define the corresponding maximal operator, $\widetilde{M}^a g = \sup_n |T_n^a g|$.

Now assuming that $\lim_n S^a f(x,t_n)$ exists a.e. for every $f \in H^s$, we see that $\widetilde{M}^a g(x) < \infty$ a.e. for every $g \in L^2(\mu_s)$. Then by Proposition 1.4, p. 529 in [6], this implies that the sublinear operator $\widetilde{M}^a : L^2(\mu_s) \to L^0(|\cdot|)$ is continuous. By the abovementioned Nikishin's theorem,

$$\int_{\{x: |\widetilde{M}^a g(x)|>\alpha\}} w(x)\, dx \leq \alpha^{-2}\|g\|_{L^2(\mu_s)}^2$$

for some weight $w$ with $w(x) > 0$ a.e. As we can replace $w$ with $\min\{w,1\}$ we may further assume that $w$ is bounded.

Next, for $f \in H^s$, $\widetilde{M}^a \hat{f} = M^a f$ and $\|\hat{f}\|_{L^2(\mu_s)} = \|f\|_{H^s}$, so

$$\int_{\{x: |M^a f(x)|>\alpha\}} w(x)\, dx \leq \alpha^{-2}\|f\|_{H^s}^2.$$

After the change of variables $x \to x + y$ and using the translation invariance of $M^a$, we can replace the integrand $w(x)$ by $w(x-y)$ for any $y$. Arguing as in Chapter VI of [6] multiply both sides of the resulting inequality by $h(y)$, where $h$ is a strictly positive continuous function with $\int h = 1$, and then integrate in $y$, to arrive at

$$\int_{\{x \,:\, |M^a f(x)| > \alpha\}} h * w(x)\, dx \leq \alpha^{-2} \|f\|_{H^s}^2.$$

Since $h * w$ is continuous it attains a minimum over any compact set. We therefore conclude that

$$\mathrm{meas}\big(\{x \in K : |M^a f(x)| > \alpha\}\big) \leq C_K \alpha^{-2} \|f\|_{H^s}^2$$

must hold true for every compact set $K$, as desired.                    $\square$

## References

[1] Jöran Bergh, Jörgen Löfstrom, Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[2] Lennart Carleson, *Some analytic problems related to statistical mechanics*, Euclidean Harmonic Analysis (Proc. Sem., Univ. Maryland, College Park, MD, 1979), Lecture Notes in Math. 779, Springer, Berlin, (1980), 5–45

[3] Chu-Hee Cho, Sanghyuk Lee, Ana Vargas, *Problems on pointwise convergence of solutions to the Schrödinger equation*, J. Fourier Anal. Appl. **18** (5) (2012), 972–994.

[4] Michael Cowling, *Pointwise behavior of solutions to Schrödinger equations.* Harmonic analysis (Cortona, 1982), 83–90, Lecture Notes in Math., 992, Springer, Berlin, 1983.

[5] Björn Dahlberg, Carlos Kenig, *A note on the almost everywhere behavior of solutions to the Schrödinger equation*, in Harmonic Analysis (Minneapolis, MN, 1981), Lecture Notes in Math. 908, Springer, Berlin, (1982), 205–209.

[6] José García-Cuerva, José L. Rubio De Francia, Weighted Norm Inequalities and Related Topics, North-Holland Mathematics Studies, 116. Notas de Matemática, 104. North-Holland Publishing Co., Amsterdam, 1985.

[7] Evangelos Dimou, Maximal estimates for solutions to dispersive equations. Thesis (Ph.D.) The University of Wisconsin - Madison. 2016. 84 pp. ISBN: 978-1339-96790-5. ProQuest LLC.

[8] Xiumin Du, Lawrence Guth, Xiaochun Li, *A sharp maximal Schrödinger estimate in $\mathbb{R}^2$*, Annals of Math. 186 (2) (2017), 607–640. Springer, Berlin (1981).

[9] Xiumin Du, Ruixiang Zhang, *Sharp $L^2$ estimate of Schrödinger maximal function in higher dimensions.* arXiv: 1805.02775v2.

[10] Carlos Kenig, Gustavo Ponce, Luis Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. 40 (1) (1991), 33–69.

[11] Carlos Kenig, Alberto Ruiz, *A strong type (2,2) estimate for a maximal operator associated to the Schrödinger equation*, Trans. of the Amer. Math. Soc. 280 (1983), 239–246.

[12] Andreas Seeger, Terence Tao, James Wright, *Endpoint mapping properties of spherical maximal operators.* J. Inst. Math. Jussieu 2 (2003), no. 1, 109–144.

[13] Andreas Seeger, Stephen Wainger, James Wright, *Pointwise convergence of spherical means*, Math. Proc. Camb. Phil. Soc. 118 (1) (1995), 115–124.

[14] Peter Sjögren, Per Sjölin, *Convergence properties for the time dependent Schrödinger equation.* Ann. Acad. Sci. Fenn. 14, 13–25 (1989).

[15] Per Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. 55 (3) (1987), 699–715.

[16] _____, *Global maximal estimates for solutions to the Schrödinger equation*, Studia Math. 110 (2) (1994), 105–114.

[17] _____, *Two Theorems on Convergence of Schrödinger Means,* Journal of Fourier Analysis and Applications, https:doi.org/10.1007/s00041-018-9644-0.

[18] Per Sjölin, Jan-Olov Strömberg, *Convergence of sequences of Schrödinger means.* arXiv:1905.05463

[19] Elias M. Stein, *On limits of sequences of operators.* Ann. Math. (2) 74, 140–170 (1961).

[20] Luis Vega, *Schrödinger equations: pointwise convergence to the initial data.* Proc. Am. Math. Soc. 102(4), 874–878 (1988).

[21] Björn Walther, *Oscillatory integrals with concave phase*, Contemp. Math. 189 (1995), 485–495.

E. Dimou, Department of Mathematics, University of Michigan, 530 Church Street Ann Arbor, MI 48109-1043, USA

*Current address*: Department of Mathematics, University of Virginia, P. O. Box 400137, Charlottesville, VA 22904

*Email address*: `ed8bg@virginia.edu`

A. Seeger, Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706, USA

*Email address*: `seeger@math.wisc.edu`