

OSCILLATORY INTEGRAL OPERATORS WITH LOW-ORDER DEGENERACIES

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ABSTRACT. We prove sharp L^2 estimates for oscillatory integral and Fourier integral operators for which the associated canonical relation $\mathcal{C} \subset T^*\Omega_L \times T^*\Omega_R$ projects to $T^*\Omega_L$ and to $T^*\Omega_R$ with corank one singularities of type ≤ 2 . This includes two-sided cusp singularities. Applications are given to operators with one-sided swallowtail singularities such as restricted X-ray transforms for well-curved line complexes in five dimensions.

Introduction

Let Ω_L, Ω_R be open sets in \mathbb{R}^d . This paper is concerned with L^2 bounds for oscillatory integral operators T_λ of the form

$$(1.1) \quad T_\lambda f(x) = \int e^{i\lambda\Phi(x,z)} \sigma(x,z) f(z) dz$$

where $\Phi \in C^\infty(\Omega_L \times \Omega_R)$ is real-valued, $\sigma \in C_0^\infty(\Omega_L \times \Omega_R)$ and λ is large. We shall also write

$$T_\lambda \equiv T_\lambda[\sigma]$$

to indicate the dependence on the symbol σ .

The decay in λ of the L^2 operator norm of T_λ is determined by the geometry of the canonical relation

$$(1.2) \quad \mathcal{C} = \{(x, \Phi_x, z, -\Phi_z) : (x, z) \in \Omega_L \times \Omega_R\} \subset T^*\Omega_L \times T^*\Omega_R,$$

specifically by the behavior of the projections $\pi_L : \mathcal{C} \rightarrow T^*\Omega_L$ and $\pi_R : \mathcal{C} \rightarrow T^*\Omega_R$,

$$(1.3) \quad \begin{aligned} \pi_L &: (x, z) \mapsto (x, \Phi_x(x, z)) \\ \pi_R &: (x, z) \mapsto (z, -\Phi_z(x, z)); \end{aligned}$$

here Φ_x and Φ_z denote the partial gradients with respect to x and z . Note that $\text{rank } D\pi_L = \text{rank } D\pi_R$ is equal to $d + \text{rank } \Phi_{xz}$ and that the determinants of $D\pi_L$ and $D\pi_R$ are equal to

$$(1.4) \quad h(x, z) := \det \Phi_{xz}(x, z).$$

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If \mathcal{C} is locally the graph of a canonical transformation, i.e., if $h \neq 0$, then $\|T_\lambda\| = O(\lambda^{-d/2})$ (see Hörmander [15], [16]). If the projections have singularities then there is less decay in λ and in various specific cases the decay has been determined. In dimension $d = 1$ Phong and Stein [21] obtained a complete description of the L^2 mapping properties, for the case of real-analytic phase functions. Similar results for C^∞ phases (which however missed the endpoints) and related L^p estimates for averaging operators in the plane are in [24]. The bounds for oscillatory integral operators in one dimension, with C^∞ phases, have recently been substantially improved by Rychkov [22], so that many endpoint estimates are now available in the C^∞ category.

Such general results are not known in higher dimensions even under the assumption of rank $\Phi_{xz} \geq d - 1$. We list some known cases. If both projections π_L and π_R have fold ($S_{1,0}$) singularities then $\|T_\lambda\| = O(\lambda^{-(d-1)/2-1/3})$ ([17], [19], [5]). If only one of the projections has fold singularities then by [8] we have $\|T_\lambda\| = O(\lambda^{-(d-1)/2-1/4})$; this is sharp if the other projection is maximally degenerate ([13]) but can be improved when that projection satisfies some finite type finite type condition (for sharp results of this sort see Comech [3]). This one-sided behavior comes up naturally when studying restricted X-ray transforms [6], [11], [14]. In [9] the authors began a study of the case of higher one-sided Morin ($S_{1,r,0}$) singularities, which are the stable singularities of corank one, and it was shown under suitable additional ("strongness") assumptions that such estimates can be deduced from sharp estimates for two-sided $S_{1,r-1,0}$ singularities. Thus the authors were able to prove that if one projection is a Whitney cusp, i.e., of type $S_{1,1,0}$, then $\|T_\lambda\| = O(\lambda^{-(d-1)/2-1/6})$; again this is only sharp if the other projection is maximally degenerate.

It is conjectured that if one of π_L or π_R has $S_{1,r,0}$ singularities then $\|T_\lambda\| = O(\lambda^{-(d-1)/2-1/(2r+2)})$ (for the discussion of some model cases where this is satisfied and sharp see [9]). Here we take up the case $r = 3$; such mappings are commonly referred to as *swallowtail* singularities. In order to prove this result it is crucial to get a sharp result for operators with two-sided cusp singularities.

Theorem.

(i) *Suppose that the only singularities of one of the projections (π_L or π_R) are Whitney folds, Whitney cusps or swallowtails. Then $\|T_\lambda\|_{L^2 \rightarrow L^2} = O(\lambda^{-(d-1)/2-1/8})$ for $\lambda \geq 1$.*

(ii) *Suppose that the only singularities of both projections π_L and π_R are Whitney folds or Whitney cusps. Then $\|T_\lambda\|_{L^2 \rightarrow L^2} = O(\lambda^{-(d-1)/2-1/4})$ for $\lambda \geq 1$.*

A slightly weaker result than (ii) was recently obtained by Comech and Cuccagna [4], who proved for two-sided cusp singularities the bound $\|T_\lambda\|_{L^2 \rightarrow L^2} \leq C_\varepsilon \lambda^{-(d-1)/2-1/4+\varepsilon}$ with $C_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

We shall prove somewhat more general results about operators of the same "type" but with the stability assumptions weakened. To formulate the hypotheses we review the definition of kernel vector fields for a map. Fix n -dimensional manifolds M, N and points $P_0 \in M$ and $Q_0 \in N$. Let $f : M \rightarrow N$ be a C^∞ map with $f(P_0) = Q_0$. Let \mathcal{U} be a neighborhood of P . A vector field V is a *kernel field* for the map f on \mathcal{U} if V is smooth on \mathcal{U} and if $Df_P V = \det(Df_P) W_{f(P)}$ for $P \in \mathcal{U}$; here W is a smooth vector field on N defined near $Q_0 = f(P_0)$ and $\det(Df_P)$ is calculated with respect to any local systems of coordinates.

Suppose now that $\text{rank } Df(P_0) \geq n - 1$. Then there is a neighborhood of P and a nonvanishing kernel vector field V for f on \mathcal{U} . If \tilde{V} is another kernel field on U then $\tilde{V} = \alpha V - \det(Df)W$ in some neighborhood of P_0 , for some vector field W and smooth function α .

This is easy to see by an elementary calculation. Indeed we may choose coordinates $x = (x', x_n)$ on M , $y = (y', y_n)$ on N vanishing at P_0 and Q_0 , respectively, so that $D'_x f = (A, b)$ and $D_{x_n} f = (c^t, d)$ where A is an invertible $(n - 1) \times (n - 1)$ matrix, b and c are vectors in $d \in \mathbb{R}$, and A, b, c, d depend smoothly on x . Define the vector field V by $V = \partial_{x_n} - \langle A^{-1}b, \partial_{x'} \rangle$. Then clearly $Df(V) =$

$(d - c^t A^{-1}b)\partial_{y_n}$ and $\det Df = (d - c^t A^{-1}b)\det A$; thus V is a kernel field. Now assume that $\tilde{V} = \langle \beta', \partial_{x'} \rangle + \beta_n \partial_{x_n}$ so that $Df(\tilde{V}) = \det(Df)Z$ with $Z = Z' + \gamma_n \partial_{y_n}$, $Z' = \langle \gamma', \partial_{y'} \rangle$; here $\beta = (\beta', \beta_n)$ are smooth functions of x and $\gamma = (\gamma', \gamma_n)$ are smooth functions of y . Then, at any x , $A\beta' + b\beta_n = \det(Df)\beta'$; therefore $\beta' = -A^{-1}b\beta_n + \det Df A^{-1}\gamma'$ and thus $\tilde{V} = \beta_n V + (\det Df)Z'$ as claimed.

Definition. Suppose that M and N are smooth n -dimensional manifolds and that $f : M \rightarrow N$ is a smooth map with $\dim \ker(Df) \leq 1$ on M . We say that f is of type k at P if there is a nonvanishing kernel field V near P so that $V^j(\det Df)_P = 0$ for $j < k$ but $V^k(\det Df)_P \neq 0$.

From the previous discussion it is clear that this definition does not depend on the choice of the nonvanishing kernel field. If one assumes that Df drops rank *simply* on the singular variety $\{\det Df = 0\}$ (i.e., if $\nabla \det Df \neq 0$) then the definition agrees with the one proposed by Comech [3].

Theorem 1.1. *Suppose that both π_L and π_R are of type ≤ 2 on \mathcal{C} . Then for $\lambda \geq 1$*

$$\|T_\lambda\|_{L^2 \rightarrow L^2} = O(\lambda^{-(d-1)/2-1/4}).$$

Theorem 1.2. *Suppose that $D\pi_L$ drops rank simply on the singular variety $\{\det D\pi_L = 0\}$ and suppose that π_L is of type ≤ 3 on \mathcal{C} . Then for $\lambda \geq 1$*

$$\|T_\lambda\|_{L^2 \rightarrow L^2} = O(\lambda^{-(d-1)/2-1/8}).$$

Of course the analogous statement holds with π_L replaced by π_R in Theorem 1.2. As a corollary of both theorems we obtain the sharp endpoint estimate for two-sided cusp and one-sided swallowtail singularities stated above.

Remark. The estimates in Theorem 1.1 and Theorem 1.2 are stable under small perturbations of Φ and σ in the C^∞ -topology.

The above theorems imply sharp L^2 -Sobolev estimates for Fourier integral operators (see [8]). Let $C \subset T^*\Omega_L \setminus \{0\} \times T^*\Omega_R \setminus \{0\}$ and let $F \in I^\mu(\Omega_L, \Omega_R; C)$ (see [15] for the definition and [8] for the reduction of smoothing estimates for Fourier integral operators to decay estimates for oscillatory integral operators). As a corollary of Theorems 1.1 and 1.2 one obtains

Theorem 1.3.

(i) *If both π_L and π_R are of type ≤ 2 , then F maps $L_{\alpha, \text{comp}}^2$ to $L_{\alpha-\mu-1/4, \text{loc}}^2$.*

(ii) *If one projection (π_L or π_R) is of type ≤ 3 and the rank of its differential drops only simply, then F maps $L_{\alpha, \text{comp}}^2$ to $L_{\alpha-\mu-3/8, \text{loc}}^2$.*

Remarks. 1. As an example of part (i) of Theorem 1.3, consider as in [9, §6] a family of curves in \mathbb{R}^4 of the form

$$\gamma_x(t) = \exp(tX + t^2Y + t^3Z + t^4W)(x)$$

for smooth vector fields X, Y, Z, W on \mathbb{R}^4 such that both of the sets of vectors

$$\left\{ X, Y, Z \mp \frac{1}{6}[X, Y], W \mp \frac{1}{4}[X, Z] + \frac{1}{24}[X, [X, Y]] \right\}$$

are linearly independent at each point x . Then the generalized Radon transform

$$Rf(x) = \int_{\mathbb{R}} f(\gamma_x(t))\chi(t)dt, \quad \chi \in C_0^\infty(\mathbb{R}),$$

belongs to $I^{-\frac{1}{2}}(\mathbb{R}^4, \mathbb{R}^4; C)$ with the canonical relation C a two-sided cusp, i.e., both π_L and π_R are Whitney cusps, and hence it follows from Theorem 1.3(i) that $R : L_{\alpha, \text{comp}}^2 \rightarrow L_{\alpha+1/4, \text{loc}}^2$, for all $\alpha \in \mathbb{R}$, generalizing the well-known fact for the translation-invariant family $\gamma_x(t) = x + (t, t^2, t^3, t^4)$.

2. Consider the translation-invariant families of curves in \mathbb{R}^3 , $\gamma_x^1(t) = x + (t, t^2, t^4)$ and $\gamma_x^2(t) = x + (t, t^3, t^4)$. Then $\{\gamma_x^1\}$ is associated with a canonical relation, C^1 , which is a two-sided cusp, while $\{\gamma_x^2\}$ is associated with a canonical relation, C^2 , for which both projections are type 2, but not Whitney cusps. In fact, the singular variety of C^2 is not smooth: it is a union of two intersecting hypersurfaces, and $\det(D\pi_L)$ and $\det(D\pi_R)$ vanish of order two at the intersection and simply away from that intersection. Averaging operators associated with any (not necessarily translation-invariant) sufficiently small C^∞ perturbation of either $\{\gamma_x^1\}$ or $\{\gamma_x^2\}$ will still have both projections of type 2 and hence map $L_{\alpha, \text{comp}}^2 \rightarrow L_{\alpha+\frac{1}{4}, \text{loc}}^2$.

3. As an instance of part (ii) of Theorem 1.3, let \mathcal{R} be the restricted X-ray transform associated to a well-curved line complex \mathfrak{C} in \mathbb{R}^5 (see [9, §5] for the definition). Then π_R has (at most) swallowtail singularities and \mathcal{R} maps $L_{\text{comp}}^2(\mathbb{R}^5)$ into $L_{1/8, \text{loc}}^2(\mathfrak{C})$. As an example consider a curve $\alpha \rightarrow \gamma(\alpha)$ in \mathbb{R}^4 with $\gamma', \gamma'', \gamma'''$ and $\gamma^{(4)}$ being linearly independent at each α and consider the X-ray transform associated to the rigid 5-dimensional line complex consisting of lines $\{\ell_{x', \alpha} : x' \in \mathbb{R}^4, \alpha \in \mathbb{R}\}$ in \mathbb{R}^5 where $\ell_{x', \alpha} = \{(x' + t\gamma(\alpha), t), t \in \mathbb{R}\}$, and perturbations of this example. For the rigid case the projection π_L is a blowdown in the sense of [13] or [14], i.e., it exhibits a maximal degeneracy; this behavior however is not invariant under small perturbations and is not required for Theorem 1.3 to apply.

4. As an example of a restricted X-ray transform in \mathbb{R}^4 which is not well-curved in the sense of [9], consider the situation as in the previous example, but with γ replaced by one of the curves $\gamma^{(1)}(\alpha) = (\alpha, \alpha^2, \alpha^4)$ or $\gamma^{(2)}(\alpha) = (\alpha, \alpha^3, \alpha^4)$ in \mathbb{R}^3 . For both examples π_R satisfies a type three condition with $\det d\pi_R$ vanishing simply; however the singularity of $d\pi_R$ for the canonical relation associated to the second line complex (defined by $\gamma^{(2)}$) is not of swallowtail type. Again \mathcal{R} and perturbations thereof map $L_{\text{comp}}^2(\mathbb{R}^4)$ into $L_{1/8, \text{loc}}^2(\mathfrak{C})$.

5. For conormal operators in two dimension the condition of type $\leq k$ for π_L corresponds to a left finite type condition of order $k+2$ in the terminology of [23], and the condition of (exact) type k corresponds to the type $(1, k+1)$ condition in the terminology of [24].

2. Bounds for operators with two-sided type two conditions

We decompose the operator according to the size of $\det \Phi_{xz}$, following Phong and Stein [20] who used this decomposition to estimate operators with fold singularities. Various extensions and refinements are in [23], [21], [5], [10], [3], [4]; in fact we will use the key estimate in [4] as the first step in our proof of Theorem 1.1. As in that work (see also [23], [3]) we shall need to localize $V_L h$ and $V_R h$ where V_L and V_R are nonvanishing kernel vector fields for π_L and π_R , respectively. We may suppose that the support of σ is small and choose coordinates $x = (x', x_d)$, $z = (z', z_d)$ in $\mathbb{R}^{d-1} \times \mathbb{R}$ vanishing at a reference point $P^0 = (x^0, z^0)$ so that

$$\Phi_{x'z'}(P^0) = I_{d-1}, \quad \Phi_{x_d z'}(P^0) = 0, \quad \Phi_{x' z_d}(P^0) = 0.$$

Write $\Phi^{z'x'} := \Phi_{x'z'}^{-1}$ and $\Phi^{x'z'} := \Phi_{z'x'}^{-1} = (\Phi_{x'z'}^t)^{-1}$. Representatives for the kernel vector fields are then given by

$$(2.1) \quad \begin{aligned} V_R &= \partial_{x_d} - \Phi_{x_d z'} \Phi^{z'x'} \partial_{x'} \\ V_L &= \partial_{z_d} - \Phi_{z_d x'} \Phi^{x'z'} \partial_{z'} \end{aligned}$$

(see [2] and the discussion in the introduction).

Let K be a fixed compact set in $\Omega_L \times \Omega_R$ which contains the support of σ in its interior. Let $A_0 \geq 10^{2d}$ so that

$$(2.2) \quad \|\Phi\|_{C^5(K)} \leq 10^{-2d} A_0.$$

We also assume that

$$(2.3) \quad |V_L^2 h| \geq A_1^{-1}, \quad |V_R^2 h| \geq A_1^{-1}.$$

for some $A_1 \geq 1$. After additional localization we may assume that σ is supported on a set of small diameter ε , for later use we choose

$$(2.4) \quad \varepsilon = 10^{-1} \min\{A_0^{-2}, A_1^{-2}\}.$$

Let $\beta_0 \in C^\infty(\mathbb{R})$ be an even function supported in $(-1, 1)$, and equal to one in $(-1/2, 1/2)$. Let $\beta(s) \equiv \beta_1(s) = \beta_0(s/2) - \beta_0(s)$ and for $j \geq 1$ let $\beta_j(s) = \beta_1(2^{1-j}s) = \beta_0(2^{-j}s) - \beta_0(2^{-j+1}s)$.

We may assume that λ is large. Let $\ell_0 = \lceil \log_2(\sqrt{\lambda}) \rceil$, that is the largest integer ℓ so that $2^\ell \leq \lambda^{1/2}$. Let then

$$(2.5) \quad \begin{aligned} \sigma_{j,k,l}(x, z) &= \sigma(x, z) \beta(2^l h(x, z)) \beta_j(2^{l/2} V_R h(x, z)) \beta_k(2^{l/2} V_L h(x, z)) \\ \sigma_{j,k,\ell_0}^0(x, z) &= \sigma(x, z) \beta_0(2_0^\ell h(x, z)) \beta_j(2^{\ell_0/2} V_R h(x, z)) \beta_k(2^{\ell_0/2} V_L h(x, z)); \end{aligned}$$

thus if $j, k > 0$ then on the support of $\sigma_{j,k,l}$ we have that $|h| \approx 2^{-l}$, $|V_L h| \approx 2^{k-l/2}$, $|V_R h| \approx 2^{j-l/2}$.

Our main technical result sharpens estimates given in [4]; we use here, as throughout, the notation $A \lesssim B$ to denote inequalities $A \leq CB$ with constants C independent of λ, j, k, l .

Theorem 2.1. *We have the following estimates:*

(i) For $0 < l < \ell_0 = \lceil \log_2(\sqrt{\lambda}) \rceil$

$$(2.6) \quad \|T_\lambda[\sigma_{j,k,l}]\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-(d-1)/2} \min\{2^{l/2} \lambda^{-1/2}; 2^{-(l+j+k)/2}\}.$$

(ii)

$$(2.7) \quad \|T_\lambda[\sigma_{j,k,\ell_0}^0]\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-(d-1)/2-1/4} 2^{-(j+k)/2}.$$

Given Theorem 2.1 we can deduce Theorem 1.2 by simply summing the estimates (2.6) and (2.7): The bound $\sum_{j,k} \|T_\lambda[\sigma_{j,k,\ell_0}^0]\| \leq \lambda^{-(d-1)/2-1/4}$ is immediate. Moreover

$$\sum_{0 \leq l \leq \log_2(\sqrt{\lambda})} \sum_{0 \leq j, k \leq l/2} \|T_\lambda[\sigma_{j,k,l}]\|_2 \leq I + II$$

where

$$\begin{aligned} I &\leq \sum_{0 \leq l \leq \log_2(\sqrt{\lambda})} \sum_{\substack{j,k \\ j+k \leq \log_2(\lambda 2^{-2l})}} 2^{l/2} \lambda^{-d/2} \\ &\lesssim \lambda^{-(d-1)/2-1/4} \sum_{0 \leq l \leq \log_2(\sqrt{\lambda})} (\lambda 2^{-2l})^{-1/4} [1 + \log(\lambda 2^{-2l})]^2 \lesssim \lambda^{-(d-1)/2-1/4} \end{aligned}$$

and

$$\begin{aligned} II &\leq \sum_{0 \leq l \leq \log_2(\sqrt{\lambda})} 2^{-l/2} \lambda^{-(d-1)/2} \sum_{\substack{j,k \\ j+k \geq \log_2(\lambda 2^{-2l})}} 2^{-(j+k)/2} \\ &\lesssim \lambda^{-(d-1)/2} \sum_{0 \leq l \leq \log_2(\sqrt{\lambda})} 2^{-l/2} (2^l \lambda^{-1/2}) (1 + \log_2(\lambda 2^{-2l})) \lesssim \lambda^{-(d-1)/2-1/4}. \quad \square \end{aligned}$$

We make some preliminary observations needed in the proof of Theorem 2.1. In what follows we always make the

Assumption: $k \leq j$.

For $k \geq j$ apply the corresponding estimates for the adjoint of $T_\lambda[\sigma_{j,k,l}]$. For the proof of Theorem 2.1 we may assume, by the known result for one-sided folds [8], that

$$(2.8) \quad 2^{k-l/2} \leq 2^{j-l/2} \leq \varepsilon$$

where ε is as in (2.4).

Affine changes of variables. Before starting with estimates we wish to mention the effect of changes of variables on (2.5). Set $x = x(u)$ and $z = z(v)$ and let $\Psi(u, v) = \Phi(x(u), z(v))$. Let $h(x, z) = \det \Phi_{xz}$ and $\tilde{h}(u, v) = \det \Psi_{uv}(u, v)$ then

$$\tilde{h}(u, v) = h(x(u), z(v)) \det \frac{Dx}{Du} \det \frac{Dz}{Dv}.$$

If $V_R = \sum s_i(x, z) \partial_{x_i}$, $V_L = \sum t_i(x, z) \partial_{z_i}$ and $\tilde{V}_R = \sum \sigma_i(u, v) \partial_{u_i}$, $\tilde{V}_L = \sum \tau_i(u, v) \partial_{v_i}$, then $V_R g(x(u), z(v)) = \tilde{V}_R(g(x(u), z(v)))$ and $V_L g(x(u), z(v)) = \tilde{V}_L(g(x(u), z(v)))$ if and only if $\vec{\sigma} = (\frac{Dx}{Du})^{-1} \vec{s}$ and $\vec{\tau} = (\frac{Dz}{Dv})^{-1} \vec{t}$.

In particular if our changes of variables are affine and of the form

$$(2.9) \quad x(u) = x^0 + (u' + a' u_d, u_d), \quad z(v) = z^0 + (v' + b' v_d, v_d)$$

with constant vectors $a', b' \in \mathbb{R}^{d-1}$ and $P = (x^0, z^0)$ and if V_L and V_R are of the form (2.1) then we have $\tilde{V}_R = \partial_{u_d} - (a^t + \Phi_{x_d z'} \Phi^{z' z'}) \partial_{u'}$ and $\tilde{V}_L = \partial_{v_d} - (b^t + \Phi_{z_d x'} \Phi^{x' z'}) \partial_{v'}$ where all coefficient functions are evaluated at $(x^0, z^0) + (u' + a' u_d, u_d, v' + b' v_d, v_d)$. Thus by choosing $a' = -\Phi^{x' z'}(P) \Phi_{z' x_d}(P)$, $b' = -\Phi^{z' x'}(P) \Phi_{x' z_d}(P)$ we achieve that $(\tilde{V}_R)_{0,0} = \partial_{u_d}$, $(\tilde{V}_L)_{0,0} = \partial_{v_d}$.

Localization. We shall perform various localizations to small boxes in (x, z) -space.

Let $P = (x^0, z^0) \in \Omega_L \times \Omega_R \subset \mathbb{R}_L^d \times \mathbb{R}_R^d$ and let $a \in \mathbb{R}_L^d$ and $b \in \mathbb{R}_R^d$ be vectors with $1 \leq \|a\|_\infty, \|b\|_\infty \leq 2$ and let π_a^\perp, π_b^\perp be the orthogonal projections to the orthogonal complement of $\mathbb{R}a$ in \mathbb{R}_L^d and $\mathbb{R}b$ in \mathbb{R}_R^d , respectively. Suppose $0 < \gamma_1 \leq \gamma_2 < 1$ and $0 < \delta_1 \leq \delta_2 \leq \varepsilon$ and let

$$(2.10) \quad B_P^{a,b}(\gamma_1, \gamma_2, \delta_1, \delta_2) = \{(x, z) : |\pi_a^\perp(x - x^0)| \leq \gamma_1, |\langle x - x^0, a \rangle| \leq \gamma_2, |\pi_b^\perp(z - z^0)| \leq \delta_1, |\langle z - z^0, b \rangle| \leq \delta_2.\}$$

Definition. We say that $\chi \in C_0^\infty$ is a *normalized cutoff function associated to* $B_P^{a,b}(\gamma_1, \gamma_2, \delta_1, \delta_2)$ if it is supported in $B_P^{a,b}(\gamma_1, \gamma_2, \delta_1, \delta_2)$ and satisfies the (natural) estimates

$$|(\pi_a^\perp \nabla_x)^{m_L} \langle a, \nabla_x \rangle^{n_L} (\pi_b^\perp \nabla_z)^{m_R} \langle b, \nabla_z \rangle^{n_R} \chi(x, z)| \leq \gamma_1^{-m_L} \gamma_2^{-n_L} \delta_1^{-m_R} \delta_2^{-n_R}$$

whenever $m_L + n_L \leq 10d$, $m_R + n_L \leq 10d$. Here $(\pi_a^\perp \nabla_x)^{n_L}$ stands for any differential operator $\langle \vec{u}_1, \partial_x \rangle \dots \langle \vec{u}_{n_L}, \partial_x \rangle$ where the vectors u_1, \dots, u_{n_L} are unit vectors perpendicular to a .

We denote by $\mathcal{Z}_P^{a,b}(\gamma_1, \gamma_2, \delta_1, \delta_2)$ the class of all normalized cutoff functions associated to $B_P^{a,b}(\gamma_1, \gamma_2, \delta_1, \delta_2)$.

We shall often localize to boxes of the form (2.10) and consider $T_\lambda[\zeta \sigma_{j,k,l}]$ where ζ is a cutoff function which is controlled by an absolute constant times a normalized cutoff function in the above sense.

Suppose now that $P = (x^0, z^0)$ and our change of variable is as in (2.9) and that $a = (a', 1)$, $b = (b', 1)$. Suppose that ζ is a normalized cutoff function associated to $B_P^{a,b}(\gamma_1, \gamma_2, \delta_1, \delta_2)$. Let $\tilde{\zeta}(u, v) = \zeta(x(u), z(v))$. Then $\tilde{\zeta}$ is supported in $\tilde{B}_{(0,0)}^{e_d, e_d}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\delta}_1, \tilde{\delta}_2)$ with $\tilde{\gamma} = (1 + |a'|)\gamma$, $\tilde{\delta} = (1 + |b'|)\delta$ and there is a positive constant C (independent of γ, δ) so that $C^{-1}\tilde{\zeta}$ is a normalized cutoff function associated to $\tilde{B}_{(0,0)}^{e_d, e_d}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\delta}_1, \tilde{\delta}_2)$.

Changing variables as in (2.9) in the expression for the operator $T_\lambda[\zeta \sigma_{j,k,l}]$ yields that

$$T_\lambda[\zeta \sigma_{j,k,l}]f(z(v)) = \int \tilde{\zeta}(u, v) \tilde{\sigma}_{j,k,l} e^{i\lambda \Psi(u,v)} dv$$

with $\tilde{\sigma}_{j,k,l}(u, v) = \sigma(x(u), z(v)) \beta(2^l \tilde{h}(u, v)) \beta_j(2^{l/2} \tilde{V}_R \tilde{h}(u, v)) \beta_k(2^{l/2} \tilde{V}_L \tilde{h}(u, v))$ and $\tilde{h}(u, v) = \det \Psi_{uv}(u, v)$.

Basic estimates.

We now give estimates for various pieces localized to (thin) boxes which will usually be longer in the directions of the kernel fields V_R and V_L .

In order to formulate our results we start with a definition.

Definition. Let $P = (x^0, z^0) \in \Omega_L \times \Omega_R$ and let

$$(2.11) \quad a_P = (-\Phi^{x'z'}(P) \Phi_{z'x_d}(P), 1), \quad b_P = (-\Phi^{z'x'}(P) \Phi_{x'z_d}(P), 1)$$

Define, for fixed j, k, l ,

$$(2.12) \quad \mathcal{A}_P(\gamma_1, \gamma_2, \delta_1, \delta_2) := \sup \left\{ \|T_\lambda[\zeta \sigma_{j,k,l}]\| : \zeta \in \mathcal{Z}_P^{a_P, b_P}(\gamma_1, \gamma_2, \delta_1, \delta_2) \right\}$$

$$(2.13) \quad \mathcal{A}_P^0(\gamma_1, \gamma_2, \delta_1, \delta_2) := \sup \left\{ \|T_\lambda[\zeta \sigma_{j,k,\ell_0}^0]\| : \zeta \in \mathcal{Z}_P^{a_P, b_P}(\gamma_1, \gamma_2, \delta_1, \delta_2) \right\}.$$

Here $\ell_0 = \lceil \log_2(\sqrt{\lambda}) \rceil$.

The main estimate in Comech-Cuccagna[4] applies to operators whose kernels are localized to boxes $B_P^{a_P, b_P}(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2})$. This result is formulated in (2.14) of the following proposition. The constants implicit in the inequalities below, do not depend on j, k, l .

Proposition 2.2. (i) For $2^l \leq \lambda^{1/2}$, $k \leq j \leq l/2$,

$$(2.14) \quad \sup_P \mathcal{A}_P(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2}) \lesssim 2^{l/2} \lambda^{-d/2}.$$

and

$$(2.15) \quad \sup_P \mathcal{A}_P(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2}) \lesssim 2^{-(l+j+k)/2} \lambda^{-(d-1)/2}.$$

(ii) Let $l = [\log_2(\sqrt{\lambda})] = \ell_0$. Then for $k \leq j \leq \ell_0/2$,

$$(2.16) \quad \sup_P \mathcal{A}_P^0(2^{-\ell_0}, 2^{-j-\ell_0/2}, 2^{-\ell_0}, 2^{-k-\ell_0/2}) \lesssim \lambda^{-(d-1)/2-1/4} 2^{-(j+k)/2}.$$

Proposition 2.2 is the starting point in our proof and is extended via orthogonality arguments. The basic steps are contained in the following Propositions 2.3-2.5.

In what follows N , denotes an integer $\leq 10d - 1$ and $l = [\log_2(\sqrt{\lambda})] = \ell_0$. Then the following estimates hold uniformly in j, k, l .

Proposition 2.3.

(i) For $2^l \leq \lambda^{1/2}$, $k \leq j \leq l/2$,

$$(2.17) \quad \begin{aligned} & \sup_P \mathcal{A}_P(2^{j+k-l}, 2^{k-l/2}, 2^{j+k-l}, 2^{k-l/2}) \\ & \lesssim \sup_P \mathcal{A}_P(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2}) + 2^{-l(2d-1)/2} 2^{-(j+k)/2} (2^{-2l} \lambda)^{-N/2}. \end{aligned}$$

(ii) For $k \leq j \leq \ell_0/2$,

$$(2.18) \quad \begin{aligned} & \sup_P \mathcal{A}_P^0(2^{j+k-\ell_0}, 2^{k-\ell_0/2}, 2^{j+k-\ell_0}, 2^{k-\ell_0/2}) \\ & \lesssim \sup_P \mathcal{A}_P^0(2^{-\ell_0}, 2^{-j-\ell_0/2}, 2^{-\ell_0}, 2^{-k-\ell_0/2}) + \lambda^{-d/2+1/4} 2^{-(j+k)/2}. \end{aligned}$$

Proposition 2.4.

(i) For $2^l \leq \lambda^{1/2}$, $k \leq j \leq l/2$,

$$(2.19) \quad \begin{aligned} & \sup_P \mathcal{A}_P(2^{j-l/2}, 2^{j-l/2}, 2^{k-l/2}, 2^{k-l/2}) \\ & \lesssim \sup_P \mathcal{A}_P(2^{j+k-l}, 2^{k-l/2}, 2^{j+k-l}, 2^{k-l/2}) + 2^{(j+k)(d-1)} 2^k 2^{-l(2d-1)/2} (2^{j+k-2l} \lambda)^{-N/2}. \end{aligned}$$

(ii) For $k \leq j \leq \ell_0/2$,

$$(2.20) \quad \begin{aligned} & \sup_P \mathcal{A}_P^0(2^{j-\ell_0/2}, 2^{j-\ell_0/2}, 2^{k-\ell_0/2}, 2^{k-\ell_0/2}) \\ & \lesssim \sup_P \mathcal{A}_P^0(2^{j+k-\ell_0}, 2^{k-\ell_0/2}, 2^{j+k-\ell_0}, 2^{k-\ell_0/2}) + 2^{(j+k) \frac{2(d-1)-N}{2} + k} \lambda^{-\frac{2d-1}{4} - \frac{N}{2}}. \end{aligned}$$

Proposition 2.5.

(i) For $2^l \leq \lambda^{1/2}$, $k \leq j \leq l/2$,

$$(2.21) \quad \|T[\sigma_{j,k,l}]\| \lesssim \sup_P \mathcal{A}_P(2^{j-l/2}, 2^{j-l/2}, 2^{k-l/2}, 2^{k-l/2}) + 2^{(j+k-l)d/2}(\lambda 2^{k-3l/2})^{-N/2}.$$

(ii) For $k \leq j \leq \ell_0/2$,

$$(2.22) \quad \|T[\sigma_{j,k,\ell_0}^0]\| \lesssim \sup_P \mathcal{A}_P^0(2^{j-\ell_0/2}, 2^{j-\ell_0/2}, 2^{k-\ell_0/2}, 2^{k-\ell_0/2}) + 2^{(j+k)d/2-kN/2} \lambda^{-d/2-N/8}.$$

Taking these estimates for granted we can give the

Proof of Theorem 2.1.

Observe that since $k \leq j \leq l/2$ and $2^l \leq \lambda^{1/2}$ the quantities $2^{-l(2d-1)/2} 2^{-(j+k)/2} (2^{-2l} \lambda)^{-N/2}$, $2^{(j+k)(d-1)/2+k-l(2d-1)/2} (2^{j+k-2l} \lambda)^{-N/2}$ and $2^{(j+k-l)d/2} (\lambda 2^{k-3l/2})^{-N/2}$ are all dominated by a constant times $\lambda^{-(d-1)/2} \min\{2^{-(l+j+k)/2}, 2^{l/2} \lambda^{-1/2}\}$, and a combination of the first parts of the Propositions 2.3-2.5 gives

$$\|T_\lambda[\sigma_{j,k,l}]\| \lesssim \sup_P \mathcal{A}_P(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2}) + \lambda^{-(d-1)/2} \min\{2^{-(l+j+k)/2}, 2^{l/2} \lambda^{-1/2}\}.$$

We estimate the quantities $\mathcal{A}_P(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2})$ by Proposition 2.2 and (2.5) follows. (2.6) is proved in the same way, using instead (2.18), (2.20) and (2.22). \square

3. Proofs of the Propositions

Preliminaries. We begin by stating two elementary Lemmas which will be used several times in the proof of Propositions 2.3-5.

Lemma 3.1. *Suppose that $\zeta \in \mathcal{Z}_P^{a_P, b_P}(\gamma_1, \gamma_2, \delta_1, \delta_2)$. Then $\zeta = \sum_{i=1}^M c_i \zeta_i$ where $\zeta_i \in \mathcal{Z}_{Q_i}(\varepsilon \gamma_1, \varepsilon \gamma_2, \varepsilon \delta_1, \varepsilon \delta_2)$ with $Q_i \in B_P^{a_P, b_P}(\gamma_1, \gamma_2, \delta_1, \delta_2)$ and so that $|c_i|, M \leq C_\varepsilon$ (independent of the specific choice of ζ and γ, δ, P).*

Proof. Immediate. \square

Lemma 3.2. *Let $Q = (x^Q, z^Q) \in B_P^{a_P, b_P}(\gamma_1, \gamma_2, \delta_1, \delta_2)$ where $\gamma_1 \leq \gamma_2 \leq \varepsilon$, $\delta_1 \leq \delta_2 \leq \varepsilon$. Suppose that $0 < \tilde{\gamma}_1 \leq \tilde{\gamma}_2 \leq \varepsilon^{-1} \gamma_2$, $0 < \tilde{\delta}_1 \leq \tilde{\delta}_2 \leq \varepsilon^{-1} \delta_2$ and assume that*

$$(3.1) \quad \min\left\{\frac{\tilde{\gamma}_1}{\tilde{\gamma}_2}, \frac{\tilde{\delta}_1}{\tilde{\delta}_2}\right\} \geq \max\{\gamma_2, \delta_2\}.$$

Then there are positive constants C, C_1 (independent of $\gamma, \tilde{\gamma}, \delta, \tilde{\delta}, P, Q$) so that for $\zeta \in \mathcal{Z}_Q^{a_P, b_P}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\delta}_1, \tilde{\delta}_2)$ the function $C_1^{-1} \zeta$ belongs to $\mathcal{Z}_Q^{a_Q, b_Q}(C \tilde{\gamma}_1, C \tilde{\gamma}_2, C \tilde{\delta}_1, C \tilde{\delta}_2)$.

Proof. Observe that

$$|a_P - a_Q| + |b_P - b_Q| \lesssim \max\{\gamma_2, \delta_2\}.$$

The relevant geometry is then that by assumption (3.1) the boxes $B_Q^{a_P, b_P}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\delta}_1, \tilde{\delta}_2)$ and $B_Q^{a_Q, b_Q}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\delta}_1, \tilde{\delta}_2)$ are contained in fixed dilates of each other. The asserted estimates are easy to check. \square

We shall denote by η a $C_0^\infty(\mathbb{R})$ function which is supported in $(-1, 1)$ and satisfies $\sum_{n \in \mathbb{Z}} \eta(\cdot - n) \equiv 1$. Moreover the $C_0^\infty(\mathbb{R}^{d-1})$ function χ is defined by $\chi(x_1, \dots, x_{d-1}) = \prod_{i=1}^{d-1} \eta(x_i)$. In the proofs of Propositions 3.3-5 we shall use dilates and translates of η and χ to decompose a suitable cutoff function ζ as

$$(3.2) \quad \zeta = \sum_{X, Z \in \mathbb{Z}^d} \zeta_{XZ}$$

the definition of ζ_{XZ} depends on the particular geometry and is given by (3.12), (3.25) and (3.30) below in the three respective cases. We shall then employ orthogonality arguments to estimate the operator norm of $T_\lambda[\zeta \sigma_{j,k,l}]$ in terms of the operator norms of

$$T_{XZ} := T_\lambda[\zeta_{XZ} \sigma_{j,k,l}].$$

This is done by using the Cotlar-Stein Lemma [25, ch. VII,2]. We then have to estimate the kernels of $T_{XW}^* T_{\tilde{X}Z}$ and $T_{XZ} T_{Y\tilde{Z}}^*$.

The kernel of $T_{XW}^* T_{\tilde{X}Z}$ is given by

$$(3.3) \quad H(w, z) \equiv H_{XW\tilde{X}Z}(w, z) = \int e^{-i\lambda(\Phi(x,w) - \Phi(x,z))} \kappa_{XW\tilde{X}Z}(x, w, z) dx$$

where

$$(3.4) \quad \kappa_{XW\tilde{X}Z}(x, w, z) = \zeta_{\tilde{X}Z}(x, z) \overline{\zeta_{XW}(x, w)} \sigma_{j,k,l}(x, z) \overline{\sigma_{j,k,l}(x, w)}.$$

The kernel of $T_{XZ} T_{Y\tilde{Z}}^*$ is given by

$$(3.5) \quad K(x, y) \equiv K_{XZY\tilde{Z}}(x, y) = \int e^{i\lambda(\Phi(x,z) - \Phi(y,z))} \omega_{XZY\tilde{Z}}(z, x, y) dz$$

with

$$(3.6) \quad \omega_{XZY\tilde{Z}}(z, x, y) = \zeta_{XZ}(x, z) \overline{\zeta_{Y\tilde{Z}}(y, z)} \sigma_{j,k,l}(x, z) \overline{\sigma_{j,k,l}(y, z)}.$$

Our localizations ζ_{XZ} will always have the property that the supports of ζ_{XW} and $\zeta_{\tilde{X}Z}$ are disjoint whenever $|X_i - \tilde{X}_i| \geq 3$ for some $i \in \{1, \dots, d\}$. Moreover the supports of ζ_{XZ} and $\zeta_{Y\tilde{Z}}$ are disjoint whenever $|Z_i - \tilde{Z}_i| \geq 3$ for some $i \in \{1, \dots, d\}$. This implies that

$$(3.7) \quad T_{XW}^* T_{\tilde{X}Z} = 0 \quad \text{if } |X - \tilde{X}|_\infty \geq 3, \quad T_{XZ} T_{Y\tilde{Z}}^* = 0 \quad \text{if } |Z - \tilde{Z}|_\infty \geq 3.$$

In what follows we shall split variables X and Z in \mathbb{Z}^d as $X = (X', X_d)$, $Z = (Z', Z_d)$. The geometric meaning of this splitting depends on the particular situation in Propositions 3.3-5.

The main orthogonality properties will always follow from either the localization properties of the operator in terms of h , $V_L h$ or $V_R h$, or by an integration by parts with respect to the directions

orthogonal to a_P or b_P . To describe this we assume that $a_P = e_d$, $b_P = e_d$ at a suitable reference point, a situation which we will always be able to achieve by an affine change of variables as described in §2. If $\Phi_{x'}(x, w) \neq \Phi_{x'}(x, z)$ for all x with $(x, w, z) \in \text{supp } \kappa_{XW\tilde{X}Z}$ then we may integrate by parts with respect to the x' variables; specifically we have

$$(3.8) \quad H(w, z) = (i/\lambda)^N \int e^{-i\lambda(\Phi(x, w) - \Phi(x, z))} \mathcal{L}^N[\kappa_{XW\tilde{X}Z}](x, w, z) dx$$

where the differential operator \mathcal{L} is defined by

$$(3.9) \quad \mathcal{L}g = \text{div}_{x'} \left(\frac{\Phi_{x'}(x, z) - \Phi_{x'}(x, w)}{|\Phi_{x'}(x, z) - \Phi_{x'}(x, w)|^2} g \right).$$

Similar formulas hold for the z' integration by parts for the integral defining $K(x, y)$.

We shall give a proof of the estimates (2.17), (2.19) and (2.21), and the proof of (2.18), (2.20) and (2.22) is similar. Here we note that the *lower* bound on $|h|$ in the localization (2.5) is used in the proof of estimate (2.14); however it is not needed for the proof of Propositions 2.3-2.5.

Remarks on the proof of Proposition 2.2. In order to prove (2.14) it suffices, by Lemma 3.1, to estimate $\mathcal{A}_P(\varepsilon 2^{-l}, \varepsilon 2^{-j-l/2}, \varepsilon 2^{-l}, \varepsilon 2^{-k-l/2})$ for small ε .

By an affine change of variable as discussed in (2.9) we may assume that $P = (0, 0)$ and that $\Phi_{z'x_d}(0, 0) = 0$, $\Phi_{x'z_d}(0, 0) = 0$, thus $\Phi_{x'z'}$ is close to the identity I_{d-1} on the support of ζ and the quantities $|\Phi_{z'x_d}(x, z)|$ and $|\Phi_{x'z_d}(x, z)|$ are bounded by $A_0 \varepsilon 2^{-k-l/2}$ for $(x, z) \in \text{supp } \zeta$ (recall that $k \leq j$). Moreover $a_P = e_d$, $b_P = e_d$; thus ζ is, up to a constant, a normalized cutoff function associated to a box where $|x'|, |z'| \leq \varepsilon 2^{-l}$, $|x_d| \leq 2^{-j-l/2}$, $|z_d| \leq 2^{-k-l/2}$. This puts us in the situation as in the proof of [4, (3.6)]. If $\mathcal{A}_P(\varepsilon 2^{-l}, \varepsilon 2^{-j-l/2}, \varepsilon 2^{-l}, \varepsilon 2^{-k-l/2})$ does not vanish identically then the function $|h(x, z)|$ is comparable to 2^{-l} on the box $B_P^{a_P, b_P}(\varepsilon 2^{-l}, \varepsilon 2^{-j-l/2}, \varepsilon 2^{-l}, \varepsilon 2^{-k-l/2})$. Set $S = T_\lambda[\zeta \sigma_{j, k, l}]$. For $j \geq k$ (assumed here) the kernel of SS^* can be estimated using integration by parts, and all the details of this argument are provided in [4].

The estimate (2.15) is more standard, but we sketch the argument for completeness. We may assume that $(V_L)_P = \partial_{z_d}$ and $(V_R)_P = \partial_{x_d}$ and then “freezing” x_d, z_d we may write

$$Sf(x', x_d) = \int_{\mathbb{R}} S^{x_d, z_d}[f(\cdot, z_d)](x') dz_d.$$

Each S^{x_d, z_d} is an oscillatory integral operator of the form (1.1) in \mathbb{R}^{d-1} and the mixed Hessian of the phase function has maximal rank $d-1$; however the amplitudes have less favorable differentiability properties. Note that each (x', z') differentiation causes a blowup of $O(2^l) = O(\lambda^{1/2})$. These estimates for the amplitudes are analogous to the differentiability properties of symbols of type $(1/2, 1/2)$, and in this situation the classical bound remains true; one can combine Hörmander’s argument in [16] with almost-orthogonality arguments in the proof of the Calderón-Vaillancourt theorem for pseudo-differential operators [2]. See also [11] for related but somewhat different arguments for Fourier integral operators associated to canonical graphs. Here it follows that the L^2 operator norm of S^{x_d, z_d} is $O(\lambda^{-(d-1)/2})$ uniformly in x_d, z_d . From the definition of $\sigma_{j, k, l}$ we see that there are intervals I and J of length $O(2^{-j-l/2})$ and $O(2^{-k-l/2})$, respectively, so that $S^{x_d, z_d} = 0$ unless $x_d \in I$ and $z_d \in J$. Thus from applications of Minkowski’s and Cauchy-Schwarz’ inequalities it follows that $\|S\| \lesssim 2^{-(j+l/2)/2} 2^{-(k+l/2)/2} \lambda^{-(d-1)/2}$. (2.16) is proved in the same way. \square

Proof of Proposition 2.3. Fix P . By Lemma 3.1 it suffices to estimate $\|T_\lambda[\zeta \sigma_{j, k, l}]\|$ where ζ belongs to $\mathcal{Z}_P^{a_P, b_P}(\varepsilon 2^{j+k-l}, \varepsilon 2^{k-l/2}, \varepsilon 2^{j+k-l}, \varepsilon 2^{k-l/2})$, with norm independent of P .

By an affine change of variable as discussed in (2.9) we may assume that $P = (0, 0)$ and that $\Phi_{z'x_d}(0, 0) = 0$, $\Phi_{x'z_d}(0, 0) = 0$, hence

$$(3.10) \quad \|\Phi_{x'z'} - I\| \leq 2^{-d}$$

$$(3.11) \quad |\Phi_{z'x_d}(x, z)| + |\Phi_{x'z_d}(x, z)| \leq A_0 \varepsilon 2^{k-l/2}$$

for $(x, z) \in \text{supp } \zeta$. Moreover $a_P = e_d$, $b_P = e_d$; thus ζ is, up to a constant, a normalized cutoff function associated to a box where $|x'|, |z'| \leq \varepsilon 2^{j+k-l}$, $|x_d|, |z_d| \leq 2^{k-l/2}$.

For $X, Z \in \mathbb{Z}^d$ let

$$(3.12) \quad \zeta_{XZ}(x, z) = \zeta(x, z) \chi(2^l \varepsilon^{-1} x' - X') \eta(2^{j+l/2} \varepsilon^{-1} x_d - X_d) \chi(2^l \varepsilon^{-1} z' - Z') \eta(2^{k+l/2} \varepsilon^{-1} z_d - Z_d)$$

and let $T_{XZ} = T_\lambda[\zeta \sigma_{j,k,l}]$. By (3.10-11) and Lemma 3.2 there are positive constants C, C_1 so that $C_1^{-1} \zeta_{XZ}$ belongs to $\mathcal{Z}_Q^{a_Q, b_Q}(C2^{-l}, C2^{-j-l/2}, C2^{-l}, C2^{-k-l/2})$. Thus

$$(3.13) \quad \|T_{XZ}\| \lesssim \sup_P \mathcal{A}_P(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2})$$

and it remains to show almost orthogonality of the pieces T_{XZ} .

By our localization the orthogonality properties (3.7) are satisfied. Therefore the assertion (2.17) follows from

$$(3.14) \quad \begin{aligned} \|T_{XW}^* T_{\tilde{X}Z}\| &\lesssim 2^{-l(2d-1)-j-k} (\lambda 2^{-2l} |W' - Z'|)^{-N} \\ &\text{if } 2A_0 |W' - Z'| \geq |W_d - Z_d| \text{ and } |W' - Z'| \geq C_1, \end{aligned}$$

$$(3.15) \quad T_{XW}^* T_{\tilde{X}Z} = 0, \quad \text{if } 2A_0 |W' - Z'| < |W_d - Z_d| \text{ and } |W_d - Z_d| \geq C_1,$$

(for suitable $C_1 \gg 1$) and

$$(3.16) \quad \begin{aligned} \|T_{XZ} T_{Y\tilde{Z}}^*\| &\lesssim 2^{-l(2d-1)-j-k} (\lambda 2^{-2l} |X' - Y'|)^{-N}, \\ &\text{if } 2A_0 |X' - Y'| \geq |X_d - Y_d| \text{ and } |X' - Y'| \geq C_1, \end{aligned}$$

$$(3.17) \quad T_{XZ} T_{Y\tilde{Z}}^* = 0, \quad \text{if } 2A_0 |X' - Y'| < |X_d - Y_d| \text{ and } |X_d - Y_d| \geq C_1.$$

We now show (3.15) and (3.14). The kernel H of $T_{XW}^* T_{\tilde{X}Z}$ is given by (3.3), (3.4). In order to see (3.15) pick points $(x, w) \in \text{supp } \zeta_{XW}$ and $(x, z) \in \text{supp } \zeta_{\tilde{X}Z}$ and also assume that (x, w) and (x, z) belong to $\text{supp } \sigma_{j,k,l}$ (if there are no two such points then $T_{XW}^* T_{\tilde{X}Z} = 0$). By definition of $\sigma_{j,k,l}$ we have

$$(3.18) \quad |h(x, z) - h(x, w)| \leq 2^{-l+2}.$$

Also for all $(x, \tilde{z}) \in \text{supp } \zeta$ we have that

$$(3.19) \quad |h_{z_d}(x, \tilde{z}) - V_L h(x, \tilde{z})| \leq A_0 \varepsilon 2^{k-l/2}$$

so that $|h_{z_d}(x, \tilde{z})| \geq 2^{k-l/2-2}$. Note that $\varepsilon 2^{-k-l/2}(|W_d - Z_d| - 2) \leq |w_d - z_d| \leq 2^{-k-l/2}(|W_d - Z_d| + 2)\varepsilon$ and $|w' - z'| \leq C_0(|W' - Z'| + 2)2^{-l}\varepsilon$. Therefore

$$(3.20) \quad \begin{aligned} |h(x, w) - h(x, z)| &\geq |h_{z_d}(x, z)||w_d - z_d| - A_0|w' - z'| \\ &\geq 2^{k-l/2-4}\varepsilon 2^{-k-l/2}|W_d - Z_d| = \varepsilon 2^{-l-2}|W_d - Z_d| \end{aligned}$$

if $|W_d - Z_d| \geq \max\{2A_0|W' - Z'|_\infty, 2A_0\}$ and $|W_d - Z_d| \geq 2A_0$. Observe that (3.18) and (3.20) can hold simultaneously only when $|W_d - Z_d|$ stays bounded; this implies (3.15).

Now assume that $|W_d - Z_d| \leq 2A_0|W' - Z'|_\infty$, and we show (3.14) if $|W' - Z'| \geq C_1$ for sufficiently large C_1 .

We perform integration by parts with respect to the x' variables in (3.3), using (3.8/3.9). Now in view of (3.10/11) we have

$$(3.21) \quad \begin{aligned} |\Phi_{x'}(x, w) - \Phi_{x'}(x, z)| &\geq |\Phi_{x'z'}(x, z)(w' - z')| - A_0\varepsilon 2^{k-l/2}|w_d - z_d| - A_0|w - z|^2 \\ &\geq \varepsilon 2^{-l}|W' - Z'| \end{aligned}$$

if $|W' - Z'| \geq C_1$ for suitable C_1 . Moreover the x -derivatives of $\Phi(x, w) - \Phi(x, z)$ are $O(2^{-l}|W' - Z'|)$, and differentiating the symbol causes a blowup of $O(2^l)$ for each differentiation. Thus for $|W' - Z'| \geq C_1$

$$|\mathcal{L}^N(\kappa_{XW\tilde{X}Z})| \lesssim (\lambda 2^{-2l}|W' - Z'|)^{-N}.$$

Taking into account the x support this yields the estimate

$$|H(w, z)| \lesssim 2^{-l(d-1)}2^{-l/2-j}(\lambda 2^{-2l}|W' - Z'|)^{-N}.$$

By Schur's test we have to bound $\sup_w \int |H(w, z)|dz$ and $\sup_z \int |H(w, z)|dw$. Since the integrals are extended over sets of measure $O(2^{-l(d-1)}2^{-l/2-k})$ we obtain the bound (3.14).

We still have to estimate the kernel K given by (3.5), (3.6). Note that $|h_{x_d}(x, \tilde{z}) - V_R h(x, \tilde{z})| \leq A_0\varepsilon 2^{k-l/2}$ so that $|h_{x_d}(x, z)| \approx 2^{j-l/2}$ (recall that $j \geq k$). Thus in place of (3.20) we have

$$(3.22) \quad |h(x, z) - h(y, z)| \geq 2^{j-l/2}|x_d - y_d| - A_0|x' - y'|$$

and in place of (3.21) we have

$$(3.23) \quad |\Phi_{z'}(x, z) - \Phi_{z'}(y, z)| \geq |\Phi_{z'x'}(y, z)(x' - y')| - A_0\varepsilon 2^{k-l/2}|x_d - y_d| - A_0|x - y|^2.$$

Since $|x_d - y_d| \approx |X_d - Y_d|$ we proceed as before to obtain (3.16) and (3.17). \square

Proof of Proposition 2.4. We continue to use the same notations as in the previous proof although our localizations are with respect to different (larger) boxes. By Lemma 3.1 it suffices to estimate the operator norm of $T_\lambda[\zeta\sigma_{j,k,l}]$ where now $\zeta \in \mathcal{Z}_P(\varepsilon 2^{j-l/2}, \varepsilon 2^{j-l/2}, \varepsilon 2^{k-l/2}, \varepsilon 2^{k-l/2})$. Again we may assume that by an affine change of variable $P = (0, 0)$ and that $\Phi_{z'x_d}, \Phi_{x'z_d}$ vanish at $(0, 0)$. It follows that

$$(3.24) \quad |\Phi_{z'x_d}(x, z)| + |\Phi_{x'z_d}(x, z)| \leq A_0\varepsilon 2^{j-l/2}, \quad (x, z) \in \text{supp } \zeta,$$

and again $a_P = e_d, b_P = e_d$.

For $X, Z \in \mathbb{Z}^d$ we now define

$$(3.25) \quad \zeta_{XZ}(x, z) = \zeta(x, z) \times \\ \chi(2^{-j-k+l}\varepsilon^{-1}x' - X')\eta(2^{-k+l/2}\varepsilon^{-1}x_d - X_d)\chi(2^{-j-k+l}\varepsilon^{-1}z' - Z')\eta(2^{-k+l/2}\varepsilon^{-1}z_d - Z_d)$$

and set $T_{XZ} = T_\lambda[\zeta_{XZ}\sigma_{j,k,l}]$. In view of (3.24), Lemma 3.1 and Lemma 3.2

$$\|T_{XZ}\| \leq \sup_P \mathcal{A}_P(2^{j+k-l}, 2^{k-l/2}, 2^{j+k-l}, 2^{k-l/2}).$$

To show the orthogonality observe that (3.7) remains valid. Moreover the width of the smaller boxes in the z_d direction is comparable to the z_d -width of the original boxes, namely $\approx 2^{k-l/2}$. This shows that

$$(3.26) \quad T_{XW}^* T_{\tilde{X}Z} = 0 \quad \text{if } |W_d - Z_d| \geq C_1$$

for sufficiently large C_1 .

This estimate is complemented by

$$(3.27) \quad \|T_{XW}^* T_{\tilde{X}Z}\| \lesssim 2^{2(j+k)(d-1)} 2^{2k} 2^{-l(2d-1)} (\lambda 2^{j+k-2l} |W' - Z'|)^{-N}, \quad \text{if } |W' - Z'| \geq C_1,$$

for large C_1 .

To see (3.27) we integrate by parts with respect to x' . Our kernel is still given by (3.3), (3.4). To perform the integration by parts we may assume that $|W_d - Z_d| \leq C_1$ by (3.26). We now see from (3.24) that

$$|\Phi_{x'}(x, w) - \Phi_{x'}(x, z)| \geq |\Phi_{x'z'}(x, z)(w' - z')| - A_0 \varepsilon 2^{j-l/2} |w_d - z_d| - A_0 |w - z|^2$$

but $|w' - z'| \approx |W' - Z'| \varepsilon 2^{j+k-l}$, and $|w_d - z_d| \leq 2^{k-l/2} \varepsilon |W_d - Z_d| \leq C \varepsilon 2^{k-l/2}$. Thus if $|W' - Z'|$ is sufficiently large we have the lower bound

$$|\Phi_{x'}(x, w) - \Phi_{x'}(x, z)| \gtrsim 2^{j+k-l} |W' - Z'|$$

for $(x, w, z) \in \text{supp } \kappa_{XW\tilde{X}Z}$. Therefore analyzing $\mathcal{L}^N(\kappa_{XW\tilde{X}Z})$ as in the proof of Proposition 3.3 we see that

$$|\mathcal{L}^N(\kappa_{XW\tilde{X}Z})| \lesssim (2^{j+k-2l} |W' - Z'|)^{-N}.$$

From this we get the pointwise bound

$$|H(w, z)| \lesssim 2^{(j+k-2l)(d-1)} 2^{2k-l/2} (\lambda 2^{j+k-2l} |W' - Z'|)^{-N}.$$

For Schur's test we have to integrate this in x or y over a set of measure $2^{(j+k-l)(d-1)} 2^{k-l/2}$ and we obtain in fact a slightly better estimate than (3.27).

Next, it remains to show that

$$(3.28) \quad \|T_{XZ} T_{Y\tilde{Z}}^*\| \lesssim 2^{2(j+k)(d-1)} 2^{2k} 2^{-l(2d-1)} (\lambda 2^{j+k-2l} |X' - Y'|)^{-N}, \\ \text{if } 2A_0 |X' - Y'| \geq |X_d - Y_d| \text{ and } |X' - Y'| \geq C_1,$$

and

$$(3.29) \quad T_{XZ}T_{Y\bar{Z}}^* = 0 \quad \text{if } 2A_0|X' - Y'| < |X_d - Y_d| \text{ and } |X_d - Y_d| \geq C_1.$$

The proof of these estimates is similar to the proof of the corresponding estimates in Proposition 2.3. The estimate (3.22) continues to hold and the estimate (3.23) is replaced by the weaker estimate

$$|\Phi_{z'}(x, z) - \Phi_{z'}(y, z)| \geq |\Phi_{z'x'}(y, z)(x' - y')| - A_0\varepsilon 2^{j-l/2}|x_d - y_d| - A_0|x - y|^2$$

which however still gives the asserted bound since $|x_d - y_d| \approx 2^{k-l/2}\varepsilon|X_d - Y_d|$ and $|x' - y'| \approx 2^{-l}|X' - Y'|$. \square

Proof of Proposition 2.5. We may assume that the support of ζ is small (*i.e.* contained in a ball of radius ε). By Lemma 3.1 it suffices to estimate the operator norm of $T_\lambda[\zeta\sigma_{j,k,l}]$ where now $\zeta \in \mathcal{Z}_P(\varepsilon, \varepsilon, \varepsilon, \varepsilon)$. By affine changes of variables we may assume that $P = (0, 0)$ and that $\Phi_{z'x_d}, \Phi_{x'z_d}$ vanish at $(0, 0)$. Thus

$$|\Phi_{z'x_d}(x, z)| + |\Phi_{x'z_d}(x, z)| \leq A_0\varepsilon, \quad (x, z) \in \text{supp } \zeta.$$

For $X, Z \in \mathbb{Z}^d$ we now consider $T_{XZ} = T_\lambda[\zeta\sigma_{j,k,l}]$ with

$$(3.30) \quad \zeta_{XZ}(x, z) = \zeta(x, z)\chi(2^{-j+l/2}\varepsilon^{-1}x' - X') \times \\ \eta(2^{-j+l/2}\varepsilon^{-1}x_d - X_d)\chi(2^{-k+l/2}\varepsilon^{-1}z' - Z')\eta(2^{-k+l/2}\varepsilon^{-1}z_d - Z_d),$$

and again $\|T_{XZ}\| \lesssim \sup_P \mathcal{A}_P(2^{j+k-l}, 2^{k-l/2}, 2^{j+k-l}, 2^{k-l/2})$.

For the orthogonality of the pieces we now use besides (3.7) the assumptions (2.3). By our choice of ε we have that

$$|V_L^2 h - \partial_{z_d} V_L h| \leq A_0\varepsilon \leq A_1/10$$

and similarly

$$|V_R^2 h - \partial_{x_d} V_R h| \leq A_1/10.$$

Thus

$$4 \cdot 2^{k-l/2} \geq |V_L h(x, w) - V_L h(x, z)| \geq (2A_1)^{-1}|w_d - z_d| - A_0|w' - z'| \\ 4 \cdot 2^{j-l/2} \geq |V_R h(x, z) - V_R h(y, z)| \geq (2A_1)^{-1}|x_d - y_d| - A_0|x' - y'|.$$

This shows that

$$(3.31) \quad T_{XW}^* T_{\bar{X}Z} = 0 \text{ if } |W_d - Z_d| \geq C$$

$$(3.32) \quad T_{XZ} T_{Y\bar{Z}}^* = 0 \text{ if } |X_d - Y_d| \geq C$$

for $C = 10A_0A_1$.

Now assume that $|W_d - Z_d| \leq C_1$. Then if $(x, w, z) \in \text{supp } \overline{\zeta_{XW}\zeta_{XZ}}$ we have

$$|\Phi_{x'}(x, w) - \Phi_{x'}(x, z)| \geq |\Phi_{x'z'}(x, z)(w' - z')| - A_0\varepsilon|w_d - z_d| - A_0|w - z|^2$$

but now $|w' - z'| \approx |W' - Z'|2^{k-l/2}$, and $|w_d - z_d| \leq 2C2^{k-l/2}$. Thus for large $|W' - Z'|$ we have the lower bound

$$|\Phi_{x'}(x, w) - \Phi_{x'}(x, z)| \geq 2^{k-l/2}|W' - Z'|$$

and it follows that

$$|\mathcal{L}^N[\kappa_{XW\tilde{X}Z}](x, w, z)| \lesssim (\lambda 2^{k-3l/2}|W' - Z'|)^{-N}.$$

Consequently $|H(w, z)| \lesssim 2^{(j-l/2)d}(\lambda 2^{k-3l/2}|W' - Z'|)^{-N}$. To apply Schur's test we observe that for fixed z the w integral is extended over a set of measure $O(2^{(k-l/2)d})$ (likewise for fixed w the z integral). We obtain the bound

$$(3.33) \quad \|T_{XW}^* T_{\tilde{X}Z}\| \lesssim 2^{(j+k-l)d}(\lambda 2^{k-3l/2}|W' - Z'|)^{-N}$$

if $|W' - Z'| \geq C'$. By a similar argument

$$(3.34) \quad \|T_{XZ} T_{Z\tilde{Z}}^*\| \lesssim 2^{(j+k-l)d}(\lambda 2^{j-3l/2}|X' - Y'|)^{-N}.$$

The asserted estimate (2.21) now follows from combining (3.31-34) and the estimate for the individual pieces. \square

4. One-sided type three singularities

In this section we discuss the proof of Theorem 1.2. The reasoning is very close to the one given by the authors in [9], but the assumptions there are somewhat different. We thus only sketch the proof and refer the reader to [8], [9] for details of some of the arguments.

First we shall need an extension of Theorem 1.1 to oscillatory integral operators of the form

$$\mathcal{T}_\mu f(x) = \int f(y) \int e^{i\mu\psi(x, y, \vartheta)} a(x, y, \vartheta) d\vartheta dy$$

where the frequency variable ϑ lives in an open set $\Theta \subset \mathbb{R}^N$ and we assume that $a \in C_0^\infty(\Omega_L \times \Omega_R \times \Theta)$. It is assumed that ψ is a nondegenerate phase function in the sense of Hörmander [15] (but not necessarily homogeneous), *i.e.* $\nabla_{\vartheta_i}(\nabla_{x, y, \vartheta}\Psi)$, $i = 1, \dots, N$ are linearly independent. The canonical relation $C_\psi \subset T^*\Omega_L \times T^*\Omega_R$ is given by

$$C_\psi = \{(x, \psi_x, y, -\psi_y) : \psi_\vartheta = 0\}.$$

Lemma 4.1. *Suppose that the projections $\pi_L : C_\psi \rightarrow T^*\Omega_L$, $\pi_R : C_\psi \rightarrow T^*\Omega_R$ are of type ≤ 2 . Then $\|\mathcal{T}_\mu\|_{L^2 \rightarrow L^2} = O(\mu^{-(d+N-1)/2-1/4})$, $\mu \rightarrow \infty$. This estimate is stable under small perturbations of ψ and a in the C^∞ -topology.*

The reduction to the situation in Theorem 1.2 involves canonical transformations on $T^*\Omega_L$ and $T^*\Omega_R$ and then as in [15] an application of the method of stationary phase to reduce the number of frequency variables (see [8] for details).

The following Lemma deals with phase functions $\Phi(x, z)$ without frequency variables.

Lemma 4.2. *Let Φ be a real-valued phase function defined near (x^0, z^0) and assume that $\nabla_x(\det \Phi_{xz}(x^0, z^0)) \neq 0$, and $|\det \Phi_{xz}(x^0, z^0)| + |V_R \det \Phi_{xz}(x^0, z^0)| \leq c|\nabla_x(\det \Phi_{xz}(x^0, z^0))|$. Let $M > 0$.*

Then, if c is sufficiently small, there are neighborhoods Ω_L^0 of x_0 , Ω_R^0 of z_0 , neighborhoods \mathcal{U} and \mathcal{V} of $(x_0, \nabla_x \Phi(x^0, z^0))$ in T^Ω_L, a canonical transformation $\chi : \mathcal{U} \rightarrow \mathcal{V}$, and a unitary operator U_λ , so that the following statements hold if σ is supported in $\Omega_L^0 \times \Omega_R^0$.*

(i) *If T_λ is the integral operator with kernel $\sigma(x, z)e^{i\lambda\Phi(x, z)}$ then*

$$U_\lambda T_\lambda = S_\lambda + R_\lambda$$

where S_λ is an integral operator with kernel $\tau(x, z)e^{i\lambda\Psi(x, z)}$ and $\|R_\lambda\|_{L^2 \rightarrow L^2} = O(\lambda^{-M})$,

(ii) *If $\mathcal{C}_\Phi = \{(x, \Phi_x, z, -\Phi_z), (x, z) \in \text{supp } \sigma\}$ then for $\mathcal{C}_\Psi = \{(x, \Psi_x, z, -\Psi_z), (x, z) \in \text{supp } \tau\}$ we have*

$$\mathcal{C}_\Psi \subset \{(\chi(x, \xi), z, \zeta) : (x, \xi, z, \zeta) \in \mathcal{C}_\Phi\}.$$

(iii) *$\nabla_z(\det \Psi_{xz}) \neq 0$ for $(x, z) \in \text{supp } \tau$.*

Proof. This can be extracted from the arguments in §4 of [9].

Proof of Theorem 1.2. We work with T_λ as in (1.1) where (x, z) is close to the origin, and the origin lies on the singular surface $\{(x, z) : \det \Phi_{xz} = 0\}$. We may assume, after a change of variable in z that

$$(4.1) \quad \Phi(0, z) = 0, \quad \Phi_{x'z'}(0, z) = I, \quad \Phi_{x'z_d}(0, z) = 0$$

(cf. the proof of Lemma 2.7 in [9]) and by a change of variable in x we may also assume

$$\Phi_{z'x_d}(0, 0) = 0.$$

We assume that π_L is of type ≤ 2 and that $\nabla_{x,z}(\det \Phi_{xz}) \neq 0$ where $\det \Phi_{xz}$ vanishes.

If $\Phi_{x_d x_d z_d} \neq 0$ or $\Phi_{x_d z_d z_d} \neq 0$ or $\Phi_{x_d z_d z_d z_d} \neq 0$ then we have a fold or cusp singularity and better results than the one claimed in Theorem 1.2 were proved in [8], [9]. Therefore assume that $\Phi_{x_d x_d z_d}$, $\Phi_{x_d z_d z_d}$ and $\Phi_{x_d z_d z_d z_d}$ are small. By (4.1) and Lemma 4.2 we may assume the more restrictive assumption that $\nabla_z(\det \Phi_{xz}) \neq 0$ which near the origin is equivalent with $\nabla_z(\Phi_{x_d z_d}) \neq 0$, again by (4.1). After a rotation we may assume

$$(4.2) \quad \Phi_{x_d z_d z_1}(0, 0) \neq 0.$$

We now consider the operator $T_\lambda T_\lambda^*$ and estimate it by the slicing technique in [9] (also familiar from the proof of Strichartz estimates). Now $T_\lambda T_\lambda^* f(x) = \int \mathcal{K}^{x_d, y_d}[f(\cdot, y_d)](x') dy_d$ where the kernel of \mathcal{K}^{x_d, y_d} as an integral operator acting on functions in \mathbb{R}^{d-1} is given by

$$K^{x_d, y_d}(x', y') = \int e^{i\lambda[\Phi(x', x_d, z) - \Phi(y', y_d, z)]} \sigma(x, z) \overline{\sigma(y, z)} dz.$$

The computation in [9] shows that after rescaling the estimation is reduced to showing that two integral operators $\mathcal{H}_\mu^\pm \equiv \mathcal{H}_{\mu, \gamma, c}^\pm$ with kernel

$$H_\mu^\pm(u, v) = \int e^{i\Psi^\pm(u, v, z; \gamma, c)} b_{\gamma, c}(u, v, z) dz$$

are bounded on $L^2(\mathbb{R}^{d-1})$ with norm $O(\mu^{-(d-1)-1/4})$, $\mu \geq 1$. Here $b_{\gamma,c}$ is C_0^∞ and

$$\Psi^\pm(u, v, z; \gamma, c) = \langle u - v, \Phi_{x'}(0, c, z) \rangle \pm \Phi'_{x_d}(0, c, z) + r_\pm(u, v, z, \gamma, c)$$

where γ and c are small parameters and $r_\pm(u, v, z, 0, c) \equiv 0$. The dependence of Ψ^\pm , r_\pm and b on γ and c is smooth and the bounds have to be uniform for small γ, c . For this it remains to show that the operators \mathcal{H}_μ^\pm are oscillatory integral operators with two-sided type two singularities to which we can apply Lemma 4.1 in $d - 1$ dimensions (with d frequency variables). It suffices to check the type two condition at $\gamma = 0, c = 0$.

The condition (4.2) guarantees that Ψ^\pm is indeed a nondegenerate phase function with critical set

$$\begin{aligned} \text{Crit}_{\Psi^\pm} &= \{(u, v, z) : \nabla_z \Psi^\pm = 0\} \\ &= \{(u, v, z) : v = u \mp \Phi^{x'z'} \Phi_{z'x_d}, \quad \Phi_{x_d z_d} - \Phi_{x_d z'} \Phi^{z'x'} \Phi_{x'z_d} = 0\} \end{aligned}$$

where the Φ derivatives are evaluated at $(0, z)$. At $x = 0$ the second equation becomes $\Phi_{x_d z_d}(0, z) = 0$ and by (4.2) we may solve this equation expressing z_1 as a function z_1^\pm of $\tilde{z} = (z'', z_d)$ with $z'' = (z_2, \dots, z_{d-1})$. Set $G^\pm(\tilde{z}) = \Phi_{x'}(0, z_1^\pm(\tilde{z}), \tilde{z})$ and $B^\pm(\tilde{z}) = \Phi^{x'z'}(0, z_1^\pm(\tilde{z}), \tilde{z}) \Phi_{z'x_d}(0, z_1^\pm(\tilde{z}), \tilde{z})$ then the canonical relation for vanishing γ, c is given by

$$C_{\Psi^\pm} = \{(u, G^\pm(\tilde{z}), u \mp B^\pm(\tilde{z}), G^\pm(\tilde{z}))\}$$

which is parametrized by the coordinates (u, \tilde{z}) . The derivative of the projection to $T^*\Omega_L$ in these coordinates is given by

$$DG^\pm = (\Phi_{x'z_1} \frac{\partial z_1}{\partial \tilde{z}} + \Phi_{x'\tilde{z}})$$

and by (4.1) we see that its determinant equals $(-1)^{d-1} \partial z_1^\pm / \partial z_d$ and $\tilde{V}_L = \partial / \partial z_d$ is a kernel vector field for the left projection. Moreover $\tilde{V}_R = \tilde{V}_L + \sum_{i=1}^{d-1} c_i(z) \partial / \partial u_i$ so that \tilde{V}_L and \tilde{V}_R coincide when acting on the determinant. By implicit differentiation we see that $\partial_{z_d}^k z_1^\pm - \Phi_{z_d x_d z_1}^{-1} \Phi_{x_d z_d^{k+1}}$ belongs to the ideal generated by $\Phi_{x_d z_d^j}$, $j \leq k$. From this one deduces that \tilde{V}_L and \tilde{V}_R are of type $\leq k - 1$ if one of the derivatives $\Phi_{x_d z_d^j}$, $j \leq k + 1$, does not vanish. We apply this for $k = 3$ to conclude the proof. \square

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