

FOURIER INTEGRAL OPERATORS WITH CUSP SINGULARITIES

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1. Introduction

Let X and Y be C^∞ manifolds of dimension d ,

$$C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$$

a homogeneous canonical relation (Lagrangian with respect to the symplectic form $\omega_{T^*X} - \omega_{T^*Y}$) and let $I^m(C; X, Y)$ denote the m^{th} order Fourier integral operators from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$ associated to C . A basic problem is to find the mapping properties of $A \in I^m(C; X, Y)$ relative to the scale of Sobolev spaces L_α^2 .

If the natural projections $\pi_L : C \rightarrow T^*X$ and $\pi_R : C \rightarrow T^*Y$ have nonsingular differentials, then C is a local canonical graph and A maps $L_{\alpha, \text{comp}}^2(Y)$ boundedly to $L_{\alpha-m, \text{loc}}^2(X)$, see Hörmander [13]. If one of the projections is singular at a point $\mathfrak{c}^0 \in C$, the other must be as well [13], and one has $\dim(\ker(d\pi_L(\mathfrak{c}^0))) = \dim(\ker(d\pi_R(\mathfrak{c}^0))) := \kappa$. Otherwise the nature of the singularities of π_L and π_R can be quite different, and the general result $A : L_{\alpha, \text{comp}}^2(Y) \rightarrow L_{\alpha-m-\frac{\kappa}{2}, \text{loc}}^2(X)$ [13] may be improved upon. In this paper we only consider cases where $\kappa = 1$.

The simplest singularity that can occur is a Whitney fold ($S_{1,0}$ in the Thom-Boardman description of singularities [1], [6].) Canonical relations C for which *both* π_L and π_R are (at most) folds arise naturally in scattering theory and were shown by Melrose and Taylor [17] to be microlocally conjugate to a single normal form, from which it follows that there is a loss of 1/6 derivative; namely operators in $I^m(C)$ map $L_{\alpha, \text{comp}}^2(Y)$ to $L_{\alpha-m-\frac{1}{6}, \text{loc}}^2(X)$; estimates on L_α^p are treated in Smith and Sogge [29]. On the other hand, canonical relations for which one projection is a fold, but with the other possibly being more degenerate, arise naturally in integral geometry [9,10,11,12] and scattering theory [3]. Under an assumption of maximal degeneracy on the other projection, it was shown in [10] that there is a loss of 1/4 derivative, and in [7] the authors extended this to all one-sided folds, with no assumption of the other projection. Estimates for one- and two-sided folds and higher singularities in a two-dimensional setting are obtained in articles by Phong and Stein [23, 24, 25, 26] and one of the authors [27, 28].

In the present work we consider Fourier integral operators for which one of the projections π_L , π_R has a Whitney cusp. These $S_{1_2,0} = S_{1,1,0}$ singularities are, after the folds, the simplest stable singularities of mappings between manifolds of the same dimension. Our main result is

Theorem 1.1. *Let $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ be a homogeneous canonical relation, so that the only singularities of the projection $\pi_L : C \rightarrow T^*X$ are Whitney folds or Whitney cusps. Let $A \in I^m(C; X, Y)$. Then A maps $L_{\alpha, \text{comp}}^2(Y)$ boundedly to $L_{\alpha-m-1/3, \text{loc}}^2(X)$.*

By a duality argument the same conclusion can be obtained if the only singularities of $\pi_R : C \rightarrow T^*Y$ are folds or cusps.

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In §5 we shall state and prove more general estimates for oscillatory integrals with not necessarily homogeneous phases. The statement of Theorem 1.1 is sharp as one makes assumptions on *one* of the projections π_L, π_R . The estimates tend to be better if one makes simultaneous assumptions on both π_L and π_R .

Our main application concerns the X-ray transform in \mathbb{R}^d , in particular when $d = 4$. Let $M_{1,d}$ be the $(2d - 2)$ -dimensional manifold of affine lines in \mathbb{R}^d and $R_{1,d} : C_0^\infty(\mathbb{R}^d) \rightarrow C_0^\infty(M_{1,d})$ be the X-ray transform,

$$R_{1,d}f(\gamma) = \int_\gamma f(\gamma(t))d\mu(t);$$

here $d\mu(t)$ denotes arclength measure.

For a line complex $\mathfrak{C} \subset M_{1,d}$, *i.e.* a smooth d -dimensional submanifold of $M_{1,d}$, the *restricted* X-ray transform is defined by

$$(1.1) \quad R_{\mathfrak{C}}f = R_{1,d}f|_{\mathfrak{C}}.$$

We shall assume that f is supported in some open subset $\Omega \subset \mathbb{R}^d$. Associated with \mathfrak{C} is the point-line incidence relation,

$$(1.2) \quad Z_{\mathfrak{C}} = \{(\gamma, y) \in \mathfrak{C} \times \Omega : y \in \gamma\}.$$

We assume that the projection $\pi_{\mathbb{R}^d} : Z_{\mathfrak{C}} \rightarrow \mathbb{R}^d$ is a submersion above Ω . Then for each $y \in \Omega$ the set $\pi_{\mathbb{R}^d}^{-1}(y) =: Z_y \subset Z_{\mathfrak{C}}$ is a smooth curve, which can be identified with $\mathfrak{C}_y = \{\gamma \in \mathfrak{C} : y \in \gamma\}$, which itself can be identified with a smooth curve in the $d - 1$ dimensional manifold $G_{1,d}^y$ of all lines through y (or its double cover \mathbb{S}^{d-1}).

Definition 1.2. A line complex $\mathfrak{C} \subset M_{1,d}$ is *well-curved* over $\Omega \subset \mathbb{R}^d$ if $\pi_{\mathbb{R}^d} : Z_{\mathfrak{C}} \rightarrow \mathbb{R}^d$ is a submersion above Ω and each $\mathfrak{C}_y, y \in \Omega$ is a nondegenerate curve; *i.e.* if $\gamma \in \mathfrak{C}_y$ and $s \rightarrow \rho(s) \in \mathfrak{C}_y$ is any smooth regular parametrization of \mathfrak{C}_y near γ with $\rho(0) = \gamma$, then the vectors $\dot{\rho}(0), \ddot{\rho}(0), \dots, \rho^{(d-1)}(0)$ in $T_\gamma G_{1,d}^y$ are linearly independent.

Clearly the definition is independent of the particular parametrization. Similar notions of well-curvedness are used in [9], [7].

In this paper we are concerned with the restricted X-ray transform in \mathbb{R}^4 . If $N^*Z_{\mathfrak{C}} \subset T^*\mathfrak{C} \times T^*\Omega$ is the conormal bundle to $Z_{\mathfrak{C}}$ then we show in §5 that the projection π_R to $T^*\Omega$ exhibits at most $S_{1,1,0}$ singularities, if $d = 4$. Since $R_{\mathfrak{C}}$ is a Fourier integral operator of order $-1/2$ part (a) of the following theorem will be an immediate consequence of Theorem 1.1. Similar statements can be made for microlocalized versions of the restricted X-ray transform in higher dimensions if one stays away from $S_{1,r}$ singularities of π_R with $r > 2$, but we omit these.

Theorem 1.3. *Suppose that the line complex $\mathfrak{C} \subset M_{1,4}$ is well curved over $\Omega \subset \mathbb{R}^4$. Then*

- a) $R_{\mathfrak{C}} : L_{s,\text{comp}}^2(\Omega) \rightarrow L_{s+\frac{1}{8},\text{loc}}^2(\mathfrak{C})$, for all $s \in \mathbb{R}$.
- b) $R_{\mathfrak{C}} : L_{\text{comp}}^p(\Omega) \rightarrow L_{\text{loc}}^q(\mathfrak{C})$ for $(\frac{1}{p}, \frac{1}{q}) \in \text{hull}\{(0,0), (1,1), (\frac{7}{12}, \frac{1}{2})\}$.

It turns out that the $L^p \rightarrow L^q$ estimates of Theorem 1.3 are sharp for $p \geq 12/7$. Consider the translation invariant complex of lines with parametrizations $\gamma(v, t) = (v_1 + v_4 t, v_2 + v_4^2 t, v_3 + v_4^3 t, t)$, $t \in \mathbb{R}$. It is not hard to see (*cf.* §5) that $R_{\mathfrak{C}}$ cannot be bounded from L^p to L^q if $(1/p, 1/q)$ belongs

to the complement of the triangle \mathcal{T} with corners $(0, 0)$, $(1, 1)$, $(\frac{7}{10}, \frac{3}{5})$. Part (b) of the Theorem establishes the boundedness for a subtriangle with vertex $(7/12, 1/2)$ on the lower edge of \mathcal{T} .

The $L^p \rightarrow L^2$ estimates of Theorem 1.3 will not hold for general $\mathcal{F} \in I^{-1/2}(X, Y; C)$ with one-sided cusp singularities. It is important that π_R satisfies an additional transversality condition with respect to the fibration in T^*Y , namely π_R being a strong cusp as defined in Definition 2.5 below; moreover the projections of the cusp surface to the fibers in T^*Y satisfy a suitable curvature assumption.

In §2, we recall some basic terminology from singularity theory, including the definition of a cusp, and more generally of $S_{1,r,0}$ or Morin singularities. We shall also introduce the notion of a strong cusp, and discuss curvature assumptions for the image of the cusp surface.

In §3, following the outlines of [7], we prove decay estimates in λ for oscillatory integral operators (with not necessarily homogeneous phases) having one-sided strong simple cusps, which then imply theorems on Radon transforms and Fourier integral operators. In [7], estimates for one-sided folds followed from those for (two-sided, of course) canonical graphs in one lower dimension; here, estimates for one-sided cusps follow from those for two-sided folds in one lower dimension. It is essential for this approach to assume the strong cusp condition. We shall also prove $L^2 \rightarrow L^q$ estimates for such operators, under various curvature assumptions on the cusp surface.

In §4 we shall use a canonical transformation to prove that a cusp need not be a strong cusp for the L^2 estimates to hold, thereby proving Theorem 1.1 and corresponding results for oscillatory integral operators.

In §5 we show that the restriction of the X-ray transform in \mathbb{R}^d to a generic line complex $\mathfrak{C} \subset M_{1,d}$ is a Fourier integral operator for which π_R exhibits strong $S_{1,d-2,0}$ singularities. In §6 we give conditions on vector fields X, Y, Z and W in \mathbb{R}^4 such that the family of curves $t \mapsto \gamma(x, t) = \exp_x(tX + t^2Y + t^3Z + t^4W)$ is associated with a strong right- or left-cusp, obtaining a formula analogous to the one found by Phong and Stein [24] for Whitney folds in \mathbb{R}^3 .

Notation: Given two quantities A_1 and A_2 we write $A_1 \lesssim A_2$ or $A_2 \gtrsim A_1$ if there is a positive constant c , such that $A_1 \leq cA_2$.

2. Morin singularities and oscillatory integrals

Let $P \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a germ of a C^∞ map at P . We say that f *drops rank simply* at P if $\text{rank } df_P = n - 1$ and if $\det df$ vanishes of order 1 at P (i.e. $d(\det df)_P \neq 0$). By the implicit function theorem the variety $S_1(f) = \{x : \text{rank } df = n - 1\}$ is (the germ of) a hypersurface. We shall say that f has an S_1 singularity at P with singularity manifold $S_1(f)$. Given diffeomorphisms Φ_1, Φ_2 , with $\Phi_1(Q) = P$ it is clear that $\Phi_2 \circ f \circ \Phi_1$ drops rank simply by 1 at Q if and only if f does at P . Therefore the notion extends to manifolds.

Next let \mathfrak{S} be a hypersurface in a manifold V and let v be a vector field defined on \mathfrak{S} with values in TV (so that $v_P \in T_P V$ for $P \in \mathfrak{S}$). We say that v is transversal to \mathfrak{S} at $P \in \mathfrak{S}$ if $v_P \notin T_P \mathfrak{S}$. We say that v is *simply tangent* to \mathfrak{S} at P if there is a one-form ω annihilating vectors tangent to \mathfrak{S} so that $\langle \omega, v \rangle|_{\mathfrak{S}}$ vanishes of first order at P . Notice that this condition does not depend on the particular choice of ω . Also let a be a smooth function nonvanishing at P ; then v is simply tangent to \mathfrak{S} if and only if av is simply tangent to \mathfrak{S} . Next let $P \rightarrow \ell(P) \subset T_P(V)$ be a smooth field of lines defined on \mathfrak{S} . Let v be a nonvanishing vector field so that $\ell(P) = \mathbb{R}v_P$. If $v_P \notin T_P \mathfrak{S}$ then ℓ is defined to be transversal to \mathfrak{S} at P . The line ℓ is defined to be *simply tangent* to \mathfrak{S} at P if v is simply tangent to \mathfrak{S} at P . Both notions do not depend on the particular choice of v .

Next consider $F : V \rightarrow W$ where $\dim V = k \geq 2$ and $\dim W = n \geq k$ and assume that

rank $dF \geq k - 1$. Suppose that \mathfrak{S} is a hypersurface in V such that rank $dF = k - 1$ on \mathfrak{S} . Suppose that $\ker dF$ is simply tangent to \mathfrak{S} at $P \in \mathfrak{S}$. Then there is a neighborhood U of P in \mathfrak{S} such that the variety $\{Q \in U : \text{rank } dF|_{T_Q \mathfrak{S}} = k - 2\}$ is a smooth hypersurface in \mathfrak{S} .

Definition 2.1. Let $1 \leq r \leq n$. For $k = 1, \dots, r$ let \mathfrak{S}_k be a submanifold of dimension $n - k$ in V so that $\mathfrak{S}_1 \supset \mathfrak{S}_2 \supset \dots \supset \mathfrak{S}_r$; we also set $\mathfrak{S}_0 := V$.

(a) We say that f has an S_{1_r} singularity in V , with a descending sequence of singularity manifolds $(\mathfrak{S}_1, \dots, \mathfrak{S}_r)$ if the following conditions hold.

(i) For $P \in V$, either df_P is bijective or f drops rank simply at P .

(ii) For $1 \leq i \leq r$, rank $d(f|_{\mathfrak{S}_{i-1}})_Q = n - i + 1$ for all $Q \in \mathfrak{S}_{i-1} \setminus \mathfrak{S}_i$.

(iii) For $2 \leq i \leq r - 1$, $\ker d(f|_{\mathfrak{S}_{i-1}})$ is simply tangent to \mathfrak{S}_i at points in \mathfrak{S}_{i+1} .

(b) Let $P \in V$. Let $1 \leq r \leq n$. We say that f has an S_{1_r} singularity at P , if there is a neighborhood U of P and submanifolds \mathfrak{S}_k of dimension $n - k$ in U so that $P \in \mathfrak{S}_r \subset \mathfrak{S}_{r-1} \subset \dots \subset \mathfrak{S}_1$ and so that $f : U \rightarrow W$ has an S_{1_r} singularity in U , with singularity manifolds $(\mathfrak{S}_1, \dots, \mathfrak{S}_r)$.

The singularity manifolds \mathfrak{S}_k are uniquely determined by f (and the choice of the open set V) and denoted by $S_{1_k}(f)$ in singularity theory, suppressing the dependence on V .

Definition 2.2. Let $P \in V$ and $1 \leq r \leq n$. We say that f has a $S_{1_r,0}$ singularity at P , if f has an S_{1_r} singularity at P and if $\ker df_P \cap T_P(S_{1_r}(f)) = \{0\}$.

An $S_{1,0}$ (or $S_{1,0}$) singularity is a Whitney fold; an $S_{1,1,0}$ (or $S_{1,2,0}$) singularity is referred to as a Whitney or simple cusp.

Remarks 2.3.

1. Suppose that f has an S_{1_r} singularity at P . Then

$$\ker df_P = \ker d(f|_{\mathfrak{S}_i})_P \text{ if } P \in S_{1_{i+1}}(f)$$

while $\ker d(f|_{S_{1_i}(f)}) = \{0\}$ at points in $S_{1_i}(f) \setminus S_{1_{i+1}}(f)$. If $\ker df$ is simply tangent to $S_{1_r}(f)$ at some point $P \in S_{1_r}(f)$, then by the implicit function theorem there is a neighborhood U_r of P and a hypersurface \mathfrak{S}_{r+1} in $S_{1_r}(f) \cap U_r$ (of dimension $n - r - 1$) so that rank $d(f|_{S_{1_r}(f)})_P = n - r$ if $P \in S_{1_r}(f) \setminus \mathfrak{S}_{r+1}$, and rank $d(f|_{S_{1_r}(f)})_P = n - r - 1$ if $P \in \mathfrak{S}_{r+1}$; this then defines the singularity manifold $S_{1_{r+1}}(f)$, and f has an $S_{1_{r+1}}$ singularity at P .

2. It is straightforward to check invariance under changes of variables. If f has an S_{1_r} singularity at P , with singularity manifolds $\mathfrak{S}_1 \supset \dots \supset \mathfrak{S}_r$ and χ_1, χ_2 are germs of diffeomorphisms, $\chi_1(Q) = P$, χ_2 defined near $f(P)$ then $\chi_2 \circ f \circ \chi_1$ has an S_{1_r} singularity at P , with singularity manifolds $\chi_1^{-1}(\mathfrak{S}_1) \supset \dots \supset \chi_1^{-1}(\mathfrak{S}_r)$.

3. It is not hard to verify the occurrence of S_{1_r} singularities when the map is given in special coordinates. Following Morin [18], we say that coordinates $t = (t', t_n)$ on V , vanishing at P and $y = (y', y_n)$ on W , vanishing at $f(P)$, are *adapted* coordinates if

$$(2.1) \quad f^* y_j = t_j, 1 \leq j \leq n - 1, \quad df^*(dy_n)|_P = 0;$$

in other words

$$(2.2) \quad f(t) = (t', h(t))$$

for some smooth $h : \mathbb{R}^n \rightarrow \mathbb{R}$ with $h(0) = 0$.

By changes of variables in source and target each map of rank $\geq d - 1$ can be put in the form (2.2).

4. f has an S_{1_r} singularity at P if and only if there is an adapted coordinate system vanishing at P with h as in (2.2) such that

$$(2.3) \quad \frac{\partial^k h}{\partial t_n^k}(0) = 0, \quad 1 \leq k \leq r$$

and if

$$(2.4) \quad \text{rank} [d_t(\frac{\partial h}{\partial t_n}), \dots, d_t(\frac{\partial^r h}{\partial t_n^r})](t^0) = r.$$

The singularity manifolds are given by

$$S_{1_k}(f) = \{x : \frac{\partial^j h}{\partial t_n^j}(0) = 0, 1 \leq j \leq k\},$$

for $k = 1, \dots, r$.

This is a straightforward consequence of Definition 2.1.

5. f has an $S_{1_r,0}$ singularity if and only if there is an adapted coordinate system with h as in (2.2) such that (2.3), (2.4) hold and moreover

$$(2.5) \quad \frac{\partial^{r+1} h}{\partial t_n^{r+1}}(0) \neq 0.$$

Equivalently, f has an $S_{1_r,0}$ singularity if and only if (2.3), (2.5) hold and (if $r > 1$)

$$(2.6) \quad \text{rank} [d_{t'}(\frac{\partial h}{\partial t_n}), \dots, d_{t'}(\frac{\partial^{r-1} h}{\partial t_n^{r-1}})](0) = r - 1.$$

6. Let $\Sigma_{\omega_r} \subset J^{r+1}(V, W)$ be the Thom-Boardman class with Boardman symbol $\omega_r = (1, 1, \dots, 1, 0)$, with r ones. Morin[18] states that for $j^{r+1}f \in (\Sigma_{\omega_r})_P$ it is necessary and sufficient that there exist adapted coordinates t on V near P , y on W near $f(P)$ such that (2.3) and (2.5) hold. If this is the case then $j^{r+1}f$ intersects Σ_{ω_r} transversally at P if and only if (2.4) holds. Therefore the definition of $S_{1_r,0}$ singularities above is equivalent with the standard description in singularity theory (cf. Boardman [1], Levine [15], Morin [18], [19]). Conditions (2.3, 2.5) and (2.4) are independent of the choice of adapted coordinate systems. It is shown in [18] that there exist adapted coordinate systems, vanishing at $P, f(P)$ such that h is given by the normal form $h(t) = t_1 t_n + t_2 t_n^2 + \dots + t_{r-1} t_n^{r-1} + t_n^{r+1}$.

In order to verify that a given map has Morin singularities we shall use the following lemma, proved by changing coordinates to adapted coordinates. In what follows we split coordinates as $x = (x', x_n)$.

Lemma 2.4. *Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth function defined near 0 and $f(0) = 0$ and $\text{rank } df_0 \geq n - 1$. Suppose that for $i, j = 1, \dots, n - 1$,*

$$(2.6) \quad \left. \frac{\partial^s f_i}{\partial x_n^s} \right|_0 = 0, \quad s = 1, \dots, r$$

$$(2.7) \quad \left. \frac{\partial^s f_i}{\partial x_j \partial x_n^{s-1}} \right|_0 = 0, \quad s = 2, \dots, r.$$

Then

(a) f has an $S_{1,r}$ singularity at 0 if and only if $\left. \frac{\partial^s f_n}{\partial x_n^s} \right|_0 = 0$, $s = 1, \dots, r$, and the set $\left\{ d \left(\left. \frac{\partial^s f_n}{\partial x_n^s} \right) \right|_0, s = 1, \dots, r \right\}$ is linearly independent.

(b) f has an $S_{1,r,0}$ singularity at 0 if and only if $\left. \frac{\partial^s f_n}{\partial x_n^s} \right|_0 = 0$, $s = 1, \dots, r$, $\left. \frac{\partial^{r+1} f_n}{\partial x_n^{r+1}} \right|_0 \neq 0$, and the set $\left\{ d_{x^i} \left(\left. \frac{\partial^s f_n}{\partial x_n^s} \right) \right|_0, s = 1, \dots, r-1 \right\}$ is linearly independent.

Proof. Let A be the $(n-1) \times (n-1)$ matrix A given by $A_{ij} = \frac{\partial f_i}{\partial x_j}$ for $i, j = 1, \dots, n-1$; then A is invertible. Let $b \in \mathbb{R}^{n-1}$ with $b_i = \frac{\partial f_i}{\partial x_n}$.

We introduce adapted coordinates $t_i(x) = f_i(x)$, $i = 1, \dots, n-1$ and $t_n = x_n$ so that with $F(t(x)) = f(x)$, $F_i(t) = t_i$ for $i = 1, \dots, n-1$. An elementary calculation yields

$$\frac{Dx}{Dt} = \begin{pmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{pmatrix}.$$

In particular

$$\begin{aligned} \frac{\partial}{\partial t_n} &= \frac{\partial}{\partial x_n} - \sum_{i=1}^{n-1} (A^{-1}b)_i \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial t_j} &= \sum_{i=1}^{n-1} (A^{-1})_{ij} \frac{\partial}{\partial x_i}, \quad j = 1, \dots, n-1. \end{aligned}$$

Let $a = A^{-1}b$. Then $\left. \frac{\partial^s a(0)}{\partial x_n^s} \right|_0 = 0$, for $s = 0, \dots, r-1$. This follows from a routine calculation using $\partial_{x_n} A^{-1} = -A^{-1} \partial_{x_n} A A^{-1}$, and the assumption (2.6). Thus, if $h(t) = f_n(x)$ then

$$\left. \frac{\partial^k h}{\partial t_n^k} \right|_0 = \left(\frac{\partial}{\partial x_n} - \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial x_i} \right)^k f_n \Big|_0 = \left. \frac{\partial^k f_n}{(\partial x_n)^k} \right|_0.$$

By a similar calculation using the assumption (2.7)

$$\begin{aligned} \left. \frac{\partial}{\partial t_j} \frac{\partial^{k-1} h}{\partial t_n^k} \right|_0 &= \sum_{\nu=1}^{n-1} (A^{-1})_{\nu j} \frac{\partial}{\partial x_\nu} \left(\frac{\partial}{\partial x_n} - \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial x_i} \right)^{k-1} f_n \Big|_0 \\ &= \sum_{\nu=1}^{n-1} (A^{-1})_{\nu j} \frac{\partial}{\partial x_\nu} \frac{\partial^{k-1} f_n}{(\partial x_n)^{k-1}}. \end{aligned}$$

for $k = 2, \dots, r$. These formulas together with the description of Morin singularities in Remarks 2.3.4 and 2.3.5 imply the statements of the Lemma. \square

We will now formulate a strengthened version of the cusp condition when the range space W is itself a fiber bundle over a base, B ; the example of relevance for oscillatory integral operators will be $W = T^*X$, the cotangent bundle of a C^∞ manifold, over $B = X$.

Assume that $W \xrightarrow{\pi_B} B$ is a fiber bundle, with $\dim(B) = q \leq n-r$, so that the fibers $W_b = \pi_B^{-1}b$ are $n-q$ dimensional manifolds.

Definition 2.5. Let $b = \pi_B(f(P))$ and let $W_b = \pi_B^{-1}b$ be the fiber through $f(P)$. f has a strong $S_{1,r,0}$ singularity at P , denoted by $S_{1,r,0}^+$, if

(a) f intersects W_b transversally, so that there is a neighborhood U of P such that the preimages $f^{-1}W_b \cap U$ are smooth manifolds of dimension $n - q$,

and if

(b) $f|_{f^{-1}(W_b) \cap U}$ has an $S_{1,r,0}$ singularity at P .

Remarks 2.6.

1. f has an $S_{1,r,0}^+$ singularity at P if and only if there exist adapted coordinate systems for f of the form $t = ((t', t''), t_n)$ on V , $y = ((y', y''), y_n)$ on W , vanishing at P , $f(P)$, with $t', y' \in \mathbb{R}^q$, $t'', y'' \in \mathbb{R}^{n-q-1}$, so that (i) (2.1) holds and furthermore, $y' = \pi_B^*x$ for some local coordinates x on B , ii) (2.3) and (2.5) hold; and, if $r > 1$, iii) the rank of the differential of the map $\mathbb{R}^{n-q-1} \ni t'' \rightarrow (\frac{\partial \phi}{\partial t_n}, \dots, \frac{\partial^{r-1} \phi}{\partial t_n^{r-1}}) \in \mathbb{R}^{r-1}$ at 0 is equal to $r - 1$.

2. Clearly, for $f : V \rightarrow W$ any cusp of order r , we may take the trivial fiber bundle $\pi_B : W \rightarrow \{\text{point}\}$, so that $q = 0$, and then the Morin singularity is strong. However, we will be interested in the nontrivial case $q > 0$, and in particular $n = 2d, q = d, r \leq d - 2$.

3. The notion of an $S_{1,r,0}^+$ singularity is invariant under diffeomorphisms of V and fiber-preserving diffeomorphisms of W .

4. The property of being an $S_{1,r,0}^+$ map is stable under perturbations in the C^{r+1} topology.

Conditions for canonical relations associated to oscillatory integral operators. Let X and Z be manifolds of dimension d and let $\omega_{T^*X}, \omega_{T^*Z}$ be the canonical two-forms on T^*X, T^*Z , respectively. Let

$$C \subset T^*X \times T^*Z$$

a submanifold, Lagrangian with respect to $\omega_{T^*X} - \omega_{T^*Z}$, i.e. a symplectic relation.

In the study of oscillatory and Fourier integral operators one is led to consider the geometry of the projections $\pi_L : C \rightarrow T^*X$, $\pi_R : C \rightarrow T^*Z$. T^*X is of course fiber bundle over X , with projection $\pi_X : T^*X \rightarrow X$ and Lagrangian fibers. Taking $V = C, W = T^*X, B = X, n = 2d$, and $q = d$, we thus have a well-defined notion of C having a left $S_{1,r,0}^+$ singularity at $c^0 \in C$. If $x^0 \in X$, $\pi_X c^0 = x^0$ and U is a (sufficiently small) neighborhood of c^0 then we can restrict π_L to $\pi_X^{-1}\{x^0\} \cap U$ and define

$$\pi_{L,x^0} = \pi_L|_{\pi_X^{-1}\{x^0\} \cap U}$$

with target space $T_{x^0}^*X$; the map π_{L,x^0} is then assumed to have an $S_{1,r,0}$ singularity at c^0 .

Similarly, we speak of C having a right $S_{1,r,0}^+$ singularity if $\pi_R : C \rightarrow T^*Z$ has an $S_{1,r,0}^+$ singularity with respect to the fibration $T^*Z \xrightarrow{\pi_Z} Z$, and define similarly the restrictions $\pi_{R,z^0} = \pi_R|_{\pi_Z^{-1}\{z^0\} \cap U}$.

We shall study oscillatory integral operators acting on functions defined in \mathbb{R}^d given by

$$(2.8) \quad T_\lambda f(x) = \int e^{i\lambda \Phi(x,z)} a(x,z) f(z) dz;$$

here $a \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and Φ is a smooth real-valued phase function defined in a neighborhood of $\text{supp } a$. The object of interest is the behavior of the $L^p \rightarrow L^q$ operator norm as $\lambda \rightarrow \infty$.

The symplectic (or canonical) relation associated to the phase Φ is given by

$$(2.9) \quad C_\Phi = \{(x, \Phi'_x(x,z); z, -\Phi'_z(x,z)) : (x,z) \in X \times Z\}.$$

Note that, since C_Φ is given as a graph over $X \times Z$, the transversality condition (a) in Definition 2.5 is always satisfied for both π_L and π_R . The $L^p \rightarrow L^q$ bounds of T_λ depend on the geometry of C_Φ , in particular on the projections $\pi_L : C \rightarrow T^*X$, $\pi_R : C \rightarrow T^*Y$, but also on the projections π_X, π_Z to X and Z and other geometric information.

We wish to give reformulations of the assumption that one of the projections, say π_L , has a (possibly strong) $S_{1,r,0}$ singularity at a point $c^0 \in C$. From Remark 2.6 (3) above, it follows that the class of strong left cusps is invariant under $\text{Diff}(X) \times \text{Can}(T^*X)$ (diffeomorphisms in X and canonical transformations in T^*X).

In proving estimates on T_λ one establishes estimates under the assumption that the amplitude a is supported in a small neighborhood of a point $P^0 = (x^0, y^0)$. This assumption can then be removed by compactness arguments.

We now fix $P^0 = (x^0, z^0)$, so that $c^0 = (x^0, \Phi'_x(x^0, z^0); z^0, -\Phi'_z(x^0, z^0))$. Clearly the operator norm of T_λ does not change by adding smooth terms depending only on x or only on z to Φ . Moreover the behavior of the operator norm in λ does not change under changes of variables in X and Z . In particular we may assume that $x^0 = 0, z^0 = 0$ and that

$$(2.10) \quad \Phi(0, z) = \Phi(x, 0) = 0.$$

Throughout the paper we shall always assume that

$$\text{rank } d\pi_L = \text{rank } d\pi_R \geq d - 1,$$

which is the case for the Morin singularities and equivalent with $\text{rank } \Phi''_{xz}(x, z) \geq d - 1$. We split variables as $x = (x', x_d), z = (z', z_d)$. By a linear transformation in the x variables we may assume that

$$(2.11) \quad \Phi''_{x_d z'}(0, 0) = 0$$

and consequently

$$\det \Phi''_{x' z'}(0, 0) \neq 0.$$

In view of Lemma 2.4 it will be advantageous to have $\Phi''_{x' z_d}$ vanish at $x = 0$.

Lemma 2.7. *Suppose that $\text{rank } (\Phi_0)''_{xz} \geq d - 1$ and Φ_0 satisfies (2.11). Then there is a smooth G with $G(0) = 0, \det(DG(0)) \neq 0$ so that for z near 0 the phase function $\Phi(x, z) = \Phi_0(x, G(z))$ satisfies (2.11),*

$$(2.12) \quad \Phi''_{x' z'}(0, 0) = I_{d-1}$$

and

$$(2.13) \quad \Phi''_{x' z_d}(0, z) = 0.$$

Proof. We set $\Psi(x, z) = \Phi_0(x, Az', z_d)$ with $A = (\Phi''_{x' z'})^{-1}(0, 0)$; then Ψ satisfies (2.10-2.12).

Define

$$Z(w) = (\Psi'_{x'}(0, w), w_d)$$

which by (2.12) is a diffeomorphism near 0 with $Z(0) = 0, DZ(0) = I_d$. Let $z \mapsto w(z) = (w'(z), z_d)$ be the inverse map and define

$$\Phi(x, z) = \Psi(x, w(z)).$$

It is immediate that Φ satisfies (2.10-2.12) and we check that (2.13) holds as well. This implies the Lemma with $G(z) = (Aw'(z), w_d)$.

We differentiate the relation $\Phi(x, Z(w)) = \Psi(x, w)$, taking into account that $\nabla_{w'} Z_d = 0$, $\partial_{w_d} Z_d = 1$ and $\frac{\partial Z'}{\partial w}(w) = \Psi''_{x'z}(0, w)$. Then

$$\begin{aligned} \Phi''_{x'z'}(x, Z(w))\Psi''_{x'z'}(0, w) &= \Psi''_{x'z'}(x, w) \\ \Phi_{x'z_d}(x, Z(w)) + \Phi''_{x'z'}(x, Z(w))\Psi''_{x'z_d}(0, w) &= \Psi''_{x'z_d}(x, w). \end{aligned}$$

Evaluating at $x = 0$ yields

$$\begin{aligned} \Phi''_{x'z'}(0, Z(w)) &= I_{d-1} \\ \Phi''_{x'z_d}(0, Z(w)) &= 0 \end{aligned}$$

and thus the assertion. \square

The proof yields more than stated in (2.12), namely $\Phi''_{x'z'}(0, z) = I_{d-1}$. However we later need to introduce changes of variables violating this condition but keeping (2.12).

Proposition 2.8. *Let $c^0 \in C$ so that $x^0 = \pi_X c^0 = 0$, $z^0 = \pi_Z c^0 = 0$. Suppose that the phase function Φ satisfies (2.11-13).*

(a) $\pi_L : C_\Phi \rightarrow T^*X$ has an $S_{1_r,0}$ singularity at c^0 if and only if

$$(2.14) \quad \frac{\partial^{k+1}\Phi}{\partial z_d^k \partial x_d}(0,0) = 0, \quad 1 \leq k \leq r, \quad \frac{\partial^{r+2}\Phi}{\partial z_d^{r+1} \partial x_d}(0,0) \neq 0$$

and, if $r \geq 2$,

$$(2.15) \quad \text{rank} \left[d_{(x,z')} \frac{\partial^2 \Phi}{\partial z_d \partial x_d}, \dots, d_{(x,z')} \frac{\partial^{r+1} \Phi}{\partial z_d^r \partial x_d} \right] (0,0) = r - 1.$$

(b) $\pi_L : C_\Phi \rightarrow T^*X$ has an $S_{1_r,0}^+$ singularity at c^0 if and only if (2.14) holds and, if $r \geq 2$,

$$(2.16) \quad \text{rank} \left[d_{z'} \frac{\partial^2 \Phi}{\partial z_d \partial x_d}, \dots, d_{z'} \frac{\partial^{r+1} \Phi}{\partial z_d^r \partial x_d} \right] (0,0) = r - 1.$$

Proof. For (a) apply Lemma 2.4 to the map $(x, z) \mapsto (x, \Phi'_x(x, z))$. For (b) apply Lemma 2.4 to the map $z \mapsto (0, \Phi'_x(0, z))$ (in fact this map is already given in adapted coordinates). \square

Note that the conditions (2.15), (2.16) are vacuous if $r = 1$; in particular, if π_L has an $S_{1,0}$ singularity (or Whitney fold) it is already strong.

Interchanging x and z , one obtains a similar statement for the projection $\pi_R : C \rightarrow T^*Z$. Obviously, C has a left $S_{1_r,0}$ singularity (or left $S_{1_r,0}^+$ singularity) if and only if the transpose relation C^* has a right $S_{1_r,0}$ singularity (right $S_{1_r,0}^+$ singularity).

The following observation will be useful when discussing curvature hypothesis for the image of cusp surfaces in the fibers. In what follows, e_1, \dots, e_d will denote the standard basis in \mathbb{R}^d .

Lemma 2.9. Let $\mathfrak{c}^0 \in C_\Phi$ so that $x^0 = \pi_X(\mathfrak{c}^0) = 0$, $z^0 = \pi_Z(\mathfrak{c}^0) = 0$. Suppose $\Phi_0(x, z)$ satisfies (2.11-13) and suppose π_L has an $S_{1_r, 0}^+$ singularity at \mathfrak{c}^0 . Then there is an invertible linear transformations $B \in GL(\mathbb{R}^{d-1})$, and smooth $W(z)$ with $W(0) = 0$ and $DW(0) \in GL(\mathbb{R}^d)$ so that $\Phi(x, z) = \Phi_0(Bx', x_d, W(z))$ satisfies (2.11), (2.12), (2.13), (2.14),

$$(2.17) \quad \nabla_z \frac{\partial^{k+1} \Phi}{\partial x_d \partial z_d^k}(\mathbf{0}, \mathbf{0}) = e_{d-r+k}, \quad k = 1, \dots, r-1,$$

and

$$(2.18_0) \quad \frac{\partial^2 \Phi}{\partial x_d \partial z_{d-r+1}}(\mathbf{0}, z_1, \dots, z_{d-r}, \mathbf{0}, \dots, \mathbf{0}) = 0,$$

$$(2.18_k) \quad \frac{\partial^{k+1} \Phi}{\partial x_d \partial z_d^k}(\mathbf{0}, z_1, \dots, z_{d-r}, \mathbf{0}, \dots, \mathbf{0}) = 0, \quad k = 1, \dots, r-1,$$

for z_i near 0.

Moreover

$$(2.19) \quad S_{1_k}(\pi_{L, x^0}) = \{((0, \Phi'_x(\mathbf{0}, z), z, -\Phi'_z(\mathbf{0}, z)) : \Phi_{x_d z_d^j}^{(j+1)}(\mathbf{0}, z) = 0, j = 1, \dots, k)\}$$

and for $\mathfrak{c} \in S_1(\pi_{L, x^0})$ near \mathfrak{c}^0 , $\pi_Z \mathfrak{c} = z$

$$(2.20) \quad \ker(d\pi_{L, x^0})_{\mathfrak{c}} = \left\{ \tau \left(\frac{\partial}{\partial z_d}, - \sum_{i=1}^d \Phi''_{z_i z_d}(\mathbf{0}, z) \frac{\partial}{\partial z_i} \right) : \tau \in \mathbb{R} \right\}.$$

Proof. By Proposition 2.8, (2.16) we may choose an invertible linear transformation B such that

$$B e_{d-r+k} = \sum_{i=1}^{d-1} \frac{\partial^{k+2} \Phi_0}{\partial x_d \partial z_d^k \partial z_i} \Big|_{(\mathbf{0}, \mathbf{0})} e_i,$$

for $k = 1, \dots, r-1$. Define $\Psi(x, z) = \Phi_0(Bx', x_d, (B^t)^{-1} z', z_d)$. Then Ψ satisfies (2.11-14), (2.16), by Proposition 2.8; moreover it satisfies (2.17). Therefore

$$\begin{aligned} \Psi_{x_d z_d^k}^{(k+1)}(\mathbf{0}, w) &= w_{d-r+k} + Q_{d-r+k}(w), \quad k = 1, \dots, r-1 \\ \Psi''_{x_d z_{d-r+1}}(\mathbf{0}, w) &= w_d + Q_d(w) \end{aligned}$$

with smooth Q_{d-r+k} vanishing of second order at 0. In what follows set $w'' = (w_1, \dots, w_{d-r})$. Define a diffeomorphism $\mathfrak{z} = (\mathfrak{z}_1, \dots, \mathfrak{z}_d)$ by

$$\mathfrak{z}_i(w) = \begin{cases} w_i, & \text{if } 1 \leq i \leq d-r \\ w_i + Q_i(w'', 0), & \text{if } d-r+1 \leq i \leq d \end{cases}$$

and let $z \mapsto w(z)$ denote its inverse. Define

$$\Phi(x, z) = \Psi(x, w(z)).$$

Clearly $w(0) = 0$, $Dw(0) = Id$ and therefore Φ satisfies (2.11-12). Since $\mathfrak{z}_i(w)$ does not depend on w_d , for $i \leq d-1$. Similarly one verifies (2.16), (2.17). To see (2.18) we differentiate the relation $\Phi(x, \mathfrak{z}(w)) = \Psi(x, w)$ and noting that $\partial_{w_{d-r+1}} \mathfrak{z}_{d-r+1} = \partial_{w_d} \mathfrak{z}_d = 1$ and $\partial_{w_d} \mathfrak{z}_i = 0$ if $i \neq d$, $\partial_{w_{d-r+1}} \mathfrak{z}_i = 0$ if $i \neq d-r+1$ we obtain

$$\begin{aligned}\Phi_{x_d z_d^k}^{(k+1)}(x, \mathfrak{z}(w)) &= \Psi_{x_d z_d^k}^{(k+1)}(x, w) \\ \Phi''_{x_d z_{d-r+1}}(x, \mathfrak{z}(w)) &= \Psi''_{x_d z_{d-r+1}}(x, w).\end{aligned}$$

Consequently

$$\begin{aligned}\Phi_{x_d z_d^k}^{(k+1)}(0, \mathfrak{z}(w'', 0)) &= \Psi_{x_d z_d^k}^{(k+1)}(0, w'', 0) = \mathfrak{z}_{d-r+k}(w'', 0) \quad 1 \leq k \leq r-1 \\ \Phi''_{x_d z_{d-r+1}}(0, \mathfrak{z}(w'', 0)) &= \Psi''_{x_d z_{d-r+1}}(0, w'', 0) = \mathfrak{z}_d(w'', 0)\end{aligned}$$

by definition of \mathfrak{z} . Since $\mathfrak{z}(w''(z'', 0), 0) = (z'', 0)$ the assertion (2.18) follows and the lemma holds with $W(z) = ((B^t)^{-1}w'(z), z_d)$.

Define $\tilde{\pi}_{L, x^0}(z) = (0, \Phi'_x(0, z))$. Then $S_{1_k}(\pi_{L, x^0})$ consists of all $(0, \Phi'_x(0, z), z, -\Phi'_{z'}(0, z))$ with $z \in S_{1_k}(\tilde{\pi}_L)$. By (2.11-12) the kernel of $d\tilde{\pi}_{L, x^0}$ is spanned by $\frac{\partial}{\partial z_d}$ from which (2.20) follows. The assertion (2.19) on $S_{1_k}(\pi_L)$ follows now from Remark 2.3.4 above, since $\tilde{\pi}_{L, x^0}$ is given in adapted coordinates. \square

Curvature conditions for strong Morin singularities. Suppose that C is a canonical relation in $T^*X \times T^*Z$ and suppose that $\pi_L : C \rightarrow T^*X$ has an $S_{1_r, 0}^+$ singularity at $\mathfrak{c}^0 \in C$. Then for a neighborhood U of \mathfrak{c}^0 , the image of the cusp surface,

$$(2.21) \quad \Sigma_{1_r}^{L, x^0} = \{\pi_{L, x^0} \mathfrak{c} : \mathfrak{c} \in S_{1_r}(\pi_{L, x^0}) \cap U\} = \pi_L(S_{1_r}(\pi_L) \cap U) \cap T_{x^0}^*X,$$

is a smooth manifold of codimension r in $T_{x^0}^*X$. The $L^2 \rightarrow L^q$ mapping properties of oscillatory integrals may depend on the curvature properties of these surfaces. Although it is possible to investigate a variety of curvature conditions we limit ourselves to two extreme cases, corresponding to having ℓ nonvanishing principal curvatures with respect to a normal n , and a weaker finite type condition.

We now give, for a submanifold of \mathbb{R}^d , the definition of *finite type* with respect to normal n . To do this recall that a (germ of a) smooth function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be of finite type k at $x^0 \in \mathbb{R}^d$ if $Q(D)f(x^0) = 0$ for all differential operators of order $\leq k-1$ and $P(D)f(x^0) \neq 0$ for some differential operator of order k .

Definition 2.10. Let M be a submanifold of \mathbb{R}^d , with codimension ℓ , $P \in M$ and $t \mapsto \Gamma(t)$ a parametrization of M near P , with $\Gamma(0) = P$. Let $n \in T_P^*\mathbb{R}^d$, so that n annihilates tangent vectors in $T_P M$. Let $k \geq 2$. M is said to be of type k at P , with respect to n if the function $\mathbb{R}^{d-\ell} \ni t \mapsto \langle n, \Gamma(t) \rangle$ is of type k at $t = 0$.

It is easy to check that the last condition is independent of the particular parametrization; so the notion of type k is well defined. Also note the invariance of this notion under linear changes of coordinates. If M is of type k at P , with respect to n , then $k \geq 2$ since n is required to annihilate tangent vectors in $T_P M$.

If M is of type k with respect to n at P , and $\Gamma(0) = P$ then there is a vector $U \in \mathbb{R}^{d-\ell}$ so that

$$\langle U, \nabla_t \rangle^k (\langle n, \Gamma(t) \rangle) \Big|_{t=0} \neq 0;$$

this follows from [30, p. 343].

The curvature condition that we shall impose on $\Sigma_{1_r}^{L,x^0}$ will be defined with respect to n in the one-dimensional cokernel of the map $(d\pi_{L,x^0})_c$; n clearly annihilates tangent vectors in $T_{\xi^0}\Sigma_{1_r}^{L,x^0}$ and the curvature conditions will be invariant under changes of coordinates in X since the induced changes of coordinates in the fibers are linear. If $\mathcal{L}_1^{L,x^0} = S_1(\pi_{L,x^0})$, the surface where π_{L,x^0} drops rank by 1, then n has the geometric interpretation of being a ‘‘normal’’ vector to the hypersurface $\pi_{L,x^0}(\mathcal{L}_1^{L,x^0})$ which is nonsmooth at the cusp points (but has a well defined tangent plane there).

Lemma 2.11. *Let $\mathbf{c}^0 = (x^0, \xi^0, z^0, \zeta^0) \in C_\Phi$ so that $x^0 = z^0 = 0$, and suppose that π_L has an $S_{1_r,0}^+$ singularity at \mathbf{c}^0 . Suppose that Φ satisfies (2.11-13), (2.14), (2.17-18). Let $\Sigma_{1_r}^{L,0} \subset T_0^*X$ be the image of $S_{1_r}(\pi_{L,x^0})$ under π_{L,x^0} , and $0 \neq n \in \text{coker}(d\pi_{L,0})_{\mathbf{c}^0}$. Then the following holds:*

(i) *For $\ell \leq d-r$ the surface $\Sigma_{1_r}^{L,0}$ has ℓ nonvanishing principal curvatures with respect to n at ξ^0 if and only if*

$$(2.22) \quad \text{rank}(\Phi'''_{x_d z_i z_j}(0,0))_{i,j=1,\dots,d-r} = \ell.$$

(ii) *$\Sigma_{1_r}^{L,0} \in T_0^*X$ is of type k at ξ^0 if and only if there is a vector $u \in \text{span}\{e_1, \dots, e_{d-r}\}$ so that*

$$(2.23) \quad (\langle u, \nabla_z \rangle)^k \Phi'_{x_d}(0,0) \neq 0$$

and

$$(2.24) \quad \partial_{z_1}^{\alpha_1} \dots \partial_{z_{d-r}}^{\alpha_{d-r}} \Phi'_{x_d}(0,0) = 0 \quad \text{if} \quad \sum_{j=1}^{d-r} |\alpha_j| < k$$

Proof. It follows from (2.13) that $\text{coker}(d\pi_{L,x^0})_c \subset T_{\xi^0}^*(T_{x^0}^*X)$ is spanned by $d\xi_d$. By (2.19)

$$\Sigma_{1_r}^{L,0} = \{\Phi'_x(0,z) : \Phi_{x_d z_d^k}^{(k+1)}(0,z) = 0 \text{ for } k = 1, \dots, r\}.$$

Writing $z = (z'', z_{d-r+1}, \dots, z_{d-1}, z_d)$ we may use (2.14), (2.17) and the implicit function theorem to obtain a function $z'' \mapsto \mathfrak{Z} = (\mathfrak{Z}_{d-r+1}, \dots, \mathfrak{Z}_d)$, with $\mathfrak{Z}(0) = 0$ so that

$$\Phi_{x_d z_d^k}^{(k+1)}(0,z) = 0 \iff z_{d-r+k} = \mathfrak{Z}_{d-r+k}(z'') \quad \text{for } 1 \leq k \leq r.$$

Implicit differentiation yields

$$(2.25) \quad \Phi_{x_d z_d^k z_i}^{(k+2)}(0, z'', \mathfrak{Z}(z'')) + \sum_{j=1}^r \Phi_{x_d z_d^k z_{d-r+j}}^{(k+2)}(0, z'', \mathfrak{Z}(z'')) \frac{\partial \mathfrak{Z}_{d-r+j}}{\partial z_i}(z'') = 0$$

for $i = 1, \dots, d-r$. Repeating this we see that for $P_\alpha(z'') = z_1^{\alpha_1} \dots z_{d-r}^{\alpha_{d-r}}$

$$(2.26) \quad P_\alpha(\partial_{z''}) \Phi_{x_d z_d^k}^{(k+1)}(0, z'', \mathfrak{Z}(z'')) + \sum_{j=1}^r \Phi_{x_d z_d^k z_{d-r+j}}^{(k+2)}(0, z'', \mathfrak{Z}(z'')) P_\alpha(\partial_{z''}) \mathfrak{Z}_{d-r+j}(z'') = R_\alpha(z)$$

where R_α belongs to the ideal of smooth functions generated by the $P_\beta(\partial_{z''})\mathfrak{Z}_{d-r+j}$ with $\sum_{j=1}^{d-r} \beta_j < \sum_{j=1}^{d-r} \alpha_j$.

Note that by (2.17)

$$\Phi_{x_d z_d^k z_{d-r+j}}^{(k+2)}(0, 0) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and it follows from (2.25) that $\partial \mathfrak{Z}_{d-r+j} / \partial z_i(0) = 0$ for $i = 1, \dots, d-r$. Inductively we use (2.26) to deduce that

$$P_\alpha(\partial_{z''})\mathfrak{Z}_{d-r+j}(0) = 0, \quad j = 1, \dots, r$$

for all multiindices α . Consequently

$$(2.27) \quad P_\alpha(\partial_{z''})(\Phi'_{x_d}(0, z'', \mathfrak{Z}(z''))) = P_\alpha(\partial_{z''})\Phi'_{x_d}(0, z'', \mathfrak{Z}(z'')) + \rho_\alpha(z'')$$

where ρ is in the ideal generated by all $P_\alpha(\partial_{z''})\mathfrak{Z}_{d-r+j}$, so that ρ vanishes at 0. The assertion of the Lemma is now an immediate consequence of (2.27). \square

Examples: For $i = 1, 2, 3, 4$ let

$$\Phi_i(x, z) = x_1 z_1 + \dots + x_{d-1} z_{d-1} + x_d h_i(x, z)$$

where

$$\begin{aligned} h_1(x, z) &= z_d^{r+1} + \sum_{k=1}^{r-1} x_{d-r+k} z_d^k \\ h_2(x, z) &= \sum_{k=1}^r z_{d-r+k} z_d^k \\ h_3(x, z) &= \sum_{k=1}^r z_{d-r+k} z_d^k + z_1^m \\ h_4(x, z) &= \sum_{k=1}^r z_{d-r+k} z_d^k + \sum_{i=1}^{\ell} z_i^2 \end{aligned}$$

with $1 \leq \ell \leq d-r$ and $k \geq 2$. Let π_L^i denote the projection of C_{Φ_i} to T^*X . Then π_L^1 has an $S_{1_r,0}$ singularity but not an $S_{1_r,0}^+$ singularity, while $\pi_L^2, \pi_L^3, \pi_L^4$ have $S_{1_r,0}^+$ singularities. For $i = 2, 3, 4$ the cokernel of $d\pi_{L,0}^i$ is generated by $n = d\xi_d$. For π_L^2 the manifold $\Sigma_{1_r}^{L,0}$ is the $d-r$ dimensional plane given by $\xi_{d-r+j} = 0, j = 1, \dots, r$. For π_L^3 this manifold is of type m (with respect to n) where $\xi_1 = 0$ and of type 2 where $\xi_1 \neq 0$. For π_L^4 it has ℓ nonvanishing principal curvatures with respect to n .

3. Estimates for oscillatory integrals with strong one-sided cusps

Let X, Z be open subsets of \mathbb{R}^d and let $\Phi \in C^\infty(X \times Z)$ be a real valued phase function and $a \in C_0^\infty(X \times Z)$. Define, for $\lambda > 0$, the oscillatory integral operator T_λ as in (2.8), and let C_Φ be the associated symplectic relation. Recall the definition (2.21) of the image $\Sigma_{1,1}^{L,x^0}$ of the cusp surface to the fibers.

Theorem 3.1. *Suppose that $(x^0, z^0) \in X \times Z$, $\mathfrak{c}^0 \in C_\Phi$ with $\pi_X \mathfrak{c}^0 = x^0$, $\pi_Z \mathfrak{c}^0 = z^0$ and let $\xi^0 = \Phi'_x(x^0, z^0)$. Then there is a neighborhood \mathcal{U} of (x^0, z^0) , depending on Φ so that the following holds provided that a is supported in \mathcal{U} .*

(i) *If π_L has an $S_{1,1,0}^+$ singularity at \mathfrak{c}^0 then*

$$(3.1) \quad \|T_\lambda\|_{L^2(Z) \rightarrow L^q(X)} \lesssim \begin{cases} \lambda^{-\frac{d-1}{q} + \frac{2}{3q} - \frac{1}{2}}, & 2 \leq q \leq \frac{10}{3} \\ \lambda^{-\frac{d}{q}}, & \frac{10}{3} \leq q \leq \infty \end{cases}.$$

(ii) *Suppose that π_L has an $S_{1,1,0}^+$ singularity at \mathfrak{c}^0 and that $\Sigma_{1,1}^{L,x^0}$ has ℓ nonvanishing principal curvatures with respect to $n \in \text{coker } d\pi_{L,x^0}$. Then*

$$(3.2) \quad \|T_\lambda\|_{L^2(Z) \rightarrow L^q(X)} \lesssim \begin{cases} \lambda^{-\frac{d-1}{q} - \frac{\ell+2}{2}(\frac{1}{2} - \frac{1}{q}) - \frac{1}{3q}}, & 2 \leq q \leq \frac{6\ell+20}{3\ell+6} \\ \lambda^{-\frac{d}{q}}, & \frac{6\ell+20}{3\ell+6} \leq q \leq \infty \end{cases}.$$

(iii) *Suppose that π_L has an $S_{1,1,0}^+$ singularity at \mathfrak{c}^0 and that $\Sigma_{1,1}^{L,x^0}$ is of finite type k at ξ^0 with respect to $n \in \text{coker } d\pi_{L,x^0}$. Then*

$$(3.3) \quad \|T_\lambda\|_{L^2(Z) \rightarrow L^q(X)} \lesssim \begin{cases} \lambda^{-\frac{d-1}{q} - \frac{k+1}{k}(\frac{1}{2} - \frac{1}{q}) - \frac{1}{3q}}, & 2 \leq q \leq \frac{10k+6}{3k+3} \\ \lambda^{-\frac{d}{q}}, & \frac{10k+6}{3k+3} \leq q \leq \infty \end{cases}.$$

We shall see that the $L^2 \rightarrow L^2$ estimates hold with just the assumption of an $S_{1,1,0}$ singularity, see §4; however the $L^2 \rightarrow L^q$ estimate (3.1) may fail to hold without the assumption of a strong cusp; see the example in Remarks 3.5 below.

The estimate should be compared with the corresponding estimates for folds which were either explicitly stated in [7] or follow by the arguments of [7]. Assuming that π_L has a fold singularity then one obtains that

$$(3.4) \quad \|T_\lambda\|_{L^2(Z) \rightarrow L^q(X)} \lesssim \begin{cases} \lambda^{-\frac{d-1}{q} - b(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2q}}, & 2 \leq q \leq q(b) \\ \lambda^{-\frac{d}{q}}, & q(b) \leq q \leq \infty \end{cases}$$

where $b = 1/2$ and $q(b) = 4$. If the image Σ_1^{L,x^0} in the fiber has ℓ nonvanishing curvatures then $b = (\ell + 1)/2$ and $q(b) = (2\ell + 4)/(\ell + 1)$, and if Σ_1^{L,x^0} is of finite type $\leq k$ with respect to the normal $n \in \text{coker } d\pi_L$ then $b = 1/2 + 1/k$ and $q(b) = 4(k + 1)/(k + 2)$.

The estimates of Theorem 3.1 can be extended to more general oscillatory integrals with nonhomogeneous phase functions depending on frequency variables. This will be useful in §4; the arguments leading to this extension are contained in [7, §3].

Let X, Y be open sets in \mathbb{R}^D , and let Ω be an open set in \mathbb{R}^N . Let S_μ be defined by

$$(3.5) \quad S_\mu f(x) = \iint_{\mathbb{R}^D \times \mathbb{R}^N} e^{i\mu\psi(x,y,\vartheta)} b(x,y,\vartheta) f(y) dy dz$$

where the phase ψ is smooth and real valued in $X \times Y \times \Omega$ and the amplitude b is smooth and compactly supported in $X \times Y \times \Omega$. We assume that $d_{x,y,\vartheta} \psi'_i, i = 1, \dots, N$ are linearly independent, so that

$$\text{Crit}_\psi = \{(x, y, \vartheta) : \psi'_\vartheta(x, y, \vartheta) = 0\}$$

is a $2D$ dimensional immersed manifold (in other words, ψ is assumed to be nondegenerate in the sense of Hörmander [13], although no homogeneity is required). Consequently

$$\mathcal{C}_\psi = \{(x, \psi'_x, y, -\psi'_y) : \psi'_\vartheta = 0\}$$

is a smooth symplectic relation.

Corollary 3.2. *Let $\mathfrak{c}^0 = (x^0, \psi'_x(x^0, y^0, \vartheta^0), y^0, -\psi'_y(x^0, y^0, \vartheta^0))$ with $(x^0, y^0) \in X \times Y, \vartheta^0 \in \Omega$, so that $\psi'_\vartheta(x^0, y^0, \vartheta^0) = 0$.*

*Suppose that the projection $\pi_L : \mathcal{C}_\psi \rightarrow T^*X$ has an $S_{1,1,0}^+$ singularity at \mathfrak{c}^0 . Then there is a neighborhood U of (x^0, y^0, ϑ^0) such that*

$$\|S_\mu\|_{L^2(Y) \rightarrow L^2(X)} \lesssim \mu^{-\frac{D+N-1}{2} - \frac{1}{6}}$$

provided that b is supported in U .

Likewise one can formulate versions of the $L^2 \rightarrow L^q$ estimates of Theorem 3.1 for the operators S_μ to obtain $L^2 \rightarrow L^q$ estimates; one sets $d = D, \mu = \lambda$ and multiplies the resulting expressions in (3.1-3.3) by $\mu^{-N/2}$.

Corresponding estimates for Fourier integral estimates with homogeneous phase functions can be deduced from (3.1-3.4) using standard arguments involving partial Fourier transforms and Littlewood-Paley type estimates [13, 24, 27, 7]. The result is

Corollary 3.3. *Let X and Y be d dimensional manifolds, $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ be a homogeneous canonical relation. Let $A \in I^m(C; X, Y)$ with compactly supported Schwartz-kernel K_A . Let $\mathfrak{c}^0 = (x^0, \xi^0, y^0, -\eta^0) \in C$ and suppose that π_L has an $S_{1,1,0}^+$ singularity at \mathfrak{c}^0 . Then the following holds provided that the wavefront relation $\text{WF}(K_A)'$ is contained in a suitable conic neighborhood of \mathfrak{c}^0 .*

(i) A maps $L_{\alpha, \text{comp}}^2(Y)$ boundedly to $L_{\beta, \text{loc}}^q(X)$ if $m + \beta - \alpha \leq d(\frac{1}{q} - \frac{1}{2}) + \frac{1}{2} - \frac{5}{3q}$ and $2 \leq q \leq 10/3$, or if $m + \beta - \alpha \leq d(\frac{1}{q} - \frac{1}{2})$ and $10/3 \leq q < \infty$;

(ii) Suppose that $\Sigma_{1,1}^{L,x^0}$ has ℓ nonvanishing principal curvatures at ξ^0 with respect to $n \in \text{coker } d\pi_{L,x^0}$. Then A maps $L_{\text{comp}}^2(Y)$ boundedly to $L_{\text{loc}}^q(X)$ if $m \leq d(\frac{1}{q} - \frac{1}{2}) + \frac{\ell+2}{4} - \frac{3\ell+10}{6q}$ and $2 \leq q \leq \frac{6\ell+20}{3\ell+6}$, or if $m \leq d(\frac{1}{q} - \frac{1}{2})$ and $\frac{6\ell+20}{3\ell+6} \leq q < \infty$.

(iii) Suppose that $\Sigma_{1,1}^{L,x^0}$ is of finite type k at ξ^0 with respect to $n \in \text{coker } d\pi_{L,x^0}$. Then A maps $L_{\text{comp}}^2(Y)$ boundedly to $L_{\text{loc}}^q(X)$ if $m \leq d(\frac{1}{q} - \frac{1}{2}) + \frac{k+1}{2k} - \frac{5k+3}{3kq}$ and $2 \leq q \leq \frac{10k+6}{3k+3}$, or if $m \leq d(\frac{1}{q} - \frac{1}{2})$ and $\frac{10k+6}{3k+3} \leq q < \infty$.

The restricted X-ray transforms discussed in the introduction are Fourier integral operators of order $-1/2$, with the projection $\pi_R : N^* \mathcal{Z}_{\mathfrak{c}} \rightarrow \Omega$ having strong cusp singularities; this is shown in §5. Thus the following can be applied to obtain estimates for these operators.

Corollary 3.4. *Suppose that X and Y are manifolds of dimension $d = 4$ and $C \subset T^*X \setminus 0 \times T^*Y \setminus 0$ is a homogeneous canonical relation such that the projection $\pi_R : C_\Phi \rightarrow T^*Y$ have singularities that are at most strong simple cusps, i.e., at every point of C_Φ , π_R is either a diffeomorphism, a Whitney fold, or an $S_{1,1,0}^+$. Suppose also that the projection $\pi_Y : S_1(\pi_R) \rightarrow Y$ has surjective differential everywhere in $S_1(\pi_R)$. Let \mathcal{R} be a Fourier integral operator in the class $I^{-1/2}(C; X, Y)$. Then*

(i) \mathcal{R} maps $L_{\alpha, \text{comp}}^2(Y)$ to $L_{\alpha + \frac{1}{3}, \text{loc}}^2(X)$ and $L_{\text{comp}}^{7/4}(Y)$ to $L_{\text{loc}}^2(X)$.

(ii) \mathcal{R} maps $L_{\text{comp}}^{12/7}(Y)$ to $L_{\text{loc}}^2(X)$, under the additional assumption that the surfaces $\Sigma_{1,1}^{L,x^0}$ are of type ≤ 3 everywhere with respect to $n \in \text{coker } d\pi_{L,x^0}$.

Corollary 3.4 follows from Corollary 3.3 by splitting $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$ where the wavefront relation of \mathcal{R}_1 is localized near the cusp surface $S_{1,1}(\pi_R)$, and therefore Corollary 3.3 can be applied to the adjoint \mathcal{R}_1^* . The operator \mathcal{R}_2^* is a Fourier integral operator of order $-1/2$ for which π_L has only fold singularities. It follows from Theorem 1.2 in [7] and interpolation that \mathcal{R}_2^* is bounded from L^2 to $L^{(4d-4)/(2d-3)}$, hence, since $d = 4$, for $q \leq 12/5$. We remark that typically the latter estimate can be improved since in view of the strong cusp assumption the images of the fold surface have curvature at least near the cusp points. However we shall not have to make use of this observation here.

Remarks 3.5. Concerning the sharpness of these estimates we consider various restricted X-ray transforms.

1. Let $R_{\mathcal{C}}$ be the restricted X-ray transform as in (1.1) for the translation-invariant complex of lines with parametrizations

$$\gamma(v, t) = (v_1 + v_4 t, v_2 + v_4^2 t, v_3 + v_4^3 t, t) : t \in \mathbb{R},$$

Test $R_{\mathcal{C}}$ on the cut-off Heaviside function $f = H(w_3)\chi(w)$. Then $f \in L_{\frac{1}{2}-\epsilon, \text{comp}}^2(\mathbb{R}^4)$, for all $\epsilon > 0$, and no better. On the other hand,

$$R_{\mathcal{C}}f(v) = \int_{v_3 + v_4^3 t \geq 0} \chi^2(v_1 + v_4 t, v_2 + v_4^2 t, v_3 + v_4^3 t, t) dt,$$

from which it is easy to see that $R_{\mathcal{C}}f(v)$ is smooth in v_1, v_2 and approximately homogeneous of degree 0 in v_3, v_4 with respect to the nonisotropic dilations $(v_3, v_4) \rightarrow (r^3 v_3, r v_4)$. Thus, its Fourier transform is Schwartz in ξ_1, ξ_2 and approximately homogeneous of degree -4 in ξ_3, ξ_4 with respect to these same dilations. From this, it is straightforward to see that $R_{\mathcal{C}}f \in L_{\frac{2}{3}-\epsilon', \text{loc}}^2$, for all $\epsilon' > 0$, and no better, and thus $R_{\mathcal{C}}$ smooths by no more than $1/6$ derivatives.

2. The $L^{7/4} \rightarrow L^2$ estimate of Corollary 3.4 cannot be improved without adding further assumptions (such as well-curvedness). To see this, consider again the restricted X-ray transform $R_{\mathcal{C}}$ with line complex given by

$$\gamma(v, t) = (v_1 + v_4 t, v_2 + v_4^2 t, v_3 + v_1 v_4^3 t, t).$$

Then, if f_δ is the characteristic function of the rectangle $\{|w_1| \leq \delta, |w_2| \leq \delta^2, |w_3| \leq \delta^4, |w_4| \leq 1\}$, we have $\|f_\delta\|_{L^p} \simeq \delta^{\frac{7}{p}}$, while $R_{\mathcal{C}}f_\delta \geq 1$ on $\{|v_1| \leq c\delta, |v_2| \leq c\delta^2, |v_3| \leq c\delta^4, |v_4| \leq c\delta\}$, so that $\|R_{\mathcal{C}}\|_{L^q} \geq c\delta^{\frac{8}{q}}$. Letting $\delta \searrow 0$, we must have $\frac{8}{q} \geq \frac{7}{p}$; in particular, for $q = 2$ we must have $p \geq \frac{7}{4}$. Hence, $R_{\mathcal{C}} : L^{\frac{7}{4}} \rightarrow L^2$, and no better.

3. For $q \neq 2$ the $L^2 \rightarrow L^q$ estimates of Corollary 3.3 cannot hold in general without the strong cusp assumption. In \mathbb{R}^3 , consider the restricted X -ray transform given by

$$R_{\mathcal{C}}f(v) = \int f(v_1 + t(v_2v_3 + v_3^3), v_2, t)\chi(t)dt.$$

The associated canonical relation is

$$C = \{(w_1 - w_3(w_2v_3 + v_3^3), w_2, v_3, \xi_1, \xi_2 + w_3v_3\xi_1, w_3(w_2 + 3v_3^2)\xi_1; \\ w_1, w_2, w_3, \xi_1, \xi_2, -(w_2v_3 + v_3^3)\xi_1) : w \in \mathbb{R}^3, v_3 \in \mathbb{R}, (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}\},$$

from which we see that $\pi_R : C \rightarrow T^*\mathbb{R}^3 \setminus \{0\}$ has at most $S_{1,1,0}$ singularities where $\xi_1 \neq 0$, but those $S_{1,1,0}$ singularities are not strong. We take $f_\delta(w) = \chi_\delta(w_2, w_3)\psi(\delta^{-3}w_1)$ where χ_δ is the characteristic function of $\{|w_2| \leq \delta^2, |w_3| \leq 1\}$, and ψ is a C_0^∞ function with cancellation. The cancellation allows us to microlocalize $R_{\mathcal{C}}$ to frequencies with $\xi_1 \neq 0$. Then $|R_{\mathcal{C}}f_\delta(v)| \geq c$ on a fixed fraction of $\{|v_1| \leq \delta^3, |v_2| \leq \delta^2, |v_3| \leq \delta\}$, so that $\|f_\delta\|_p \sim \delta^{\frac{5}{p}}$ and $\|R_{\mathcal{C}}f_\delta\|_2 \sim \delta^3$. Hence, the $L^p \rightarrow L^2$ boundedness of (the microlocalized version of) $R_{\mathcal{C}}$ implies $p \geq 5/3$ and $\mathcal{R}_{\mathcal{C}}^* : L^2 \rightarrow L^q$ only if $q \leq 5/2$. The corresponding result for Fourier integral operators with *strong* cusps (Corollary 3.3) would imply a better $L^2 \rightarrow L^{8/3}$ estimate. \square

In order to prove Theorem 3.1 we now wish to follow the proof [7] of the corresponding results for Whitney folds. Some of the arguments are in fact valid under the assumption that π_L is a strong cusp of order $r \leq d - 2$ everywhere, so we will work under this assumption at first. We will have to be able to prove an estimate for particular families of oscillatory integrals of the form (3.5) with phases and amplitudes depending smoothly on a parameter γ ; the canonical relations will have two-sided $S_{1,r-1}$ singularities. The conjectured L^2 estimates $\|T_\lambda\| \lesssim \lambda^{-d-1-1/(2r+2)}$ could be proved if one could show that $\|S_\mu\|_{L^2 \rightarrow L^2} \lesssim \mu^{-\frac{d+N-1}{2}-a}$ with $a = 1/(r+1)$ (see (3.14_a) below.) If \mathcal{C}_ψ is a folding symplectic relation (so $r-1 = 1$ and both π_L, π_R have only Whitney folds or $S_{1,0}$ singularities) then this estimate does hold with $a = 1/3$. This is shown in [22] (see also [17] for the corresponding result for homogeneous canonical relations, and [24], [29], [4], [8] for different proofs); these bounds are uniform with respect to parameters as follows for example by combining arguments in [7, §3] and [8].

We shall perform the change of variables discussed in §2, and from now on work close to the origin; the general assumption is that Φ satisfies (2.10-2.13) and C_Φ has a strong $S_{1,r}$ singularity at \mathfrak{c}^0 above $(0,0)$. According to Lemma 2.9 we can assume that (2.14), (2.17) and (2.18) hold. The amplitude a is supposed to be supported where $|x| + |z| \leq \varepsilon_0 \ll \varepsilon$, the parameter ε is small (the argument below determines how small these parameters are to be chosen).

We now argue as in §2 of [7, p.42 ff.], to reduce matters to estimates involving oscillatory integral operators such as in (3.5) acting on functions in \mathbb{R}^{d-1} . To bound $\|T_\lambda\|_{L^2 \rightarrow L^{p'}} \leq \|T_\lambda T_\lambda^*\|_{L^p \rightarrow L^{p'}}^{1/2}$, one writes $T_\lambda T_\lambda^* f(x', x_d) = \int \mathcal{K}_{x_d, y_d} [f(\cdot, y_d)](x') dy_d$ where

$$(3.6) \quad \mathcal{K}_{x_d, y_d} g(x') = \int K_\lambda(x', x_d, y', y_d) g(y') dy'$$

with

$$(3.7) \quad K_\lambda(x', x_d, y', y_d) = \int e^{i\lambda[\Phi(x', x_d, z) - \Phi(y', y_d, z)]} a(x, z) \overline{a(y, z)} dz.$$

The kernel of \mathcal{K}_{x_d, y_d} can be split as $H_{x_d, y_d}(x', y') + R_{x_d, y_d}(x', y')$ so that $H_{x_d, y_d}(x', y') = 0$ if $|x' - y'| \geq \epsilon|x_d - y_d|$ or if $|x_d - y_d| \leq \lambda^{-1}$. Here $\epsilon \ll 1$ but $\epsilon \gg \epsilon_0$. Observe that by (2.12), (2.13)

$$\Phi'_{x'}(x', x_d, z) - \Phi'_{x'}(y', y_d, z) = x' - y' + O(\epsilon_0|x - y|);$$

therefore we integrate by parts with respect to z' and obtain that the operator \mathcal{R}_{x_d, y_d} with kernel R_{x_d, y_d} is $L^p \rightarrow L^{p'}$ bounded with

$$(3.8) \quad \|\mathcal{R}_{x_d, y_d}\|_{L^p \rightarrow L^{p'}} \lesssim \lambda^{(1-d)/p'} (1 + \lambda|x_d - y_d|)^{-N}, \quad 1 \leq p \leq 2.$$

By (2.17), $\Phi'''_{x_d z_d - r + 1 z_d}(0, 0) \neq 0$. Therefore we can apply the method of stationary phase to obtain that

$$|K_\lambda(x', x_d, y', y_d)| \lesssim (1 + \lambda|x_d - y_d|)^{-1}, \quad \text{if } |x' - y'| \geq \epsilon|x_d - y_d|;$$

a better estimate is valid in the complementary region by (3.8). Therefore we have

$$(3.9_\beta) \quad \|K_{x_d, y_d}\|_{L^1 \rightarrow L^\infty} \lesssim (1 + \lambda|x_d - y_d|)^{-\beta},$$

with $\beta = 1$. This may be improved if one imposes additional curvature assumptions on the images of cusp surfaces (see Lemma 3.8 below).

We now turn to L^2 estimate of H_{x_d, y_d} ; recall that only x_d, y_d with $|x_d| + |y_d| \ll \epsilon$ are of interest. For fixed x_d, y_d one splits $H_{x_d, y_d} = \sum_{n \in \mathbb{Z}^{d-1}} H_{x_d, y_d}^n$ where

$$H_{x_d, y_d}^n(x', y') = \beta(\varepsilon_1^{-1}|x_d - y_d|^{-1}x' - n)H(x', y'), \quad n \in \mathbb{Z}^{d-1}$$

and ε_1 is small. The kernels H_{x_d, y_d}^n are localized to cubes with center $c_n = n\varepsilon_1|x_d - y_d|$ and diameter $O(\varepsilon_1|x_d - y_d|)$, so as in [7], because of the localization and therefore by (almost) orthogonality, it suffices to prove the required bounds for the individual operators \mathcal{H}_{x_d, y_d}^n (with kernels $H_{x_d, y_d}^n(x', y')$); in fact

$$\left\| \sum_n \mathcal{H}_{x_d, y_d}^n \right\|_{L^2 \rightarrow L^2} \lesssim \varepsilon_1^{-2d} \sup_n \|\mathcal{H}_{x_d, y_d}^n\|_{L^2 \rightarrow L^2}.$$

One introduces rescaled operators $\tilde{\mathcal{H}}_{x_d, y_d}^n$ with kernels

$$\tilde{H}_{x_d, y_d}^n(u, v) = H^n(c_n + u|x_d - y_d|, c_n + v|x_d - y_d|);$$

then $\mathcal{H}_{x_d, y_d}^n g(x) = |x_d - y_d|^{d-1} \tilde{\mathcal{H}}_{x_d, y_d}^n [f(|x_d - y_d| \cdot + n)](\frac{x}{|x_d - y_d|} - n)$ and

$$(3.10) \quad \|H_{x_d, y_d}^n\|_{L^p(\mathbb{R}^{d-1}) \rightarrow L^{p'}(\mathbb{R}^{d-1})} = |x_d - y_d|^{2(d-1)/p'} \|\mathcal{H}_{x_d, y_d}^n\|_{L^p(\mathbb{R}^{d-1}) \rightarrow L^{p'}(\mathbb{R}^{d-1})};$$

notice also that $\tilde{H}_{x_d, y_d}^n(u, v) = 0$ for $|u - v| \geq C\varepsilon_1$.

The kernels \tilde{H}_{x_d, y_d}^n can be imbedded in two families of oscillatory integrals $h_{\mu; \gamma, c}^\pm(u, v)$ depending on the large parameter

$$\mu = \lambda|x_d - y_d|$$

and the small parameters

$$\gamma = (\gamma_1, \gamma_2) = (|x_d - y_d|, y_d); \quad c = n\varepsilon_1 |x_d - y_d|$$

notice that $\gamma = O(\varepsilon_0)$, $c = O(\varepsilon_0)$. The oscillatory integrals are given by

$$(3.11) \quad h_{\mu, \gamma, c}^{\pm}(u, v) = \int e^{i\mu\Psi_{\pm}(u, v, z; \gamma, c)} b_{\gamma, c}(u, v, z) dz$$

where the amplitudes $b_{\gamma, c}(u, v, z)$ belong to bounded subsets of C^{∞} , and depend smoothly on the parameters c, γ ; the phases are given by

$$\begin{aligned} \Psi^{\pm}(u, v, z; \gamma, c) &= \frac{\Phi(u\gamma_1 + c, \gamma_2 \pm \gamma_1, z) - \Phi(v\gamma_1 + c, \gamma_2, z)}{\gamma_1} \\ &= \int_0^1 \langle u - v, \Phi'_{x'} \rangle (v\gamma_1 + s(u - v)\gamma_1 + c\gamma_2 \pm s\gamma_1, z) \pm \Phi'_{x_d}(v\gamma_1 + s(u - v)\gamma_1 + c, \gamma_2 \pm s\gamma_1, z) ds. \end{aligned}$$

Of course this last formula makes sense for $\gamma_1 = 0$. Expanding in γ_1 and γ_2 about 0 yields

$$(3.12) \quad \begin{aligned} \Psi^{\pm}(u, v, z; \gamma, c) &= \langle u - v, \Phi'_{x'} \rangle(0, c, z) \pm \Phi'_{x_d}(0, c, z) \\ &\quad + \gamma_1 \rho_1^{\pm}(u, v, z; \gamma, c) + \gamma_2 \rho_2^{\pm}(u, v, z; \gamma, c), \end{aligned}$$

so that Ψ^{\pm} are perturbations of phases which occur in a translation invariant situation.

Define

$$(3.13) \quad S_{\mu; \gamma, c}^{\pm} g(u) = \int h_{\mu; \gamma, c}^{\pm}(u, v) g(v) dv$$

with $h_{\mu; \gamma, c}^{\pm}$ as in (3.11).

We shall work under the following

$$(3.14_a) \quad \textit{Hypothesis:} \quad \|S_{\mu; \gamma, c}\|_{L^2 \rightarrow L^2} \lesssim \mu^{-(d-1-a)}, \quad \text{for small } \gamma \text{ and } c, \text{ and large } \mu.$$

As mentioned above it is conjectured that (3.14) holds with $a = 1/(r+1)$; in fact we shall prove this estimate for the limiting case $\gamma = 0$ (see Lemma 3.9 below). Moreover we shall verify that the operators $S_{\mu; \gamma, c}$ are oscillatory integrals associated to smooth symplectic relations, with the projections π_L, π_R being $S_{1, r-1, 0}^+$ singularities (see Lemma 3.7 below). In particular we know then that (3.14_a) holds if $r = 2$ and $a = 1/3$ (of course it also holds with $r = 1$ and $a = 1/2$; this is the situation of [7]).

Lemma 3.6. *Hypotheses (3.14_a) and (3.9_{\beta}) imply that*

$$(3.15) \quad \|T_{\lambda}\|_{L^2 \rightarrow L^q} \lesssim \lambda^{-\frac{d-1}{q} - \beta(\frac{1}{2} - \frac{1}{q}) - \frac{a}{q}}, \quad 2 \leq q \leq \frac{2(\beta + 1 - a)}{\beta}.$$

$$(3.16) \quad \|T_{\lambda}\|_{L^2 \rightarrow L^q} \lesssim \lambda^{-\frac{d}{q}}, \quad \frac{2(\beta + 1 - a)}{\beta} \leq q \leq \infty.$$

Proof. Continuing the reasoning of [7, §2] we obtain, using (3.14)_a and taking (3.10) into account, that

$$(3.17) \quad \|H_{x_d, y_d}\|_{L^2 \rightarrow L^2} \lesssim \lambda^{1-d} (1 + \lambda|x_d - y_d|)^{-a}$$

and hence the same estimate for $\|K_{x_d, y_d}\|_{L^2 \rightarrow L^2}$.

Interpolating between (3.9 $_\beta$) and (3.17) for H_{x_d, y_d} replaced by K_{x_d, y_d} yields

$$(3.18) \quad \|K_{x_d, y_d}\|_{L^p \rightarrow L^{p'}} \lesssim \lambda^{\frac{2}{p}(1-d)} (1 + \lambda|x_d - y_d|)^{-\beta(p)}, \quad 1 \leq p \leq 2,$$

where

$$\beta(p) = \beta\left(\frac{2}{p} - 1\right) + \frac{2}{p'a}.$$

If $\frac{1}{p} - \frac{1}{p'} \leq 1 - \beta(p)$ one may use fractional integration as in [7] to obtain that

$$(3.19) \quad \|T_\lambda T_\lambda^*\|_{L^p \rightarrow L^{p'}} \lesssim \lambda^{\frac{2}{p}(1-d) - \beta(p)}, \quad \frac{2(1 + \beta - a)}{2 + \beta - 2a} \leq p \leq 2;$$

which implies (3.15). Interpolating the resulting $L^2 \rightarrow L^{\frac{2(\beta+1-a)}{\beta}}$ estimate with the trivial estimate $\|T_\lambda\|_{L^2 \rightarrow L^\infty} \leq C$, we obtain (3.16). \square

We shall now analyze the oscillatory integrals in (3.11).

Lemma 3.7. *Let Ψ^\pm be as in (3.12) and let*

$$C_{\gamma, c}^\pm = \{(u, (\Psi^\pm)'_u, v, -(\Psi^\pm)'_v) : (\Psi^\pm)'_z = 0, u, v, z \text{ near } 0\}.$$

For c near 0, γ near 0, $C_{\gamma, c}^\pm$ is a smooth symplectic relation, with two sided $S_{1, r-1, 0}^\pm$ singularities.

Proof. In order to simplify the notation we write Ψ for Ψ^\pm . We shall also assume that $\gamma = 0, c = 0$ which is no loss of generality in view of the invariance of our statements under small perturbations.

We first show that the set of critical points $\text{Crit}_{\Psi^\pm} = \{(u, v, z) : \Psi'_{z_j}(u, v, z) = 0, j = 1, \dots, d\}$ is a smooth manifold. By (2.12), we know that $\Psi''_{uz'} = \Phi''_{x'z'}$ is nonsingular, moreover since $u - v = O(\varepsilon_1)$

$$(3.20) \quad (\Psi^\pm)''_{z_d z_{d-r+1}} = \pm \Phi'''_{x_d z_d z_{d-r+1}}(0, 0) + O(\varepsilon).$$

By (2.17) $(\Psi^\pm)''_{z_d z_{d-r+1}}$ is bounded below so the gradients $\nabla_{u, v, z} \Psi'_{z_j}$ are linearly independent and Crit_{Ψ^\pm} are smooth manifolds. Note that we had to use the assumption of a *strong* $S_{1, r, 0}$ singularity here.

In view of the symmetry at $\gamma = 0, c = 0$, it suffices to check that, $\pi_L : C_{0, 0}^\pm \rightarrow T^*X$ has $S_{1, r-1}^\pm$ singularities at the point $\mathbf{c}^0 = (0, 0, 0, 0)$; notice that $\Phi'_x(0, 0) = \Phi'_z(0, 0) = 0$ by (2.10).

We solve the equations $\Psi'_{z_d}(0, 0, z) = \Phi''_{x_d z_d}(0, 0, z) = 0$ and in view of (2.17) we see that z_{d-r+1} can be expressed as a function of $\tilde{z} = (z_1, \dots, z_{d-r}, z_{d-r+2}, \dots, z_d)$, so that

$$(3.21) \quad \Phi''_{x_d z_d}(0, 0, z_1, \dots, z_{d-r}, z_{d-r+1}^\pm(\tilde{z}), z_{d-r+2}, \dots, z_d) = 0.$$

We then have to show that

$$\tilde{z} \mapsto F(\tilde{z}) := \Phi'_{x'}(0, z_1, \dots, z_{d-r}, z_{d-r+1}^\pm(\tilde{z}), z_{d-r+2}, \dots, z_d)$$

has an $S_{1,r-1}$ singularity at 0. To do this we use Lemma 2.4 (for functions of $d-1$ variables), the appropriate version of (2.6), (2.7) being

$$(3.22) \quad \frac{\partial^k F_\nu}{\partial z_d^k}(0) = 0, \quad 1 \leq k \leq r-1, \nu \neq d-r+1$$

$$(3.23) \quad \frac{\partial^k F_\nu}{\partial z_d^{k-1} \partial z_i}(0) = 0, \quad 2 \leq k \leq r-1, \nu \neq d-r+1.$$

Given (3.22), (3.23) we shall then have to verify that

$$(3.24) \quad \frac{\partial^k F_{d-r+1}}{\partial z_d^k}(0) = \begin{cases} 0, & \text{if } 1 \leq k \leq r-1 \\ 1, & \text{if } k = r \end{cases}.$$

and

$$(3.25) \quad \frac{\partial^k F_{d-r+1}}{\partial z_d^{k-1} \partial z_i}(0) = \begin{cases} 0, & \text{if } 2 \leq k \leq r-1, i \neq d-r+k \\ 1, & \text{if } 2 \leq k \leq r-1, i = d-r+k \end{cases}.$$

Differentiating (3.21) yields

$$(3.26) \quad \Phi'''_{x_d z_d z_{d-r+1}} \frac{\partial z_{d-r+1}^\pm}{\partial z_d} + \Phi'''_{x_d z_d^2} = 0$$

where the derivatives of Φ are evaluated at $z_{d-r+1} = z_{d-r+1}^\pm(\tilde{z})$. Similarly

$$(3.27) \quad \Phi'''_{x_d z_d z_{d-r+1}} \frac{\partial^k z_{d-r+1}^\pm}{\partial z_d^k} + \Phi^{(k+2)}_{x_d z_d^{k+1}} \in \mathcal{J}_{k-1}$$

where \mathcal{J}_{k-1} is the ideal generated by the functions $\frac{\partial z_{d-r+1}^\pm}{\partial z_d^j}$, for $1 \leq j \leq k-1$. (3.26), (3.27) imply together with (2.14), (2.17) that

$$(3.28) \quad \frac{\partial^k z_{d-r+1}^\pm}{\partial z_d^k}(0) = \begin{cases} 0, & \text{if } 1 \leq k \leq r-1 \\ 1, & \text{if } k = r \end{cases}.$$

Differentiating the relation (3.26) with respect to z_i yields

$$\Phi'''_{x_d z_d z_{d-r+1}} \frac{\partial^k z_{d-r+1}^\pm}{\partial z_d^{k-1}} + \Phi^{(k+1)}_{x_d z_d^{k-1} z_i} \in \mathcal{J}_{k-1}$$

so that

$$(3.29) \quad \frac{\partial^k z_{d-r+1}^\pm}{\partial z_d^{k-1} \partial z_i} = \begin{cases} 0, & \text{if } 2 \leq k \leq r, i \neq d-r+k \\ 1, & \text{if } 2 \leq k \leq r, i = d-r+k \end{cases}.$$

We now differentiate F and use that $\Phi''_{x'z_d}(0, z) \equiv 0$ (by (2.13)). It follows that

$$(3.30) \quad \frac{\partial^k F_\nu}{\partial z_d^k} - \Phi''_{x_\nu z_{d-r+1}} \frac{\partial^k z_{d-r+1}^\pm}{\partial z_d^k} \in \mathcal{J}_{k-1}, \quad 1 \leq k \leq r$$

$$(3.31) \quad \frac{\partial^k F_\nu}{\partial z_d^k} - \Phi''_{x_\nu z_{d-r+1}} \frac{\partial^k z_{d-r+1}^\pm}{\partial z_d^{k-1} \partial z_i} \in \mathcal{J}_{k-1}, \quad 2 \leq k \leq r.$$

Now (3.30), (3.31) together with (2.12) imply (3.22), (3.23), (3.24) and (3.25). \square

We now show how our various curvature assumption imply improved decay estimates in (3.9); this is analogous to the role of curvature in proving restriction theorems for the Fourier transform.

Lemma 3.8. *Assuming $r \geq 2$ then*

(i) *Estimate (3.9 $_\beta$) holds with $\beta = 1$.*

(ii) *If $\Sigma_{1,1}^{L,x^0}$ is of finite type k at ξ^0 with respect to $n \in \text{coker } d\pi_{L,x^0}$ then estimate (3.9 $_\beta$) holds with $\beta = (k+1)/k$.*

(iii) *If $\Sigma_{1,1}^{L,x^0}$ has ℓ nonvanishing principal curvatures with respect to $n \in \text{coker } d\pi_{L,x^0}$ at ξ^0 then (3.9 $_\beta$) holds with $\beta = (\ell+2)/2$.*

Proof. We split variables as $z = (z'', z_{d-r+1}, \tilde{z}, z_d)$ (so that the \tilde{z} -part is not present if $r = 2$). We begin by solving in (z_{d-r+1}, z_d) the equations $\Phi''_{x_d z_{d-r+1}} = \Phi''_{x_d z_d} = 0$ for $\tilde{z} = 0$. This is possible since $\text{rank } \frac{\partial(\Phi''_{x_d z_{d-r+1}}, \Phi''_{x_d z_d})}{\partial(z_{d-r+1}, z_d)} = 2$, by (2.17), (2.14).

We obtain functions Z_{d-r+1}, Z_d depending on z'' and vanishing at 0 so that

$$\begin{aligned} \Phi''_{x_d z_{d-r+1}}(z'', Z_{d-r+1}(z''), 0, Z_d(z'')) &= 0 \\ \Phi''_{x_d z_d}(z'', Z_{d-r+1}(z''), 0, Z_d(z'')) &= 0 \end{aligned}$$

We examine the derivatives of Z_{d-r+1} and Z_d . Implicit differentiation yields that for $P_\alpha(z'') = z_1^{\alpha_1} \cdots z_{d-r}^{\alpha_{d-r}}$

$$(3.32) \quad P_\alpha(\partial_{z''}) \Phi''_{x_d z_{d-r+1}} + \Phi'''_{x_d z_{d-r+1} z_{d-r+1}} P_\alpha(\partial_{z''}) Z_{d-r+1} + \Phi'''_{x_d z_{d-r+1} z_d} P_\alpha(\partial_{z''}) Z_d = R_{d-r+1, \alpha}$$

$$(3.33) \quad P_\alpha(\partial_{z''}) \Phi''_{x_d z_d} + \Phi'''_{x_d z_d z_{d-r+1}} P_\alpha(\partial_{z''}) Z_{d-r+1} + \Phi'''_{x_d z_d^2} P_\alpha(\partial_{z''}) Z_d = R_{d, \alpha}$$

where $R_{d-r+1, \alpha} = R_{d, \alpha} = 0$ if $|\alpha| = 1$ and where otherwise $R_{d-r+1, \alpha}$ and $R_{d, \alpha}$ belong to the ideal generated by all $P_\beta(\partial_{z''}) Z_{d-r+1}$, $P_\beta(\partial_{z''}) Z_d$ with $|\beta| < |\alpha|$. Applying (3.33) for $|\alpha| = 1$ yields

$\frac{\partial Z_{d-r+1}}{\partial z_i}(0) = 0$, in view of (2.17) and (2.14). Applying then (3.32) yields $\frac{\partial Z_d}{\partial z_i}(0) = 0$, in view of (2.17) and (2.18)₀; here $i = 1, \dots, d-r$. Inductively we obtain for all multiindices α

$$(3.34) \quad P_\alpha(\partial_{z''})Z_{d-r+1}(0) = P_\alpha(\partial_{z''})Z_d(0) = 0.$$

In view of (3.8) we have to verify the estimate (3.9) _{β} only for $|x' - y'| \leq \varepsilon|x_d - y_d|$. Expand

$$(3.35) \quad \varphi(x, y, z) := \frac{\Phi(x', x_d, z) - \Phi(y', y_d, z)}{x_d - y_d} = \Phi'_{x_d}(0, z'', z_{d-r+1}, 0, z_d) + \rho(x, y, z)$$

where

$$\rho(x, y, z) = \left\langle \frac{x' - y'}{x_d - y_d}, A_1(x, y, z) \right\rangle + \langle x', A_2(x, y, z) \rangle + \langle y', A_3(x, y, z) \rangle + \langle \tilde{z}, A_4(x, y, z) \rangle$$

the latter is a small perturbation as a function of (z'', z_{d-r+1}, z_d) , in the C^∞ topology.

We can solve the equations $\varphi'_{z_{d-r+1}} = \varphi'_{z_d} = 0$ obtaining functions $\mathfrak{Z}_{d-r+1}, \mathfrak{Z}_d$ of (x, y, z'', \tilde{z}) so that

$$\begin{aligned} \varphi'_{z_{d-r+1}}(x, y, z'', \mathfrak{Z}_{d-r+1}, \tilde{z}, \mathfrak{Z}_d) &= 0 \\ \varphi'_{z_d}(x, y, z'', \mathfrak{Z}_{d-r+1}, \tilde{z}, \mathfrak{Z}_d) &= 0; \end{aligned}$$

here the functions $\mathfrak{Z}_{d-r+1} - Z_{d-r+1}$ and $\mathfrak{Z}_d - Z_d$ and their z'' derivatives are $O(|x'| + |y'| + \frac{|x' - y'|}{|x_d - y_d|} + |\tilde{z}|)$. Moreover we have $\varphi''_{z_d z_{d-r+1}}(0, 0) \neq 0$ and $\varphi''_{z_d z_d}(0, 0) = 0$ so that $\text{rank} \frac{\partial^2 \varphi}{\partial(z_{d-r+1}, z_d)} = 2$. We may therefore apply the method of stationary phase. Set

$$\psi(x, y, \tilde{z}, z'') = \varphi(x, y, z'', \mathfrak{Z}_{d-r+1}(x, y, z'', \tilde{z}), \tilde{z}, \mathfrak{Z}_d(x, y, z'', \tilde{z})).$$

Then

$$K_\lambda(x', x_d, y', y_d) = \sum_{j=0}^M (\lambda|x_d - y_d|)^{-1-j} \int I_j(x, y, \tilde{z}, \lambda) d\tilde{z} + O((\lambda|x_d - y_d|)^{-M})$$

where

$$I_j(x, y, \tilde{z}, \lambda) = \int e^{i\lambda(x_d - y_d)\psi(x, y, \tilde{z}, z'')} \chi_j(x, y, \tilde{z}, z'') dz''$$

with compactly supported smooth χ_j .

By Lemma 2.11 and (3.34), (3.35) the assumption of ℓ nonvanishing principal curvature of $\Sigma_{1,1}^{L,0}$ implies that the Hessian of $z'' \mapsto \psi(x, y, \tilde{z}, z'')$ has rank ℓ . Another application of the method of stationary phase yields $I_j(x, y, \lambda) = O((\lambda|x_d - y_d|)^{-\ell/2})$. Similarly the assumption of finite type k and an application of van der Corput's lemma using (2.23) yields $I_j(x, y, \lambda) = O((\lambda|x_d - y_d|)^{-1/k})$ in this case. Putting the previous estimates together yields the assertion of the lemma. \square

Proof of Theorem 3.1. We specialize to the case of a strong one-sided cusp ($r = 2$) and apply Lemma 3.6. Since $C_{\gamma,c}^\pm$ is a folding canonical relation, by Lemma 3.7, we know that the required (3.14_a) holds with $a = 1/3$. The appropriate bounds (3.9) _{β} are given in Lemma 3.8. \square

We shall now conclude this section showing that the inequality (3.14_a) for $a = 1/(r+1)$ holds at least if $\gamma = 0$. In order to prove the sharp bound $\|T_\lambda\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-(d-1)-1(2r+2)}$ for operators with one sided $S_{1^+,0}^+$ singularities, $r \geq 3$, one would have to extend this result to small values of γ .

Lemma 3.9. *Let $S_{\mu,\gamma,c}^{\pm}$ be as in (3.13) and suppose that c is small. Then for $\mu \geq 1$*

$$\|S_{\mu,0,c}\|_{L^2 \rightarrow L^2} \lesssim \mu^{-(d-1-\frac{1}{r+1})}.$$

Proof. Consider the multiplier

$$m_{\mu,\pm}(\eta) = \int \chi(u, z) e^{i[\langle u, \mu \Phi_{x'}(0, c, z) - \eta \rangle \pm \mu \Phi'_{x_d}(0, c, z)]} dudz$$

where $\chi \in C_0^\infty$. Let $\psi(u, z) = \langle u, \Phi_{x'}(0, c, z) \rangle \pm \Phi'_{x_d}(0, c, z)$ then the rank of the Hessian of ψ with respect to the variables u, z' is $2(d-1)$, and this Hessian is equal to the Hessian of $\psi(u, z) - \langle u, \eta/\mu \rangle$. We apply the method of stationary phase in these variables to see that

$$m_{\mu,\pm}(\eta) = \sum_{j=0}^M \mu^{-(d-1+j)} \int \chi_j(z_d, \eta) e^{i\mu\phi(z_d, \eta)} dz_d + O(\mu^{-M})$$

where $\sum_{l=1}^{r+1} |\partial_{z_d}^l \phi(z_d, \eta)| \neq 0$ and the bounds are uniform in η . This follows from the assumption (2.14) (cf. also the calculation in the proof of Lemma 3.7). Van der Corput's Lemma shows that the integrals are $O(\mu^{-1/(r+1)})$. Therefore

$$(3.36) \quad \|m_{\mu,\pm}\|_\infty \lesssim \mu^{-(d-1-\frac{1}{r+1})}.$$

To obtain the conclusion of the Lemma write

$$b_{0,c}(u, v, z) = \frac{1}{(2\pi)^{d-1}} \int \widehat{b}_{0,c}(u, \theta, z) e^{i\langle \theta, v \rangle} d\theta$$

where the C^N norms of $z \mapsto \widehat{a}(\theta, z)$ are $O((1 + |\theta|)^{-M})$ for all M, N . We now apply the estimate (3.36) for the multiplier with cutoff function $\chi(u, z) = \widehat{b}_{0,c}(u, \theta, c)$. \square

4. L^2 estimates for oscillatory integrals with nonstrong cusp singularities

The purpose of this section is to prove Theorem 1.1. By [7, §3] it is an immediate consequence of the corresponding estimate for oscillatory integral operators (2.8) which we shall now formulate.

Theorem 4.1. *Let T_λ be as in (2.8), C_Φ as in (2.9). Suppose that $(x^0, z^0) \in X \times Z$, $\mathfrak{c}^0 \in C_\Phi$ with $\pi_X \mathfrak{c}^0 = x^0$, $\pi_Z \mathfrak{c}^0 = z^0$ and let $\xi = \Phi'_x(x^0, z^0)$. Suppose that π_L has a Whitney cusp at \mathfrak{c}^0 . Then there is a neighborhood \mathcal{U} of (x^0, z^0) , so that*

$$\|T_\lambda\|_{L^2(Z) \rightarrow L^2(X)} \lesssim \lambda^{-\frac{d}{2} + \frac{1}{3}}$$

provided that the amplitude a is supported in \mathcal{U} .

Proof. We may assume that $x^0 = z^0 = 0$ and that Φ satisfies (2.10-2.13). By Proposition 2.8 we have that (2.14) and (2.15) hold for $r = 2$. We may assume that $\Phi'''_{x_d x z_d}(0, 0)$ is large compared to $\Phi'''_{x_d z' z_d}(0, 0)$ since otherwise we can use Theorem 3.1. Moreover we may assume that $\Phi'''_{x_d x' z_d}(0, 0)$

is large compared to $\Phi'''_{x_d^2 z_d}(0,0)$ since otherwise π_R has a fold singularity at $(0,0)$ and the estimates are better (at least $\|T_\lambda\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-\frac{d}{2} + \frac{1}{4}}$, by [7]).

Replacing Φ by $\Phi(B^t x', x_d, B^{-1} z', z_d)$ for a suitable rotation B in \mathbb{R}^{d-1} we can assume that

$$(4.1) \quad |\Phi'''_{x_d x_1 z_d}(0,0)| \geq 10|\Phi'''_{x_d z' z_d}(0,0)| + 10|\Phi'''_{x_d^2 z_d}(0,0)|$$

then the properties (2.10-2.13), (2.14), (2.15) are still satisfied. We shall also assume that the neighborhood \mathcal{U} in the statement of Theorem 4.1 is chosen as a subset of $\{(x, z) : |x| + |z| \leq 10^{-2}\}$.

We will construct a unitary operator which reduces the study of C_Φ to the strong cusp situation we already understand by the results of §3. First split the variables as $x = (x_1, \tilde{x})$ and define

$$\Gamma_\lambda^\pm f(x) = \frac{1}{2\pi} \iint e^{i\lambda[\langle x_1 - w_1, \xi_1 \rangle \pm \frac{\xi_1^2}{2}]} d\xi_1 f(w_1, \tilde{x}) dw_1$$

where the ξ_1 integral is to be interpreted as conditionally convergent oscillatory integral. By rescaling one reduces the L^2 behavior for Γ_λ^\pm to the study of the Fourier multipliers $\exp(\pm i\lambda \xi_1^2/2)$ and it is easy to see that $\lambda \Gamma_\lambda^\pm$ are in fact unitary operators on $L^2(\mathbb{R}^d)$, and $\Gamma_\lambda^- \Gamma_\lambda^+ = \lambda^{-2} Id$. It therefore suffices to show that

$$(4.2) \quad \|\Gamma_\lambda^+ T_\lambda\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-1 - \frac{d}{2} + \frac{1}{3}}.$$

Note that composing T_λ with Γ_λ^+ corresponds to applying a linear canonical transformation in T^*X and composing its graph with the relation C_Φ .

Let $\chi_0 \in C^\infty(\mathbb{R})$ so that $\chi_0(s) = 1$ for $|s| \leq 1/20$ and $\chi_0(s) = 0$ for $|s| \geq 1/10$. We localize in the x_1 and ξ_1 variables and split $\Gamma_\lambda^+ = \Gamma_{\lambda,1}^+ + \Gamma_{\lambda,2}^+$ where

$$2\pi \Gamma_{\lambda,1}^+ f(x) = \chi_0(x_1) \iint \chi_0(\xi_1) e^{i\lambda[\langle x_1 - w_1, \xi_1 \rangle + \frac{\xi_1^2}{2}]} d\xi_1 f(w_1, \tilde{x}) dw_1$$

We first show that the operator $\Gamma_{\lambda,2}^+ T_\lambda$ is negligible. Let $\chi_j(\xi_1) = \chi_0(2^{-j} \xi_1) - \chi_0(2^{-j+1} \xi_1)$. The kernel of $\Gamma_{\lambda,2}^+$ can be decomposed as

$$R_\lambda(x, z) = G_{\lambda,1}(x, z) + \sum_{j=1}^{\infty} G_{\lambda,2,j}(x, z) + \sum_{j=1}^{\infty} G_{\lambda,3,j}(x, z)$$

where

$$\begin{aligned} 2\pi G_{\lambda,1}(x, z) &= (1 - \chi_0(x_1)) \iint \chi_0(\xi_1) e^{i\lambda[\langle x_1 - w_1, \xi_1 \rangle + \frac{\xi_1^2}{2} + \Phi(w_1, \tilde{x}, z)]} a(w_1, \tilde{x}, z) d\xi_1 dw_1 \\ 2\pi G_{\lambda,2,j}(x, z) &= \chi_0(x_1) \iint \chi_j(\xi_1) e^{i\lambda[\langle x_1 - w_1, \xi_1 \rangle + \frac{\xi_1^2}{2} + \Phi(w_1, \tilde{x}, z)]} a(w_1, \tilde{x}, z) d\xi dw \\ 2\pi G_{\lambda,3,j}(x, z) &= (1 - \chi_0(x_1)) \iint \chi_j(\xi_1) e^{i\lambda[\langle x_1 - w_1, \xi_1 \rangle + \frac{\xi_1^2}{2} + \Phi(w_1, \tilde{x}, z)]} a(w_1, \tilde{x}, z) d\xi dw. \end{aligned}$$

First consider $G_{\lambda,1}$. Integrating by parts in w_1 we gain arbitrary negative powers of $\lambda|x_1 - w_1|$ and in view of the support properties of a and χ_0 we have $|x_1 - w_1| \geq 1/100$ here. From this it is easy to see that $|G_{\lambda,1}(x, z)| \leq C_{M_1, M-2} \lambda^{-M_1} (1 + |x_1|)^{-M_2}$.

The kernel $G_{\lambda,2,j}$ is $\leq C_M(2^j\lambda)^{-M}$ since we can integrate by parts with respect to ξ , the ξ_1 derivative of the phase function is $\lambda(x_1 - w_1 + \xi_1)$ which is of the order of λ in view of the support properties of a and χ_0 .

To handle $G_{\lambda,3,j}$ we integrate by parts first with respect to x_1 and, if $|x_1| \leq 2^j/100$, with respect to ξ_1 and x_1 . The result is that $|G_{\lambda,3,j}(x, z)| \leq C_{M_1, M_2} 2^{-j} \lambda^{-M_1} (1 + |x_1|)^{-M_2}$. These estimates clearly imply that $\|\Gamma_\lambda^+ T_\lambda\|_{L^2 \rightarrow L^2} = O(\lambda^{-M})$ for any M .

We now consider the operator $S_\lambda := \Gamma_{\lambda,1} T_\lambda$; then S_λ is of the form (3.5) with frequency variables $\vartheta = (w_1, \xi_1) \in \mathbb{R}^2$ and phase

$$\psi(x, z, w_1, \xi_1) = \langle x_1 - w_1, \xi_1 \rangle + \frac{\xi_1^2}{2} + \Phi(w_1, \tilde{x}, z).$$

The canonical relation \mathcal{C}_ψ is given by

$$\mathcal{C}_\psi = \{(x, \Phi'_x(w_1, \tilde{x}, z); z, -\Phi'_z(w_1, \tilde{x}, z)) : x_1 - w_1 + \Phi'_{x_1}(w_1, \tilde{x}, z) = 0\}$$

We solve

$$(4.3) \quad x_1 - w_1 + \Phi'_{x_1}(w_1, \tilde{x}, z) = 0 \iff w_1 = g(x, z)$$

with $g(0) = 0$. This is possible since Φ'_x is small near $(0, 0)$ by (2.10) (we assume that \mathcal{U} is chosen so small that (4.3) holds for $(w, z) \in \mathcal{U}$ and $x_1 \in \text{supp } \chi_0$).

We verify that $\pi_L : \mathcal{C}_\psi \rightarrow T^*X$ has a strong cusp at $\mathfrak{c}^0 = (0, 0, 0, 0)$. In order to do this we have to show that the map

$$F : z \mapsto \Phi'_x(g(0, z), 0, z)$$

is a cusp at $z = 0$; to do this we use Lemma 2.4.

Let \mathcal{I}_m is the ideal generated by $\frac{\partial^j g}{\partial z_d^j}$, $1 \leq j \leq m$. Then (4.3) yields

$$\begin{aligned} -\frac{\partial g}{\partial z_d} + \Phi''_{x_1 x_1} \frac{\partial g}{\partial z_d} + \Phi''_{x_1 z_d} &= 0 \\ -\frac{\partial^2 g}{\partial z_i \partial z_d} + \Phi''_{x_1 x_1} \frac{\partial^2 g}{\partial z_d \partial z_i} + \Phi'''_{x_1 z_d z_i} &\in \mathcal{I}_1, \quad i = 1, \dots, d \\ -\frac{\partial g}{\partial z_1} + \Phi''_{x_1 x_1} \frac{\partial g}{\partial z_1} + \Phi''_{x_1 z_1} &= 0 \end{aligned}$$

and all derivatives of Φ are evaluated at $x_1 = g(0, z)$.

Using (2.13)

$$(4.4) \quad g(0) = \frac{\partial g}{\partial z_d}(0) = \frac{\partial^2 g}{\partial z_d \partial z_i}(0) = 0$$

for $i = 1, \dots, d$, and

$$(4.5) \quad \frac{\partial g}{\partial z_1}(0) = 1.$$

Differentiating F yields

$$\begin{aligned}\frac{\partial F_d}{\partial z_d} &= \Phi''_{x_d x_1} \frac{\partial g}{\partial z_d} + \Phi''_{x_d z_d} \\ \frac{\partial^2 F_d}{\partial z_d^2} &= \Phi''_{x_d x_1} \frac{\partial^2 g}{\partial z_d^2} + \Phi'''_{x_d z_d^2} + R_1 \\ \frac{\partial^3 F_d}{\partial z_d^3} &= \Phi''_{x_d x_1} \frac{\partial^3 g}{\partial z_d^3} + \Phi^{(4)}_{x_d z_d^3} + R_2\end{aligned}$$

where $R_1 \in \mathcal{I}_1$ and $R_s \in \mathcal{I}_2$. From (4.4), (4.5) and (2.10), (2.14) it follows that $\frac{\partial F_d}{\partial z_d}$ and $\frac{\partial^2 F_d}{\partial z_d^2}$ vanish at 0, but $\frac{\partial^3 F_d}{\partial z_d^3}$ does not (that is (2.3), (2.5) hold).

Next

$$\frac{\partial^2 F_d}{\partial z_1 \partial z_d} = \Phi'''_{x_d z_d x_1} \frac{\partial g}{\partial z_1} + \Phi'''_{x_d z_d z_1} + \Phi''_{x_d x_1} \frac{\partial^2 g}{\partial z_d \partial z_1} + R_3$$

where $R_3 \in \mathcal{I}_1$. Since we assume that $\Phi'''_{x_d z_d x_1}(0, 0)$ is large compared to $\Phi'''_{x_d z_d z_1}$ it follows from (4.5), (2.10) that $\frac{\partial^2 F_d}{\partial z_1 \partial z_d}(0) \neq 0$, *i.e.* (2.4) is satisfied. By Corollary 3.2 the operator norm of $\Gamma_{\lambda, 1}^+ T_\lambda$ is $O(\lambda^{-1 - \frac{d}{2} + \frac{1}{3}})$ which implies the required (4.2). \square

5. The X-ray transform for well curved line complexes

In this section we shall show that the strong cusp assumptions holds for the canonical relations associated to line complexes in \mathbb{R}^4 that are well curved in the sense of Definition 1.2, and then prove Theorem 1.3. Finally we shall show optimal $L^p \rightarrow L^2$ estimates for a translation invariant line complex in higher dimension which serves as a model example for the class of well-curved line complexes (see (5.8), (5.9) below).

Proposition 5.1. *If $\mathfrak{C} \subset M_{1,d}$ is a well-curved line complex over $\Omega \subset \mathbb{R}^d$, then the singularities of the projection $\pi_R : N^* Z'_\mathfrak{C} \rightarrow T^* \Omega$ are all $S_{1,r,0}^+$ singularities with $r \leq d - 2$.*

Proof. By choosing local coordinates v on \mathfrak{C} vanishing at $\gamma_0 \in \mathfrak{C}_{w^0}$ and (linear) w on \mathbb{R}^d , we may assume that $w^0 = 0$ and that the line γ^0 is the w_d -axis, so that locally the incidence relation is given by

$$Z_\mathfrak{C} = \{(v, w) \in \mathbb{R}^d \times \mathbb{R}^d : w' = v' + w_d \sigma(v'; v_d)\},$$

with $\sigma : \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^{d-1}$ a smooth family of nondegenerate curves parametrized by the last variable. By the well-curvedness of the line complex we have

$$(5.1) \quad \text{rank} \left(\frac{\partial \sigma}{\partial v_d}, \dots, \frac{\partial^{d-1} \sigma}{\partial v_d^{d-1}} \right) = d - 1$$

By a linear change of variables in v', w' , we replace $\sigma(v', v_d)$ by $A^{-1} \sigma(Av', v_d)$. We can therefore assume that

$$(5.2) \quad \frac{\partial^j \sigma}{\partial v_d^j}(0) = e_j, \quad j = 1, \dots, d - 1.$$

We may then write the restricted X -ray transform as

$$(5.3) \quad \mathcal{R}_{\mathcal{C}} f(v) = \iint_{\mathbb{R}^{d-1} \times \mathbb{R}^d} e^{i(v' - w' + w_d \sigma(v'; v_d)) \cdot \theta'} a(v, w, \theta') f(w) dw d\theta' \quad ,$$

with amplitude $a \in S^0(\mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}^{d-1} \setminus \{0\}))$ and σ satisfying (5.2).

The nondegenerate phase function

$$\psi(v, w, \theta') = (v' - w' + w_d \sigma(v'; v_d)) \cdot \theta'$$

has critical manifold $\text{Crit}_\psi = \{(v', v_d, v' + w_d \sigma(v'; v_d), w_d, \theta') : v \in \mathbb{R}^d, w_d \in \mathbb{R}, \theta' \in \mathbb{R}^{d-1} \setminus \{0\}\}$ and thus parametrizes the canonical relation

$$(5.4) \quad C = N^* Z'_{\mathcal{C}} = \{(v', v_d, (I + w_d d_{v'} \sigma)^* \theta', w_d d_{v_d} \sigma \cdot \theta'; v' + w_d \sigma, w_d, \theta', -\sigma \cdot \theta')\}.$$

For $|w_d|$ sufficiently small, we may solve for v' in terms of w , and thus obtain

$$(5.5) \quad \pi_R C = \{(w', w_d, \xi', -\tilde{\sigma}(w, v_d) \cdot \xi') : w \in \mathbb{R}^d, v_d \in \mathbb{R}, \xi' \in \mathbb{R}^{d-1} \setminus \{0\}\}$$

for suitable $\tilde{\sigma}$. Notice that

$$\frac{\partial^j \tilde{\sigma}}{\partial v_d^j}(w', w_d, v_d) = \frac{\partial^j \sigma}{\partial v_d^j}(v', v_d) + O(w_d) \quad \text{if } w' = v' + w_d \sigma(v', v_d)$$

Therefore

$$(5.6) \quad \frac{\partial^j \tilde{\sigma}}{\partial v_d^j}(w^0, 0) = e_j, \quad j = 1, \dots, d-1.$$

To show that $\pi_R \in S_{1_r, 0}^+$, we need to show that the map

$$F : (\xi', v_d) \mapsto (\xi', -\tilde{\sigma}(w^0, v_d) \cdot \xi')$$

has only $S_{1_r, 0}$ singularities with $r \leq d-2$, at $v_d = 0$. This map is given in adapted coordinates (2.2) with $h(\xi', v_d) = -\tilde{\sigma}(w^0, v_d) \cdot \xi'$.

Let ξ'_0 be fixed. By (5.6), either $\dot{\tilde{\sigma}}(w^0; 0) \cdot \xi'_0 \neq 0$, in which case F is a diffeomorphism near $(\xi'_0, 0)$, or there is a least integer r , $1 \leq r \leq d-2$, such that $\tilde{\sigma}^{(k)}(w^0; 0) \cdot \xi'_0 = 0$ for $1 \leq k \leq r$ and $\tilde{\sigma}^{(r+1)}(w^0; 0) \cdot \xi'_0 \neq 0$, so that (2.3) and (2.5) are satisfied. Denoting derivatives of $\tilde{\sigma}$ with respect to v_d by $\dot{\tilde{\sigma}}$ etc., we also have

$$\begin{aligned} \text{rank} \left[d\left(\frac{\partial h}{\partial v_d}\right), \dots, d\left(\frac{\partial^r h}{\partial v_d^r}\right) \right] &= \text{rank} \left[d(\dot{\tilde{\sigma}} \cdot \xi'), \dots, d(\tilde{\sigma}^{(r)} \cdot \xi') \right] \\ &\geq \text{rank} \left[d_{\xi'}(\dot{\tilde{\sigma}} \cdot \xi'), \dots, d_{\xi'}(\tilde{\sigma}^{(r)} \cdot \xi') \right] = \text{rank} [\dot{\tilde{\sigma}}, \dots, \tilde{\sigma}^{(r)}] = r \end{aligned}$$

by (5.6). Thus, (2.4) is also satisfied, and F has only $S_{1_r, 0}$ singularities, with $r \leq d-2$. \square

Proof of Theorem 1.3. By Corollary 3.4 we have to verify that the image of the cusp surface in the fibers of $T^*\Omega$ satisfies a finite type condition. We shall use the notation in the proof of Proposition 5.1. By (5.5) we see that $d\pi_{R,w^0}$ drops rank by 1 where $\dot{\sigma} \cdot \theta' = 0$, and there $\ker(d\pi_{R,w^0})$ is generated by $\frac{\partial}{\partial v_4}$. Thus

$$(5.7) \quad \Sigma_{1,1}^{R,w^0} = \{(\theta, -\tilde{\sigma} \cdot \theta) : v_d \text{ near } 0, \theta \in \mathbb{R}^3 \setminus 0, \dot{\sigma}(w^0, v_4) \cdot \theta = \ddot{\sigma}(w^0, v_4) \cdot \theta = 0\}.$$

By (5.6)

$$\tilde{\sigma}(w; v_4) = \left(v_4 + O(|w| + v_4^2), \frac{1}{2}v_4^2 + O(|w| + |v_4|^3), \frac{1}{6}v_4^3 + O(|w| + v_4^4) \right),$$

and therefore

$$\dot{\sigma}(w; v_4) = \left(1 + O(|w| + |v_4|), v_4 + O(|w| + v_4^2), \frac{1}{2}v_4^2 + O(|w| + |v_4|^3) \right),$$

and

$$\ddot{\sigma}(w; v_4) = \left(O(1), 1 + O(|w| + v_4), v_4 + O(|w| + v_4^2) \right).$$

From this, it follows that $\dot{\sigma} \cdot \theta' = 0$ implies that for $|v_4|$ small,

$$\theta_1 = -v_4\theta_2 - \frac{1}{2}v_4^2\theta_3 + O(|w| + \sum_{j=1}^3 |v_4^j||\theta_j|)$$

and thus $\ddot{\sigma} \cdot \theta' = 0$ implies that

$$\theta_2 = -v_4\theta_3 + O(|w| + |v_4|^3|\theta_3|).$$

Therefore

$$\Sigma_{1,1}^{R,w^0} = \left\{ \left(\frac{1}{2}v_4^2\theta_3 + \mathcal{O}_1, -v_4\theta_3 + \mathcal{O}_2, \theta_3, -\frac{1}{6}v_4^3\theta_3 + \mathcal{O}_4 \right) : (w, v_4) \text{ near } (w^0, 0), \theta_3 \in \mathbb{R} \setminus 0 \right\}$$

where $\mathcal{O}_j = O(|w| + |v_4|^j)$. Intersecting $\Sigma_{1,1}^{R,w^0}$ with any transverse hyperplane in $T_0^*\Omega$ yields a curve with nonzero curvature and torsion; this is stronger than the type ≤ 3 condition required in Corollary 3.4. \square

Remark. Our estimates can also be applied to operators with two-sided cusp singularities. Consider the translation invariant operator defined on functions in \mathbb{R}^4 by

$$Af(x) = \int f(x - \Gamma(t))\chi(t)dt$$

with $\Gamma(t) = (t, t^2, t^3, t^4)$. Then $A \in I^{-\frac{1}{2}}(C; \mathbb{R}^4, \mathbb{R}^4)$. Writing $\xi' = (\xi_2, \xi_3, \xi_4)$ and $g(t, \xi') = -(2t\xi_2 + 3t^3\xi_3 + 4t^3\xi_4)$ one computes that

$$C = \{(x, -g(t, \xi'), \xi'; x - \Gamma(t), -g(t, \xi'), \xi')\}$$

which exhibits π_R and π_L as strong cusps. The fold surface $S_1(\pi_L)$ is given by $2\xi_2 + 6t\xi_3 + 12t^2\xi_4 = 0$ and the cusp surface $S_{1,1}(\pi_L)$ by $\xi_2 = 6t^2\xi_4, \xi_3 = -4t\xi_4$ so that for any $x_0 \in \mathbb{R}^4$,

$$\Sigma_{1,1}^{L,x^0} = \{(-4t^3\xi_4, 6t^2\xi_4, -4t\xi_4, \xi_4) : t \in \mathbb{R}, \xi_4 \in \mathbb{R} \setminus 0\}.$$

Intersecting this with $H_{\xi^0} = \{\xi_4 = \xi_4^0\}$, we get a curve with nonzero curvature and torsion, so that the type ≤ 3 condition of Corollary 3.4 is satisfied. Since the same applies to π_R we deduce from Corollary 3.4 and interpolations that A maps L^p to L^q if $(\frac{1}{p}, \frac{1}{q}) \in \text{hull}\{(0, 0), (1, 1), (\frac{7}{12}, \frac{1}{2}), (\frac{1}{2}, \frac{5}{12})\}$. This falls substantially short of the recent sharp estimates for A obtained by Oberlin [21], but it does apply to non-translation-invariant variants of A . We shall consider one way of describing such variants in §6.

Appendix: A translation invariant line complex. Let $d \geq 3$ and consider the operator defined by

$$(5.8) \quad Rf(v', v_d) = \chi(v) \int f(v' + t\gamma(v_d))\eta(t)dt$$

where

$$(5.9) \quad \gamma(s) = (s, s^2, \dots, s^{d-1}, 1);$$

moreover $\chi \in C_0^\infty(\mathbb{R}^d)$ with $\chi(0) = 1$ and $\eta \in C_0^\infty(\mathbb{R})$, both are assumed to be real valued without loss of generality. Carrying out the analysis in §3 leads to a translation invariant situation as in Lemma 3.9 and one concludes that R maps L^2 to $L_{1/(2d-2)}^2$.

The natural $L^p \rightarrow L^q$ conjecture is that R is bounded from L^p to L^q if and only if $(1/p, 1/q)$ belongs to the triangle $\mathcal{T} = \text{hull}(A, B, C)$ where $A = (0, 0)$, $B = (1, 1)$, $C = (\frac{d^2-d+2}{d^2+d}, \frac{d^2-d}{d^2+d})$. It is necessary that $(1/p, 1/q)$ is on or above \overline{CB} , that is $d/p \leq 1 + (d-1)/q$. To see this one tests R on the characteristic function of the ball $\{|w| \leq \delta\}$ and lets $\delta \rightarrow 0$. To check that $(1/p, 1/q)$ has to be on or above \overline{AC} , that is $(d^2-d)/p \leq (d^2-d+2)/q$, one tests R on characteristic functions of the rectangle $\{w : |w_j| \leq \delta^j, j = 1, \dots, d-1, |w_d| \leq 1\}$. The following proposition establishes the $L^p \rightarrow L^q$ estimate for $(1/p, 1/q)$ in a subtriangle with vertex D on the lower edge AC of \mathcal{T} , so that the sharp $L^p \rightarrow L^q$ estimate is obtained for $p \geq 2 - \frac{4}{d^2-d+2}$.

Proposition 5.2. *Let R be as in (5.8), (5.9). R maps L^p to L^q if $\mathcal{T} = \text{hull}(A, B, D)$ where $A = (0, 0)$, $B = (1, 1)$, $D = (\frac{d^2-d+2}{2d^2-2d}, \frac{1}{2})$.*

Proof. It suffices to show that R maps L^p to L^2 where $p = \frac{2d(d-1)}{d^2-d+2}$, $d \geq 3$. For this it suffices to show that R^*R maps L^p to $L^{p'}$. A computation yields

$$R^*Rf(x) = \iint h(\alpha, \beta, x) f(x' - \beta\gamma(\alpha), x_d - \beta) d\alpha d\beta$$

with a suitable C^∞ function h with compact support. Now $R^*Rf(x) \leq T(|f|)(x)$ where

$$Tf(x) = \chi_1(x')\chi_2(x_d) \iint \chi_3(\alpha)\chi_4(\beta) f(x' - \beta\gamma(\alpha), x_d - \beta) d\alpha d\beta$$

with suitable smooth and positive cutoff functions χ_i . For functions defined in \mathbb{R}^{d-1} let

$$S_\beta g(x') = \int \chi_3(\alpha) g(x' - \beta\gamma(\alpha)) d\alpha.$$

Then

$$Tf(x', x_d) = \chi_1(x')\chi_2(x_d) \int \chi_4(\beta) S_\beta[f(\cdot, x_d - \beta)](x') d\beta$$

For $\gamma(t) = (t, \dots, t^n)$ McMichael [16] proved that S_1 is bounded from $L^p(\mathbb{R}^n \rightarrow L^{p'}(\mathbb{R}^n))$ provided that $2n(n+1)/(n^2+n+2) \leq p \leq 2$, here $n = d-1$. Since $S_\beta g(x') = S_1[f(\beta \cdot)](x'/\beta)$ the $L^p \rightarrow L^{p'}$ operator norm of S_β is $\leq C\beta^{(d-1)(1-2/p)}$, for $2(d^2-d)/(d^2-d+2) \leq p \leq 2$. We use the

(by now) standard slicing argument due to Oberlin [20] (see also Strichartz [31].) By Minkowski's inequality

$$(5.10) \quad \begin{aligned} \|Tf\|_{p'} &\leq \left(\int \chi_2(x_d) \left[\int \chi_4(\beta) \|S_\beta[f(\cdot, x_d - \beta)]\|_{p'} d\beta \right]^{p'} dx_d \right)^{1/p'} \\ &\lesssim \left(\int \chi_2(x_d) \left[\int \chi_4(\beta) \beta^{(d-1)(1-2/p)} \|f(\cdot, x_d - \beta)\|_p d\beta \right]^{p'} dx_d \right)^{1/p'}. \end{aligned}$$

In order to apply the theorem on fractional integration we need the restriction $(d-1)(1-\frac{2}{p}) \geq \frac{1}{p} - \frac{1}{p'} - 1$ or $d(1-\frac{2}{p}) + 1 > 0$; notice that $d(1-\frac{2}{p}) + 1 \geq \frac{d-2}{d-1}$ for $p \geq 2(d^2-d)/(d^2-d+2)$. Since we are assuming $d \geq 3$ the right hand side of (5.10) is bounded by $C\|f\|_p$. \square

6. Strong cusps and exponentials of vector fields

We next examine the strong cusp condition in the context of families of curves in \mathbb{R}^4 given by exponentials of vector fields; the setup is as in [2]. Let X, Y, Z and W be smooth vector fields on \mathbb{R}^4 and

$$\gamma_x(t) = \exp(tX + t^2Y + t^3Z + t^4W)(x),$$

so that $\{\gamma_x : x \in \mathbb{R}^4\}$ is a smooth family of curves, with $\gamma_x(0) = x$. Let $\chi \in C_0^\infty(\mathbb{R})$. The generalized Radon transform,

$$\mathcal{R}f(x) = \int_{\mathbb{R}} f(\gamma_x(t))\chi(t)dt$$

belongs to $I^{-\frac{1}{2}}(\mathbb{R}^4, \mathbb{R}^4; C)$, where $C = N^*\Gamma'$, the conormal bundle of

$$\Gamma = \{(x, \gamma_x(t)) : x \in \mathbb{R}^4, t \in \text{supp}(\chi)\}.$$

We assume that $\text{supp}(\chi)$ is small and are concerned with the behavior of the Schwartz kernel of \mathcal{R} close to the diagonal. The following is the analogue for $S_{1^r,0}^+$ singularities of a result in [24] for folds in three variables.

Proposition 6.1. *Let $\mathfrak{c}^0 \in C$ be above the diagonal, $\mathfrak{c}^0 = (x^0, \xi^0, x^0, -\eta^0)$.*

(i) *Suppose that the vectors fields*

$$(6.1) \quad X, \quad Y, \quad Z - \frac{1}{6}[X, Y], \quad W - \frac{1}{4}[X, Z] + \frac{1}{24}[X, [X, Y]]$$

are linearly independent at x^0 . Then the only possible singularities of the projection $\pi_R : C \rightarrow T^\mathbb{R}^4$ at \mathfrak{c}^0 are Whitney folds and strong Whitney cusps.*

(ii) *Suppose that the vectors fields*

$$(6.2) \quad X, \quad Y, \quad Z + \frac{1}{6}[X, Y], \quad W + \frac{1}{4}[X, Z] + \frac{1}{24}[X, [X, Y]]$$

are linearly independent at x^0 . Then the only possible singularities of the projection $\pi_L : C \rightarrow T^\mathbb{R}^4$ at \mathfrak{c}^0 are Whitney folds and strong Whitney cusps.*

Proof. We will approximate Γ , and thus C , to high order near the diagonal, by using the first terms of the Baker-Campbell-Hausdorff formula: for vector fields A, B ,

$$(6.3) \quad \exp(B)\exp(A)(x) = \exp(A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A - B, [A, B]] + \dots)(x),$$

where ... denotes commutators of four or more terms. Note the order of the product, since composition of diffeomorphisms is right-to-left. Using (6.3), an elementary calculation leads to

$$(6.4) \quad \exp(tX + t^2Y + t^3Z + t^4W)(x) = \exp(t^3\tilde{Z} + t^4\tilde{W} + \mathcal{O}^5)\exp(t^2Y)\exp(tX)(x),$$

with

$$\tilde{Z} = Z - \frac{1}{2}[X, Y] \quad \text{and} \quad \tilde{W} = W - \frac{1}{2}[X, Z] + \frac{1}{6}[X, [X, Y]].$$

Here, \mathcal{O}^j denotes terms which are $\mathcal{O}(t^j)$ as $t \rightarrow 0$ (or $\mathcal{O}((y_1 - x_1)^j)$ below.) We will also need the second order Taylor polynomial of $\exp(tA)$:

$$(6.5) \quad \exp(tA)(x) = x + tA(x) + \frac{t^2}{2}DA(x)(A(x)) + \mathcal{O}^3.$$

Since $X \neq 0$, by a local change of variables we can take $X = \frac{\partial}{\partial y_1}$. Write $x = (x_1, x')$, $y = (y_1, y')$ and vector fields as $A = (A_1, A')$. Fixing a basepoint y^0 , one can also assume that $Y(y^0) = \frac{\partial}{\partial y_2}$. Before applying (6.5) to the vector field Y below, we note that the $DY(Y)$ term is not invariantly defined. In fact, if Ψ is a diffeomorphism and $\tilde{Y}(y) = \Psi^*Y(y) = (D\Psi(y))^{-1}(Y(\Psi(y)))$ denotes the pullback of Y , then a calculation yields

$$D\tilde{Y}(\tilde{Y}) = (D\Psi)^{-1}\left(DY(Y) - D^2\Psi(\tilde{Y}, \tilde{Y})\right).$$

Setting $v = DY(Y)(y^0)$ (and assuming $y^0 = 0$), we take $\Psi(y) = y + \frac{y_2^2}{2}v$. This preserves the conditions $X = \frac{\partial}{\partial y_1}$ and $Y(y^0) = e_2$, and also $D^2\Psi(e_2, e_2) = v$ hence $D\tilde{Y}(\tilde{Y})(y^0) = 0$, so that we can assume

$$(6.6) \quad X = \frac{\partial}{\partial y_1}, \quad Y_1(y^0) = 0, \quad DY(Y)(y^0) = 0.$$

Since $X = \frac{\partial}{\partial y_1}$, by (6.3) we have

$$\Gamma = \{(x, \exp(t^3\tilde{Z} + t^4\tilde{W} + \mathcal{O}^5)\exp(t^2Y)(x_1 + t, x')) : x \in \mathbb{R}^4, t \in \mathbb{R}\},$$

which by (6.5) with t replaced by t^2 equals

$$\begin{aligned} & \left\{ (x, \exp(t^3\tilde{Z} + t^4\tilde{W} + \mathcal{O}^5)(x + (t, 0) + t^2Y(x_1 + t, x') + \frac{t^4}{2}(DY(Y))(x_1 + t, x'))) \right\} \\ & = \left\{ (x, x + (t, 0) + t^2Y(x_1 + t, x') + t^3\tilde{Z}(x_1 + t, x') + t^4(\tilde{W} + \frac{1}{2}DY(Y) + \mathcal{O}^5)) \right\}. \end{aligned}$$

But,

$$\begin{aligned} \tilde{Z}(x_1 + t, x') &= \tilde{Z}(x) + t[X, \tilde{Z}](x) + \mathcal{O}^2 \\ Y(x_1 + t, x') &= Y(x) + t[X, Y](x) + \frac{t^2}{2}[X, [X, Y]](x) + \mathcal{O}^3 \end{aligned}$$

and therefore

$$\Gamma = \left\{ (x, x + (t, 0) + t^2Y + t^3(\tilde{Z} + [X, Y]) + t^4(\tilde{W} + [X, \tilde{Z}] + \frac{1}{2}[X, [X, Y]] + \frac{1}{2}DY(Y) + \mathcal{O}^5)) \right\},$$

where all the vector fields are evaluated at x . Now change coordinates on Γ from (x, t) to (x, y_1) via

$$y_1 = x_1 + t + t^2 Y_1(x) + t^3 (\tilde{Z} + [X, Y])_1(x);$$

by the inverse function theorem, we may then express

$$t = (y_1 - x_1) - (y_1 - x_1)^2 Y_1 - (y_1 - x_1)^3 ((\tilde{Z} + [X, Y])_1 - 2Y_1^2) + \mathcal{O}^4,$$

$$t^2 = (y_1 - x_1)^2 - (y_1 - x_1)^3 \cdot 2Y_1 + (y_1 - x_1)^4 (5Y_1^2 - 2(\tilde{Z} + [X, Y])_1) + \mathcal{O}^5,$$

and

$$t^3 = (y_1 - x_1)^3 - (y_1 - x_1)^4 3Y_1 + \mathcal{O}^5,$$

with all vector fields being evaluated at x . Thus,

$$\begin{aligned} \Gamma = \left\{ \right. & \left(x, y_1 + \mathcal{O}^4, x' + (y_1 - x_1)^2 Y' + (y_1 - x_1)^3 (-2Y_1 Y + \tilde{Z} + [X, Y])' \right. \\ & \left. + (y_1 - x_1)^4 (5Y_1^2 Y - 2(\tilde{Z} + [X, Y])_1 Y - 3Y_1 (\tilde{Z} + [X, Y]) \right. \\ & \left. + \tilde{W} + [X, \tilde{Z}] + \frac{1}{2} DY(Y))' + \mathcal{O}^5 \right) : x \in \mathbb{R}^4, y_1 \in \mathbb{R} \left. \right\}. \end{aligned}$$

To change the point where the vector fields are evaluated to (y_1, x') , we expand $Y \bmod \mathcal{O}^3$ as above and $\tilde{Z}, [X, Y]$ and $Y_1 \cdot Y \bmod \mathcal{O}^2$. This leads to

$$\begin{aligned} \Gamma = \left\{ \right. & \left(x, y_1 + \mathcal{O}^4, x' + (y_1 - x_1)^2 Y' + (y_1 - x_1)^3 (-2Y_1 Y + \tilde{Z})' \right. \\ & \left. + (y_1 - x_1)^4 (5Y_1^2 Y - 2(\tilde{Z} + [X, Y])_1 Y - 3Y_1 (\tilde{Z} + [X, Y]) \right. \\ & \left. + \tilde{W} + [X, \tilde{Z}] + \frac{1}{2} [X, [X, Y]] + \frac{1}{2} DY(Y) \right. \\ & \left. + \frac{1}{2} [X, [X, Y]] - [X, \tilde{Z}] - [X, [X, Y]] \right. \\ (6.7) \quad & \left. + 2[X, Y]_1 Y + 2Y_1 [X, Y])' + \mathcal{O}^5 \right) : x \in \mathbb{R}^4, y_1 \in \mathbb{R} \left. \right\} \\ = \left\{ \right. & \left(x, y_1 + \mathcal{O}^4, x' + (y_1 - x_1)^2 Y' + (y_1 - x_1)^3 (-2Y_1 Y + \tilde{Z})' \right. \\ & \left. + (y_1 - x_1)^4 ((5Y_1^2 - 2(\tilde{Z} + [X, Y])_1) Y + 2Y_1 [X, Y] \right. \\ & \left. - 3Y_1 (\tilde{Z} + [X, Y]) + \tilde{W} + [X, \tilde{Z}] + \frac{1}{2} DY(Y))' \right) : x \in \mathbb{R}^4, y_1 \in \mathbb{R} \left. \right\}, \end{aligned}$$

with the vector fields evaluated henceforth at (y_1, x') . Using (6.7), we may give a set of approximate defining functions, $f'(x, y) = (f_2(x, y), f_3(x, y), f_4(x, y))$, for Γ . These will be specified modulo \mathcal{O}^5 ; since we only need to check the strong cusp condition at y^0 , the $f'(x, y)$ will suffice for our purpose.

One has

$$\Gamma \simeq \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : f'(x, y) = 0\}$$

where

$$\begin{aligned} f'(x, y) = & x' - y' + (y_1 - x_1)^2 \cdot Y' + (y_1 - x_1)^3 (\tilde{Z} - 2Y_1 Y)' \\ & + (y_1 - x_1)^4 \left(\tilde{W} + (5Y_1^2 - 2\tilde{Z})_1 Y - Y_1 [X, Y] - 3Y_1 \tilde{Z} + \frac{1}{2} DY(Y) \right)'. \end{aligned}$$

To calculate the conormal bundle $C = N^* \Gamma'$, we really just need to find $-d_{y_1} f' \cdot \eta'$ for $\eta' = (\eta_2, \eta_3, \eta_4) \in \mathbb{R}^3 \setminus \{0\}$:

$$\begin{aligned} -d_{y_1} f' \cdot \eta' = & - \left(2(y_1 - x_1)Y + (y_1 - x_1)^2 (3\tilde{Z} - 6Y_1 Y + [X, Y]) \right. \\ & + (y_1 - x_1)^3 (4\tilde{W} + (20Y_1^2 - 8\tilde{Z}_1)Y - 4Y_1 [X, Y] \quad \text{mod } ((y_1 - x_1)^4). \\ & \left. - 12Y_1 \tilde{Z} + 2DY(Y) + [X, \tilde{Z}] - 2Y_1 [X, Y] - 2[X, Y]_1 Y \right)' \cdot \eta' \end{aligned}$$

Changing variables $y' = x' + \dots$ and $s = y_1 - x_1$, we have (not worrying about the (x, ξ) terms)

$$\begin{aligned} C = & \left\{ \left(*, *, y, - \left(2sY + s^2 (3\tilde{Z} + [X, Y] - 6Y_1 Y) \right. \right. \right. \\ & \left. \left. + s^3 (4\tilde{W} + [X, \tilde{Z}] - 6Y_1 [X, Y] + (20Y_1^2 - 8\tilde{Z}_1 - 2[X, Y]_1)Y \right. \right. \\ & \left. \left. - 12Y_1 \tilde{Z} + 2DY(Y) \right) \right)' \cdot \eta', \eta' \Big\} \quad : y \in \mathbb{R}^4, s \in \mathbb{R}, \eta' \in \mathbb{R}^3 \setminus \{0\}, \end{aligned}$$

with the vector fields evaluated at $(y_1, x') = (y_1, y' + \dots)$. Thus, the singularities of $\pi_R : C \rightarrow T^* \mathbb{R}^4$ are completely determined by $\frac{\partial \eta_1}{\partial s}$. The derivatives $\frac{\partial^j \eta_1}{\partial s^j} \text{ mod } s^{4-j}$, $1 \leq j \leq 3$, are given by

$$\begin{aligned} -\frac{\partial \eta_1}{\partial s} = & \left(2Y + 2s(3\tilde{Z} + [X, Y] - 6Y_1 Y) + 3s^2 \left(4\tilde{W} + [X, \tilde{Z}] - 6Y_1 [X, Y] \right. \right. \\ & \left. \left. + (20Y_1^2 - 8\tilde{Z}_1 - 2[X, Y]_1)Y - 12Y_1 \tilde{Z} + 2DY(Y) \right) \right)' \cdot \eta', \\ -\frac{\partial^2 \eta_1}{\partial s^2} = & 2 \left((3\tilde{Z} + [X, Y] - 6Y_1 Y) + 3s(\dots) \right)' \cdot \eta', \\ -\frac{\partial^3 \eta_1}{\partial s^3} = & 6(\dots)' \cdot \eta', \end{aligned}$$

where \dots denotes the expression above beginning with $4\tilde{W} + [X, \tilde{Z}] + \dots$. Evaluating at $y = y^0, s = 0$, these three derivatives are (using $Y_1(y^0) = 0$ and $DY(Y)(y^0) = 0$) given by functions $g_1 = 2Y' \cdot \eta'$, $g_2 = 6(Z - \frac{1}{6}[X, Y])' \cdot \eta'$ and

$$g_3 = 24 \left(W - \frac{1}{4}[X, Z] + \frac{1}{24}[X, [X, Y]] - (2\tilde{Z}_1 + \frac{1}{2}[X, Y]_1)Y \right)' \cdot \eta'.$$

If the vectors in (6.1) are linearly independent, then $d_{\eta'} g_1, d_{\eta'} g_2, d_{\eta'} g_3$ are linearly independent ($d_{\eta'} g_3$ differing from the last term in (6.1) by at most a multiple of Y .) Thus, (η', η_1) on $T_y^* \mathbb{R}^4$ and (η', s) on $\pi_R^{-1}(\{y\})$ are adapted coordinates and thus π_R has at most $S_{1,0}$ or $S_{1,1,0}^+$ singularities. To obtain the corresponding statement for π_L one simply repeats the above argument with (X, Y, Z, W) replaced by $(-X, -Y, -Z, -W)$. \square

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