NECESSARY CONDITIONS FOR VECTOR-VALUED OPERATOR INEQUALITIES IN HARMONIC ANALYSIS

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Abstract. Via a random construction we establish necessary conditions for $L^p(\ell^q)$ inequalities for certain families of operators arising in harmonic analysis. In particular we consider dilates of a convolution kernel with compactly supported Fourier transform, vector maximal functions acting on classes of entire functions of exponential type, and a characterization of Sobolev spaces by square functions and pointwise moduli of smoothness.

1. Introduction

For $r > 0$ let $\mathcal{E}(r)$ be the space of all (smooth) distributions on $\mathbb{R}^d$ whose Fourier transforms are supported in $\{\xi : |\xi| \le r\}$. Also let $\mathcal{E}_o(r)$ be the space of functions in $\mathcal{E}(r)$ whose Fourier transforms are supported in the annulus $\{\xi : r/2 \le |\xi| \le r\}$.

Let us first consider a convolution kernel K whose Fourier transform is compactly supported, say $K \in \mathcal{E}(1)$. We are concerned with vector valued inequalities involving dilates of K, of the form

$$\Big\| \Big(\sum_k |r_k^d K(r_k \cdot) * f_k|^q \Big)^{1/q} \Big\|_p \le A \Big\| \Big(\sum_k |f_k|^q \Big)^{1/q} \Big\|_p . \tag{1.1}$$

An immediate necessary, but not sufficient, condition for (1.1) to hold is that $K \in L^s$ for all $s \ge p$. This is seen by setting all but one f_k to 0 and (after possibly a rescaling) convolving K with a Schwartz function whose Fourier transform is equal to 1 on the support of $\widehat{K}$. In the case $p > q$ we get a further necessary condition:

Theorem 1.1. *Suppose $0 < q \le p < \infty$ and let $\{r_k\}_{k=1}^\infty$ be a fixed sequence of positive numbers. Suppose that $K \in \mathcal{E}(1)$ and that (1.1) holds for all choices of $f_k \in \mathcal{E}(2r_k)$ with $\{f_k\} \in L^p(\ell^q)$.*

Then $K \in L^q$ and there exists a constant $C = C(p,q,d)$ so that

$$\|K\|_q \le C(p,q,d) A; \tag{1.2}$$

in particular $C(p,q,d)$ does not depend on the choice of the sequence $\{r_k\}$.

As an application consider the Bochner-Riesz means defined by

$$\widehat{S_r^\lambda f}(\xi) = (1 - r^{-2}|\xi|^2)_+^\lambda \widehat{f}(\xi).$$

Let K_λ be the convolution kernel for S_1^λ. From the well known formula for K_λ ([18]) we know that $K_\lambda \in L^q$ if and only if $\lambda > d(1/q - 1/2) - 1/2$. Consequently if $q < p < 2$ then the operator

$$\{f_k\} \mapsto \{S_{r_k}^\lambda f_k\}$$

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fails to be bounded on $L^p(\ell^q)$ if $\lambda \le d(1/q-1/2)-1/2$, as well as on the corresponding subspace with the restrictions $f_k \in \mathcal{E}(2r_k)$. This complements the familiar necessary condition $\lambda > \max\{d(1/p-1/2)-1/2, 0\}$ ([18], [4]), which is known also to be sufficient for certain p; for some refinements and implications to known multiplier theorems see the remark at the end of §3 below.

We shall prove Theorem 1.1 by a random construction which will be described in the next section. This construction applies also to other situations, in particular to maximal functions which arise in the theory of function spaces. As the most basic such example we consider a maximal operator acting on functions of exponential type, which was introduced by Peetre [9], following earlier related research by Fefferman and Stein [6].

For $r > 0$ and $\sigma \ge 0$ set

$$\mathfrak{M}_{\sigma,r} g(x) = \sup_y \frac{|g(x+y)|}{(1+r|y|)^\sigma}. \tag{1.3}$$

As shown in [9] one has the majorization

$$\mathfrak{M}_{\sigma,r} g(x) \lesssim [M_{HL}(|g|^s)]^{1/s}, \quad \forall \sigma \ge d/s, \quad \text{if } g \in \mathcal{E}(r); \tag{1.4}$$

here M_{HL} denotes the Hardy-Littlewood maximal operator. Now by the Fefferman-Stein vector-valued maximal theorem ([5])

$$\Big\| \Big(\sum_k |\mathfrak{M}_{\sigma,r_k} f_k|^q \Big)^{1/q} \Big\|_p \lesssim \Big\| \Big(\sum_k |f_k|^q \Big)^{1/q} \Big\|_p, \quad \sigma > \max\{\frac{d}{p}, \frac{d}{q}\}, \tag{1.5}$$

provided that $f_k \in \mathcal{E}(r_k)$ and $\{r_k\}_{k=1}^\infty$ is any sequence of positive radii.

It is well known that the condition $\sigma > d/p$ is necessary — again to see this one simply chooses a fixed Schwartz function for g_1 and sets $g_k = 0$ for $k \ge 2$. Moreover if $r_k = 1$ for all k the inequality clearly fails for all $q \le p$; this is the same example that disproves an $L^p(\ell^1)$ inequality for the Hardy-Littlewood maximal function [5]. Indeed let $\eta \in \mathcal{E}(1) \cap \mathcal{S}$, let A be a large positive integer, and let $\{x(k)\}_{k=1}^{(2A+1)^d}$ be an enumeration of all integer lattice points in the cube Q_A of sidelength $2A$ centered at the origin. Define $f_k(x) = \eta(x - x(k))$ if $1 \le k \le (2A+1)^d$ and $f_k(x) = 0$ otherwise. Then $\|\{f_k\}\|_{L^p(\ell^q)} \lesssim A^{d/p}$. Also $\mathfrak{M}_{\sigma,1} f_k(x) \gtrsim (1+|x-x(k)|)^{-\sigma}$ and a computation shows that $\|\{\mathfrak{M}_{\sigma,1} f_k\}\|_{L^p(\ell^q)} \gtrsim A^{d/p}\log A$ if $\sigma = d/q$ and $\gtrsim A^{d/p+d/q-\sigma}$ if $\sigma < d/q$. Thus the condition $\sigma > \max\{\frac{d}{p}, \frac{d}{q}\}$ in (1.5) is sharp if $r_k \equiv 1$.

The preceding example does not immediately apply to cases where the sequence of radii r_k is sparse (say lacunary), which happens in many of the interesting cases for which (1.5) is used. Nevertheless we show that the condition $\sigma > d/q$ is necessary for (1.5) to hold:

Theorem 1.2. *Let $\{r_k\}$ be any sequence of radii and suppose that $0 < q \le p < \infty$. Suppose $\sigma \le d/q$. Then there is a positive constant $c(p,q,\sigma,d)$ such that for every $L \in \mathbb{N}$ there are functions $f_k \in \mathcal{E}_o(r_k) \cap L^p$, for $k = 1, \dots, L$, so that*

$$\Big\| \Big(\sum_{k=1}^L |\mathfrak{M}_{\sigma,r_k} f_k|^q \Big)^{1/q} \Big\|_p \ge c(p,q,\sigma,d) \max\{L^{-\sigma+d/q}, \log^{1/q} L\} \Big\| \Big(\sum_{k=1}^L |f_k|^q \Big)^{1/q} \Big\|_p. \tag{1.6}$$

Note that this lower bound holds for functions in $\mathcal{E}_o(r_k)$, not merely in $\mathcal{E}(r_k)$.

Next we shall state a result on a characterization of Sobolev spaces (or more general Triebel-Lizorkin spaces) by means of pointwise moduli of continuity. For $h \in \mathbb{R}^d$ let

$\Delta_h f(x) = f(x+h) - f(x)$ and define higher difference operators inductively by $\Delta^0 f = f$, $\Delta_h^m f = \Delta_h(\Delta_h^{m-1} f)$, $m \ge 1$. For suitable classes of functions let

$$\mathfrak{D}_m^{\sigma,q} f(x) = \Big(\int_0^1 \frac{\sup_{|h|\le t} |\Delta_h^m f(x)|^q}{t^{1+\sigma q}} dt \Big)^{1/q}. \tag{1.7}$$

It is known that if $m > \sigma$, $q = 2$, and $1 < p < \infty$ one can characterize Sobolev spaces $\mathcal{L}_\sigma^p$ using $\mathfrak{D}_m^{\sigma,2}$, namely $\|f\|_{\mathcal{L}_\sigma^p} := \big\|\mathcal{F}^{-1}[(1+|\cdot|^2)^{\sigma/2}\widehat{f}]\big\|_p \approx \|f\|_p + \|\mathfrak{D}_m^{\sigma,2} f\|_p$ provided that $\sigma > \max\{d/p, d/2\}$. This is a special case of a result on Triebel-Lizorkin spaces $F_{\sigma,q}^p$ ([20], [21]). We recall that $F_{\sigma,q}^p$ is defined by dyadic frequency decompositions; namely if $\beta_0 \in \mathcal{E}(1)$ so that $\widehat{\beta_0}$ is equal to 1 in a neighborhood of the origin, and if $\beta_k = 2^{kd}\beta_0(2^k\cdot) - 2^{(k-1)d}\beta_0(2^{k-1}\cdot)$ for $k \ge 1$ then

$$\|f\|_{F_{\sigma,q}^p} \approx \Big\| \Big(\sum_{k=0}^{\infty} 2^{k\sigma q} |\beta_k * f|^q \Big)^{1/q} \Big\|_p ;$$

thus $F_{\sigma,2}^p = \mathcal{L}_\sigma^p$, $1 < p < \infty$, by the usual Littlewood-Paley inequalities. Now by [20], §2.5.10 we have for $m > \sigma$, $0 < p < \infty$, $0 < q \le \infty$ and $\sigma > \max\{d/p, d/q\}$

$$\|f\|_p + \|\mathfrak{D}_m^{\sigma,q} f\|_p \approx \|f\|_{F_{\sigma,q}^p}. \tag{1.8}$$

Again the condition $\sigma > d/p$ is necessary in (1.8), but it was apparently open whether for $p > q$ the characterization (1.8) could hold without the additional restriction $\sigma > d/q$ (*cf.* [21]). This was pointed out to the second author by Herbert Koch and Winfried Sickel at an Oberwolfach meeting some years ago. We show that the restriction $\sigma > d/q$ is indeed necessary and in the range $d/p < \sigma \le d/q$ we quantify the failure of (1.8) in terms of the support of the Fourier transform.

Theorem 1.3. *Suppose that $0 < \sigma < m$ and $0 < q < p < \infty$. For $r \ge 100$ let*

$$\mathcal{A}_{p,q,\sigma}(r) = \sup\big\{ \|\mathfrak{D}_m^{\sigma,q} f\|_p : \|f\|_{F_{\sigma,q}^p} \le 1, f \in \mathcal{E}(r) \big\}.$$

Then

$$\mathcal{A}_{p,q,\sigma}(r) \approx (\log r)^{\frac{d}{q}-\sigma} \qquad \text{if } d/p < \sigma < d/q, \tag{1.9}$$

and, for $\sigma = d/q$,

$$\mathcal{A}_{p,q,d/q}(r) \approx (\log\log r)^{1/q} \qquad \text{if } q \le 1. \tag{1.10}$$

Moreover,

$$C^{-1}(\log\log r)^{1/q} \le \mathcal{A}_{p,q,d/q}(r) \le C \log\log r \qquad \text{if } 1 < q < p. \tag{1.11}$$

In (1.9) the notation $a_1 \approx a_2$ means that there is a positive constant $C = C(p,q,d,\sigma,m)$ which does not depend on r so that $C^{-1}a_1 \le a_2 \le Ca_1$. An application of the Banach-Steinhaus theorem (*cf.* Theorems 2.5, 2.6 in [12]) shows that for $\sigma \le d/q$ there is an $f \in F_{\sigma,q}^p(\mathbb{R}^d)$ for which $\mathfrak{D}_m^{\sigma,q} f$ does not belong to $L^p(\mathbb{R}^d)$ (in fact this holds for a class of second category in $F_{\sigma,q}^p$).

Finally we settle an endpoint question about oscillatory multipliers on the F-spaces. Consider the operator given by

$$\widehat{T_{\gamma,b} f}(\xi) = \frac{e^{i|\xi|^\gamma}}{(1+|\xi|^2)^{b/2}} \widehat{f}(\xi), \tag{1.12}$$

for $0 < \gamma < 1$. It is well known that $T_{\gamma,b}$ maps the Besov spaces $B_{\alpha,q}^p$ into itself if and only $b/\gamma \ge d|1/p - 1/2|$, and by a simple application of Hölder's inequality the same result holds for $F^p_{\alpha,q}$ with the strict inequality $b/\gamma > d|1/p-1/2|$. If $1 \le p \le q \le p'$ (if $p>1$), or $p \le q \le \infty$, $p<1$ the endpoint result with $b/\gamma = d|1/p-1/2|$ holds for the F-spaces (see [6], and for more general multiplier theorems [1], [15]).

We show that for the endpoint result the restriction on q is necessary.

Theorem 1.4. *Let $0<q<p\le 2$, $\alpha\in\mathbb{R}$ and let $0<\gamma<1$, $b=\gamma d(1/p-1/2)$. Then for $r\ge 2$*

$$\sup\big\{\|T_{\gamma,b}f\|_{F^p_{\alpha,q}} : \|f\|_{F^p_{\alpha,q}} \le 1, f\in\mathcal{E}(r)\big\} \approx \big(\log r\big)^{1/q-1/p} \tag{1.13}$$

In §2 we shall give the basic random construction that underlies the proofs of all the theorems. Theorem 1.1 is proved in §3. Theorem 1.2 will be proved in §4 and a second deterministic proof of the lacunary case will be given in §5. Theorem 1.3 will be proved in §6 and Theorem 1.4 in §7.

2. A random construction

For each $n\in\{0,1,2,\cdots\}$ let $\mathcal{Q}(n)$ be the set of all dyadic cubes of sidelength 2^{-n} in $[0,1)^d$; more specifically all cubes of the form $\prod_{i=1}^d [j_i 2^{-n}, (j_i+1)2^{-n})$ where the j_i are integers, $0\le j_i < 2^n$, for $i=1,\dots,d$. For any dyadic cube Q let χ_Q denote the characteristic function of Q.

Let $a\in(0,1)$ be a parameter to be specified. Let Ω be a probability space with probability measure μ, on which there is a family $\{\theta_{Q,a}\}$ of independent random variables indexed by the dyadic subcubes of $[0,1]^d$, each of which takes the value 1 with probability a and the value 0 with probability $1-a$. If $B\subset\Omega$ we denote by $\mu(B)$ the probability of B and the expectation of a function g on Ω (i.e. a random variable) is given by the integral $\mathbb{E}(g)=\int_\Omega g(\omega)\,d\mu(\omega)$.

In what follows we fix a sequence $\{n_k\}_{k=0}^\infty$ of nonnegative integers. We consider random functions

$$h_k^{\omega,a}(x) = \sum_{Q\in\mathcal{Q}(n_k)} \theta_{Q,a}(\omega)\chi_Q(x); \tag{2.1}$$

these are supported on $[0,1]^d$. Note that $h_k^{\omega,a}(x)\in\{0,1\}$ for all x. The parameter a will be mostly fixed (except in §7), and we use the notation $h_k^\omega \equiv h_k^{\omega,a}$, $\theta_Q=\theta_{Q,a}$ if the value of a is clear.

Lemma 2.1. *Suppose $p,q\in(0,\infty)$ and $0<a<C_1L^{-1}$. Let $\sigma>\max\{d/p,d/q\}$. Then*

$$\Big(\int_\Omega \Big\|\Big(\sum_{k=1}^L [\mathfrak{M}_{\sigma,2^{n_k}} h_k^{\omega,a}]^q\Big)^{1/q}\Big\|_p^p d\mu\Big)^{1/q} \le C(p,q,C_1) \tag{2.2}$$

Proof. We first observe that for $r>0$ and every $x\in[0,1]^d$,

$$\int_\Omega \Big(\sum_{k=1}^L |h_k^\omega(x)|^q\Big)^r d\mu = \sum_{n=0}^L \binom{L}{n} a^n(1-a)^{L-n}n^r. \tag{2.3}$$

To see this let $x\in[0,1)^d$ and observe that for each k, $h_k^\omega(x)=\theta_Q(\omega)\chi_Q(x)$ for a single $Q=Q(k,x)\in\mathcal{Q}(n_k)$ and thus also $h_k^\omega(x)=[h_k^\omega(x)]^q$. One has then 2^L possible events, indexed by all subsets $S\subset\{1,2,\cdots,L\}$; the event $\Omega(S,x)$ that $\theta_{Q(k,x)}(\omega)$ equals 1 for all $k\in S$ and equals 0 for all $k\notin S$ has probability

$$\mu(\Omega(S,x)) = a^{\mathrm{card}(S)}(1-a)^{L-\mathrm{card}(S)},$$

by independence. The function $(\sum_{k=1}^{L} h_k(x,\omega))^r$ has value $\mathrm{card}(S)^r$ at such an event. Lastly the number of subsets S having cardinality n is $\binom{L}{n}$. Thus, for every x,

$$\int_\Omega \Big(\sum_{k=1}^{L} h_k^\omega(x)\Big)^r d\mu = \sum_{n=0}^{L} \sum_{\mathrm{card}(S)=n} \int_{\Omega(S,x)} \Big(\sum_{k\in S} h_k^\omega(x)\Big)^r d\mu$$

$$= \sum_{n=0}^{L} \binom{L}{n} a^n (1-a)^{L-n} n^r \tag{2.4}$$

which gives (2.3).

We set $r = p/q$ in (2.3) and let r_0 be the smallest positive integer $\geq p/q$. Then

$$\sum_{n=0}^{L} \binom{L}{n} a^n (1-a)^{L-n} n^{p/q} \leq \sum_{n=1}^{\infty} \frac{L^n}{n!} a^n n^{r_0} = (t\frac{d}{dt})^{r_0} e^t\Big|_{t=La} \leq C(r_0, C_1).$$

By (2.3), integration in x and Fubini's theorem the last inequality implies

$$\Big(\int_\Omega \Big\|\Big(\sum_{k=1}^{L} |h_k^\omega|^q\Big)^{1/q}\Big\|_p^p d\mu\Big)^{1/p} \leq C(p/q, C_1) \quad \text{if } La \leq C_1. \tag{2.5}$$

The conclusion of the lemma now follows from (1.5), but we repeat the derivation since it involves an estimate that will be needed later. Observe that since h_k^ω assumes only the values 1 and 0 and is constant on dyadic cubes of length 2^{-n_k} there is the estimate

$$\sup_{2^{-n_k+l} \leq |y| \leq 2^{-n_k+l+1}} |h_k^\omega(x+y)| \leq C_s 2^{l\frac{d}{s}} \big(M([h_k^\omega]^s)\big)^{1/s} \tag{2.6}$$

for any $s \leq 1$. Consequently $\mathfrak{M}_{\sigma,r_k}[h_k^\omega](x) \leq C_\sigma\big(M([h_k^\omega]^s)\big)^{1/s}$ if $\sigma > d/s$ and the vector Fefferman-Stein inequality [5] can be applied if $p/s > 1$, $q/s > 1$. Thus the asserted maximal inequality follows from (2.5). $\square$

An immediate consequence is

Corollary 2.2. *Suppose $p, q \in (0,\infty)$, $\sigma > \max\{d/p, d/q\}$, $L \in \mathbb{N}$ and $0 < a < C_1 L^{-1}$.*

Let η be a Schwartz function and $\eta_k(x) = 2^{n_k d}\eta(2^{n_k}x)$. Denote by F^ω the random vector-valued function defined by $F_k^\omega(x) = \eta_k * h_k^{\omega,a}(x)$ if $1 \leq k \leq L$, and $F_k^\omega(x) = 0$ if $k > L$. Then

$$\Big(\int_\Omega \|F^\omega\|_{L^p(\ell^q)}^p d\mu\Big)^{1/p} \leq \widetilde{C}(p,q,C_1) \tag{2.7}$$

Remark. The quantity (2.4) is bounded by $C_r La$ if $L^{-1} \leq a \leq 1$, see a calculation in Bourgain [3]. There is also a corresponding lower bound for $r \geq 1$, in fact there is the identity $\sum_{n=0}^{L} \binom{L}{n} b^n (1-b)^{L-n} n = Lb$, $\quad 0 < b < 1$. To see this observe that the left hand side is equal to $(1-b)^L t\frac{d}{dt}(1+t)^L$ when evaluated at $t = b/(1-b)$. One also has $(\sum_{n=0}^{L} \binom{L}{n} b^n (1-b)^{L-n} n^r)^{1/r} \geq Lb$ if $r \geq 1$; this follows from Hölder's inequality since $\sum_{n=0}^{L} \binom{L}{n} b^n (1-b)^{L-n} = 1$.

3. Proof of Theorem 1.1

For $z \in \mathbb{R}^d$, $\ell \in \mathbb{Z}$ denote by $\mathfrak{Q}(\ell, z)$ the family of all cubes of the form $z + Q$, with Q any dyadic cube of sidelength $2^{-\ell}$ in $\mathbb{R}^d$. We shall use the important Plancherel-Pólya theorem for entire functions of exponential type ([11], [20]). It says that there are absolute positive constants C, m depending only on $q \in (0, \infty)$ and d so that for all ℓ, z

(3.1)
$$C^{-1}\Big(\sum_{Q\in\mathfrak{Q}(\ell+m,z)} |f(x_Q)|^q\Big)^{1/q} \le 2^{\ell d/q}\|f\|_q \le C\Big(\sum_{Q\in\mathfrak{Q}(\ell+m,z)} |f(\widetilde{x}_Q)|^q\Big)^{1/q}, \quad f \in \mathcal{E}(2^\ell);$$

here $x_Q \in Q$, $\widetilde{x}_Q \in Q$ and the constants in (3.1) are independent of the specific choices of x_Q, $\widetilde{x}_Q$.

An equivalent formulation is

(3.2)
$$C^{-1}\Big(\int \sup_{|x-y|\le u2^{-k}} |f(y)|^q dx\Big)^{1/q} \le \|f\|_q \le C\Big(\int \inf_{|x-y|\le u2^{-k}} |f(y)|^q dx\Big)^{1/q}, \quad f \in \mathcal{E}(2^k);$$

here C and $u \in (0, 1)$ depend only on q and d.

As the statement of Theorem 1.1 is trivial for $p \le q$ we shall assume $p \ge q$ in what follows. If $K \in \mathcal{E}(1)$ satisfies condition (1.1) with $q \le p$ we shall show that for all $N \in \mathbb{N}$

$$\Big(\int_{|x|\le 2^N} \inf_{|x-y|\le u} |K(y)|^q dx\Big)^{1/q} \le C(q, d, u)A. \tag{3.3}$$

Here we may pass to the limit as $N \to \infty$ and then, choosing $u = u(q, d)$ sufficiently small, we may apply the second inequality in (3.2) to deduce the assertion of Theorem 1.1. In what follows we pick an integer M so that $2^{-M+d+1} \le u < 2^{-M+d+2}$.

In order to show (3.3) we may use (1.1) for functions $\{f_k\}_{k=1}^L$ indexed by a finite family of radii; we put $L = 2^{Nd}$ and by a scaling we may assume that

$$r_k \ge 2^{10d+10N}, \qquad k = 1, \dots, L. \tag{3.4}$$

It will be useful to replace K with a kernel which vanishes for $|x| \ge 2^{N+2}$. Let ζ be a C^∞ function with compact support in $\{x : |x| < 4\}$ which equals 1 for $|x| \le 2$. Let $\zeta_N(x) = \zeta(2^{-N}x)$ and let $K^N = K\zeta_N$. Clearly (3.3) follows from

$$\Big(\int \inf_{|x-y|\le u} |K^N(y)|^q dx\Big)^{1/q} \le C'(q, d, u)A. \tag{3.5}$$

We first deduce from (1.1) a vector-valued inequality for the dilates of K^N. We define positive integers n_k as in the previous section, namely by

$$2^{n_k-M-d-1} \le r_k < 2^{n_k-M-d}. \tag{3.6}$$

With these specifications on r_k, n_k we prove

Lemma 3.1. *Suppose that $q \le p$ and that* (1.1) *and* (3.4) *hold. Set $K_k^N(x) = r_k^d K^N(r_k x)$, $a = L^{-1} = 2^{-Nd}$ and define $h_k^\omega \equiv h_k^{\omega,a}$ as in* (2.1). *Then*

$$\Big(\sum_{k=1}^L \int_{[0,1]^d} \int_\Omega |K_k^N * h_k^{\omega,a}|^q d\mu\, dx\Big)^{1/q} \le CA \tag{3.7}$$

Proof. By Hölder's inequality and Fubini's theorem

$$\Big(\sum_{k=1}^{L}\int_{[0,1]^d}\int_{\Omega}|K_k^N*h_k^\omega|^q d\mu dx\Big)^{1/q}\le\Big(\int_\Omega\int_{[0,1]^d}\Big(\sum_{k=1}^{L}|K_k^N*h_k^\omega|^q\Big)^{p/q}dxd\mu\Big)^{1/p}. \tag{3.8}$$

Let $e_z(x)=e^{i\langle x,z\rangle}$. Then for any compactly supported bounded function g

$$K_k^N*g(x)=(2\pi)^{-d}\int\widehat{\zeta_N}(\xi)e_{r_k\xi}(x)K_k*[ge_{-r_k\xi}](x)d\xi. \tag{3.9}$$

Let η be a Schwartz function in $\mathcal{E}(2)$ with the property that $\widehat{\eta}(\xi)=1$ for $|\xi|\le 1$. Let $\eta_k=r_k^d\eta(r_k\cdot)$ and $K_k=r_k^dK(r_k\cdot)$, then

$$K_k*\eta_k=K_k. \tag{3.10}$$

Now suppose $1\le q\le p$. Then (3.9), (3.10), Minkowski's inequality and the assumption (1.1) imply for fixed ω

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^{L}|K_k^N*h_k^\omega|^q\Big)^{1/q}\Big\|_p&\le\int|\widehat{\zeta_N}(\xi)|\Big\|\Big(\sum_{k=1}^{L}\big|K_k*\eta_k*[h_k^\omega e_{-r_k\xi}]\big|^q\Big)^{1/q}\Big\|_p d\xi\\&\le A\int|\widehat{\zeta_N}(\xi)|\Big\|\Big(\sum_{k=1}^{L}\big|\eta_k*[h_k^\omega e_{-r_k\xi}]\big|^q\Big)^{1/q}\Big\|_p d\xi\\&\le C_\rho A\Big\|\Big(\sum_{k=1}^{L}\Big[\sup_y\frac{|h_k^\omega(\cdot+y)|}{(1+r_k|y|)^\rho}\Big]^q\Big)^{1/q}\Big\|_p\end{aligned}$$

for any $\rho>0$. We have used that $\|\widehat{\zeta_N}\|_1=O(1)$. We choose $\rho>d/q$, take pth powers, and integrate over $\omega\in\Omega$. By Lemma 2.1 we obtain

$$\Big(\int_\Omega\Big\|\Big(\sum_{k=1}^{L}|K_k^N*h_k^\omega|^q\Big)^{1/q}\Big\|_p^p d\mu\Big)^{1/p}\le C(p,q,d)A \tag{3.11}$$

and (3.7) follows from (3.11) and (3.8) (in the case $q\ge 1$).

It remains to prove (3.11) in the case $q\le 1$. Since ζ has compact support we can apply the Plancherel-Pólya theorem in L^q. Let $\{Q_\nu^N\}$ denote the collection of dyadic cubes of sidelength 2^{-M-N} (where $2^{-M}\approx u=u(q)$ as in (3.2)). For each such cube choose $\xi_\nu\in Q_\nu^N$. Then for fixed ω

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^{L}|K_k^N*h_k^\omega|^q\Big)^{1/q}\Big\|_p&\le\Big(\int\Big(\sum_{k=1}^{L}\Big(\int\big|\widehat{\zeta_N}(\xi)K_k*[h_k^\omega e_{-r_k\xi}](x)\big|d\xi\Big)^q\Big)^{p/q}dx\Big)^{1/p}\\&\lesssim\Big(\int\Big(\sum_{k=1}^{L}\Big(\sum_\nu 2^{-Nd}|\widehat{\zeta_N}(\xi_\nu)K_k*[h_k^\omega e_{-r_k\xi_\nu}](x)|\Big)^q\Big)^{p/q}dx\Big)^{1/p}\end{aligned}$$

and by the imbedding $\ell^q\subset\ell^1$ and Minkowski's inequality ($p/q\ge 1$) this is dominated by

$$\begin{aligned}&\Big(\int\Big(\sum_{k=1}^{L}\sum_\nu 2^{-Ndq}\big|\widehat{\zeta_N}(\xi_\nu)K_k*[h_k^\omega e_{-r_k\xi_\nu}](x)\big|^q\Big)^{p/q}dx\Big)^{1/p}\\&\lesssim\Big(2^{-Ndq}\sum_\nu|\widehat{\zeta_N}(\xi_\nu)|^q\Big(\int\Big(\sum_{k=1}^{L}\big|K_k*[h_k^\omega e_{-r_k\xi_\nu}(x)]\big|^q\Big)^{p/q}dx\Big)^{q/p}\Big)^{1/q}.\end{aligned}$$

By (3.10) and (1.1) the last expression is in turn dominated by

$$\Big(2^{-Ndq}\sum_\nu |\widehat{\zeta_N}(\xi_\nu)|^q A^q\Big(\int\Big(\sum_{k=1}^L \big|\eta_k * [h_k^\omega e_{-r_k\xi_\nu}]\big|^q\Big)^{p/q}dx\Big)^{q/p}\Big)^{1/q}$$

$$\lesssim C_M A\Big(2^{-Ndq}\sum_\nu |\widehat{\zeta_N}(\xi_\nu)|^q\Big)^{1/q}\Big\|\Big(\sum_{k=1}^L \big|\sup_y \frac{|h_k^\omega(\cdot+y)}{(1+r_k|y|)^\rho}\big|^q\Big)^{1/q}\Big\|_p. \tag{3.12}$$

To eliminate the ν-summation we observe that by the Plancherel-Pólya theorem

$$2^{-Ndq}\sum_\nu |\widehat{\zeta_N}(\xi_\nu)|^q \lesssim 2^{-Nd(q-1)}\int |\widehat{\zeta_N}(\xi)|^q d\xi = \int |\widehat{\zeta}(\xi)|^q d\xi.$$

Thus we may apply Lemma 2.1 (choosing $\rho > d/q$) to bound (3.12) and obtain (3.11) in the case $q < 1$ as well. ☐

Proof of Theorem 1.1, conclusion. Let $Q_k^{N+M+2}(x)$ be the unique dyadic cube of sidelength $2^{-n_k+N+M+2}$ containing x and let $V_k^{N,M}(x)$ be the union of all dyadic cubes of sidelength $2^{-n_k+N+M+2}$ whose boundaries have nonempty intersection with the boundary of $Q_k^{N+M+2}(x)$. Then $V_k^{N,M}(x) \subset [0,1]^d$ provided that $x \in [1/4, 3/4]^d$. Let $\mathcal{V}_k^{N,M}(x)$ be the family of all dyadic cubes in $\mathcal{Q}(n_k)$ which are contained in the closure of $V_k^{N,M}(x)$.

One of the obstacles to be overcome in our proofs is that unwanted cancellations could conceivably arise between the different terms contributing to expressions such as

$$\sum_{Q\in\mathcal{Q}(n_k)} \theta_Q(\omega) K_k^N * \chi_Q(x).$$

We will handle this by considering the contributions of events in which all terms but one in the sum are either small, or have coefficients $\theta_Q(\omega) = 0$. To this end, for each $Q \in \mathcal{V}_k^{N,M}(x)$ define the event

$$\Omega(k,x,Q) = \{\omega\in\Omega:\ \theta_Q(\omega) = 1 \text{ and } \theta_{Q'}(\omega) = 0 \text{ for all } Q' \in \mathcal{V}_k^{N,M}(x)\setminus\{Q\}\}\ . \tag{3.13}$$

If $Q \in \mathcal{Q}(n_k)$ but $Q \notin \mathcal{V}_k^{N,M}(x)$ then $r_k|x-y| \ge r_k 2^{-n_k+N+M+2} \ge 2^{N+2}$ for all $y \in Q$ and thus $K_k^N * \chi_Q = 0$. For fixed $1 \le k \le L$, $x \in [1/4, 3/4]^d$,

$$\begin{aligned}\int_\Omega |K_k^N * h_k^\omega(x)|^q d\mu &= \int_\Omega \Big|\sum_{Q'\in\mathcal{V}_k^{N,M}(x)} \theta_{Q'}(\omega) K_k^N * \chi_{Q'}(x)\Big|^q d\mu \\ &\ge \sum_{Q\in\mathcal{V}_k^{N,M}(x)} \int_{\Omega(k,x,Q)} \Big|\sum_{Q'\in\mathcal{V}_k^{N,M}(x)} \theta_{Q'}(\omega) K_k^N * \chi_{Q'}(x)\Big|^q d\mu \\ &= \sum_{Q\in\mathcal{V}_k^{N,M}(x)} \mu(\Omega(k,x,Q))\big|K_k^N * \chi_Q(x)\big|^q.\end{aligned}$$

Now

$$\mu(\Omega(k,x,Q)) = a(1-a)^{2^{(N+M+2)d}3^d-1}$$

with $a = 2^{-Nd}$. Thus

$$\mu(\Omega(k,x,Q)) \ge c_M 2^{-Nd} \tag{3.14}$$

and therefore

$$\mu(\Omega(k,x,Q))\big|K_k^N * \chi_Q(x)\big|^q \ge c_M 2^{-Nd} 2^{-n_k dq} \inf_{y\in Q} |r_k^d K^N(r_k(x-y))|^q.$$

We have thus proved that

$$\int_\Omega |K_k^N * h_k^\omega(x)|^q d\mu \ge c'_M 2^{-Nd} \sum_{Q\in\mathcal{V}_k^{N,M}(x)} \inf_{z\in r_kQ} |K^N(r_kx-z))|^q.$$

Now the disjoint cubes $r_kx - r_kQ$ cover the ball of radius 2^{N+2} as Q ranges over the cubes in $\mathcal{V}_k^{N,M}(x)$. The diameter of $r_kx - r_kQ$ is bounded by $\sqrt{d}r_k 2^{-n_k} \le 2^{-M} \le u$ and therefore

$$\int_\Omega |K_k^N * h_k^\omega(x)|^q d\mu \ge c'_M 2^{-Nd} \int \inf_{|y-z|\le u} |K^N(z)|^q dy. \tag{3.15}$$

Now integrate over $x \in [1/4/3/4]^d$ and sum in $k = 1, \dots, 2^{Nd}$ and the assertion (3.5) follows from (3.15) and (3.7). □

Remark. Theorem 1.1 can be applied to the case of Bochner-Riesz multipliers mentioned in the introduction. A refinement of this example is as follows. Let χ be supported in $\{\xi : 3/4 < |\xi| < 5/4\}$ and be equal to 1 in a neighborhood of the unit circle and consider the multiplier

$$m_{\lambda,\delta}(\xi) = \sum_{k\in\mathbb{Z}} \chi(2^{-k}\xi)(1-2^{-2k}|\xi|^2)_+^\lambda [\log(1-2^{-k}|\xi|)^{-1}]^{-\delta}$$

Then $f \mapsto \mathcal{F}^{-1}[m_{\lambda,\delta}\widehat{f}]$ fails to be bounded on the homogeneous Triebel-Lizorkin space $\dot{F}_0^{pq}$ if $\lambda < d(1/q - 1/2) - 1/2$, or $\lambda = d(1/q-1/2) - 1/2$, $\delta \le 1/q$. These examples show that the restriction $p \le q \le p'$ in some multiplier theorems for Triebel-Lizorkin spaces stated in [13], [15] is needed; moreover, if $q \le 1$ then the condition on q in the analogue of the Mikhlin-Hörmander multiplier theorem stated on p.75 in [20] is necessary.

4. Proof of Theorem 1.2

We use the random construction of §2. Fix a real valued Schwartz function η so that $\widehat{\eta}$ is supported in $\{\xi : 1/2 < |\xi| < 1\}$ and so that $\eta(x) \ge 1$ for $|x| \le 2^{-M+2+d}$ (with some positive M which is fixed in the proof).

Let $\sigma \le d/q$ and $q \le p$ and let L be large. We may assume that $L = 2^{Nd}$ for some large $N \in \mathbb{N}$. To show the lower bound (1.6) we may assume that the r_k's, $k = 1, \dots, L$ are large. This follows by scaling, namely if $\delta_t f(x) := f(tx)$, and if $f_k \in \mathcal{E}_o(r)$ then $\delta_t f_k \in \mathcal{E}_o(tr)$; moreover $\delta_t^{-1}\mathfrak{M}_{\sigma,rt}\delta_t = \mathfrak{M}_{\sigma,r}$. Thus the operator norms of $\{\mathfrak{M}_{\sigma,r_k}\}$ and $\{\mathfrak{M}_{\sigma,tr_k}\}$ are the same.

We may assume

$$2^{n_k - M} \le r_k < 2^{n_k+1-M}, \quad n_k \in \mathbb{Z}, \quad n_k \ge 100d + M + N,$$

for $k = 1, \dots, L$. Define $\eta_k(x) = r_k^d \eta(r_k x)$ and

$$g_k^{\omega,a} = \eta_k * h_k^{\omega,a}, \quad a = 2^{-Nd}, \tag{4.1}$$

with $h_k^{\omega,a}$ as in (2.1). Note that $g_k^{\omega,a} \in \mathcal{E}_o(r_k)$. We omit the superscript a in what follows.

Since $p \ge q$ we see by Hölder's inequality and Fubini's theorem that

$$\Big(\int_\Omega \Big\| \Big(\sum_{k=1}^L |\mathfrak{M}_{\sigma,r_k} g_k^\omega|^q \Big)^{1/q} \Big\|_p^p d\mu\Big)^{1/p} \ge \Big(\sum_{k=1}^L \int_{[0,1]^d} \int_\Omega |\mathfrak{M}_{\sigma,r_k} g_k^\omega|^q d\mu\, dx\Big)^{1/q}. \tag{4.2}$$

Let $x \in [1/4, 3/4]^d$ and let $Q_k^j(x)$ be the unique dyadic cube of sidelength 2^{-n_k+j} which contains x. Let

$$\mathcal{M}_{j,k} f(x) = \sup_{y \in Q_k^j(x) \setminus Q_k^{j-1}(x)} |f(y)|$$

Then

$$\mathfrak{M}_{\sigma, r_k}[g_k^\omega](x) \ge c_{M,d} \sup_{2 \le j \le N} 2^{-j\sigma} \mathcal{M}_{j,k} g_k^\omega(x). \tag{4.3}$$

Thus, in view of Corollary 2.2, (4.2) and (4.3) it suffices to show that for $\sigma \le d/q$

$$\Big(\sum_{k=1}^L \int_{[\frac14, \frac34]^d} \int_\Omega \sup_{2 \le j \le N} |2^{-j\sigma} \mathcal{M}_{j,k} g_k^\omega(x)|^q d\mu dx \Big)^{1/q} \ge c \max\{2^{N(-\sigma + d/q)}, N^{1/q}\}. \tag{4.4}$$

To show (4.4) we let $V_k^N(x)$ be the union of all dyadic cubes of sidelength 2^{-n_k+N+1} whose boundaries have nonempty intersection with the boundary of $Q_k^{N+1}(x)$. Split

$$\mathcal{M}_{j,k} g_k^\omega(x) = I_j^\omega(k,x) + II_j^\omega(k,x)$$

where

$$I_j^\omega(k,x) = \sup_{y \in Q_k^j(x) \setminus Q_k^{j-1}(x)} \Big| \sum_{\substack{Q \in \mathcal{Q}(n_k) \\ Q \subset V_k^N(x)}} \theta_Q(\omega) \eta_k * \chi_Q(y) \Big|$$

$$II_j^\omega(k,x) = \sup_{y \in Q_k^j(x) \setminus Q_k^{j-1}(x)} \Big| \sum_{\substack{Q \in \mathcal{Q}(n_k) \\ Q \subset [\frac14, \frac34]^d \setminus V_k^N(x)}} \theta_Q(\omega) \eta_k * \chi_Q(y) \Big|$$

The terms $II_j^\omega(k,x)$ are error terms; indeed if $y \in Q_k^j(x)$, $j \le N$ and $z \in [\frac14, \frac34]^d \setminus V_k^N(x)$ then $|y - z| \ge c 2^{-n_k + N}$ and from this it is easy to see that

$$\sup_{2 \le j \le N} \big| II_j^\omega(k,x) \big| \le C_{M,\rho} \mathfrak{M}_{\rho, 2^{-n_k}}[h_k^\omega]$$

for any $\rho > 0$. Thus by Lemma 2.1

$$\Big(\sum_{k=1}^L \int_{[\frac14, \frac34]^d} \int_\Omega \sup_{2 \le j \le N} |2^{-j\sigma} II_j^\omega(k,x)|^q d\mu \, dx \Big)^{1/q} \le C(p,q,M). \tag{4.5}$$

We show for almost every $x \in [1/4, 3/4)^d$, $1 \le k \le L$ the uniform lower bound

$$\int_\Omega \sup_{2 \le j \le N} |2^{-j\sigma} I_j^\omega(k,x)|^q d\mu \ge c' 2^{-Nd} \max\{2^{N(d - q\sigma)}, N\}. \tag{4.6}$$

Clearly (4.4) follows from (4.6) after integrating in x and then summing in k (recall that $L = 2^{Nd}$); the error term (4.5) changes this lower bound only by a small constant if N is large.

Next, to prove (4.6) we observe that if $Q \in \mathcal{Q}(n_k)$ and if y_Q is the center of Q and $z \in Q$ then $|y_Q - z| \le \sqrt{d} 2^{-n_k} \le 2^{-M+d+1} r_k^{-1}$ and since $\eta(w) \ge 1$ for $|w| \le 2^{-M+d+1}$ it follows that

$$\eta_k * \chi_Q(y_Q) = \int r_k^d \eta(r_k(y_Q - z)) \chi_Q(z) dz \ge r_k^d 2^{-n_k d} \ge 2^{-Md}. \tag{4.7}$$

Now assume $Q \in \mathcal{Q}(n_k)$ is contained in $V_k^N(x)$. For this Q let $\Omega(k,x,Q)$ be the event that $\theta_Q(\omega) = 1$, but $\theta_{Q'}(\omega) = 0$ for all other $Q' \in \mathcal{Q}(n_k)$ contained in $V_k^N(x)$. The probability of this event is $\mu(\Omega(k,x,Q)) = a(1-a)^{3^d 2^{(N+1)d}}$ and since $a = 2^{-Nd}$ we get the uniform lower bound

$$\mu(\Omega(k,x,Q)) \geq c_d 2^{-Nd}. \tag{4.8}$$

Moreover, if $2 \leq l \leq N$ and $Q \subset Q_k^l(x) \setminus Q_k^{l-1}(x)$, $Q \subset V_k^N(x)$ then

$$\begin{aligned}
&\int_{\Omega(k,x,Q)} \sup_{2\leq j\leq N} |2^{-j\sigma} I_j^\omega(k,x)|^q d\mu \\
&= \int_{\Omega(k,x,Q)} 2^{-lq\sigma} \sup_{y\in Q_k^l(x)\setminus Q_k^{l-1}(x)} \big|\eta_k * \chi_Q(y)\big|^q d\mu \\
&\geq \mu(\Omega(k,x,Q)\, 2^{-lq\sigma} |\eta_k * \chi_Q(y_Q)|^q \;\geq\; 2^{-Nd} 2^{-lq\sigma} 2^{-Mdq}.
\end{aligned}$$

For fixed k, x the events $\Omega(k,x,Q)$ are disjoint and we can sum over Q. Thus

$$\begin{aligned}
&\int_\Omega \sup_{2\leq j\leq N} |2^{-j\sigma} I_j^\omega(k,x)|^q d\mu \geq \sum_{\substack{Q\in\mathfrak{Q}(n_k)\\ Q\subset V_k^N(x)}} \int_{\Omega(k,x,Q)} \sup_{2\leq j\leq N} |2^{-j\sigma} I_j^\omega(k,x)|^q d\mu \\
&\geq c_1 \sum_{2\leq l\leq N} \sum_{\substack{Q\in\mathfrak{Q}(n_k)\\ Q\subset Q_k^l(x)\setminus Q_k^{l-1}(x)}} 2^{-lq\sigma} 2^{-Nd} \geq c_2 \sum_{2\leq l\leq N} (2^{ld} - 2^{(l-1)d}) 2^{-lq\sigma} 2^{-Nd} \\
&\geq c_3 2^{-Nd} \max\{2^{N(d-q\sigma)}, N\}
\end{aligned}$$

where the constants depend only on d, σ and M. This proves (4.6) and (4.4) follows. □

5. Deterministic examples

We return to Theorem 1.2 and give a nonprobabilistic proof for the lower bound in the case where $r_k = 2^{-k}$, $k > 0$. With small modifications the argument can be made to apply in the general lacunary case, where $\inf_k r_{k+1}/r_k > 1$, but we leave this to the reader.

Fix $M > 0$ sufficiently large and let $\eta \in \mathcal{S} \cap \mathcal{E}_o(1)$ be a Schwartz function such that $\eta(x) \geq 1$ if $|x_i| \leq 2^{-M}$ for $i = 1, \dots, d$. Let $\eta_k = 2^{kd}\eta(2^k\cdot)$.

We fix N large and set $L = 2^{Nd}$. For $k \geq N$, let $\mathcal{Z}_{k,N}^d = \{0, 1, \dots, 2^{k-N} - 1\}^d$ and for $j = (j_1, \dots, j_d) \in \mathcal{Z}_{k,N}^d$ we set

$$Q_{k,j} = [j_1 2^{-k+N}, j_1 2^{-k+N} + 2^{-k-M}] \times \cdots \times [j_d 2^{-k+N}, j_d 2^{-k+N} + 2^{-k-M}].$$

Denote by $h_{k,j}$ be the characteristic function of $Q_{k,j}$, and let $h_k = \sum_{j\in\mathcal{Z}_{k,N}^d} h_{k,j}$. Let $f_k = h_k * \eta_k$ so that $f_k \in \mathcal{E}_o(2^k)$.

Proposition 5.1. *For $0 < q \leq p < \infty$, there is N_0 so that for $N > N_0$*

$$\Big\| \Big(\sum_{k=N}^{2^{Nd}} |f_k|^q \Big)^{1/q} \Big\|_p \leq C(p,q), \tag{5.1}$$

and, for $\sigma \leq d/q$,

$$\Big\| \Big(\sum_{k=N}^{2^{Nd}} |\mathfrak{M}_{\sigma,2^k} f_k|^q \Big)^{1/q} \Big\|_p \geq c(p,q,M) \max\{2^{N(\frac{d}{q}-\sigma)}, N^{1/q}\}. \tag{5.2}$$

Proof. It is easy to see that $|f_k| \le C_s(M_{HL}[|h_k|^s])^{1/s}$, for $s > 0$; see the argument for (2.6) in the proof of Lemma 2.1. Thus it suffices to prove (5.1) with f_k replaced by h_k. For the proof we may assume that $p \ge q$, and in fact $p = nq$ for some integer n (the intermediate cases follow by interpolation). Thus we have to show that the $L^1([0,1]^d)$ norm of $(\sum_{k=N}^{2^{Nd}} h_k)^n$ has an upper bound depending only on n. Since each h_k is nonnegative, this follows from

$$\sum_{\substack{k_1,\dots,k_n\in[N,2^{Nd}]\\ k_1\le k_2\le\dots\le k_n}} \int \prod_{i=1}^n h_{k_i}(x)dx \le C(n). \tag{5.3}$$

In comparison with the random case, we have lost independence; the correlation between h_{k_i} and $h_{k_{i+1}}$ is strongest when $k_{i+1}-k_i$ is small. To estimate (5.3) observe that the support of h_k has measure $2^{-(N+M)d}$ and that

$$\text{meas}\Big(\bigcup_{Q_{k_{i+1},\nu}\subset Q_{k_i,j}} Q_{k_{i+1},\nu}\Big) = \begin{cases} |Q_{k_i,j}|2^{-(N+M)d} & \text{if } k_{i+1}\ge k_i+N+M,\\ |Q_{k_i,j}|2^{(k_i-k_{i+1})d} & \text{if } k_i\le k_{i+1}\le k_i+N+M.\end{cases}$$

Thus

$$\int \prod_{i=1}^n h_{k_i}(x)dx = \text{meas}\big(\text{supp }(\prod_{i=1}^n h_{k_i})\big) = 2^{-(N+M)d}\prod_{i=1}^{n-1}\max\{2^{(k_i-k_{i+1})d}, 2^{-(N+M)d}\}.$$

We sum in $k_n, k_{n-1},\dots,k_1$ (each ranging over the integers in $[N, 2^{Nd}]$) and (5.3) follows.

We now show for $\sigma \le d/q$ the lower bound (5.2). [1] Let

$$\mathfrak{M}_{\sigma,2^k,N}f(x) := \sup_{y:|y|\le 2^{-k+N-2}} \frac{|f(x+y)|}{(1+2^k|y|)^\sigma}.$$

Then $\mathfrak{M}_{\sigma,2^k}(x) \ge \mathfrak{M}_{\sigma,2^k,N}(x)$.

Let $y_{k,j}$ be the center of $Q_{k,j}$ and observe that $\eta_k * h_{k,j}(y_{k,j}) \ge 2^{-Md}$. Thus also

$$\mathfrak{M}_{\sigma,2^k,N}[\eta_k * h_{k,j}](x) \ge 2^{-Md}(1+2^k|x-y_{k,j}|)^{-\sigma} \text{ if } |x-y_{k,j}|\le 2^{-k+N-2}. \tag{5.4}$$

We derive an upper bound for $\mathfrak{M}_{\sigma,2^k,N}[\eta_k * \sum_{j'\ne j} h_{k,j'}](x)$ for $|x-y_{k,j}|\le 2^{-k+N-2}$. With this restriction and with $A \gg d$ we get

$$\mathfrak{M}_{\sigma,2^k,N}[\eta_k * \sum_{j'\ne j} h_{k,j'}](x)$$
$$\lesssim \sup_{|y|\le 2^{-k+N-2}} (1+2^k|y|)^{-\sigma} C_A \int \frac{2^{kd}}{(1+2^k|x+y-w|)^A}\Big|\sum_{j'\ne j} h_{k,j'}(w)\Big| dw$$

and since for $w \in \text{supp }(h_{k,j'})$ we have $|x+y-w| \ge |y_{k,j}-y_{k,j'}| - 3\cdot 2^{-k+N-2}$ we dominate the last expression by

$$C_A 2^{-Md}\sum_{j'\ne j}(2^{N-4}|j-j'|)^{-A} \lesssim C_{A,M}2^{-NA}.$$

Fix $A \gg d$. We may choose N_0 such that for $N \ge N_0$

$$C_{A,M}2^{-N(A-d)} \le \frac12 2^{-Md}2^{(2-N)\sigma} \le \frac12 \inf_{|x-y_{k,j}|\le 2^{-k+N-2}} 2^{-Md}(1+2^k|x-y_{k,j}|)^{-\sigma} \tag{5.5}$$

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[1] We thank Gustavo Garrigós for pointing out a sloppy argument in the published manuscript.

By Hölder's inequality

$$\Big(\int_{[0,1]^d}\Big(\sum_{k=N}^{2^{Nd}}|\mathfrak{M}_{\sigma,2^k}f_k(x)|^q\Big)^{p/q}dx\Big)^{1/p}\geq\Big(\sum_{k=N}^{2^{Nd}}\int_{[0,1]^d}|\mathfrak{M}_{\sigma,2^k,N}f_k(x)|^q dx\Big)^{1/q}$$

$$\geq\Big(\sum_{k=N}^{2^{Nd}}\sum_{j\in\mathcal{Z}^d_{k,N}}\int_{\substack{x\in[0,1]^d\\|x-y_{k,j}|\leq 2^{-k+N-2}}}\Big||\mathfrak{M}_{\sigma,2^k,N}[\eta_k*h_{k,j}](x)|-|\mathfrak{M}_{\sigma,2^k,N}[\eta_k*\sum_{j'\neq j}h_{k,j'}](x)|\Big|^q dx\Big)^{1/q}$$

$$\geq 2^{-Md}\Big(\sum_{k=N}^{2^{Nd}}\sum_{j\in\mathcal{Z}^d_{k,N}}\int_{\substack{x\in[0,1]^d\\|x-y_{k,j}|\leq 2^{-k+N-2}}}2^{-q}(1+2^k|x-y_{k,j}|)^{-\sigma q}dx\Big)^{1/q},$$

by (5.5). One easily verifies that the last term is bounded below by $c2^{N(d-\sigma q)/q}$ if $\sigma<d/q$ and by $N^{1/q}$ if $\sigma=d/q$. Thus (5.2) follows. □

6. Proof of Theorem 1.3

We shall first use arguments from singular integral theory to establish the upper bounds. Then we show the lower bounds by somewhat more technical variants of the ideas used above to prove Theorem 1.2.

6.1. **Upper bounds.** In this section we set $L_kf=\eta_k*f$ where $\eta_k=2^{kd}\eta(2^k\cdot)$ and η is a Schwartz function whose Fourier transform is supported in $\{\xi:1/2\leq|\xi|\leq 2\}$.

It suffices to set $r=2^{2^{Nd}}$ and the claimed upper bound follows easily from

$$\Big\|\Big(\int_0^1\sup_{|h|\leq t}\Big|\Delta_h^m\sum_{k=1}^{2^{Nd}}L_kf(x)|^q t^{-1-\sigma q}dt\Big)^{1/q}\Big\|_p\lesssim A_N(p,q,\sigma)\Big\|\Big(\sum_{k=1}^{2^{Nd}}2^{\sigma qk}|L_kf|^q\Big)^{1/q}\Big\|_p \tag{6.1}$$

for $q\leq p$, where

$$A_N(p,q,\sigma)=2^{N(-\sigma+d/q)}\quad\text{if } d/p<\sigma<d/q,\quad A_N(p,q,d/q)=\max\{N,N^{1/q}\}.$$

The contributions for the terms with $|2^kh|\leq 1$ can be dealt with by standard arguments using Peetre's maximal function. One obtains

$$\Big\|\Big(\int_0^1\sup_{|h|\leq t}\Big|\sum_{\substack{1\leq k\leq 2^{Nd}\\2^k|h|\leq 1}}\Delta_h^mL_kf(x)|^q t^{-1-\sigma q}dt\Big)^{1/q}\Big\|_p\lesssim\Big\|\Big(\sum_{k=1}^{2^{Nd}}2^{k\sigma q}|L_kf|^q\Big)^{1/q}\Big\|_p; \tag{6.2}$$

it is only here that the more detailed structure of the difference operator Δ_h^m and in particular the condition $m>\sigma$ is used. Therefore (after a change of variable and application of the triangle inequality) matters are reduced to the inequality

$$\Big\|\Big(\int_0^1\sup_{|h|\leq t}\Big|\sum_{\substack{1\leq k\leq 2^{Nd}\\2^k|h|\geq 1}}L_kf(x+h)|^q t^{-1-\sigma q}dt\Big)^{1/q}\Big\|_p\lesssim A_N(p,q,\sigma)\Big\|\Big(\sum_{k=1}^{2^{Nd}}2^{\sigma qk}|L_kf|^q\Big)^{1/q}\Big\|_p. \tag{6.3}$$

For $n\geq 0$ define

$$\mathcal{M}_k^nf_k(x)=\sup_{|h|\leq 2^{n-k+1}}|f_k(x+h)|. \tag{6.4}$$

Proposition 6.1.1. *Let $0 < q \le p < \infty$. Then if $f_k \in \mathcal{E}(2^k)$*

$$\big\|\{\mathcal{M}_k^n f_k\}\big\|_{L^p(\ell^q)} \lesssim 2^{nd/q}\big\|\{f_k\}\big\|_{L^p(\ell^q)}. \tag{6.5}$$

Remarks.

(i) Note that $\mathcal{M}_k^n f_k \lesssim 2^{nd/\rho}\mathfrak{M}_{\rho,2^k} f_k$ so that the non-endpoint $L^p(\ell^q)$ bound with constant $C_\varepsilon 2^{n(d/q+\varepsilon)}$ follows from Peetre's maximal theorem.

(ii) There is also an endpoint inequality when $p < q$, namely

$$\big\|\{\mathcal{M}_k^n f_k\}\big\|_{L^p(\ell^q)} \lesssim 2^{nd/p}\big\|\{f_k\}\big\|_{L^p(\ell^q)}, \qquad 0 < p \le q < \infty.$$

This bound is not needed here and can be proved using arguments in §3 of [15].

Proof that Proposition 6.1.1 implies (6.3).

Assuming $q \ge 1$ we estimate

$$\Big(\int_0^1 \sup_{|h|\le t}\Big|\sum_{\substack{1\le k\le 2^{Nd}\\ 2^k|h|\ge 1}} L_k f(x+h)\Big|^q t^{-1-\sigma q}dt\Big)^{1/q}$$
$$\lesssim \sum_{m\ge 0}\sum_{n\ge 0}\Big(\sum_{l=1}^{2^{Nd}} 2^{l\sigma q}\sup_{2^{-l-m}\le |h|\le 2^{-l-m+2}}|L_{l+m+n}f(x+h)|^q\Big)^{1/q}$$

and using Proposition 6.1.1 we obtain

$$\Big\|\Big(\sum_{l=1}^{2^{Nd}} 2^{l\sigma q}\sup_{2^{-l-m}\le |h|\le 2^{-l-m+2}}|L_{l+m+n}f(x+h)|^q\Big)^{1/q}\Big\|_p$$
$$\lesssim 2^{-m\sigma}2^{n(\frac{d}{q}-\sigma)}\Big\|\Big(\sum_l 2^{(l+m+n)\sigma q}|L_{l+m+n}f|^q\Big)^{1/q}\Big\|_p.$$

The contributions of very large parameters n are negligible, but an alternative bound is needed to quantify this. One such bound can derived by invoking Hölder's inequality to get

$$\Big\|\Big(\sum_{l=1}^{2^{Nd}} 2^{l\sigma q}\sup_{2^{-l-m}\le |h|\le 2^{-l-m+2}}|L_{l+m+n}f(x+h)|^q\Big)^{1/q}\Big\|_p$$
$$\le 2^{Nd(1/q-1/p)}\Big\|\Big(\sum_{l=1}^{2^{Nd}} 2^{l\sigma p}\sup_{2^{-l-m}\le |h|\le 2^{-l-m+2}}|L_{l+m+n}f(x+h)|^p\Big)^{1/p}\Big\|_p$$
$$\lesssim 2^{-m\sigma}2^{n(\frac{d}{p}-\sigma)}2^{Nd(1/q-1/p)}\Big\|\Big(\sum_l 2^{(l+m+n)\sigma q}|L_{l+m+n}f|^q\Big)^{1/q}\Big\|_p.$$

Consequently after summing in m, n we obtain

$$\mathcal{A}_{p,q,\sigma}(2^{2^{Nd}}) \lesssim \sum_{n\ge 0}\min\{2^{n(\frac{d}{p}-\sigma)}2^{Nd(1/q-1/p)},\, 2^{n(\frac{d}{q}-\sigma)}\}. \tag{6.6}$$

In Theorem 1.3 we have the hypotheses $q < p$ and $\sigma > d/p$. Thus the series is $O(2^{N(d/q-\sigma)})$ if $\sigma < d/q$ and is $O(N)$ when $\sigma = d/q$.

If $q < 1$ we have to bound the L^p norm of

$$\Big(\sum_{m\geq 0}\sum_{n\geq 0}\sum_{l=1}^{2^{Nd}} 2^{l\sigma q} \sup_{2^{-l-m}\leq|h|\leq 2^{-l-m+2}} |L_{l+m+n}f(x+h)|^q\Big)^{1/q}$$

and now $\mathcal{A}^q_{p,q,\sigma}(2^{2^{Nd}}) \lesssim \sum_{n=0}^{\infty} \min\{2^{nq(\frac{d}{p}-\sigma)}2^{Nd(1-q/p)}, 2^{nq(\frac{d}{q}-\sigma)}\}$ which is $O(2^{N(d-q\sigma)})$ if $\sigma < d/q$ and $O(N)$ when $\sigma = d/q$. Thus we have shown that the upper bound in Theorem 1.3 is implied by Proposition 6.1.1.

Proof of Proposition 6.1.1.

We first observe that the known arguments in Peetre's maximal inequality yield the assertion for $p = q$. Indeed a small modification of the proof in [20], p. 20, shows that for $g \in \mathcal{E}(1)$, $0 < r < \infty$, $\rho > d/r$

$$\sup_z \frac{|\nabla g(x+z)|}{(1+2^{-n}|z|)^\rho} \lesssim \sup_z \frac{|g(x+z)|}{(1+2^{-n}|z|)^\rho} \tag{6.7}$$

and that this can be used to obtain

$$\sup_z \frac{|g(x+z)|}{(1+2^{-n}|z|)^\rho} \lesssim 2^{nd/r} \sum_{m=0}^{\infty} 2^{-m(\rho-d/r)}\Big(2^{-d(n+m)}\int_{|z|\leq 2^{n+m}} |g(x+z)|^r dz\Big)^{1/r}. \tag{6.8}$$

(6.8) implies for $0 < r < \infty$ the inequality

$$\big\|\mathcal{M}^n_k g\big\|_r \lesssim 2^{nd/r}\|g\|_r, \quad g \in \mathcal{E}(2^k), \tag{6.9}$$

first for $k = 0$ and then by scaling also for general k. Thus we obtain (6.5) for $p = q$.

We now consider the assertion for $p > q$. First observe that the case $1 \leq q < p$ can be proved by interpolation with the $L^p(\ell^p)$ bound once the cases $q \leq p \leq 1$ and $q \leq 1, p > 1$ are settled. We consider these cases in what follows and use rather standard arguments from singular integral theory, namely the Fefferman-Stein #-function estimate ([6]) which is valid for Banach-space valued functions; it is applied here to $L^{p/q}$ functions which take values in the Banach space $\ell^1(L^\infty)$.

In what follows the slashed integral $\fint_Q$ will denote an average over the cube Q. By the Fefferman-Stein theorem it suffices to bound

$$\Big(\int \Big[\sup_{Q\ni x} \fint_Q \sum_k \sup_{|h|\leq 2^{n-k}} \Big||f_k(w+h)|^q - \fint_Q |f_k(z+h)|^q dz\Big| dw\Big]^{p/q} dx\Big)^{1/p}$$

by $2^{nd/q}\big\|\{f_k\}\big\|_{L^p(\ell^q)}$. Since $|a|^q - |b|^q \leq |a-b|^q$ for $q \leq 1$ this bound follows from

$$\Big(\int \Big[\sup_{Q\ni x} \fint_Q \sum_k \sup_{|h|\leq 2^{n-k}} \fint_Q |f_k(w+h) - f_k(z+h)|^q dz dw\Big]^{p/q} dx\Big)^{1/p} \lesssim 2^{nd/q}\big\|\{f_k\}\big\|_{L^p(\ell^q)} \tag{6.10}$$

In what follows we denote by $\ell(Q)$ the integer ℓ for which the sidelength of Q is in $[2^{\ell-1}, 2^\ell)$. Moreover we let $\mathcal{R}_0(Q)$ be the region of all points x which have distance $\leq d2^{\ell(Q)}$ from Q and for $m > 0$ let $\mathcal{R}_m(Q)$ of all points x for which $d2^{\ell(Q)+m-1} \leq \operatorname{dist}(x,Q) < d2^{\ell(Q)+m}$. Let η be a Schwartz function in $\mathcal{E}(2)$ whose Fourier transform is equal to 1 in $\{\xi : |\xi| \leq 1\}$. Let $\eta_k = 2^{kd}\eta(2^k\cdot)$ and $P_k f = \eta_k * f$ and observe that $P_k f_k = f_k$ if $f \in \mathcal{E}(2^k)$. The estimate (6.10) is a consequence of the following three inequalities:

$$\Big(\int\Big[\sup_{Q\ni x}\fint_Q\sum_{k>n-\ell(Q)}\sup_{|h|\le 2^{n-k+1}}|f_k(w+h)|^q dw\Big]^{p/q}dx\Big)^{1/p}\lesssim 2^{nd/q}\big\|\{f_k\}\big\|_{L^p(\ell^q)},\tag{6.11}$$

$$\Big(\int\Big[\sup_{Q\ni x}\fint_Q\sum_{k=-C_0n-\ell(Q)}^{n-\ell(Q)}\sup_{|h|\le 2^{n-k+1}}|f_k(w+h)|^q dw\Big]^{p/q}dx\Big)^{1/p}\tag{6.12}$$
$$\lesssim (n+1)^{1/q-1/p}2^{nd/p}\big\|\{f_k\}\big\|_{L^p(\ell^q)},$$

and, if $C_0>d/p$,

$$\Big(\int\Big[\sup_{Q\ni x}\fint_Q\sum_{k<-C_0n-\ell(Q)}\sup_{|h|\le 2^{n-k}}\fint_Q|f_k(w+h)-f_k(z+h)|^q dz dw\Big]^{p/q}dx\Big)^{1/p}\tag{6.13}$$
$$\lesssim\big\|\{f_k\}\big\|_{L^p(\ell^q)}.$$

First the estimation of the main term (6.11) is rather analogous to the standard "good-function estimate" in Calderón-Zygmund theory. We split

$$f_k=P_kf_k=\sum_{m=0}^{\infty}P_k[\chi_{\mathcal R_m(Q)}f_k]$$

and estimate for fixed x and Q using (6.9)

$$\fint_Q\sum_{k>n-\ell(Q)}\sup_{|h|\le 2^{n-k+1}}|P_k\chi_{\mathcal R_0(Q)}f_k](w+h)|^q dw\tag{6.14}$$
$$\lesssim 2^{nd}|Q|^{-1}\sum_{k>n-\ell(Q)}\int_{\mathbb R^d}|P_k\chi_{\mathcal R_0(Q)}f_k](w)|^q dw.$$

It is straightforward to estimate for any ρ, ρ'

$$|P_k[\chi_{\mathcal R_0(Q)}f_k](w)|\le C_\rho\mathfrak M_{\rho,2^k}f_k(w)\qquad\text{if } w\in\mathcal R_0(Q), k+\ell(Q)\ge 0$$

and

$$|P_k[\chi_{\mathcal R_0(Q)}f_k](w)|\le C_{\rho_1,\rho_2}2^{-(\ell(Q)+k+m)\rho'}\mathfrak M_{\rho,2^k}f_k(w)$$
$$\text{if } w\in\mathcal R_m(Q),\ m\ge 1, k+\ell(Q)+m\ge 0.$$

Therefore from (6.14)

$$\sup_{Q\ni x}\fint_Q\sup_{|h|\le 2^{n-k+1}}|P_k[\chi_{\mathcal R_0(Q)}f_k](w+h)|^q dw\le C2^{nd}M_{HL}(|\mathfrak M_{\rho,2^k}f_k|^q)(x)\tag{6.15}$$

and we may use the $L^{p/q}$ boundedness of M_{HL} and the Peetre maximal theorem to deduce

$$\Big\|\Big[\sup_{Q\ni x}\fint_Q\sum_{k>n-\ell(Q)}\sup_{|h|\le 2^{n-k+1}}|P_k[\chi_{\mathcal R_0(Q)}f_k](w+h)|^q dw\Big]^{1/q}\Big\|_p\lesssim 2^{nd/q}\big\|\{f_k\}\big\|_{L^p(\ell^q)}\tag{6.16}$$

We also obtain for $m\ge 1$

$$|P_k[\chi_{\mathcal R_m(Q)}f_k](x')|\le C_\rho 2^{-(k+\ell(Q)+m)\rho}\mathfrak M_{\rho,2^k}f_k(x)$$

if $x \in \mathcal{R}_0(Q)$, $x' \in \mathcal{R}_0(Q)$, $k + \ell(Q) + m \geq 0$. This can be applied to bound the expression $\sup_{|h| \leq 2^{n-k+1}} |P_k[\chi_{\mathcal{R}_m(Q)} f_k](w+h)|$ when $w \in Q$ and $k \geq n - \ell(Q)$. We obtain

$$\Big\| \Big[\sup_{Q \ni x} \fint_Q \sum_{k > n-\ell(Q)} \sup_{|h| \leq 2^{n-k+1}} |P_k[\chi_{\mathcal{R}_m(Q)} f_k](w+h)|^q dw \Big]^{1/q} \Big\|_p \lesssim 2^{-m\rho} \big\| \{f_k\} \big\|_{L^p(\ell^q)} \tag{6.17}$$

and (6.11) follows from (6.16) and (6.17).

By Hölder's inequality the left hand side of (6.12) is controlled by

$$\begin{aligned} & C \Big(\int \Big[\sup_{Q \ni x} \fint_Q \Big(\sum_{k=-C_0 n - \ell(Q)}^{n-\ell(Q)} |\mathcal{M}_k^n f_k(w)|^q(w) \Big)^{q/p} (1+n)^{1-q/p} dw \Big]^{p/q} dx \Big)^{1/p} \\ & \lesssim (1+n)^{1/q-1/p} \Big(\int \Big[M_{HL} \big((\sum_k |\mathcal{M}_k^n f_k|^p)^{q/p} \big) \Big]^{p/q} dx \Big)^{1/p} \\ & \lesssim (1+n)^{1/q-1/p} \Big(\int \sum_k |\mathcal{M}_k^n f_k|^p dx \Big)^{1/p}. \end{aligned} \tag{6.18}$$

By (6.9) for $r = p$ we bound (6.18) by a constant times

$$(1+n)^{1/q-1/p} 2^{nd/p} \Big(\sum_k \int |f_k|^p dx \Big)^{1/p} \lesssim (1+n)^{1/q-1/p} 2^{nd/p} \big\| \{f_k\} \big\|_{L^p(\ell^q)}$$

and (6.12) is proved.

Finally to see (6.13) we simply observe that $\mathcal{M}_k^n(\nabla f_k) \lesssim 2^k \mathcal{M}_k^n(f_k)$ and thus for $x \in Q$

$$\fint_Q \sup_{|h| \leq 2^{n-k}} \fint_Q |f_k(w+h) - f_k(w+h+z)|^q dz dw \lesssim 2^{kq} (\mathrm{diam}(Q))^q |\mathcal{M}_k^n f_k(x)|^q.$$

Therefore the left hand side of (6.13) is bounded by

$$\Big(\int \big(M_{HL}[2^{-C_0 nq} |\mathcal{M}_k^n f_k|^q] \big)^{p/q} \Big)^{1/p} \lesssim 2^{n(\frac{d}{p} - C_0)} \big\| \{f_k\} \big\|_{L^p(\ell^q)}. \quad \square$$

Remarks. (i) For the sequence of dyadic radii $r_k = 2^k$ consider the maximal operators $\mathfrak{M}_{\sigma, r_k}$. Proposition 6.1.1 can be used to show a converse to the lower bound in Theorem 1.2 in this case; *i.e.* if

$$\mathcal{B}_{p,q,\sigma}(L) = \sup \Big\{ \big\| \big(\sum_{k=1}^L |\mathfrak{M}_{\sigma, 2^k} f_k|^q \big)^{1/q} \big\| : \| \{f_k\} \|_{L^p(\ell^q)} \leq 1, f_k \in \mathcal{E}(2^k), k = 1, \dots, L \Big\}$$

then $\mathcal{B}_{p,q,\sigma}(L) \approx L^{-\sigma + d/q}$ if $d/p < \sigma < d/q$ and $\mathcal{B}_{p,q,d/q}(L) \approx \log^{1/q} L$ if $p > q$.

(ii) Proposition 6.1.1 and the preceding remark remain valid for a general lacunary sequence ($r_{k+1}/r_k \geq \gamma > 1$).

6.2. **Lower bounds.** We shall work with the Schwartz function η defined as in the proof of Theorem 1.2 (i.e. with $\widehat{\eta}$ vanishing identically outside of $\{1/2 < |\xi| < 1\}$ and with $|\eta(x)| \geq 1$ for $|x| \leq 2^{-M+d+2}$, for suitable M). We shall need a C^∞ function ϕ supported in $[-2^{-2M-4}, 2^{-2M-4}]^d$ so that

$$|\phi * \eta(z)| \geq c_0(M) > 0 \text{ if } |z| \leq 2^{-M+d+1} \tag{6.19}$$

Let R be a large positive integer, to be chosen later. We may assume that $R \ge 10d(1 + 1/p + 1/q)$. It clearly suffices to prove the lower bound in (1.9) for r of the form

$$r = 2^{R2^{Nd}} \tag{6.20}$$

uniformly for all large positive integers N.

We let

$$n_k = kR, \quad k = 1, 2, \dots, \tag{6.21}$$

$$r_k = 2^{n_k - M}. \tag{6.22}$$

Set $\phi_k(x) = r_k^d \phi(r_k x)$ and $\eta_k(x) = r_k^d \eta(r_k x)$. Specify

$$a = 2^{-Nd} = L^{-1}, \tag{6.23}$$

let $g_k^\omega \equiv g_k^{\omega,a}$ as in (4.1), and set

$$G_k^\omega(x) = 2^{-n_k\sigma} g_k^\omega(x), \qquad G^\omega(x) = \sum_{k=1}^{2^{Nd}} G_k^\omega(x). \tag{6.24}$$

We need the following estimates for convolutions with the functions g_k^ω and G^ω.

Lemma 6.2.1. *(i) Let H be a Schwartz function so that $\int H(x) x^\alpha dx = 0$ for all multi-indices α with $\max_i |\alpha|_i \le N_0$. Let $H_\ell = 2^{\ell d} H(2^\ell \cdot)$. Then*

$$|H_\ell * g_l^\omega(x)| \lesssim 2^{-|\ell - n_l|(N_0 - d/s)} \big(M((\sum_{Q \in \mathcal{Q}(n_l)} \theta_Q(\omega) \chi_Q)^s)(x) \big)^{1/s}. \tag{6.25}$$

(ii) For $0 < p, q < \infty$

$$\Big(\int_\Omega \|G^\omega\|^p_{F^p_{\sigma,q}} d\mu(\omega) \Big)^{1/p} \le C_1 \tag{6.26}$$

and

$$\Big(\int_\Omega \int_{\mathbb{R}^d} \Big[\int_0^1 \sup_{|h| \le t} \sum_{l : t2^{n_l} \le 1} \big| \Delta_h^m G_l^\omega(x) \big|^q t^{-1-\sigma q} dt \Big]^{p/q} dx d\mu(\omega) \Big)^{1/p} \le C_2. \tag{6.27}$$

Moreover

$$\Big(\int_\Omega \int_{\mathbb{R}^d} \Big[\int_0^1 \Big(\sum_{l : t2^{n_l} \ge 1} |G_l^\omega(x)| \Big)^q t^{-1-\sigma q} dt \Big]^{p/q} dx d\mu(\omega) \Big)^{1/p} \le C_3. \tag{6.28}$$

Here C_1, C_2, C_3 depend only on p,q, m and d.

Proof. (6.25) is straightforward, *cf.* the reasoning for inequality (2.6). The other assertions follow in a straightforward manner from the basic estimates (2.6) and (6.25), a suitable application of Minkowski's inequality, and Lemma 2.1. □

Using somewhat nonstandard notation we define $\widetilde{\Delta}_h^m$ by

$$\widetilde{\Delta}_h^m f(x) = \Delta_h^m f(x) - (-1)^m f(x) = \sum_{\nu=1}^m (-1)^{m-\nu} \binom{m}{\nu} f(x + \nu h) \tag{6.29}$$

and let

$$I_{k,j} = [2^{-n_k + j + 2d}, 2^{-n_k + j + 2d + 1}].$$

In view of Lemma 6.2.1 and Hölder's inequality on $\Omega\times[0,1]^d$ (with $p/q\ge 1$), in order to prove the lower bounds in (1.9), (1.10), (1.11) it suffices to prove that

$$\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\sum_k\sum_{j=d+M}^{N-M-d}\int_{I_{k,j}}\sup_{|h|\le mt}\Big|\sum_{l\ge k}\widetilde\Delta^m_{\frac{h}{m}}G^\omega_l(x)\Big|^q t^{-1-\sigma q}dtdxd\mu\Big)^{1/q} \ge c_0\max\{2^{N(\frac{d}{q}-\sigma)},N^{1/q}\} \tag{6.30}$$

for some $c_0>0$.

Let

$$\Gamma^{l,Q}_{k,m}(x,h):=\int\phi_k(y)\widetilde\Delta^m_{\frac{h-y}{m}}\eta_l*\chi_Q(x)dy,\qquad Q\in\mathcal{Q}(n_l) \tag{6.31}$$

and

$$\Gamma^l_{k,m}(x,h,\omega):=\int\phi_k(y)\widetilde\Delta^m_{\frac{h-y}{m}}G^\omega_l(x)dy=2^{-n_l\sigma}\sum_{Q\in\mathcal{Q}(n_l)}\theta_Q(\omega)\Gamma^{l,Q}_{k,m}(x,h). \tag{6.32}$$

We use the elementary inequality

$$|\phi_k*a(h)|\le C\sup_{|h-u|\le 2^{-n_k-2}}|a(u)|$$

to deduce that (6.30) follows from the existence of a constant $c_1>0$ such that

$$\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\sum_k\sum_{j=M+d}^{N-M-d}2^{(n_k-j)\sigma q}\sup_{|h|\le m2^{-n_k+j+d}}\Big|\sum_{l\ge k}\Gamma^l_{k,m}(x,h,\omega)\Big|^q dxd\mu\Big)^{1/q} \ge c_1\max\{2^{N(\frac{d}{q}-\sigma)},N^{1/q}\}. \tag{6.33}$$

We show now that the only relevant terms in (6.33) are those with $l=k$; it is here where we have to choose R sufficiently large.

Lemma 6.2.2. *For $0<q\le p<\infty$,*

$$\Big(\int_\Omega\int\sum_k\sum_{j=M+d}^{N-M-d}2^{(n_k-j)\sigma q}\sup_{|h|\le m2^{-n_k+j+d}}\Big|\sum_{l\ge k+1}\Gamma^l_{k,m}(x,h,\omega)\Big|^q dxd\mu\Big)^{1/q} \le C_4 2^{-R}\max\{2^{N(\frac{d}{q}-\sigma)},N^{1/q}\} \tag{6.34}$$

Proof. [2] Let $\widetilde\eta$ be a Schwartz function whose Fourier transform is supported in $\{\xi: 1/4<|\xi|<4\}$ and is equal to 1 on $\{\xi:1/2\le|\xi|\le 2\}$. Define $\widetilde\eta_l=2^{n_ld}\widetilde\eta(2^{n_l}\cdot)$ and observe that $\widetilde\eta_l*\eta_l=\eta_l$. Write

$$\Gamma^l_{k,m}(x,h,\omega)=\sum_{\nu=1}^m(-1)^{m-\nu}\binom{m}{\nu}\Gamma^l_{k,m,\nu}(x,h,\omega)$$

where

$$\Gamma^l_{k,m,\nu}(x,h,\omega)=2^{-n_l\sigma}\sum_{Q\in\mathcal{Q}(n_l)}\theta_Q(\omega)\int\phi_k(y)\widetilde\eta_l*\eta_l*\chi_Q(x+\tfrac{\nu}{m}(h-y))dy.$$

[2]We thank Tino Ullrich who tactfully raised questions about the validity of the proof in the published version of this paper.

This expression can be rewritten as

$$\Gamma^l_{k,m,\nu}(x,h,\omega) = 2^{-n_l\sigma}\int \mathcal{J}_{k,l}(x,h,z)F^\omega_l(z)dz$$

where

$$F^\omega_l(z) = \sum_{Q\in\mathcal{Q}(n_l)} \theta_Q(\omega)\eta_l * \chi_Q(z)$$

and

$$\mathcal{J}_{k,l}(x,h,z) = \int \phi_k(y)\widetilde{\eta}_l(x+\tfrac{\nu}{m}(h-y)-z)\,dy\,.$$

We analyze $\mathcal{J}_{k,l}(x,h,z)$. Fixing j we have the assumption $|h|\le m2^{-n_k+j+d}$. We then distinguish the case where $x-z$ is larger or smaller than $10m\cdot 2^{-n_k+j+d}$. In the first case we have $|x+\frac{\nu}{m}(h-y)-z)|\approx|x-z|$ (by the support property of ϕ_k) and thus

$$|\mathcal{J}_{k,l}(x,h,z)| \le C(N_1)\frac{2^{n_l d}}{(1+2^{n_l}|x-z|)^{N_1}} \text{ if } |x-z|\ge 10m\cdot 2^{-n_k+j+d}. \tag{6.35a}$$

In the opposite case we use that $\widetilde{\eta}_l$ has vanishing moments of arbitrary order. Thus we may subtract the Taylor polynomial of order N_0 of ϕ_k, expanded about

$$y_{x,h,z} = h-(x-z)\tfrac{m}{\nu}.$$

We shall later need $N_0 > d + d/q$. We now get

$$\mathcal{J}_{k,l}(x,h,z) = \int_0^1 \frac{(1-s)^{N_0-1}}{N_0!}\int \langle y-y_{x,h,z},\nabla\rangle^{N_0}\phi_k((1-s)y_{x,h,z}+sy)\widetilde{\eta}_l(x-z\frac{\nu}{m}(h-y))\,dy.$$

Note that $|x-z+\frac{\nu}{m}(h-y)| = \frac{\nu}{m}|y-y_{x,h,z}|$. Thus we get with $N_1 > N_0+d$

$$\begin{aligned}|\mathcal{J}_{k,l}(x,h,z)| &\lesssim 2^{n_k d}\int \frac{2^{n_k N_0}|y-y_{x,h,z}|^{N_0}}{(1+2^{n_l}|y-y_{x,h,z}|)^{N_1}}dy\\ &\lesssim 2^{-(n_l-n_k)(N_0-d)}\,.\end{aligned} \tag{6.35b}$$

Putting the two cases together we obtain

$$\begin{aligned}|\Gamma^l_{k,m}(x,h,\omega)| \lesssim\;& 2^{-n_l\sigma}2^{-(n_l-n_k)(N_0-d)}\sup_{|x-z|\le 10m\cdot 2^{-n_k+j+d}}|F^\omega_l(z)|\\ &+2^{-n_l\sigma}2^{-(n_l-n_k)(N_1-d)}\sum_{\tau=1}^\infty 2^{-(j+\tau)(N_1-d)}\sup_{|x-z|\le C_d 2^{-n_k+j+\tau}}|F^\omega_l(z)|\,.\end{aligned}$$

Thus (for sufficiently large C_d)

$$\begin{aligned}\sup_{|h|\le m2^{-n_k+j+d}}|\Gamma^l_{k,m}(x,h,\omega)| \lesssim\;& 2^{-n_l\sigma}2^{(n_k-n_l)(N_0-d)}\sup_{|u|\le C_d 2^{-n_k+j}}|F^\omega_l(x+u)|\\ &+2^{-n_l\sigma}2^{-(n_l-n_k)(N_1-d)}\sum_{\tau=1}^\infty 2^{-(j+\tau)(N_1-d)}\sup_{|u|\le C_d 2^{-n_k+j+\tau}}|F^\omega_l(x+u)|\,.\end{aligned} \tag{6.36}$$

By (6.9) we have for $\tau = 0,1,\dots$

$$\Big\|\sup_{|y|\le 2^{-n_k+j+\tau}}|F^\omega_l(\cdot+y)|\Big\|_q \lesssim 2^{(n_l-n_k+j+\tau)d/q}\|F^\omega_l\|_q \tag{6.37}$$

and hence

$$\Big\|\sup_{|h|\le m2^{-n_k+j+d}}|\Gamma^l_{k,m}(\cdot,h,\omega)|\Big\|_q \lesssim 2^{-n_l\sigma}2^{-(n_l-n_k)(N_0-d-d/q)}2^{jd/q}\|F^\omega_l\|_q\,.$$

Therefore the left hand side of (6.34) is dominated (when $q \geq 1$) by

$$\sum_{s=1}^{\infty}\Big(\int_\Omega\int\sum_{k=1}^{2^{Nd}}\sum_{j=M+d}^{N-M-d}2^{(n_k-j)\sigma q}\sup_{|h|\le m2^{-n_k+j+d}}\Big|\Gamma^{k+s}_{k,m}(x,h,\omega)\Big|^q dxd\mu\Big)^{1/q}$$

$$\lesssim\sum_{s=1}^{\infty}\Big(\int_\Omega\int\sum_{k=1}^{2^{Nd}}\sum_{j=M+d}^{N-M-d}2^{(n_k-j)\sigma q}\big[2^{-n_{k+s}\sigma}2^{-(n_{k+s}-n_k)(N_0-d-d/q)}2^{jd/q}\|F^\omega_{K+s}\|_q\big]^q d\mu\Big)^{1/q}$$

$$\lesssim\sum_{s=1}^{\infty}2^{-Rs(N_0-d-d/q+\sigma)}\Big(\sum_{j=M+d}^{N-M-d}2^{-j(\sigma q-d)}\sum_{k=1}^{2^{Nd}}\int_\Omega\|F^\omega_{k+s}\|_q^q d\mu\Big)^{1/q}.$$

Here we have used Minkowski's inequality, (6.36) and (6.37).

By Corollary 2.2 we have

$$\Big(\sum_{k=1}^{2^{Nd}}\int_\Omega\|F^\omega_{k+s}\|_q^q d\mu\Big)^{1/q}\lesssim C.$$

Therefore the left hand side of (6.34) is bounded by a constant times

$$\begin{cases}2^{-R}2^{(-\sigma+d/q)N} & \text{if } 0<\sigma<d/q\,,\\ 2^{-R}N^{1/q} & \text{if } \sigma=d/q\,.\end{cases}$$

If $q<1$ we use the ℓ^q triangle inequality in place of Minkowski's inequality and we have to bound

$$\Big(\sum_{s=1}^{\infty}\int_\Omega\int\sum_{k=1}^{2^{Nd}}\sum_{j=M+d}^{N-M-d}2^{(n_k-j)\sigma q}\sup_{|h|\le m2^{-n_k+j+d}}\Big|\Gamma^{k+s}_{k,m}(x,h,\omega)\Big|^q dxd\mu\Big)^{1/q}$$

and the outcome is the same. □

Given Lemma 6.2.2 the lower bound (6.33) follows from a corresponding lower bound for the expression only involving the $\Gamma^k_{k,m}(x,h,\omega)$ and it remains to show for $\sigma\le d/q$:

Lemma 6.2.3.

$$(6.38)\quad \Big(\sum_k\sum_{j=M+d}^{N-M-d}2^{-j\sigma q}\int_{[\frac14,\frac34]^d}\int_\Omega\sup_{|h|\le m2^{-n_k+j+d}}\Big|2^{n_k\sigma}\Gamma^k_{k,m}(x,h,\omega)\Big|^q d\mu dx\Big)^{1/q}$$
$$\geq c_2\max\{2^{N(\frac{d}{q}-\sigma)},N^{1/q}\}$$

for some $c_2>0$.

Proof. In what follows we fix $x\in[1/4,3/4]^d$ and $1\le k\le 2^{Nd}$. As in the proof of Theorem 1.2 define $V_k^N(x)$ to be the union of all dyadic cubes of sidelength 2^{-n_k+N+1} whose boundaries intersect the boundary of $Q_k^{N+1}(x)$. Let $\mathcal{V}_k^N(x)$ be the set of all $Q\in\mathcal{Q}(n_k)$ that are contained in the closure of $V_k^N(x)$. Denote by $\Omega(k,x,Q)$ the event that $\theta_Q(\omega)=1$, but $\theta_{Q'}(\omega)=0$ for all $Q'\in\mathcal{V}_k^N(x)\setminus\{Q\}$. For the probability of this event there is the lower bound $\mu(\Omega(k,x,Q))\ge c_d2^{-Nd}$; see (4.8).

Now let $\mathcal{W}(k,j,x)$ be the set of all cubes $Q\in\mathcal{Q}(n_k)$ for which

$$2^{-n_k+j}\le\operatorname{dist}(x,Q)\le 2^{-n_k+j+1}.$$

For $Q\in\mathcal{W}(k,j,x)$ denote by y_Q the center of Q and set $h_{Q,x}=y_Q-x$ so that $|h_{Q,x}|\lesssim 2^{-n_k+j+1}$.

Thus the left hand side of (6.38) is bounded below by

$$c\Big(\sum_k\sum_{j=M+d}^{N-M-d}2^{-j\sigma q}\int_{[\frac14,\frac34]^d}\sum_{Q\in\mathcal{W}(k,j,x)}\int_{\Omega(k,x,Q)}\Big|2^{n_k\sigma}\Gamma^k_{k,m}(x,h_{Q,x},\omega)\Big|^q d\mu dx\Big)^{1/q}. \tag{6.39}$$

For $Q\in\mathcal{W}(k,j,x)$ and $\omega\in\Omega(k,x,Q)$ we decompose further

$$2^{n_k\sigma}\Gamma^k_{k,m}(x,h_{Q,x},\omega)=\sum_{\nu=1}^m(-1)^{m-\nu}\binom{m}{\nu}I^\omega_\nu(k,x,Q)+II^\omega(k,x)$$

where

$$I^\omega_\nu(k,x,Q)=\int\phi_k(y)\eta_k*\chi_Q(x+\frac{\nu}{m}(h_{Q,x}-y))dy \tag{6.40}$$

and

$$II^\omega(k,x,Q)=\sum_{\substack{Q'\in\mathcal{Q}(n_k)\\ Q'\notin\mathcal{V}^N_k(x)}}\theta_{Q'}(\omega)\int\phi_k(y)\widetilde\Delta^m_{\underline{\frac{h(Q,x)-y}{m}}}\eta_l*\chi_{Q'}(x)dy. \tag{6.41}$$

We prove a lower bound for I^ω_m and upper bounds for II^ω and I^ω_ν, $\nu\le m-1$.

Notice that for $\omega\in\Omega(k,x,Q)$

$$\begin{aligned}I^\omega_m(k,x,Q)&=\int\phi_k(y)\eta_k*\chi_Q(x+h_{Q,x}-y)dy\\&=\int r_k^d(\eta*\phi)(r_k(y_Q-z))\chi_Q(z)dz\end{aligned}$$

and since $|r_k(y_Q-z)|\le\sqrt{d}2^{-n_k}\cdot2^{n_k-M}$ for $z\in Q$ it follows from (6.19) that

$$|I^\omega_m(k,x,Q)|\ge c(M)r_k^d2^{-n_kd}\ge c'(M)\qquad\text{if }\omega\in\Omega(k,x,Q),\ Q\in\mathcal{W}(k,j,x). \tag{6.42}$$

Since $\mu(\Omega(k,x,Q))\ge c2^{-Nd}$ and $\operatorname{card}(\mathcal{W}(k,j,x))\approx2^{jd}$,

$$\begin{aligned}&\Big(\sum_{k=1}^{2^{Nd}}\sum_{j=M+d}^{N-M-d}2^{-j\sigma q}\int_{[\frac14,\frac34]^d}\sum_{Q\in\mathcal{W}(k,j,x)}\int_{\Omega(k,x,Q)}|I^\omega_m(k,x,Q)|^q d\mu dx\Big)^{1/q}\\&\gtrsim\Big(\int_{[\frac14,\frac34]^d}\sum_{k=1}^{2^{Nd}}2^{-Nd}\sum_{j=M+d}^{N-M-d}2^{j(d-\sigma q)}\Big)^{1/q}\gtrsim c_q\big(\max\{2^{N(d-\sigma q)},N\}\big)^{1/q}.\end{aligned} \tag{6.43}$$

Next notice that for $\omega\in\Omega(k,x,Q)$, $Q\in\mathcal{W}(k,j,x)$, $y\in\operatorname{supp}\phi_k$ and $\nu\le m-1$,

$$|x+(h_{Q,x}-y)\nu/m-y_Q|\ge|x-y_Q|(1-\nu/m)-\nu/m|y|\gtrsim2^{-n_k+j}$$

which shows that $|\eta_k*\chi_Q(x+\frac{\nu}{m}(h_{Q,x}-y))|\le C_\rho2^{-j\rho}$ for all ρ and consequently we get the estimate

$$|I^\omega_\nu(k,x,Q)|\le C_{M,\rho}\mathfrak{M}_{\rho,2^{-n_k}}[h^\omega_k],\quad\nu\le m-1.$$

Similarly for $II^\omega(k,x,Q)$ we can argue as for the corresponding term in the proof of Theorem 1.2 and see that

$$\sup_{2\le j\le N}\big|II^\omega(k,x)\big|\le C_{M,\rho}\mathfrak{M}_{\rho,2^{-n_k}}[h^\omega_k]\qquad\text{if }\omega\in\Omega(k,x,Q)$$

for any $\rho > 0$. Thus

$$\Big(\sum_{k=1}^{2^{Nd}}\sum_{j=M+d}^{N-M-d} 2^{-j\sigma q}\int_{[\frac14,\frac34]^d}\sum_{Q\in\mathcal{W}(k,j,x)}\int_{\Omega(k,x,Q)}\Big[\sum_{\nu=1}^{m-1}|I_\nu^\omega(k,x,Q)|+|II^\omega(k,x,Q)|\Big]^q d\mu dx\Big)^{1/q}$$

(6.44)

$$\lesssim \Big(\sum_{k=1}^{2^{Nd}}\int_\Omega\int \big(\mathfrak{M}_{\rho,2^{-n_k}}h_k^\omega\big)^q dx d\mu\Big)^{1/q}\le C$$

by Lemma 2.1. We combine the estimates (6.43) and (6.44) to see that the expression (6.38) is bounded below by $\max\{2^{N(d-\sigma/q)}, N^{1/q}\}$. □

We now combine the various estimates and note that the lower bound in Lemma 6.2.3 is independent of R. Thus if R is chosen to be sufficiently large, the upper bounds in (6.34) can be absorbed by the lower bound (6.38), and (6.33) consequently follows. All told, we have shown the lower bound $\mathcal{A}_{p,q,\sigma}(2^{R2^{Nd}}) \gtrsim \max\{2^{N(d-\sigma/q)}, N^{1/q}\}$, which implies the asserted lower bound for large $r = 2^{R2^{Nd}}$. □

Remark. One can also consider the more regular variant

$$\mathfrak{D}^{\sigma,q}_{s,m}f(x) = \Big(\int_0^1\Big[\fint_{|h|\le t}|\Delta_h^m f(x)|^s dh\Big]^{q/s} t^{-1-\sigma q}dt\Big)^{1/q}.$$

where the slashed integral denotes the average over the ball $\{h : |h| \le t\}$. Then $\|f\|_{F^p_{\sigma,q}} \approx \|f\|_p + \|\mathfrak{D}^{\sigma,q}_{s,m}f\|_p$ provided that $\sigma > \max\{0, d(1/p-1/s), d(1/q-1/s)\}$. A modification of our argument shows that the characterization fails when $\sigma \le d(1/q - 1/s)$.

7. Proof of Theorem 1.4

Let η be a Schwartz function in $\mathcal{E}(2)$ such that $\widehat{\eta}$ vanishes identically in a neighborhood of the origin and $\widehat{\eta}(\xi) = 1$ if $2^{-1/2} \le |\xi| \le 2^{1/2}$. In what follows we fix $q < p \le 2$ and assume that $0 < \gamma < 1$ and that $b = \gamma d(1/p - 1/2)$. Define for $k > 1$ the operator T_k by

$$\widehat{T_k f}(\xi) = e^{i|\xi|^\gamma}\widehat{\eta}(2^{-k}\xi)\widehat{f}(\xi). \tag{7.1}$$

It is easy to see, using (1.5), that the statement of the Theorem is equivalent with the statement that the best constant $\mathcal{A}_L$ in the inequality

$$\Big\|\Big(\sum_{k=1}^{L}|2^{-kb}T_k f_k|^q\Big)^{1/q}\Big\|_p \le \mathcal{A}_L\Big\|\Big(\sum_{k=1}^{L}|f_k|^q\Big)^{1/q}\Big\|_p, \qquad \text{with } f_k\in\mathcal{E}(2^{k+1}), \tag{7.2}$$

satisfies $\mathcal{A}_L \approx L^{1/q-1/p}$. As the operators $2^{-kb}T_k$ map $L^p \cap \mathcal{E}(2^{k+1})$ to L^p, with bounds uniform in $k > 0$ (*cf.* [6]), the upper bound $\mathcal{A}_L \lesssim L^{1/q-1/p}$ is immediate by Hölder's inequality and the embedding $\ell^q \subset \ell^p$. In what follows we prove the lower bound.

We use a variant of the random construction of §2 and define $\theta_{Q,a}$ and $h_k^{\omega,a}$ as in (2.1); however we now let a depend on k and require that

$$a_k = 2^{-k\gamma d}. \tag{7.3}$$

Also let $\widetilde{\eta}$ be a Schwartz function in $\mathcal{E}(2)$ whose Fourier transform equals 1 on the support of $\widehat{\eta}$. Define (using the notation in §2 with $n_k = k$)

$$f_k^\omega(x) = \beta_k\sum_{Q\in\mathcal{Q}(k)}\theta_{Q,a_k}(\omega)\widetilde{\eta}(2^k(x-x_Q)) \tag{7.4}$$

where x_Q is the center of Q and

$$\beta_k = a_k^{-1/p}. \tag{7.5}$$

We claim

$$\Big(\int_\Omega \Big\|\Big(\sum_{k=1}^L |f_k^\omega|^q\Big)^{1/q}\Big\|_p^p d\mu(\omega)\Big)^{1/p} \lesssim L^{1/p}; \tag{7.6}$$

this inequality will use only (7.5) and the fact that the β_k increase at least in a geometric progression; the specific choice (7.3) is not yet needed.

A straightforward estimate yields $|f_k^\omega(x)| \le C_s\big(M_{HL}(|\beta_k h_k^{\omega,a_k}|^s)\big)^{1/s}$ for all $s>0$ and therefore it suffices to prove (7.6) with f_k^ω replaced by $\beta_k h_k^{\omega,a_k}$.

Now since h_k^{ω,a_k} takes only values 1 and 0 and the β_k increase at least geometrically we see that for all x

$$\Big(\sum_{k=1}^L |\beta_k h_k^{\omega,a_k}(x)|^q\Big)^{1/q} \le C_\gamma \sup_{1\le k\le L} |\beta_k h_k^{\omega,a_k}(x)|,$$

with a finite constant C_γ independent of x. After replacing the supremum by an ℓ^p norm we see that the left hand side of (7.6) can be estimated by

$$\Big(\int_{[0,1]^d}\sum_{k=1}^L\int_\Omega \big|\beta_k h_k^{\omega,a_k}\big|^p d\mu(\omega)dx\Big)^{1/p} \le C\Big(\int_{[0,1]^d}\sum_{k=1}^L \beta_k^p a_k dx\Big)^{1/p} = CL^{1/p}.$$

It remains to show the lower bound

$$\Big(\int_\Omega \Big\|\Big(\sum_{k=1}^L |T_k f_k^\omega|^q\Big)^{1/q}\Big\|_p^p d\mu\Big)^{1/p} \ge cL^{1/q}. \tag{7.7}$$

Now let K_k be the convolution kernel of T_k. Then

$$T_k f_k^\omega(x) = \beta_k \sum_{Q\in\mathcal{Q}(k)} \theta_{Q,a_k}(\omega) 2^{-kd} K_k(x-x_Q).$$

A stationary phase calculation shows that for suitable $\varepsilon_1>0$ there is the uniform estimate for large k

$$|K_k(x)| \ge 2^{k(d-d\gamma/2)} \quad \text{if } (1-\varepsilon_1)2^{-k(1-\gamma)} \le |x| \le (1+\varepsilon_1)2^{-k(1-\gamma)}; \tag{7.8}$$

moreover for any $\rho<\infty$

$$|K_k(x)| \le C_\rho 2^{kd}(2^k|x|)^{-\rho} \text{ if } |x| \ge B2^{-k(1-\gamma)} \tag{7.9}$$

for suitable B (≥ 2); this is seen by using integration by parts for the oscillatory integral, which has a nonstationary phase when $|x|\ge B2^{-k(1-\gamma)}$. Now apply Hölder's inequality (as in all previous examples):

$$\Big(\int_\Omega\int_{[0,1]^d}\Big(\sum_{k=1}^L \big|2^{-kb}T_k f_k^\omega\big|^q\Big)^{p/q} dx d\mu\Big)^{1/p}$$
$$\ge \Big(\int_{[\frac14,\frac34]^d}\int_\Omega \sum_{k=1}^L \Big|2^{-k(d+b)}\beta_k \sum_{Q\in\mathcal{Q}(k)} \theta_{Q,a_k}(\omega)K_k(x-x_Q)\Big|^q d\mu dx\Big)^{1/q}.$$

Fix $x\in[1/4,3/4]^d$ and let $\mathcal{V}_{k,\gamma}(x)$ be the set of all cubes $Q\in\mathcal{Q}(k)$ whose distance to x is $\le C_2 2^{-k(1-\gamma)}$ where $C_2 \gg B$ for a sufficiently large constant B. Let $\Omega(k,x,Q)$ be the event that $\theta_Q(\omega) = 1$ but $\theta_{Q'}(\omega) = 0$ for all $Q' \in \mathcal{V}_{k,\gamma}(x) \setminus \{Q\}$. The probability of this event satisfies

$$\mu(\Omega(k, x, Q)) \geq a_k(1 - a_k)^{\mathrm{card}(\mathcal{V}_{k,\gamma}(x))-1} \geq ca_k,$$

by our choice (7.3).

By the upper bound (7.9) we get

$$\int_\Omega \Big| 2^{-k(d+b)}\beta_k \sum_{Q\in\mathcal{Q}(k)\setminus\mathcal{V}_{k,\gamma}(x)} \theta_{Q,a_k}(\omega)K_k(x - x_Q)\Big|^q d\mu(\omega)$$
$$\lesssim 2^{kd} \int_{|z-x|\geq 2^{-k(1-\gamma)}} (\beta_k 2^{-k(b+d)})^q (2^k|z - x|)^{-\rho q}\, dz \leq \beta_k^q 2^{-kbq}2^{k\gamma(d-\rho q)}$$

provided that $\rho > d/q$. Consequently by choosing ρ large enough we find that

$$\Big(\int_{[\frac14,\frac34]^d} \int_\Omega \sum_{k=1}^{L} \Big|2^{-k(d+b)}\beta_k \sum_{Q\in\mathcal{Q}(k)\setminus\mathcal{V}_{k,\gamma}(x)} \theta_{Q,a_k}(\omega)K_k(x - x_Q)\Big|^q d\mu dx\Big)^{1/q} \leq C \tag{7.10}$$

uniformly in L.

By (7.10) we may estimate

$$\Big(\int_{[\frac14,\frac34]^d} \int_\Omega \sum_{k=1}^{L} \Big|2^{-k(d+b)}\beta_k \sum_{Q} \theta_{Q,a_k}(\omega)K_k(x - x_Q)\Big|^q d\mu dx\Big)^{1/q}$$
$$\geq \Big(\int_{[\frac14,\frac34]^d} \sum_{k=1}^{L} \int_\Omega \Big|2^{-k(d+b)}\beta_k \sum_{Q'\in\mathcal{V}_{k,\gamma}(x)} \theta_{Q',a_k}(\omega)K_k(x - x_{Q'})\Big|^q d\mu dx\Big)^{1/q} - C$$

and the main term is

$$\geq \Big(\int_{[\frac14,\frac34]^d} \sum_{k=1}^{L} \sum_{Q\in\mathcal{V}_{k,\gamma}(x)} \int_{\Omega(k,x,Q)} \Big|2^{-k(d+b)}\beta_k \sum_{Q'\in\mathcal{V}_{k,\gamma}(x)} \theta_{Q',a_k}(\omega)K_k(x - x_{Q'})\Big|^q d\mu dx\Big)^{1/q}$$
$$= \Big(\int_{[\frac14,\frac34]^d} \sum_{k=1}^{L} \sum_{Q\in\mathcal{V}_{k,\gamma}(x)} \int_{\Omega(k,x,Q)} \Big|2^{-k(d+b)}\beta_k K_k(x - x_Q)\Big|^q d\mu dx\Big)^{1/q}. \tag{7.11}$$

Now let $\mathcal{W}_{k,\gamma}(x)$ be the family of all cubes in $\mathcal{Q}(k)$ which are contained in the set $\{y : (1 - \varepsilon_1)2^{-k(1-\gamma)} \leq |x - y| \leq (1 + \varepsilon_1)2^{-k(1-\gamma)}\}$. These cubes are also in $\mathcal{V}_{k,\gamma}(x)$ and if $Q \in \mathcal{W}_{k,\gamma}(x)$ then we may use the lower bound (7.8) for the term $K_k(x - x_Q)$. Note also that $\mathrm{card}(\mathcal{W}_{k,\gamma}(x)) \gtrsim 2^{kd\gamma} \approx a_k^{-1}$ for large k.

Thus the term (7.11) is bounded below for large L by

$$\Big(\int_{[\frac14,\frac34]^d} \sum_{k=1}^{L} \sum_{Q\in\mathcal{W}_{k,\gamma}(x)} \int_{\Omega(k,x,Q)} \big[2^{-k(d+b)}\beta_k 2^{k(d-\gamma d/2)}\big]^q d\mu\, dx\Big)^{1/q}$$
$$\geq c\Big(\int_{[\frac14,\frac34]^d} \sum_{k=C}^{L} \mathrm{card}(\mathcal{W}_{k,\gamma}(x))a_k dx\Big)^{1/q} \gtrsim L^{1/q}$$

and consequently we obtain (7.7). □

Remark: The case $\gamma = 1$ which is relevant for the wave equation is an exceptional case (see [8], [10]), as the critical b is given by $b = (d - 1)(1/p - 1/2)$, $1 < p \leq 2$. However if these parameters are chosen in Theorem 1.4 then a modification of the above argument, with $a_k = 2^{-k(d-1)}$, shows that (1.13) remains valid.

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