

CRM NOTES ON SINGULAR INTEGRALS AND A PROBLEM ON MIXING FLOWS

Acknowledgement. Sections 1-4 of these lecture notes rely on material from joint work with Charles Smart and Brian Street [37], and work in progress with Mahir Hadžić, Charles Smart, and Brian Street [24].

1. MIXING FLOWS AND BRESSAN'S PROBLEM

1.1. *Mixing of subsets in $\mathbb{T}^d$.* Let $r \in (0, 1/2)$, $x \in \mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$ the d -dimensional torus, and let $B_r(x)$ denote the ball of radius r centered at x , with respect to the natural geodesic distance on $\mathbb{T}^d$. A measurable set $E \subset \mathbb{T}^d$ is mixed at scale r , with mixing constant $\kappa \in (0, 1/2)$ if

$$(1) \quad \kappa \leq \frac{|E \cap B_r(x)|}{|B_r(x)|} \leq 1 - \kappa \text{ for all balls } B_r(x).$$

1.2. *Vector fields on $\mathbb{T}^d$.* Let v be a time-dependent a priori smooth vector field, defined on $\mathbb{T}^d \times [0, T]$ with values in the tangent bundle of the torus. The vector field can be considered a vector field $(x, t) \mapsto v(x, t)$ on $\mathbb{R}^d$ which is periodic in x , i.e.

$$v(x + k, t) = v(x, t) \text{ for all } (x, t) \in \mathbb{R}^d \times \mathbb{R}, \ k \in \mathbb{Z}^d.$$

We assume that v is divergence-free

$$\sum_{i=1}^d \frac{\partial v_i}{\partial x_i} = 0.$$

Let ϕ be the flow generated by v , satisfying

$$\begin{aligned} \frac{\partial}{\partial t} \phi(x, t) &= v(\phi(x, t), t), \\ \phi(x, 0) &= x. \end{aligned}$$

For every t the map $x \mapsto \phi(x, t)$ is a diffeomorphism on $\mathbb{R}^d$ satisfying

$$\phi(x + k, t) - k = \phi(x, t),$$

for all $x \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$. In what follows we shall also use the notation $\phi_t(x) = \phi(x, t)$. Since v is divergence free the flow is incompressible, i.e. ϕ_t preserves the volume.

1.3. *Bressan's problem.* Split $\mathbb{T}^d$ as $\Omega_L \cup \Omega_R$ with $\Omega_R = \Omega_L^c$ where

$$(2) \quad \Omega_L = \{x : 0 \leq x_1 < \frac{1}{2}\}, \quad \Omega_R = \{x : \frac{1}{2} \leq x_1 < 1\}.$$

Let $0 < \varepsilon < 1/4$. Consider a periodic flow generated by a smooth time dependent divergence free vector field, and assume that at time $t = T$ the flow mixes up Ω_L at scale ε , i.e. the set

Date: May 9, 2016.

2010 Mathematics Subject Classification. 42B20.

Supported in part by the National Science Foundation.

$E = \Phi_T(\Omega_L)$ satisfies (1) with $r = \varepsilon$. Bressan [5] asks (setting $\kappa = 1/3$) whether *there is a universal constant $c > 0$ (independent of ε) such that*

$$(3) \quad \int_0^T \int_{[0,1]^d} |D_x v(x, t)| dx dt \geq c \log(1/\varepsilon).$$

As noted in [5] it suffices to consider the case $T = 1$, replacing $v(x, t)$ by $Tv(x, t/T)$. In [4], Bressan formulated a more general conjecture for mildly compressible flows.

Bressan's conjecture is still open, at the time of this writing. Therefore it is of interest to ask for corresponding lower bounds if the $L^1(\mathbb{T}^d)$ norm is replaced by a larger norm (for example an $L^p(\mathbb{T})$ norm). I.e. under the assumption that the flow generated by v mixes the set up at scale ε with mixing constant γ , do we have a universal lower bound of the form

$$(4) \quad \int_0^T \int_{[0,1]^d} \|D_x v(\cdot, t)\|_{\mathcal{Y}} dt \geq c_{\mathcal{Y}}(\kappa) \log(1/\varepsilon)?$$

for suitable function spaces $\mathcal{Y} \subset L^1(\mathbb{T}^d)$ or even $\mathcal{Y} \subset M(\mathbb{T}^d)$ with $M(\mathbb{T}^d)$ the space of bounded Borel measures on $\mathbb{T}^d$. Crippa and DeLellis [15] showed this for $\mathcal{Y} = L^p(\mathbb{T}^d)$, $1 < p < \infty$ and also for the space $\mathcal{Y}$ consisting of functions for which the Hardy-Littlewood maximal function Mf belongs to $L^1(\mathbb{T}^d)$, i.e. for $\mathcal{Y} = L \log L(\mathbb{T}^d)$. The norm in $L \log L$ is given by

$$(5) \quad \|f\|_{L \log L} = \inf \left\{ t > 0 : \int \frac{|f(x)|}{t} \log \left(e + \frac{|f(x)|}{t} \right) dx \leq 1 \right\}.$$

Some model classes for which the conjecture is true and optimal are considered in [24]. It should also be noted that the lower bound $\log(1/\varepsilon)$ cannot be improved by working with L^p spaces, see recent results by Yao and Zlatoš [43] and by Alberti, Crippa and Mazzucato [1].

1.4. On the result by Crippa and de Lellis. We formulate a slight extension of their result. It involves the local Hardy space $h^1(\mathbb{T}^d)$ which contains $L \log L(\mathbb{T}^d)$ as a proper subspace.

Theorem 1.1. *Let v satisfy the hypothesis in Bressan's conjecture. Then*

$$(6) \quad \int_0^T \|D_x v(\cdot, t)\|_{h^1(\mathbb{T}^d)} dt \geq c \log(1/\varepsilon).$$

We follow the elegant method outlined in [29]. For

$$(7) \quad \varepsilon \leq 2^{-6/\kappa}$$

let

$$\mathcal{V}(\varepsilon) = \{(x, y) : x \in \Omega_L, y \in \Omega_R, |\phi_T(x) - \phi_T(y)| \leq \varepsilon\}$$

and

$$\Gamma(\varepsilon) = \iint_{\mathcal{V}(\varepsilon)} \log \left(\frac{|x - y|}{|\phi_T(x) - \phi_T(y)|} \right) dx dy.$$

Let Ω_d be the d -dimensional volume of the unit ball in $\mathbb{R}^d$ and

$$\mathcal{V}_x(\varepsilon) = \{y : (x, y) \in \mathcal{V}(\varepsilon)\}, \quad \mathcal{V}^y(\varepsilon) = \{x : (x, y) \in \mathcal{V}(\varepsilon)\}.$$

Then for all x, y

$$(8) \quad \begin{aligned} |\mathcal{V}^x(\varepsilon)| &\leq \Omega_d \varepsilon^d \\ |\mathcal{V}^y(\varepsilon)| &\leq \Omega_d \varepsilon^d \end{aligned}$$

by the incompressibility of the flow.

It is not hard to see that

$$\left| \iint_{\mathcal{V}(\varepsilon)} \log \left(\frac{1}{|x-y|} \right) dx dy \right| \lesssim \varepsilon^d.$$

On the other hand, by the mixing condition at time T and the incompressibility of the flow,

$$\left| \iint_{\mathcal{V}(\varepsilon)} \log \left(\frac{1}{|\phi_T(x) - \phi_T(y)|} \right) dx dy \right| \gtrsim \kappa \varepsilon^d \log(1/\varepsilon).$$

Hence for small ε

$$(9) \quad \Gamma(\varepsilon) \geq \frac{\kappa}{2} \varepsilon^d \log(1/\varepsilon).$$

We write

$$\Gamma(\varepsilon) = \iint_{\mathcal{V}(\varepsilon)} \int_0^T \frac{d}{dt} \log \left(\frac{1}{|\phi_t(x) - \phi_t(y)|} \right) dt dx dy$$

and use the differential equation to see that

$$\left| \frac{d}{dt} \log \left(\frac{1}{|\phi_t(x) - \phi_t(y)|} \right) \right| \lesssim \frac{|v(\phi_t(x), t) - v(\phi_t(y), t)|}{|\phi_t(x) - \phi_t(y)|}.$$

To handle this term one proves that for vector fields $x \mapsto b(x)$

$$(10) \quad \frac{|b(x) - b(y)|}{|x - y|} \leq \mathfrak{M}b(x) + \mathfrak{M}b(y)$$

where

$$(11) \quad \|\mathfrak{M}b\|_{L^1} \lesssim \|\nabla b\|_{h_1}.$$

with h^1 the local Hardy space.

It suffice to assume that $|x - y| \leq 10^{-2}$. Now let $\phi \in C_c^\infty$ supported on $\{y : |y| \leq 1/4\}$ such that $\int \phi(y) dy = 1$, and $\int y_i \phi(y) dy = 0$ for $i = 1, \dots, n$. Let $\phi_k(x) = 2^{kd} \phi(2^k x)$, and

$$\psi_k = \phi_{k+1} - \phi_k$$

so that for any $\ell > 0$, $b = \phi_\ell * b + \sum_{k=\ell}^\infty \psi_k * b$. Now assume $2^{-\ell-1} \leq |x - y| \leq 2^{-\ell}$.

$$\begin{aligned} \frac{|\phi_\ell * b(x) - \phi_\ell * b(y)|}{|x - y|} &= \left| \left\langle \frac{x - y}{|x - y|}, \int_0^1 \phi_\ell * \nabla b((1-s)x + sy) ds \right\rangle \right| \\ &\leq \mathcal{M}_0(\nabla b)(x) + \mathcal{M}_0(\nabla b)(y) \end{aligned}$$

where

$$\mathcal{M}_0 g(x) = \sup_{\ell > 4} \sup_{|h| \leq 2^{-\ell}} |\phi_\ell * g(x + h)|.$$

By standard Hardy space theory,

$$\|\mathcal{M}_0 g\|_{L^1} \lesssim \|g\|_{h_1}$$

(which will be applied here to $g = \partial b / \partial x_j$).

Secondly, for $k \geq \ell$,

$$\begin{aligned} \frac{|\psi_k * b(x) - \psi_k * b(y)|}{|x - y|} &\leq 2^{\ell+2} \sup_k (|\psi_k * b(x)| + |\psi_k * b(y)|) \\ &\leq M_1 b(x) + M_1 b(y) \end{aligned}$$

with

$$M_1 b(x) = \sup_{k > 0} 2^k |\psi_k * b(x)|.$$

Now, by the cancellation property of ψ , $\int \psi(y)l(y)dy = 0$ for all affine linear functions l , we have

$$\|M_1 b\|_1 \leq \left\| \left(\sum_{k=1}^{\infty} [2^k |\psi_k * b|]^2 \right)^{1/2} \right\|_1 \lesssim \|\nabla b\|_{h^1}$$

which gives (10) with

$$\mathfrak{M}b(x) = \mathcal{M}_0(\nabla b) + M_1(b).$$

Now set $b^t(x) := v(x, t)$. We have

$$\begin{aligned} \Gamma(\varepsilon) &\lesssim \int_0^T \iint_{\mathcal{V}(\varepsilon)} (\mathfrak{M}b^t(\phi_t(x)) + \mathfrak{M}b^t(\phi_t(y))) dx dy dt \\ &\lesssim \varepsilon^d \int_0^T \int \mathfrak{M}b^t(\phi_t(x)) dx dt \lesssim \varepsilon^d \int_0^T \int \mathfrak{M}b^t(x) dx dt \\ &\lesssim \varepsilon^d \int_0^T \|\nabla_x v(\cdot, t)\|_{h^1} dt. \end{aligned}$$

We compare this to (9) and obtain for sufficiently small ε

$$\int_0^T \|\nabla_x v(\cdot, t)\|_{h^1} dt \gtrsim \log(1/\varepsilon).$$

1.5. *An approach to Bressan's problem via the Bianchini semi-norm.* For $\varepsilon < 1/8$ we define the truncated Bianchini semi-norm by

$$\|f\|_{\mathcal{B}(\varepsilon)} = \int_{\varepsilon}^{1/4} \int_{\mathbb{T}^d} |f(x) - \int_{B_r(x)} f(y) dy| dx \frac{dr}{r}$$

and the *Bianchini space* be consist of all $L^1(\mathbb{T}^d)$ functions for which

$$\|f\|_{\mathcal{B}} = \sup_{\varepsilon < 1/4} \|f\|_{\mathcal{B}(\varepsilon)}.$$

is finite.

This space was proposed in [3] as a measure for mixing in a one-dimensional shuffling problem. There it was denoted $\dot{B}^{0,1,1}$ in reference to Besov although this space does not actually belong to the usual scale of Besov spaces.

Observation: If E is mixed at scale $\varepsilon > 0$ then

$$\left| \mathbb{1}_E(x) - \int_{B_r(x)} \mathbb{1}_E(y) dy \right| \geq \kappa \text{ a.e. } r > \varepsilon.$$

Hence integrating in r and x we get

$$\|\mathbb{1}_E\|_{\mathcal{B}} \gtrsim \log(1/\varepsilon).$$

Also by straightforward computation

$$\|\mathbb{1}_{A_0}\|_{\mathcal{B}} \lesssim 1$$

for A_0 as in (2). Thus the following result gives an alternative proof of the Crippa-de Lellis $L \log L$ lower bound.

Theorem 1.2. *Let v, ϕ be as above. Then for any measurable subset $A \subset \mathbb{T}^d$*

$$\|\mathbb{1}_{\phi_T(A)}\|_{\mathcal{B}} \leq \|\mathbb{1}_A\|_{\mathcal{B}} + C_{\mathcal{Y}} \int_0^T \|Dv(\cdot, t)\|_{\mathcal{Y}} dt$$

for $\mathcal{Y} = L^p(\mathbb{T}^d)$, $1 < p < \infty$, or $\mathcal{Y} = L \log L(\mathbb{T}^d)$.

Given $A \subset \mathbb{T}^d$ we define

$$f_A(x) = \mathbb{1}_A(x) - \mathbb{1}_{A^c}(x).$$

Since constants (and thus $\mathbb{1}_A + \mathbb{1}_{A^c}$) have semi-norm equal to 0 in $\mathcal{B}(\varepsilon)$ we have

$$(12) \quad \|\mathbb{1}_A\|_{\mathcal{B}(\varepsilon)} = \frac{1}{2} \|f_A\|_{\mathcal{B}(\varepsilon)}.$$

Finally given for a periodic vector field b and a function g on $\mathbb{T}^d$ we define

$$(13) \quad \mathfrak{S}_\varepsilon[g, b](x) = \int_{\varepsilon \leq |x-y| \leq 1/4} \frac{\langle x-y, b(x) - b(y) \rangle}{|x-y|^{d+2}} g(y) dy.$$

Proposition 1.3. *Let v, ϕ be as above. Then*

$$\|\mathbb{1}_{\phi_T(A)}\|_{\mathcal{B}_\varepsilon} - \|\mathbb{1}_A\|_{\mathcal{B}_\varepsilon} = \frac{1}{2V_d} \int_0^T \int_Q f_{\phi_t(A)}(x) \mathfrak{S}_\varepsilon[f_{\phi_t(A)}, v(\cdot, t)](x) dx dt$$

Proof. We compute using the incompressibility of the flow,

$$\begin{aligned} & \|f_A \circ \phi_T^{-1}\|_{\mathcal{B}(\varepsilon)} - \|f_A\|_{\mathcal{B}(\varepsilon)} \\ &= \int_\varepsilon^{1/4} \left[\int_{\mathbb{T}} |f_A(\phi_T^{-1}(x)) - \int_{B_r(x)} f_A(\phi_T^{-1}(y)) dy| dx \right. \\ & \quad \left. - \int |f_A(x) - \int_{B_r(x)} f_A(y) dy| dx \right] \frac{dr}{r} \\ &= \int_\varepsilon^{1/4} \left[\int_{\mathbb{T}} |f_A(z) - \int_{\phi_T^{-1}B_r(\phi_T(z))} f_A(w) dw| dz \right. \\ & \quad \left. - \int |f_A(x) - \int_{B_r(x)} f_A(y) dy| dx \right] \frac{dr}{r}. \end{aligned}$$

Now $f_A(x) = 1$ for $x \in A$ and $f_A(x) = -1$ for $x \in A^c$. Thus from the above we obtain

$$\begin{aligned} & \|f_A \circ \phi_T^{-1}\|_{\mathcal{B}(\varepsilon)} - \|f_A\|_{\mathcal{B}(\varepsilon)} = \\ & \int_\varepsilon^{1/4} \left\{ \int_A [f_A(x) - \int_{\phi_T^{-1}B_r(\phi_T(x))} f_A(y) dy] dx \right. \\ & \quad - \int_A [f_A(x) - \int_{B_r(x)} f_A(y) dy] dx \\ & \quad + \int_{A^c} [\int_{\phi_T^{-1}B_r(\phi_T(x))} f_A(y) dy - f_A(x)] dx \\ & \quad \left. - \int_{A^c} [\int_{B_r(x)} f_A(y) dy - f_A(x)] dx \right\} \frac{dr}{r} \end{aligned}$$

and this implies

$$\begin{aligned} & \|f_A \circ \phi_T^{-1}\|_{\mathcal{B}(\varepsilon)} - \|f_A\|_{\mathcal{B}(\varepsilon)} \\ &= \int_\varepsilon^{1/4} \int f_A(x) \left[\int_{\phi_T^{-1}B_r(\phi_T(x))} f_A(y) dy - \int_{B_r(x)} f_A(y) dy \right] dx \frac{dr}{r} \\ (14) \quad &= \int f_A(x) \int_0^T \frac{d}{dt} \left[\int_\varepsilon^{1/4} \int_{\phi_t^{-1}B_r(\phi_t(x))} f_A(y) dy \frac{dr}{r} \right] dt dx \end{aligned}$$

Now let V_d denote the measure of the unit ball in $\mathbb{R}^d$. Then

$$\begin{aligned} & \int_{\varepsilon}^{1/4} \int_{\phi_t^{-1}B_r(\phi_t(x))} f_A(y) dy \frac{dr}{r} \\ &= V_d^{-1} \int_{\varepsilon}^{1/4} r^{-d-1} \int_{y: |\phi_t(x) - \phi_t(y)| \leq r} f_A(y) dy dr \\ &= \int H_{\varepsilon}(\phi_t(x) - \phi_t(y)) f_A(y) dy \end{aligned}$$

where

$$H_{\varepsilon}(u) = \begin{cases} d^{-1}V_d^{-1}(\varepsilon^{-d} - (1/4)^{-d}) & \text{if } |u| \leq \varepsilon \\ d^{-1}V_d^{-1}(|u|^{-d} - (1/4)^{-d}) & \text{if } \varepsilon < |u| \leq 1/4 \\ 0 & \text{if } |u| > 1/4 \end{cases}$$

Now H_{ε} is a Lipschitz function, thus has a bounded gradient given by

$$\nabla H_{\varepsilon}(u) = -V_d^{-1} \frac{u}{|u|^{d+2}} \chi_{\mathcal{A}(\varepsilon, 1/4)}(u)$$

where $\mathcal{A}(\varepsilon, 1/4)(u) = \{u \in \mathbb{R}^d : \varepsilon \leq |u| \leq 1/4\}$. Thus

$$\begin{aligned} & \frac{d}{dt} \left[\int_{\varepsilon}^{1/4} \int_{\phi_t^{-1}B_r(\phi_t(x))} f_A(y) dy \frac{dr}{r} \right] \\ &= \int \left\langle \frac{d}{dt}(\phi_t(x) - \phi_t(y)), \nabla H_{\varepsilon}(\phi_t(x) - \phi_t(y)) \right\rangle f_A(y) dy \\ &= \int_{\varepsilon \leq |\phi_t(x) - \phi_t(y)| \leq \frac{1}{4}} f_A(y) \frac{\langle v(\phi_t(x), t) - v(\phi_t(y), t), \phi_t(x) - \phi_t(y) \rangle}{V_d |\phi_t(x) - \phi_t(y)|^{d+2}} dy dx. \end{aligned}$$

Using this in (14) and changing variables we obtain

$$\begin{aligned} & \|f_{\phi_T(A)}\|_{\mathcal{B}(\varepsilon)} - \|f_A\|_{\mathcal{B}(\varepsilon)} \\ &= \int_0^T \iint_{\substack{(x,y): \\ \varepsilon \leq |x-y| \leq \frac{1}{4}}} f_{\phi_t(A)}(x) f_{\phi_t(A)}(y) \frac{\langle v(x, t) - v(y, t), x - y \rangle}{V_d |x - y|^{d+2}} dy dx dt. \end{aligned}$$

□

In order to complete the proof of Theorem 1.2 it suffices to prove, for divergence free vector fields b , the inequality

$$(15) \quad \left| \int_{\mathbb{T}^d} \mathfrak{S}_{\varepsilon}(\mathbb{1}_A, b) \mathbb{1}_B(x) dx \right| \lesssim \|Db\|_{L \log L(\mathbb{T})}$$

for measurable subsets $A, B \subset \mathbb{T}^d$ and apply Proposition 1.3. Without loss of generality (after localization) one can assume that the diameters of A and B are small. We can then transfer the problem to $\mathbb{R}^d$ and look at the analogous singular integral form on $\mathbb{R}^d$. One then needs to prove for cubes Q of side length 1, measurable sets $A, B \subset Q$ and for divergence free vector fields b the inequality

$$(16) \quad \left| \iint_{\varepsilon \leq |x-y| \lesssim 1} \frac{\langle x-y, b(x) - b(y) \rangle}{|x-y|^{d+2}} \mathbb{1}_A(x) \mathbb{1}_B(y) dx dy \right| \leq \|Db\|_{\mathcal{X}}$$

with $\mathcal{X} = L^p$ or $\mathcal{X} = L \log L(Q)$.

1.6. *Reduction to Christ-Journé operators.* There is a close relation with a special case of the Christ-Journé operators considered in [9]. For $a \in L^1_{\text{loc}}$ we can define for almost every pair $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$(17) \quad m_{x,y}[a] = \int_0^1 a(sx + (1-s)y) ds,$$

the mean of a over the line segment connecting the points x and y . Given a Calderón-Zygmund convolution kernel K in $\mathbb{R}^d$, $d \geq 2$, and $a \in L^\infty(\mathbb{R}^d)$ (or $L^q(\mathbb{R}^d)$) the so called d -commutator of first order is defined by

$$(18) \quad \mathcal{C}_{K,1}[f, a](x) = \text{p.v.} \int_{\mathbb{R}^d} K(x-y) m_{x,y}[a] f(y) dy.$$

More generally, for $a_1, \dots, a_n \in L^\infty(\mathbb{R}^d)$ we call

$$(19) \quad \mathcal{C}_{K,n}[f, a_1, \dots, a_n](x) = \text{p.v.} \int_{\mathbb{R}^d} K(x-y) \prod_{i=1}^n m_{x,y}[a_i] f(y) dy$$

a *Christ-Journé operator of order n* . If $a_1 = \dots = a_n$ we get a d -commutator of order n . The case $n = 1$ is relevant for (16). Answering a question by Yves Meyer, Christ and Journé [9] established that $\mathcal{C}_{K,n}$ (under suitable conditions on K) maps $L^p(\mathbb{R}^d) \times (L^\infty(\mathbb{R}^d))^n$ to $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, and with bound $C_{n,p} \prod_{i=1}^n \|a_i\|_\infty$ where C_n is polynomial in n . In one dimension, $d = 1$, one gets the more regular Calderón commutators [6].

Back to (16). Now we set $a_{ij} = \frac{\partial b_i}{\partial x_j}$ and thus

$$b_i(x) - b_i(y) = \sum_{j=1}^d (x_j - y_j) \int_0^1 a_{ij}((1-s)x + sy) ds.$$

The assumption that $\text{div}(b) = 0$ means that the trace of (a_{ij}) is zero, i.e.,

$$a_{dd}(x) = - \sum_{i=1}^{d-1} a_{ii}(x).$$

Hence

$$\begin{aligned} \frac{\langle x-y, b(x) - b(y) \rangle}{|x-y|^{d+2}} &= \sum_{i=1}^{d-1} K_i(x-y) \int_0^1 a_{ii}((1-s)x + sy) ds \\ &\quad + \sum_{\substack{1 \leq i, j \leq d \\ i \neq j}} K_{ij}(x) \int_0^1 a_{ij}((1-s)x + sy) ds \end{aligned}$$

where

$$(20) \quad K_i(x) = \frac{(x_i - y_i)^2 - (x_d - y_d)^2}{|x-y|^{d+2}}$$

and

$$(21) \quad K_{ij}(x) = \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^{d+2}}.$$

Thus we have reduced our problem to a question on Christ-Journé operators (with $n = 1$). Note that $K_i(x)$ and $K_{ij}(x)$ above are of the form

$$(22a) \quad \frac{\Omega\left(\frac{x}{|x|}\right)}{|x|^d}$$

where $\Omega \in C^\infty(S^{d-1})$ is an *even* function with

$$(22b) \quad \int_{S^{d-1}} \Omega(\theta) d\theta = 0.$$

Let $h \in L^p(\mathbb{R}^d)$. Let A, B be measurable subsets of $[-\frac{1}{2}, \frac{1}{2}]^d$ and let K be a homogeneous Calderón-Zygmund convolution operator with kernel satisfying (22). Then for the L^p question we need to prove

$$(23) \quad \left| \iint K(x-y) \int_0^1 h((1-s)x + sy) ds \mathbb{1}_A(x) \mathbb{1}_B(y) dx dy \right| \lesssim \|h\|_{L^p(\mathbb{R}^d)}$$

for $1 < p < \infty$.

Since

$$(24) \quad \begin{aligned} & \iint K(x-y) \int_0^{1/2} h((1-s)x + sy) ds \mathbb{1}_A(x) \mathbb{1}_B(y) dx dy \\ &= \iint K(y-x) \int_{1/2}^1 h((1-s)x + sy) ds \mathbb{1}_B(x) \mathbb{1}_A(y) dx dy. \end{aligned}$$

it suffices to establish

$$(25) \quad \left| \iint K(x-y) \int_{1/2}^1 h((1-s)x + sy) ds \mathbb{1}_A(x) \mathbb{1}_B(y) dx dy \right| \lesssim \|h\|_{L^p(\mathbb{R}^d)}.$$

We rewrite the form on the left hand side of (25). Interchange first the y and s integration and then, for fixed x, s , change variables $z = (1-s)x + sy$.

Let $s \mapsto \gamma(s)$ be defined on $(0, \infty)$ (above $\gamma(s) = \chi_{[1/2, 1]}(s)$) and write with $g = \mathbb{1}_A$, $\alpha = \mathbb{1}_B$

$$\begin{aligned} & \iint K(x-y) \int h((1-s)x + sy) \gamma(s) ds g(x) \alpha(y) dx dy \\ &= \int \int \int K(x - \frac{z-(1-s)x}{s}) h(z) \alpha(\frac{z-(1-s)x}{s}) s^{-d} dz \gamma(s) ds g(x) dx \\ &= \int \int \int \sigma^d K(\sigma(x-z)) h(z) \int \alpha((1-\sigma)x + \sigma z) \sigma^{-2} \gamma(\sigma^{-1}) d\sigma dz g(x) dx \end{aligned}$$

where we have changed variables $\sigma = s^{-1}$. If we use the homogeneity property of K and set $\gamma(s) = \chi_{[1/2, 1]}(s)$ we see that this equals

$$\int T_\alpha h(x) g(x) dx$$

with

$$(26) \quad T_\alpha h(x) = \int K(x-z) h(z) \int \alpha((1-\sigma)x + \sigma z) \sigma^{-2} \mathbb{1}_{[\frac{1}{2}, 1]}(\sigma^{-1}) d\sigma dz.$$

For $\alpha \in L^\infty$ the operator T_α can be considered as a variant of the d -commutator used in the paper by Christ and Journé. One can check that their proof applies to give

$$\left| \int T_\alpha h(x) g(x) dx \right| \leq C_p \|h\|_p \|g\|_{p'} \|\alpha\|_\infty, \quad 1 < p < \infty,$$

and since $\alpha = \mathbb{1}_A$, and $g = \mathbb{1}_B$ with $|B| \lesssim 1$, this yields the desired bound for (25). Moreover one can check that the arguments in [36] apply to prove that T_α is of weak type $(1, 1)$ and thus map compactly supported functions in $L \log L$ to L^1 .

The above discussion led the authors of [37] to the question whether there is a symmetric estimate of the form

$$(27) \quad \|C_{K,n}[f, a_1, \dots, a_n]\|_q \leq C_{q,n}(K) \|f\|_{p_1} \prod_{j=2}^n \|a_j\|_{p_j}$$

with $\sum_{j=1}^n p_j^{-1} = q^{-1}$. This had been an open problem, formulated explicitly *e.g.* in [17], and it was answered recently in [37]. Some ideas will be discussed in §2 below.

1.7. A discouraging result regarding Bressan's problem. Deviating slightly from our previous notation let $\Omega_L = (-1, 0) \times (-1, 1)$, $\Omega_R = (0, 1) \times (-1, 1)$. For a resolution of Bressan's problem on $\mathbb{T}^2$ it would be relevant if

$$(28) \quad \left| \iint \frac{\langle x - y, b(x) - b(y) \rangle}{|x - y|^4} \chi_A(x) \chi_B(y) dx dy \right| \lesssim \|Db\|_1$$

held for subsets $A \subset \Omega_L$, $B \subset \Omega_R$ and divergence free vector fields b . In particular we could consider smoothed versions of

$$b(x) = \begin{cases} (0, 1) & \text{for } x_1 < 0, \\ (0, -1) & \text{for } x_1 > 0. \end{cases}$$

Notice that

$$Db(x) = \begin{pmatrix} 0 & 0 \\ -2\delta(x_1) & 0 \end{pmatrix}$$

where δ is the Dirac measure in one dimension, and thus $\text{div}(b)=0$. For this choice of b the above expression becomes $|\mathcal{I}(A, B)|$ with

$$(29a) \quad \mathcal{I}(A, B) = \iint_{(x,y) \in A \times B} K_{|x_1 - y_1|}(x_2 - y_2) dx dy$$

where

$$(29b) \quad K_r(s) = \frac{s}{(r^2 + s^2)^2} = -\frac{1}{2} \frac{d}{ds} \frac{1}{r^2 + s^2}.$$

Charlie Smart had asked whether $\mathcal{I}(A, B)$ is bounded independently of A and B as long as $A \subset \Omega_L$, $B \subset \Omega_R$. Unfortunately this turns out to be not true, and one gets a precise upper and lower bound in terms of some separation condition on A and B .

Proposition 1.4. *Let*

$$\mathcal{M}(\varepsilon) = \sup \{ |\mathcal{I}(A, B)| : \text{dist}(A, B) \geq \varepsilon, A \subset \Omega_L, B \subset \Omega_R \}.$$

Then for $0 < \varepsilon < 1/2$ we have

$$\mathcal{M}(\varepsilon) \approx \log(1/\varepsilon).$$

Upper bounds. Suppose g satisfies $\sup_s (1 + |s|)^{\delta+1} |g(s)| < \infty$, for some $\delta > 0$. Note that $K_r(s) = r^{-2} r^{-1} g(s/r)$ if we take $g(s) = s(1 + s^2)^{-2}$, and thus the following estimate gives the upper bound in the proposition.

Lemma 1.5. *Suppose $A \subset \Omega_L$, $B \subset \Omega_R$ and $\text{dist}(A, B) > \varepsilon$. Then, with $g(s) = s(1 + s^2)^{-2}$,*

$$\iint_{\Omega_L \times \Omega_R} |x_1 - y_1|^{-3} |g(\frac{x_2 - y_2}{|x_1 - y_1|})| \chi_B(y) \chi_A(x) dx dy \lesssim \log(1/\varepsilon).$$

Proof. Observe that for $x \in \Omega_L$, $y \in \Omega_R$ we have $|x_1 - y_1| = |x_1| + |y_1|$.

We consider separately the regions with (i) $|x_2 - y_2| \leq |x_1 - y_1|$ (which for $x \in A$, $y \in B$ implies $|x_1 - y_1| \geq \varepsilon/2$) and (ii) $2^{m-1}|x_1 - y_1| \leq |x_2 - y_2| < 2^m|x_1 - y_1|$ for some $m \geq 1$ (which for $x \in A$, $y \in B$ implies $|x_1 - y_1| \geq 2^{-m-2}\varepsilon$).

First,

$$\begin{aligned} & \iint_{\substack{(x,y) \in \Omega_L \times \Omega_R \\ |x_2 - y_2| \leq |x_1 - y_1|}} |x_1 - y_1|^{-3} |g(\frac{x_2 - y_2}{|x_1 - y_1|})| \chi_B(y) \chi_A(x) dx dy \\ & \lesssim \iint_{\substack{(x_1, y_1) \in [-1, 0] \times [0, 1] \\ |x_1 - y_1| \geq \varepsilon/2}} |x_1 - y_1|^{-2} \iint_{[-1, 1]^2} \frac{1}{|x_1 - y_1|} |g(\frac{x_2 - y_2}{|x_1 - y_1|})| dx_2 dy_2 dx_1 dy_1 \\ & \lesssim \|g\|_{L^1(\mathbb{R})} \iint_{\varepsilon/2 < |x_1| + |y_1| \leq 2} \frac{1}{(|x_1| + |y_1|)^2} dx_1 dy_1 \lesssim \log(1/\varepsilon) \end{aligned}$$

Next, when $|x_2 - y_2| \approx 2^m|x_1 - y_1|$ we have $|g(\frac{x_2 - y_2}{|x_1 - y_1|})| \lesssim 2^{-m(1+\delta)}$ and thus

$$\begin{aligned} & \iint_{\substack{(x,y) \in \Omega_L \times \Omega_R \\ |x_2 - y_2| \approx 2^m|x_1 - y_1|}} |x_1 - y_1|^{-3} |g(\frac{x_2 - y_2}{|x_1 - y_1|})| \chi_B(y) \chi_A(x) dx dy \\ & \lesssim 2^{-m\delta} \iint_{\substack{(x_1, y_1) \in [-1, 0] \times [0, 1] \\ |x_1 - y_1| \geq 2^{-m-2}\varepsilon}} |x_1 - y_1|^{-2} dx_1 dy_1 \\ & \lesssim 2^{-m\delta} \iint_{2^{-m-2}\varepsilon \leq |x_1| + |y_1| \leq 2} \frac{1}{(|x_1| + |y_1|)^2} dx_1 dy_1 \lesssim 2^{-m\delta} \log(2^m/\varepsilon). \end{aligned}$$

Now sum in m to finish the proof. $\square$

Lower bounds. We now take $g(s) = \frac{s}{(1+s^2)^2}$ and construct a specific pair A, B for which $\text{dist}(A, B) \geq \varepsilon$ and $|\mathcal{I}(A, B)| \gtrsim \log(1/\varepsilon)$. It suffices to take $\varepsilon = 2^{-LM}$ for some integer L (and M be a sufficiently large fixed integer, $M > 10$).

Define

$$\begin{aligned} I_k^L &= [-2^{-kM}, -2^{-kM-1}], \\ I_k^R &= [2^{-kM-1}, 2^{-kM}], \\ J_{k,n}^L &= [(Mn+2)2^{-kM}, (Mn+3)2^{-kM}], \\ J_{k,n}^R &= [Mn2^{-kM}, (Mn+1)2^{-kM}] \end{aligned}$$

and

$$\begin{aligned} A &= \bigcup_{1 \leq k \leq L-1} \bigcup_{0 \leq n \leq \frac{2^{kM}}{M+1}} I_k^L \times J_{k,n}^L, \\ B &= \bigcup_{1 \leq k \leq L-1} \bigcup_{0 \leq n \leq \frac{2^{kM}}{M+1}} I_k^R \times J_{k,n}^R. \end{aligned}$$

Clearly $A \in \Omega_L$, $B \in \Omega_R$ and $\text{dist}(A, B) \geq 2^{-LM}$.

Let

$$\mathcal{I}(k_L, k_R, n_L, n_R) = \int_{x \in I_{k_L}^L \times J_{k_L, n_L}^L} \int_{y \in I_{k_R}^R \times J_{k_R, n_R}^L} K_{|x_1 - y_1|}(x_2 - y_2) dy dx$$

and split $\mathcal{I}(A, B) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$ where

$$\begin{aligned} \mathcal{E}_1 &= \sum_{1 \leq k \leq L-1} \sum_{0 \leq n \leq \frac{2^{kM}}{M+1}} \mathcal{I}(k, k, n, n) \\ \mathcal{E}_2 &= \sum_{1 \leq k \leq L-1} \sum_{\substack{0 \leq n_L, n_R \leq \frac{2^{kM}}{M+1} \\ n_L \neq n_R}} \mathcal{I}(k, k, n_L, n_R) \\ \mathcal{E}_3 &= \sum_{\substack{1 \leq k_L, k_R \leq L-1 \\ k_L \neq k_R}} \sum_{0 \leq n_L, n_R \leq \frac{2^{kM}}{M+1}} \mathcal{I}(k_L, k_R, n_L, n_R). \end{aligned}$$

We prove a lower bound for $\mathcal{E}_1$ and upper bounds for $\mathcal{E}_2, \mathcal{E}_3$.

For the lower bound observe

$$2^{-kM} \leq x_2 - y_2 \leq 2^{2-kM} \text{ for } x_2 \in J_{k,n}^L, y_2 \in J_{k,n}^L.$$

Thus

$$\begin{aligned} & \int_{I_k^L \times J_{k,n}^L} \int_{I_k^R \times J_{k,n}^L} K_{|x_1 - y_1|}(x_2 - y_2) dx dy \\ &= \iiint \frac{x_2 - y_2}{((x_1 + y_1)^2 + (x_2 - y_2)^2)^2} dy_2 dx_2 dy_1 dx_1 \\ & \quad \substack{(x_1, y_1) \in [2^{-kM-1}, 2^{-kM}] \\ (Mn+2)2^{-kM} \leq x_2 \leq (Mn+3)2^{-kM} \\ Mn2^{-kM} \leq y_2 \leq (Mn+1)2^{-kM}} \\ & \geq \frac{2^{-kM}}{1000} \end{aligned}$$

and thus

$$\mathcal{E}_1 \geq 10^{-3} \sum_{k=1}^{L-1} \sum_{0 \leq n_L \leq \frac{2^{kM}}{M+1}} 2^{-kM} \geq \frac{L-1}{10^3(M+1)}.$$

If $n_L \neq n_R$ we have

$$\int_{I_k^L \times J_{k,n_L}^L} \int_{I_k^R \times J_{k,n_R}^L} |K_{|x_1 - y_1|}(x_2 - y_2)| dx dy \lesssim \frac{2^{-kM}}{M^3 |n_L - n_R|^3}$$

and thus

$$|\mathcal{E}_2| \leq \sum_{1 \leq k < L} \sum_{0 \leq n_L \leq \frac{2^{kM}}{M+1}} \sum_{n_R \neq n_L} \frac{2^{-kM}}{M^3 |n_L - n_R|^3} \leq C \frac{L}{M^4}.$$

Next, set $g(s) = |s|(1 + s^2)^{-2}$, and

$$G(x_1, y_1) = \iint_{-1 \leq x_2, y_2 \leq 1} \frac{1}{|x_1 - y_1|} |g(\frac{x_2 - y_2}{|x_1 - y_1|})| dy_2 dx_2$$

so that $G(x_1, y_1)$ is nonnegative and uniformly bounded. We have $|\mathcal{E}_3| \leq \mathcal{E}_{3,1} + \mathcal{E}_{3,2}$ where

$$\mathcal{E}_{3,1} = \sum_{1 \leq k_L < k_R \leq L-1} \iint_{I_{k_L}^L \times I_{k_R}^R} |x_1 - y_1|^{-2} G(x_1, y_1) dy_1 dx_1$$

and $\mathcal{E}_{3,2}$ is the corresponding term with the (k_L, k_R) summation extended over $1 \leq k_R < k_L \leq L-1$. The two terms are symmetric and it suffices to estimate $\mathcal{E}_{3,1}$.

Now $|x_1 - y_1| \approx 2^{-k_L M}$ if $x_1 \in I_{k_L}^L$, $y_1 \in I_{k_R}^L$, and $k_L < k_R$. Therefore

$$\mathcal{E}_{3,1} \lesssim \sum_{1 \leq k_L < k_R \leq L-1} 2^{2k_L M} |I_{k_L}^L| |I_{k_R}^R| \lesssim \sum_{1 \leq k_L < k_R \leq L-1} 2^{(k_L - k_R)M} \lesssim L 2^{-M}$$

and similarly we also get $\mathcal{E}_{3,2} \lesssim L 2^{-M}$. Combining the estimates we get

$$\mathcal{M}(2^{-LM}) \geq \mathcal{E}_1 - |\mathcal{E}_2| - |\mathcal{E}_3| \geq 10^{-3} \frac{L-1}{M+1} - C_1 L M^{-3} - C_2 L 2^{-M}$$

and the assertion follows by choosing M sufficiently large.

2. CLASSES OF SINGULAR MULTILINEAR FORMS OF CHRIST-JOURNÉ TYPE

2.1. Christ-Journé multilinear forms. Let $0 < \epsilon < 1$ and let $K \in \mathcal{S}'(\mathbb{R}^d) \cap L_{\text{loc}}^1(\mathbb{R}^d \setminus \{0\})$ be a regular Calderón-Zygmund convolution kernel on $\mathbb{R}^d$, satisfying the standard size and regularity assumptions,

$$(30a) \quad |K(x)| \leq C|x|^{-d}, \quad x \neq 0,$$

$$(30b) \quad |K(x+h) - K(x)| \leq C \frac{|h|^\epsilon}{|x|^{d+\epsilon}}, \quad x \neq 0, |h| \leq \frac{|x|}{2},$$

and the L^2 boundedness condition

$$(30c) \quad \|\widehat{K}\|_\infty \leq C < \infty.$$

Let $\|K\|_{CZ(\epsilon)}$ be the smallest constant C for which the three inequalities (30) hold.

Define the *Christ-Journé multilinear form* by

$$(31) \quad \Lambda_{\text{CJ}}(f_1, \dots, f_{n+2}) = \iint K(x-y) \left(\prod_{i=1}^n m_{x,y} f_i \right) f_{n+1}(y) f_{n+2}(x) dx dy$$

where K is a regular Calderón-Zygmund convolution operator and $m_{x,y}$ is as in (17). Inequality (27), for $q \geq 1$ is equivalent with

$$(32) \quad |\Lambda_{\text{CJ}}(f_1, \dots, f_{n+2})| \leq C_{p,n}(K) \prod_{i=1}^{n+2} \|f_i\|_{p_i}.$$

Theorem 2.1. Suppose that $d \geq 1$, $1 < p_i \leq \infty$, $i = 1, \dots, n+2$, and $\sum_{i=1}^{n+2} p_i^{-1} = 1$. Let $\epsilon > 0$ and $\min\{p_1, \dots, p_{n+2}\} \geq 1 + \delta$. Then inequality (33) holds with

$$(33) \quad C_{p,n}(K) \leq C(\delta) \|K\|_{CZ(\epsilon)} n^2 \log^3(2+n).$$

2.2. Previous results. We list some previous results on the $n+2$ -linear form Λ_{CJ} , including many in dimension $d = 1$, covering the classical Calderón commutators.

(i) Christ and Journé [9] who established (33) for $p_1 = \dots = p_n = \infty$ and $C(n) = O(n^{2+\epsilon})$. Several ideas in our proof can be traced back to their work.

(ii) The first estimates of the form (33), for the case $d = 1$ and $n = 1$ were proved in the seminal paper by A.P. Calderón [6].

(iii) More generally, still in dimension $d = 1$, Coifman, McIntosh and Meyer [13] proved estimates of the form (33) for arbitrary n , with $p_1 = \dots = p_n = \infty$ and polynomial bounds $C(n) = O(n^4)$ as $n \rightarrow \infty$. This allowed them to establish the L^2 boundedness of the Cauchy integral operator on general Lipschitz curves. See also [11] for other applications to related

problems of Calderón. Christ and Journé [9] were able to improve the Coifman-McIntosh-Meyer bounds to $C(n) = O(n^{2+\varepsilon})$ (and to $O(n^{1+\varepsilon})$ for odd kernels κ).

(iv) Duong, Grafakos and Yan [17] developed a rough version of the multisingular integral theory in [23] to cover the estimates (33) with general exponents for $d = 1$, however their arguments yield constants $C(n)$ which are of exponential growth in n .

One should note that [17] also treats the higher Calderón commutators $\mathcal{C}[f_1, \dots, f_n]$, with target space L^p where $p > 1/2$. For the bilinear version this had been first done by C.P. Calderón [7]. It would be interesting to obtain appropriate similar results for the d -commutators.

(v) Muscalu [33] recently developed a new approach for proving (33) in dimension $d = 1$, see also [34, Theorem 4.11]. An explicit bound for the constant as $A(n, \ell)$ where ℓ is the number of indices j such that $p_j \neq \infty$ and, for fixed ℓ , $n \mapsto A(n, \ell)$ is of polynomial growth. However, by using complex interpolation (as in §15) to the case when $p_j = \infty$ for all but two j , one may remove the dependence of A on ℓ . This yields polynomial bounds for all admissible sets of exponents, as in our results.

(vi) Some results for $d \geq 2$, when K is a homogeneous *odd* convolution kernel can be obtained by the method of rotation, cf. the discussion in §2.3.

(vii) Hofmann [26] obtained estimates (33) for operators with less regular kernels K , and an extension to weighted norm inequalities; however the induction argument in [26] only gives exponential bounds as $n \rightarrow \infty$.

2.3. An simple special case: Application of the method of rotations. The existence of estimates such as (27) is plausible by an easy proof for a special case. In [17], Duong, Grafakos and Yan consider the case of odd K , i.e.

$$K(x) = -K(-x)$$

and derive L^p estimates for the first order commutator. They prove that the inequality

$$\|\mathcal{C}_{K,1}[f, a]\|_q \lesssim \|f\|_{q_1} \|a\|_{q_2}$$

holds for $1/q_1 + 1/q_2 = 1/q$ under the additional restriction that $(1/q_1, 1/q_2, 1/q_3)$ lies in the interior of the convex hull of the points $(0, 1/2, 1/2)$, $(0, 0, 0)$, $(1, 0, 1)$, $(1/2, 1/2, 1)$ and $(1/6, 4/6, 5/6)$. They use the classical method of rotation to reduce the estimates for the Christ-Journé operators to uniform estimates for the bilinear Hilbert transforms

$$H_\alpha[f, g](x) = \int_{-\infty}^{\infty} f(x + \alpha t) g(x - t) \frac{dt}{t}$$

on the real line. The known range of *uniform* estimates, under some side conditions, [41], [22], [32], is smaller than the one for each fixed α in [30], [31]. This yields the result for $(1/q_1, 1/q_2, 1/q)$ in the above restricted range. Here we are interested in bounds for choices of $(1/q_1, 1/q_2, 1/q)$ near $(0, 1, 1)$, and observe that the method of rotation can be used to get a better range, and also an estimate for higher order commutators can be achieved.

Proposition 2.2. *Let $K_\Omega(x) = |x|^{-d} \Omega(x/|x|)$ where $\Omega \in L^1(S^{d-1})$ and*

$$\Omega(\theta) = -\Omega(-\theta).$$

Define

$$\mathcal{C}_{K_\Omega}[f_1, \dots, f_{n+1}](x) = p.v. \int K_\Omega(x - y) f_{n+1}(y) \prod_{i=1}^n \int_0^1 f_i((1 - s_i)x + s_i y) ds_i dy$$

Then for $1 \leq q < \infty$, $1 < p_i \leq \infty$ with $\sum_{i=1}^{n+1} 1/p_i = 1/q$

$$\|\mathcal{C}_{K_\Omega}[f_1, \dots, f_{n+1}]\|_q \lesssim C_n \prod_{i=1}^{n+1} \|f_i\|_{p_i}$$

Proof. There is little (if anything) new here. It is just the observation that instead of reducing to the hard estimates for bilinear Hilbert transforms (or their even harder analogues in the multilinear case for which nothing is known) one can reduce to the more classical estimates on the Calderón commutator $\mathcal{C}$ on $\mathbb{R}$. It is defined, with $A'_i = a_i$, by

$$\begin{aligned} \mathcal{C}[f, a_1, \dots, a_n](x) &= \int_{\mathbb{R}} \frac{\prod_{i=1}^n (A_i(x) - A_i(y))}{(x-y)^{n+1}} f(y) dy \\ &= \int_{\mathbb{R}} f(y) \frac{1}{x-y} \prod_{i=1}^n \int_0^1 a_i((1-s_i)x + s_i y) ds_i dy \end{aligned}$$

and it is known (see [6], [12], [7], [9], [17]) that for $1 < 1/q = \sum_{i=1}^{n+1} 1/q_i$ and $1 \leq q_i \leq \infty$ the operator $\mathcal{C}$ maps $L^{q_1}(\mathbb{R}) \times \dots \times L^{q_{n+1}}(\mathbb{R})$ to $L^q(\mathbb{R})$. There are also estimates for $q < 1$, see [7], [17] (but these cannot be used for the method of rotation). By [17] the operator $\mathcal{C}$ maps $L^{q_1}(\mathbb{R}) \times \dots \times L^{q_{n+1}}(\mathbb{R})$ to $L^{q,\infty}(\mathbb{R})$ provided that $0 < 1/q = \sum_{i=1}^{n+1} 1/q_i$ and $1 \leq q_i \leq \infty$. The estimates are polynomial in m .

Now for the Christ-Journé operators with odd Ω we write¹

$$\begin{aligned} \mathcal{C}_{K_\Omega}[f_1, \dots, f_{n+1}](x) &= \int K_\Omega(x-y) f_{n+1}(y) \prod_{i=1}^n \int_0^1 f_i((1-s_i)x + s_i y) ds_i dy \\ &= \int K_\Omega(w) f_{n+1}(x-w) \prod_{i=1}^n \int_0^1 f_i(x - s_i w) ds_i dy \\ &= \int_{S^{d-1}} \Omega(\theta) \int_0^\infty f_{n+1}(x-r\theta) \prod_{i=1}^n \int_0^1 f_i(x - s_i r\theta) ds_i \frac{dr}{r} \end{aligned}$$

and the right hand side is equal to

$$\begin{aligned} &\frac{1}{2} \int_{S^{d-1}} \Omega(\theta) \int_0^\infty f_{n+1}(x-r\theta) \prod_{i=1}^n \int_0^1 f_i(x - s_i r\theta) ds_i \frac{dr}{r} \\ &\quad + \frac{1}{2} \int_{S^{d-1}} \Omega(-\theta) \int_0^\infty f_{n+1}(x+r\theta) \prod_{i=1}^n \int_0^1 f_i(x + s_i r\theta) ds_i \frac{dr}{r} \\ &= \frac{1}{2} \int_{S^{d-1}} \Omega(\theta) \int_0^\infty \left[f_{n+1}(x-r\theta) \prod_{i=1}^n \int_0^1 f_i(x - s_i r\theta) ds_i \right. \\ &\quad \left. + f_{n+1}(x+r\theta) \prod_{i=1}^n \int_0^1 f_i(x + s_i r\theta) ds_i \right] \frac{dr}{r} \\ &= \frac{1}{2} \int_{S^{d-1}} \Omega(\theta) \mathcal{C}_\theta[f_1, \dots, f_{n+1}](x) d\theta \end{aligned}$$

where

$$\mathcal{C}_\theta[f_1, \dots, f_{n+1}](x) = \int_{-\infty}^\infty f_{n+1}(x-r\theta) \prod_{i=1}^n \int_0^1 f_i(x - s_i r\theta) ds_i \frac{dr}{r}$$

¹In order to make these computations rigorous we should replace the convolution kernel $K(w)$ with $K(w)\chi_\varepsilon(w)$ where χ_ε is the characteristic function of the annulus $w : \varepsilon < |w| < \varepsilon^{-1}$ and then let $\varepsilon \rightarrow 0$.

Now if R_θ is a rotation with $R_\theta e_1 = \theta$ we have

$$\mathcal{C}_\theta[f_1, \dots, f_{n+1}](x) = \mathcal{C}_{e_1}[f_1 \circ R_\theta, \dots, f_{n+1} \circ R_\theta](R_\theta^{-1}x)$$

and thus the operator norms of $\mathcal{C}_\theta$ are independent of θ . Hence

$$\|\mathcal{C}_{K_{\Omega,n}}\|_{L^{p_1} \times \dots \times L^{p_{n+1}} \rightarrow L^q} \leq \|\Omega\|_{L^1(S^{d-1})} \|\mathcal{C}_{e_1}\|_{L^{p_1} \times \dots \times L^{p_{n+1}} \rightarrow L^q};$$

here we used Minkowski's inequality which restricts the range to $q \geq 1$. Finally observe after changing variables $y_1 = x_1 - r$

$$\begin{aligned} \mathcal{C}_{e_1}[f_1, \dots, f_{n+1}](x) = \\ \int_{-\infty}^{\infty} f_{n+1}(y_1, x_2, \dots, x_n) \prod_{i=1}^n \int_0^1 f_i((1-s_i)x_1 + s_i y_1, x_2, \dots, x_d) ds_i \frac{1}{x_1 - y_1} dy_1 \end{aligned}$$

and we have $\|\mathcal{C}_{e_1}\|_{L^{p_1} \times \dots \times L^{p_{n+1}} \rightarrow L^p} \leq C_n$, by the known results for the Calderón commutator. $\square$

Remark. Unlike in the classical case for convolution operators the method of rotation does not seem to be of any use if Ω is not odd. Note that the functions Ω of interest in (20), (21) have cancellation but they are even. While Proposition 2.2 strongly suggests a more general result the proof above does not give us a clue how such a result should be proved.

2.4. Towards a more general result for Christ-Journé operators. In order to prove Theorem 2.1 it suffices to prove estimate (33) for the cases where two of the exponents, say p_i, p_j , $1 \leq i < j \leq n+2$ belong to $(1, \infty)$ and the other n exponents are equal to ∞ . Equivalently, if ϖ is a permutation of $\{1, \dots, n+2\}$ and

$$\Lambda_{\text{CJ}}^{\varpi}(f_1, \dots, f_{n+2}) = \Lambda_{\text{CJ}}(f_{\varpi(1)}, \dots, f_{\varpi(n+2)})$$

one needs to show the inequalities

$$(34) \quad |\Lambda_{\text{CJ}}^{\varpi}[f_1, \dots, f_{n+2}]| \leq C_{\delta,p} n^2 (\log n)^3 \|K\|_{CZ(\delta)} \left(\prod_{i=1}^n \|f_i\|_{\infty} \right) \|f_{n+1}\|_p \|f_{n+2}\|_{p'},$$

for $1 < p < \infty$, with $C_{\delta,p}$ independent of n, ϖ . The general case is then proved by complex interpolation for multilinear forms, see [2, Theorem 4.4.1].

$\Lambda_{\text{CJ}}^{\varpi}$ takes the form

$$(35) \quad \Lambda_{\text{CJ}}^{\varpi}(f_1, \dots, f_{n+2}) = \iint \int_{\mathbb{R}^n} \kappa^{\varpi}(s, x-y) f_{n+2}(x) f_{n+1}(y) \prod_{i=1}^n f_i(x - s_i(x-y)) ds dx dy$$

with a suitable kernel κ^{ϖ} depending also on $s \in \mathbb{R}^n$. This expression $\Lambda_{\text{CJ}}(f_1, \dots, f_{n+2})$ is not symmetric, but there are partial symmetries. By the symmetry in $\{1, \dots, n\}$ and essential symmetry in $\{n+1, n+2\}$ (with a change of variable $s_j \mapsto (1-s_j)$) two cases are of particular interest:

- (1) If ϖ^i is the permutation that interchanges i and $n+1$ and leaves all $k \notin \{i, n+1\}$ fixed then κ^{ϖ^i} is given by

$$\kappa^{\varpi^i}(s, v) = \begin{cases} |s_i|^{d-n-1} \kappa(s_i v) & \text{if } s_i \geq 1, 0 \leq s_j \leq s_i, j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

- (2) If $1 \leq i, j \leq n$, $i \neq j$ and ϖ^{ij} is the permutation with $\varpi^{ij}(i) = n+1$, $\varpi^{ij}(j) = n+2$ and $\varpi^{ij}(k) = k$ for $k \notin \{i, j, n+1, n+2\}$ then the kernel $\kappa^{\varpi^{ij}}$ is given by

$$\begin{aligned} \kappa^{\varpi^{ij}}(s, v) &= |s_i - s_j|^{d-n-1} \kappa((s_i - s_j)(x - y)) \\ &\quad \text{either if } s_i \leq 0, s_j \geq 1, s_i \leq s_k \leq s_j \text{ for } k \neq i, j; \\ &\quad \text{or if } s_j \leq 0, s_i \geq 1, s_j \leq s_k \leq s_i \text{ for } k \neq i, j; \\ \kappa^{\varpi^{ij}}(s, v) &= 0 \text{ otherwise.} \end{aligned}$$

The above formulas suggest that one should study multilinear forms of the type

$$(36) \quad \Lambda[\kappa](f_1, \dots, f_{n+2}) = \iiint \kappa(s, x - y) f_{n+2}(x) f_{n+1}(y) \prod_{i=1}^n f_i(x - s_i(x - y)) ds dx dy,$$

where $x \in \mathbb{R}^d$, $s \in \mathbb{R}^n$, and $\kappa(s, x)$ is a Calderón-Zygmund kernel in the x variable which depends on a parameter $s \in \mathbb{R}^n$. We note that if $\kappa \in L^1(\mathbb{R}^n \times \mathbb{R}^d)$ the form $\Lambda[\kappa]$ is well defined for $(f_1, \dots, f_{n+2}) \in L^{p_1} \times \dots \times L^{p_{n+2}}$ with $\sum_{i=1}^{d+2} p_i^{-1} = 1$, by Hölder's inequality. In what follows the reader might find it helpful to keep $\kappa \in L^1$ as an a priori assumption as long it is not used in a substantial quantitative way.

We will impose some regularity conditions in the s variable. The basic example, corresponding to the Christ-Journé multilinear forms, is

$$\kappa(s, x) = \mathbb{1}_{[0,1]^n}(s) K(x)$$

where K is a regular Calderón convolution kernel satisfying the conditions (30).

Our task is to

- To introduce a reasonably general class $\mathcal{K}_\varepsilon$ of kernels $K(s, x)$, for which linear forms of type (36) are closed under adjoints. If ϖ is a permutation of $\{1, \dots, n+2\}$, then the multilinear form $\Lambda[K](f_{\varpi(1)}, \dots, f_{\varpi(n+2)})$ should be written as $\Lambda[K^\varpi](f_1, \dots, f_{n+2})$ for a suitable K^ϖ , with appropriate bounds on K^ϖ in the class $\mathcal{K}_\varepsilon$.
- To prove estimates for this same class of kernels that cover the estimates for the d -commutators in Theorem 2.1.

2.5. The classes $\mathcal{K}_\varepsilon$. We first introduce certain classes of tempered distributions on $\mathbb{R}^n \times \mathbb{R}^d$ which satisfy integrability properties in the first (s -)variable and contain all kernels allowable in (36).² For each $N \in \mathbb{N}_0$ consider the normed space $MS'_N(\mathbb{R}^n \times \mathbb{R}^d)$ consisting of tempered distributions κ on $\mathbb{R}^n \times \mathbb{R}^d$ for which there is $C > 0$ so that for all $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^d)$

$$(37) \quad |\langle \kappa, f \rangle| \leq C \sup_{s \in \mathbb{R}^n, x \in \mathbb{R}^d} \sum_{|\gamma| \leq N} (1 + |x|)^N |\partial_x^\gamma f(s, x)|.$$

Here $\langle \kappa, f \rangle$ denotes the pairing between distributions and test functions and the minimal C in (37) is the norm in $MS'_N(\mathbb{R}^n \times \mathbb{R}^d)$. The space $MS'(\mathbb{R}^n \times \mathbb{R}^d)$ is the space of tempered distributions κ on $\mathbb{R}^n \times \mathbb{R}^d$ for which (37) holds for some $N \in \mathbb{N}$. The locally convex space $MS'(\mathbb{R}^n \times \mathbb{R}^d)$ is the inductive limit of the normed spaces $MS'_N(\mathbb{R}^n \times \mathbb{R}^d)$. A net $\{f_i\}_{i \in \mathbb{J}}$ is Cauchy in this topology if there exists an N so that all f_i belong to $MS'_N(\mathbb{R}^n \times \mathbb{R}^d)$ for some fixed N and so that f_i is Cauchy in the norm topology of $MS'_N(\mathbb{R}^n \times \mathbb{R}^d)$. Since the normed spaces $MS'_N(\mathbb{R}^n \times \mathbb{R}^d)$ are complete the space $MS'(\mathbb{R}^n \times \mathbb{R}^d)$ is also complete. Let $M(\mathbb{R}^n)$ be

²The reader may want to skip the functional analysis definitions that follow and continue with Definition 2.3 below.

the space of bounded Borel measures on $\mathbb{R}^n$. A distribution $\kappa \in M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ gives rise to a continuous linear operator $\beta_\kappa : \mathcal{S}(\mathbb{R}^d) \rightarrow M(\mathbb{R}^n)$ defined by

$$\langle \beta_\kappa(\phi_2), \phi_1 \rangle := \langle \kappa, \phi_1 \otimes \phi_2 \rangle \text{ for } \phi_1 \in \mathcal{S}(\mathbb{R}^n), \phi_2 \in \mathcal{S}(\mathbb{R}^d).$$

Let $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ be the subspace of $M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ consisting of those κ for which $\beta_\kappa(\phi_2) \in L^1(\mathbb{R}^n)$, for all $\phi_2 \in \mathcal{S}(\mathbb{R}^d)$. $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ is a closed subspace of $M\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ and inherits its complete locally convex topology.

We now define a Banach space $\mathcal{K}_\varepsilon$ consisting of kernels of singular integral operators. For $\kappa \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^d)$, let

$$\kappa^{(t)}(s, x) := t^d \kappa(s, tx).$$

This is extended to $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ by continuity. Also, the convolution $\kappa(s, \cdot) * \eta$ with $\eta \in \mathcal{S}(\mathbb{R}^d)$ makes sense for $\kappa \in L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$; it yields an L^1 function in the s variable, which depends smoothly on x .

Fix $\eta \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$(38) \quad \inf_{\theta \in S^{d-1}} \sup_{\tau > 0} |\widehat{\eta}(\tau\theta)| > 0,$$

where $\widehat{\eta}$ denotes the Fourier transform of η .

Definition 2.3. Let η be as in (38), and $0 < \varepsilon \leq 1$. We define five seminorms by

$$(39a) \quad \|\kappa\|_{\mathcal{K}_{\varepsilon,1}}^\eta := \sup_{\substack{1 \leq i \leq n \\ t > 0}} \int (1 + |s_i|)^\varepsilon \|\eta * \kappa^{(t)}(s, \cdot)\|_{L^2(\mathbb{R}^d)} ds,$$

$$(39b) \quad \|\kappa\|_{\mathcal{K}_{\varepsilon,2}}^\eta := \sup_{\substack{1 \leq i \leq n \\ t > 0 \\ 0 < h \leq 1}} h^{-\varepsilon} \int \|\eta * [\kappa^{(t)}(s + he_i, \cdot) - \kappa^{(t)}(s, \cdot)]\|_{L^2(\mathbb{R}^d)} ds,$$

$$(39c) \quad \|\kappa\|_{\mathcal{K}_{\varepsilon,3}} := \sup_{\substack{1 \leq i \leq n \\ R > 0 \\ R \leq |x| \leq 2R}} \iint (1 + |s_i|)^\varepsilon |\kappa(s, x)| dx ds,$$

$$(39d) \quad \|\kappa\|_{\mathcal{K}_{\varepsilon,4}} := \sup_{\substack{1 \leq i \leq n \\ R > 0 \\ 0 < h \leq 1}} h^{-\varepsilon} \iint_{R \leq |x| \leq 2R} |\kappa(s + he_i, x) - \kappa(s, x)| dx ds,$$

$$(39e) \quad \|\kappa\|_{\mathcal{K}_{\varepsilon,5}} := \sup_{\substack{R > 2 \\ y \in \mathbb{R}^d}} R^\varepsilon \iint_{|x| \geq R|y|} |\kappa(s, x - y) - \kappa(s, x)| dx ds.$$

and denote by $\mathcal{K}_\varepsilon$ the subspace of $L\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^d)$ consisting of those κ for which the norm $\|\kappa\|_{\mathcal{K}_\varepsilon} := \sum_{i=1}^5 \|\kappa\|_{\mathcal{K}_{\varepsilon,i}}^\eta$ is finite.

The definition of $\|\cdot\|_{\mathcal{K}_\varepsilon}$ depends on a choice of $\eta \in \mathcal{S}(\mathbb{R}^d)$ satisfying (38). However, the equivalence class of the norm does not depend on the choice, and the constants in the equivalences of different choices of η will not depend on n .

Example 2.4. Let $\epsilon \in (0, 1)$ and let $K \in \mathcal{S}'(\mathbb{R}^d) \cap L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ be a convolution kernel in $\mathbb{R}^d$ satisfying

$$(40a) \quad \|\widehat{K}\|_\infty \leq C$$

and

$$(40b) \quad \sup_{R \geq 2} R^\epsilon \sup_{y \in \mathbb{R}^d} \int_{|x| \geq R|y|} |K(x - y) - K(x)| dx \leq C.$$

Let

$$\kappa(x, s) = \chi_{[0,1]^n}(s)K(x).$$

Then $\kappa \in \mathcal{K}_\delta(\mathbb{R}^n \times \mathbb{R}^d)$ for $\delta < \epsilon$ and it is not hard to check that

$$\|\kappa\|_{\mathcal{K}_\delta} \lesssim_{\delta, \epsilon} C_{(40)}.$$

We can now state a preliminary version of our boundedness result (see Theorem 2.10 below for a more definitive version).

Theorem 2.5. (i) *There is a constant $C = C(p, d, \epsilon, \eta)$ such that the following statement holds a priori for all kernels in $\mathcal{K}_\epsilon$ which also belong to $L^1(\mathbb{R}^n \times \mathbb{R}^d)$. The multilinear form $\Lambda[\kappa]$ as defined in (36) satisfies for all $f_i \in L^{p_i}(\mathbb{R}^d)$, $1 < p_i < \infty$, $\sum_{i=1}^{n+2} p_i^{-1} = 1$,*

$$(41) \quad |\Lambda[\kappa](f_1, \dots, f_{n+2})| \leq Cn^2 \log^3(1+n) \|\kappa\|_{\mathcal{K}_\epsilon} \prod_{i=1}^{n+2} \|f_i\|_{p_i}.$$

(ii) *The multilinear form $(\kappa, f_1, \dots, f_{n+2}) \mapsto \Lambda[\kappa](f_1, \dots, f_{n+2})$ uniquely extends to a bounded multilinear form on $\mathcal{K}_\epsilon \times L^{p_1} \times \dots \times L^{p_{n+2}}$ satisfying (41) for all $\kappa \in \mathcal{K}_\epsilon$.*

Singular integral kernels can usually be understood as sums of dilates of more basic building blocks satisfying some kind cancellation conditions. In this spirit the proof of Theorem 2.5 relies heavily on a decomposition theorem for the class $\mathcal{K}_\epsilon$. This decomposition will specify further part (ii) of the theorem, i.e. describe how to extend the result from part (i) to all kernels in $\mathcal{K}_\epsilon$.

2.6. Related spaces of Besov type. In the following definition $e_1, \dots, e_n$ will denote the standard basis of $\mathbb{R}^n$.

Definition 2.6. For $n, d \in \mathbb{N}$ and $0 \leq \epsilon \leq 1$ we define four (semi-)norms

$$(42a) \quad \|\varsigma\|_{\mathcal{B}_{\epsilon,1}} := \max_{1 \leq i \leq n} \iint (1 + |s_i|)^\epsilon |\varsigma(s, v)| ds dv,$$

$$(42b) \quad \|\varsigma\|_{\mathcal{B}_{\epsilon,2}} := \sup_{\substack{0 < h \leq 1 \\ 1 \leq i \leq n}} h^{-\epsilon} \iint |\varsigma(s + he_i, v) - \varsigma(s, v)| ds dv,$$

$$(42c) \quad \|\varsigma\|_{\mathcal{B}_{\epsilon,3}} := \sup_{0 < |h| \leq 1} |h|^{-\epsilon} \iint |\varsigma(s, v + h) - \varsigma(s, v)| ds dv,$$

$$(42d) \quad \|\varsigma\|_{\mathcal{B}_{\epsilon,4}} := \iint (1 + |v|)^\epsilon |\varsigma(s, v)| ds dv.$$

Let $\mathcal{B}_\epsilon(\mathbb{R}^n \times \mathbb{R}^d)$ be the space of those $\varsigma \in L^1(\mathbb{R}^n \times \mathbb{R}^d)$ such that the norm $\|\varsigma\|_{\mathcal{B}_\epsilon} := \sum_{i=1}^4 \|\varsigma\|_{\mathcal{B}_{\epsilon,i}}$ is finite.

For $0 < \epsilon < 1$ the space $\mathcal{B}_\epsilon$ is a type of *weighted Besov space*. In [37] it is also realized as a kind of *Besov space on differential forms in real projective space*. The following result helps us understand the behavior of our multilinear forms under adjoints.

Theorem 2.7. *Let $\epsilon > 0$. There exists $c > 0$ (independent of n) such that for $\epsilon' \geq c\epsilon$ and any permutation ϖ of $\{1, \dots, n+2\}$ there exists a bounded linear transformation*

$$\ell_\varpi : \mathcal{B}_\epsilon(\mathbb{R}^n \times \mathbb{R}^d) \rightarrow \mathcal{B}_{\epsilon'}(\mathbb{R}^n \times \mathbb{R}^d)$$

satisfying $(\ell_\varpi \varsigma)^{(t)} = \ell_\varpi(\varsigma^{(t)})$ for $t > 0$, and $\Lambda^\varpi[\varsigma] = \Lambda[\ell_\varpi \varsigma]$, such that

$$(43) \quad \|\ell_\varpi \varsigma\|_{\mathcal{B}_{\epsilon'}} \lesssim n^2 \|\varsigma\|_{\mathcal{B}_\epsilon}$$

and

$$\|\ell_{\varpi}\varsigma\|_{L^1} = \|\varsigma\|_{L^1}.$$

Furthermore, if $\int \varsigma(s, v) dv = 0$ a.e. then also $\int \ell_{\varpi}\varsigma(s, v) dv = 0$ a.e.

Every permutation of $\{1, \dots, n+2\}$ is a composition of at most four permutations of the following three forms, with the permutation in (iii) occuring at most twice.

- (i) A permutation of $\{1, \dots, n\}$, leaving $n+1$ and $n+2$ fixed.
- (ii) The permutation which switches $n+1$ and $n+2$, leaving all other elements fixed.
- (iii) The permutation which switches $n+1$ and 1, leaving all other elements fixed.

Case (i). If ϖ is a permutation of $\{1, \dots, n\}$, leaving $n+1$ and $n+2$ fixed, then $\ell_{\varpi}\varsigma(s, v) = \varsigma(s_{\varpi^{-1}(1)}, \dots, s_{\varpi^{-1}(n)}, v)$, and thus $\|\ell_{\varpi}\varsigma\|_{\mathcal{B}_\varepsilon} = \|\varsigma\|_{\mathcal{B}_\varepsilon}$ and $\|\ell_{\varpi}\varsigma\|_{L^1} = \|\varsigma\|_{L^1}$.

Case (ii). If ϖ is the permutation which switches $n+1$ and $n+2$, leaving all other elements fixed, then it is immediate to verify that $\ell_{\varpi}\varsigma(s, v) = \varsigma(1 - s_1, \dots, 1 - s_n, v)$. We have $\|\varsigma_{\varpi}\|_{\mathcal{B}_\varepsilon} \approx \|\varsigma\|_{\mathcal{B}_\varepsilon}$ and $\|\varsigma_{\varpi}\|_{L^1} = \|\varsigma\|_{L^1}$.

In both of the above cases, if $\int \varsigma(s, v) dv = 0 \forall s$ then $\int \varsigma_{\varpi}(s, v) dv = 0 \forall s$.

Case (iii). This is the only interesting situation. We have

$$\begin{aligned} & \Lambda[\varsigma](f_n, f_2, \dots, f_n, f_1, f_{n+2}) \\ &= \iiint \varsigma(s, v) f_{n+1}(x - s_1 v) \left(\prod_{i=2}^n f_i(x - s_i v) \right) f_1(x - v) f_{n+2}(x) dv dx ds \\ &= \iiint |s_1|^{-d} \varsigma(s, s_1^{-1} w) f_{n+1}(x - w) \left(\prod_{i=2}^n f_i(x - s_i s_1^{-1} w) \right) f_1(x - s_1^{-1} w) f_{n+2}(x) dx dw ds \\ &= \iiint \beta^{d-n-1} \varsigma(\beta_1^{-1}, \beta_1^{-1} \beta_2, \dots, \beta_1^{-1} \beta_n, \beta_1 w) \prod_{i=1}^n f_i(x - \beta_i v) f_{n+1}(x - w) f_{n+2}(x) dx dw d\beta \end{aligned}$$

where we have first changed variables $v = s_1^{-1}u$, then interchanged the order of integration, and changed variables $s_1 = \beta_1^{-1}$, $s_i = \beta_i \beta_1^{-1}$ for $i = 2, \dots, n$. Hence if ϖ is the transposition interchanging 1 and $n+1$ and leaving $2, \dots, n, n+2$ fixed then $\Lambda^\varpi[\varsigma] = \Lambda[\ell_{\varpi}\varsigma]$ with

$$(44) \quad \ell_{\varpi}\varsigma(s_1, \dots, s_n, v) = \varsigma(s_1^{-1}, s_1^{-1} s_2, \dots, s_1^{-1} s_n, s_1 v).$$

Now if we define the inversion J , with respect to the s_1 -variable, and multiplicative shears M_{n-1} , $\widetilde{M}_d$ by

$$\begin{aligned} Jg(s_1, \dots, s_n, v) &= s_1^{-2} g(s_1^{-1}, s_2, \dots, s_n, v) \\ M_{n-1}g(s_1, \dots, s_n, v) &= s_1^{n-1} g(s_1, s_1 s_2, \dots, s_1 s_n, v) \\ \widetilde{M}_d g(s_1, \dots, s_n, v) &= s_1^d g(s_1, \dots, s_n, s_1 v) \end{aligned}$$

then ℓ_{ϖ} in (44) can be factorized as

$$(45) \quad \ell_{\varpi} = J \circ \widetilde{M}_d \circ J \circ M_{n-1} \circ J.$$

It is shown in §4 of [37] that for $\varepsilon' \leq \varepsilon/3$

$$(46a) \quad \|Jg\|_{\mathcal{B}_{\varepsilon'}} \lesssim \|g\|_{\mathcal{B}_\varepsilon},$$

$$(46b) \quad \|M_{n-1}g\|_{\mathcal{B}_{\varepsilon'}} \lesssim n \|g\|_{\mathcal{B}_\varepsilon},$$

$$(46c) \quad \|\widetilde{M}_d g\|_{\mathcal{B}_{\varepsilon'}} \lesssim \|g\|_{\mathcal{B}_\varepsilon}.$$

The proofs of these inequalities are somewhat lengthy but fairly straightforward. Hence one gets $\|\ell_{\varpi}\varsigma\|_{\mathcal{B}_\delta} \lesssim n \|\varsigma\|_{\mathcal{B}_\varepsilon}$, in case (iii) at least when $\delta < 3^{-5}\varepsilon$. Finally if ϖ is a general permutation

than we can split $\varpi = \varpi_1 \circ \varpi_2 \circ \varpi_3 \circ \varpi_4$, each ϖ_i of the form in (i), (ii) or (iii), with at most two of the form in (iii). Hence we get $\Lambda^\varpi[\varsigma] = \Lambda[\ell_\varpi \varsigma]$ where $\|\ell_\varpi \varsigma\|_{\mathcal{B}_\delta} \lesssim n^2 \|\varsigma\|_{\mathcal{B}_\varepsilon}$, at least for $\delta < 3^{-10}\varepsilon$. This is clearly not the best δ in terms of ε but improvements of the δ -range are irrelevant for our purpose.

2.7. Decomposition of kernels in $\mathcal{K}_\varepsilon$. We decompose distributions in $\mathcal{K}_\varepsilon$ via a Littlewood-Paley decomposition. For what follows we use the notation $\varsigma^{(t)}(s, x) := t^d \varsigma(s, tx)$.

Definition 2.8. (i) Let $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\int \phi(x) dx = 1$, let Q_j denote the operator of convolution with $2^{jd} \phi(2^j \cdot) - 2^{(j-1)d} \phi(2^{j-1} \cdot)$. When acting on $\kappa \in LS'(\mathbb{R}^n \times \mathbb{R}^d)$, we define $Q_j \kappa$ by taking the convolution in $\mathbb{R}^d$.

(ii) Set

$$(47) \quad \varsigma_j[\kappa] := (Q_j \kappa)^{(2^{-j})}.$$

(iii) For $\kappa \in LS'(\mathbb{R}^n \times \mathbb{R}^d)$ let

$$(48) \quad \|\kappa\|_{\mathfrak{K}_*^0} = \sup_{j \in \mathbb{Z}} \|\varsigma_j[\kappa]\|_{L^1(\mathbb{R}^n \times \mathbb{R}^d)}.$$

(iv) For $\varepsilon > 0$ let $\mathfrak{K}_*^\varepsilon$ be the space of all $\kappa \in LS'(\mathbb{R}^n \times \mathbb{R}^d)$ such that

$$\|\kappa\|_{\mathfrak{K}_*^\varepsilon} := \sup_{j \in \mathbb{Z}} \|\varsigma_j[\kappa]\|_{\mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)}$$

is finite.

We now have the following decomposition theorem which also gives a relation between the spaces $\mathcal{K}_\varepsilon$ and $\mathfrak{K}_\varepsilon$, in particular it shows that $\bigcup_{0 < \varepsilon < 1} \mathcal{K}_\varepsilon = \bigcup_{0 < \varepsilon < 1} \mathfrak{K}_*^\varepsilon$.

Theorem 2.9. (i) A distribution $\kappa \in LS'(\mathbb{R}^n \times \mathbb{R}^d)$ belongs to $\bigcup_{0 < \varepsilon < 1} \mathcal{K}_\varepsilon$ if and only if there exists an $\varepsilon > 0$ and a bounded set $\{\varsigma_j : j \in \mathbb{Z}\} \subset \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ satisfying

$$\int \varsigma_j(s, v) dv = 0$$

for all j, s and

$$\kappa = \sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)},$$

holds with convergence in the topology on $LS'(\mathbb{R}^n \times \mathbb{R}^d)$ (and thus also in the sense of distributions). For given $\kappa \in \mathcal{K}_\varepsilon$ one can choose $\varsigma_j = \varsigma_j[\kappa]$ as in (47).

(ii) Let $\kappa \in \mathcal{K}_\varepsilon$. Then for $\delta < \varepsilon$,

$$\|\kappa\|_{\mathfrak{K}_*^\delta} \leq C_{\delta, \varepsilon, d} \|\kappa\|_{\mathcal{K}_\varepsilon}.$$

(iii) Let $\kappa \in \mathfrak{K}_*^\varepsilon$. Then for $\delta < \varepsilon/2$

$$\|\kappa\|_{\mathcal{K}_\delta} \leq C_{\delta, \varepsilon, d} \|\kappa\|_{\mathfrak{K}_*^\varepsilon}.$$

Theorem 2.9 suggests to define, for $\kappa \in \mathcal{K}_\varepsilon$, the expression $\Lambda[\kappa](f_1, \dots, f_{n+2})$ as the limit of partial sums

$$(49) \quad \sum_{j=-N}^N \Lambda[\varsigma_j^{(2^j)}](f_1, \dots, f_{n+2})$$

as $N \rightarrow \infty$; here $\varsigma_j^{(2^j)} = Q_j \kappa$. A sharper version of Theorem 2.5 is

Theorem 2.10. *Let $0 < \delta < 1$, let $p_1, \dots, p_{n+2} \in [1 + \delta, \infty]$ with $\sum_{l=1}^{n+2} p_l^{-1} = 1$. Let $\kappa \in \mathcal{K}^\varepsilon$ so that $\kappa = \sum_{j \in \mathbb{Z}} \varsigma_j^{(2^j)}$ in $LS'(\mathbb{R}^n \times \mathbb{R}^n)$ with $\int \varsigma_j(s, x) dx = 0$ for almost all $s \in \mathbb{R}^n$. Let $\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon} < \infty$, $f_1 \in L^{p_1}$, ..., $f_{n+2} \in L^{p_{n+2}}$. Then $\sum_{j=-\infty}^\infty \Lambda[\varsigma_j^{(2^j)}]$ converges in the operator topology of $(n+2)$ -linear functionals to a limit $\Lambda[K]$ satisfying*

$$|\Lambda[\kappa](f_1, \dots, f_{n+2})| \leq C_{p_0, \varepsilon, d} n^2 \|\kappa\|_{\mathfrak{K}_0} \log^3 \left(2 + n \frac{\|\kappa\|_{\mathfrak{K}_\varepsilon}}{\|\kappa\|_{\mathfrak{K}_0}} \right) \prod_{l=1}^{n+2} \|f_l\|_{p_l}.$$

We note that in this bound we may replace $\log(2 + n(\|\kappa\|_{\mathfrak{K}_\varepsilon}/\|\kappa\|_{\mathfrak{K}_0}))$ with $\log(2 + n^M(\|\kappa\|_{\mathfrak{K}_\varepsilon}/\|\kappa\|_{\mathfrak{K}_0}))$ for any fixed M and get an equivalent expression. It suffices to prove the estimate

$$(50) \quad |\Lambda[\kappa](f_1, \dots, f_{n+2})| \leq C_{\delta, \varepsilon, d} n^2 \|\kappa\|_{\mathfrak{K}_0} \log^3 \left(2 + n \frac{\|\kappa\|_{\mathfrak{K}_\varepsilon}}{\|\kappa\|_{\mathfrak{K}_0}} \right) \|f_{n+1}\|_p \|f_{n+2}\|_{p'} \prod_{l=1}^n \|f_l\|_\infty.$$

Indeed we can use Theorem 2.7 (observing that the factor n^2 in (43) is now irrelevant) and get the bounds

$$|\Lambda[\kappa](f_{\varpi(1)}, \dots, f_{\varpi(n+2)})| \leq C_{\delta, \varepsilon, d} n^2 \|\kappa\|_{\mathfrak{K}_0} \log^3 \left(2 + n \frac{\|\kappa\|_{\mathfrak{K}_\varepsilon}}{\|\kappa\|_{\mathfrak{K}_0}} \right) \|f_{n+1}\|_p \|f_{n+2}\|_{p'} \prod_{l=1}^n \|f_l\|_\infty.$$

for $p, p' \geq 1 + \delta$. Then complex interpolation for multilinear forms gives Theorem 2.10.

3. STRUCTURE OF THE PROOF OF BOUNDEDNESS

The structure of the proof of (50) is already anticipated in the paper by Christ and Journé [9]. Let ϕ be an even C_0^∞ function supported in $\{|x| < 1\}$ such that $\phi \geq 0$ and $\int \phi = 1$. For $j \in \mathbb{Z}$ define $\phi^{(2^j)}(x) := 2^{jd} \phi(2^j x)$ and define the operator $P_j f = f * \phi^{(2^j)}$. Furthermore, we choose ϕ to be even so that P_j is its adjoint and transpose. We have $\lim_{j \rightarrow \infty} P_j = I$, and, by the nonnegativity of ϕ ,

$$(51) \quad \|P_j\|_{L^\infty \rightarrow L^\infty} = 1.$$

In [9], [37] the property (51) is crucial for obtaining bounds which are polynomial in n .

In (50) we are given a family

$$\vec{\varsigma} = \{\varsigma_j : j \in \mathbb{Z}\} \subset \mathcal{B}_\varepsilon,$$

bounded in $\mathcal{B}_\varepsilon$. For some estimates we also need the cancellation condition satisfying the cancellation condition

$$(52) \quad \int \varsigma_j(\alpha, v) dv = 0 \text{ for all } j \in \mathbb{Z}.$$

Of particular interest are the choices $\varsigma_j = (Q_j \kappa)^{(2^{-j})}$ in (47). Our estimates will involve the following quantities depending on the family $\vec{\varsigma}$. First

$$(53) \quad \Gamma_\varepsilon \equiv \Gamma_\varepsilon[\vec{\varsigma}] := \frac{\sup_j \|\varsigma_j\|_{\mathcal{B}_\varepsilon}}{\sup_j \|\varsigma_j\|_{L^1}},$$

and then, for $n \geq 1$, $\nu \geq 0$,

$$(54) \quad \mathfrak{M}_\nu^{n, \varepsilon} \equiv \mathfrak{M}_\nu^{n, \varepsilon}[\vec{\varsigma}] := \sup_j \|\varsigma_j\|_{L^1} \log^\nu(1 + n \Gamma_\varepsilon(\vec{\varsigma})).$$

For

$$(55) \quad \Lambda(f_1, \dots, f_{n+2}) = \lim_{N \rightarrow \infty} \sum_{j=-N}^N \Lambda[\varsigma_j^{(2^j)}](f_1, \dots, f_{n+2}),$$

we wish to establish the bound

$$(56) \quad |\Lambda(f_1, \dots, f_{n+2})| \leq C_{\delta, \epsilon, d} n^2 \mathfrak{M}_3^{n, \epsilon} \|f_{n+1}\|_p \|f_{n+2}\|_{p'} \prod_{l=1}^n \|f_l\|_\infty.$$

for $p, p' \geq 1 + \delta$. We have not yet shown that the sum defining Λ converges in any reasonable sense though this is not hard to see if all f_j belong to $C_0^\infty(\mathbb{R}^d)$. One first establishes a priori estimates which are valid under the assumption that all but finitely many of the ς_j are zero and then examines the convergence issues in the general case. Throughout we take $n \geq 1$, as the result for $n = 0$ is classical.

We split Λ into various terms which are separately studied. For $1 \leq l_1 < l_2 \leq n + 2$, we define

$$(57) \quad \begin{aligned} & \Lambda_{l_1, l_2}^1(f_1, \dots, f_{n+2}) \\ &:= \sum_{j \in \mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](f_1, \dots, f_{l_1-1}, (I - P_j)f_{l_1}, P_j f_{l_1+1}, \dots, P_j f_{l_2-1}, (I - P_j)f_{l_2}, P_j f_{l_2+1}, \dots, P_j f_{n+2}). \end{aligned}$$

For $1 \leq l \leq n + 2$, we define

$$(58) \quad \Lambda_l^2(f_1, \dots, f_{n+2}) := \sum_{j \in \mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](P_j f_1, \dots, P_j f_{l-1}, (I - P_j)f_l, P_j f_{l+1}, \dots, P_j f_{n+2}).$$

Finally, we define

$$(59) \quad \Lambda^3(f_1, \dots, f_{n+2}) := \sum_{j \in \mathbb{Z}} \Lambda[\varsigma_j^{(2^j)}](P_1 f_1, \dots, P_{n+2} f_{n+2}).$$

One verifies

$$(60) \quad \begin{aligned} & \Lambda(f_1, \dots, f_{n+2}) \\ &= \sum_{1 \leq l_1 < l_2 \leq n+2} \Lambda_{l_1, l_2}^1(f_1, \dots, f_{n+2}) + \sum_{1 \leq l \leq n+2} \Lambda_l^2(f_1, \dots, f_{n+2}) + \Lambda^3(f_1, \dots, f_{n+2}). \end{aligned}$$

Theorem 3.1. *Let $p \in (1, 2]$ and $p' \in [2, \infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Assume that $\varsigma_j = 0$ for all but finitely many j . Then (with $\mathfrak{M}_\nu^{n, \epsilon}[\varsigma]$ as in (54))*

(I)

$$|\Lambda_{n+1, n+2}^1(f_1, \dots, f_{n+2})| \lesssim \mathfrak{M}_2^{n, \epsilon}[\varsigma] \left(\prod_{i=1}^n \|f_i\|_{L^\infty} \right) \|f_{n+1}\|_{L^p} \|f_{n+2}\|_{L^{p'}}.$$

(II) For $1 \leq l_1 \leq n, l_2 \in \{n+1, n+2\}$,

$$|\Lambda_{l_1, l_2}^1(f_1, \dots, f_{n+2})| \lesssim \mathfrak{M}_{5/2}^{n, \epsilon}[\varsigma] \left(\prod_{i=1}^n \|f_i\|_\infty \right) \|f_{n+1}\|_p \|f_{n+2}\|_{p'}.$$

(III) For $1 \leq l_1 < l_2 \leq n$,

$$|\Lambda_{l_1, l_2}^1(f_1, \dots, f_{n+2})| \lesssim \mathfrak{M}_3^{n, \epsilon}[\varsigma] \left(\prod_{i=1}^n \|f_i\|_\infty \right) \|f_{n+1}\|_p \|f_{n+2}\|_{p'}.$$

(IV) Under the additional cancellation condition (52) we have, for $1 \leq l \leq n + 2$,

$$|\Lambda_l^2(f_1, \dots, f_{n+2})| \lesssim n \mathfrak{M}_3^{n, \epsilon}[\varsigma] \left(\prod_{i=1}^n \|f_i\|_\infty \right) \|f_{n+1}\|_p \|f_{n+2}\|_{p'}.$$

(V) Suppose that (52) holds. Then

$$|\Lambda^3(f_1, \dots, f_{n+2})| \lesssim n^2 \mathfrak{M}_3^{n,\varepsilon}[\zeta] \left(\prod_{i=1}^n \|f_i\|_\infty \right) \|f_{n+1}\|_p \|f_{n+2}\|_{p'}.$$

In the above the implicit constants depend only on $p \in (1, 2]$, $d \in \mathbb{N}$, and $\epsilon > 0$.

Summing up the estimates for the five parts yields (56).

The proof of Theorem 3.1 in [37] gives information about the convergence in (55). The sums defining the above five functionals converge in the operator topology of multilinear functionals and the limits satisfy the above estimates. Moreover for fixed $b_1, \dots, b_n \in L^\infty(\mathbb{R}^d)$ fixed, we can associate to the multilinear form Λ an operator $T \equiv T[b_1, \dots, b_n]$ via

$$\int g(x) T[b_1, \dots, b_n] f(x) dx := \Lambda(b_1, \dots, b_n, f, g).$$

In this way we associate operators T_{l_1, l_2}^1 , T_l^2 and T^3 to the forms Λ_{l_1, l_2}^1 , Λ_l^2 and Λ^3 . Then the sums defining these operators converge, for fixed $b_1, \dots, b_n \in L^\infty(\mathbb{R}^d)$, in the strong operator topology as operators $L^p \rightarrow L^p$.

The proof of Theorem 3.1 in [37] is rather technical and lengthy. We extend arguments in the paper by Christ and Journé [9]. The estimation for the forms Λ_l^2 and Λ^3 relies on arguments related to the David-Journé theorem and generalizations. In order to estimate $\Lambda_{n+1, n+2}^1$, and prove (I), one uses almost orthogonality arguments based on a crucial L^2 estimate for the forms $\Lambda[\zeta]$ with $\zeta \in \mathcal{B}_\varepsilon$. This estimate will be discussed in the next section. It also plays an important role in the technical proofs of (II) and (III), for details see [37].

4. A BASIC L^2 ESTIMATE FOR ROUGH KERNELS

The following is from §7 in [37].

Theorem 4.1. *Let u be a continuous function supported in $\{y \in \mathbb{R}^d : |y| \leq 1/4\}$ such that $\|u\|_\infty \leq 1$ and*

$$\int u(y) dy = 0.$$

Let $\mathfrak{Q}_k$ be the operator of convolution with $u^{(2^k)}$. Let $0 < \varepsilon < 1$, $\zeta \in \mathcal{B}_\varepsilon(\mathbb{R}^n \times \mathbb{R}^d)$ and assume that $\text{supp}(\zeta) \subset \{(s, v) : |v| \leq 1/4\}$. Then for all $k \in \mathbb{N}$, for $b_{n+1}, b_{n+2} \in L^2(\mathbb{R}^d)$, $b_i \in L^\infty(\mathbb{R}^d)$, $i = 1, \dots, n$,

$$|\Lambda[\zeta](b_1, \dots, \mathfrak{Q}_k b_{n+1}, b_{n+2})| \lesssim 2^{-k\varepsilon/(3d+3)} n \|\zeta\|_{B_\varepsilon} \|b_{n+1}\|_2 \|b_{n+2}\|_2 \prod_{i=1}^n \|b_i\|_\infty.$$

Write

$$(61) \quad \Lambda[\zeta](b_1, \dots, \mathfrak{Q}_k b_{n+1}, b_{n+2}) = \iint F_k[\zeta](x, y) b_{n+1}(y) b_{n+2}(x) dx dy,$$

where, using the cancellation of u , we have

$$F_k[\zeta](x, y) = \iint \left[\zeta(s, x - z) \prod_{i=1}^n b_i(x - s_i(x - z)) - \zeta(s, x - y) \prod_{i=1}^n b_i(x - s_i(x - y)) \right] u^{(2^k)}(z - y) dz ds.$$

Here $u^{(2^k)} = 2^{kd}u(2^k \cdot)$. Using a Leibniz rule for differences one can write

$$F_k[\varsigma] = \sum_{i=0}^n F_{k,i}[\varsigma]$$

where

$$\begin{aligned} F_{k,0}[\varsigma](x, y) &= \iint [\varsigma(s, x - z) - \varsigma(s, x - y)] \prod_{j=1}^n b_j(x - s_j(x - z)) u^{(2^k)}(z - y) dz ds, \\ F_{k,i}[\varsigma](x, y) &= \iint \varsigma(s, x - y) \prod_{j=1}^{i-1} b_j(x - s_j(x - y)) \times \\ &\quad \left(b_i(x - s_i(x - z)) - b_i(x - s_i(x - y)) \right) \prod_{j=i+1}^n b_j(x - s_j(x - z)) u^{(2^k)}(z - y) dz ds, \end{aligned}$$

with the convention that the products $\prod_{j=1}^0$ and $\prod_{j=n+1}^n$ stand for the number 1. We let $T_k[\varsigma]$, $T_{k,i}[\varsigma]$ denote the operator with Schwartz kernels $F_k[\varsigma]$, $F_{k,i}[\varsigma]$. We thus must estimate the $L^2 \rightarrow L^2$ operator norms for the operators $T_{k,i}[\varsigma]$.

The case $i = 0$ is straightforward; we use the standard Schur test and the estimation relies just on the assumption $\varsigma \in \mathcal{B}_\varepsilon$:

$$\begin{aligned} &\sup_x \int |F_{k,0}[\varsigma](x, y)| dy \\ &\leq \sup_x \prod_{j=1}^n \|b_j\|_\infty \int_{|h| \leq 2^{-k}} |u^{(2^k)}(h)| \int |\varsigma(s, x - y - h) - \varsigma(s, x - y)| dy ds dh \\ &\lesssim \prod_{j=1}^n \|b_j\|_\infty \sup_{|h| \leq 2^{-k}} \int \|\varsigma(s, \cdot - h) - \varsigma(s, \cdot)\| ds \lesssim 2^{-k\varepsilon} \prod_{j=1}^n \|b_j\|_\infty \|\varsigma\|_{\mathcal{B}_\varepsilon}. \end{aligned}$$

Similarly

$$\sup_y \int |F_{k,0}[\varsigma](x, y)| dx \lesssim 2^{-k\varepsilon} \prod_{j=1}^n \|b_j\|_\infty \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

Hence the Schur test implies

$$(63) \quad \|T_{k,0}[\varsigma]\|_{L^2 \rightarrow L^2} \lesssim 2^{-k\varepsilon} \prod_{j=1}^n \|b_j\|_\infty \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

The Schur test applied to $T_{k,i}[\varsigma]$, $i = 1, \dots, n$ gives

$$(64) \quad \|T_{k,i}[\varsigma]\|_{L^p \rightarrow L^p} \lesssim \|\varsigma\|_{L^1(\mathbb{R}^n \times \mathbb{R}^d)} \prod_{j=1}^n \|b_j\|_\infty.$$

Now we need to get a nontrivial bound for $\|T_{k,i}[\varsigma]\|_{L^2 \rightarrow L^2}$, when $i = 1, \dots, n$ and $k \gg 1$. We begin with a regularization of ς , in the x and the s_i variables, depending on a parameter R to be chosen later. Here $1 \ll R \ll 2^k$ (we'll see later that $R = 2^{k/(3d+3)}$ turns out to be a good choice).

Let $\phi \in C^\infty(\mathbb{R}^d)$ supported in $\{x : |x| \leq 1/2\}$ so that $\int \phi(x) dx = 1$. Let $\varphi \in C^\infty(\mathbb{R})$ be supported in $\{u : |u| \leq 1/2\}$ so that $\int \varphi(u) du = 1$. Define

$$\varsigma_R^i(s, v) = \iint \chi_{[-R, R]}(s - \rho e_i) \varsigma(s - \rho e_i, v - z) R \varphi(R \rho) R^d \phi(R z) dz d\rho.$$

Lemma 4.2. For $i = 1, \dots, n$,

(i)

$$\|\varsigma - \varsigma_R^i\|_{L^1(\mathbb{R}^n \times \mathbb{R}^d)} \lesssim R^{-\varepsilon}$$

(ii)

$$\|T_{k,i}[\varsigma - \varsigma_R^i]\|_{L^2 \rightarrow L^2} \lesssim R^{-\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

Proof. (ii) follows from (i) by the trivial bound (64) applied with $\varsigma - \varsigma_R^i$. We expand $\varsigma - \varsigma_R^i = I + II + III$ where

$$\begin{aligned} I(s, v) &= \int [\varsigma(s, v) - \varsigma(s, v - z)] R^d \phi(Rz) dz, \\ II(s, v) &= \iint [\varsigma(s, v - z) - \varsigma(s - \rho e_i, v - z)] R \varphi(R\rho) R^d \phi(Rz) dz d\rho, \\ III(s, v) &= \iint \chi_{[-R, R]} \mathbb{C}(s - \rho e_i) \varsigma(s - \rho e_i, v - z) R \varphi(R\rho) R^d \phi(Rz) dz d\rho. \end{aligned}$$

Then

$$\|I\|_{L^1(\mathbb{R}^n \times \mathbb{R}^d)} \lesssim \int R^d |\phi(Rz)| \iint |\varsigma(s, v) - \varsigma(s, v - z)| ds dv |R^d \phi(Rz)| dz \lesssim R^{-\varepsilon} \|\varsigma\|_{\mathcal{B}_{\varepsilon, 3}}.$$

For the second term,

$$\|II\|_{L^1(\mathbb{R}^n \times \mathbb{R}^d)} \lesssim \int R |\varphi(R\rho)| \iint |\varsigma(s, v) - \varsigma(s - \rho e_i, v)| ds dv d\rho \lesssim R^{-\varepsilon} \|\varsigma\|_{\mathcal{B}_{\varepsilon, 2}}.$$

Finally

$$\|III\|_{L^1(\mathbb{R}^n \times \mathbb{R}^d)} \lesssim \int \int_{[-R, R] \mathbb{C}} |\varsigma(s, v)| ds dv \lesssim R^{-\varepsilon} \|\varsigma\|_{\mathcal{B}_{\varepsilon, 1}}$$

and part (i) follows. The second part follows from Lemma 64 applied to $\varsigma - \varsigma_R^i$, and the first part. $\square$

In order to estimate the term $T_{k,i}[\varsigma_R^i]$ we shall need L^2 inequalities for ς_R^i .

Lemma 4.3. Let $0 < \varepsilon < 1$, $d \geq 2$. Then

(i)

$$\int \left(\int |\varsigma_R^i(s, v)|^2 dv \right)^{\frac{1}{2}} ds \lesssim R^{\frac{d}{2} - \varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

(ii) Let $\theta \in S^{d-1}$ and let $\theta^\perp$ the orthogonal complement of $\mathbb{R}\theta$. Then

$$\int \sup_{\theta^\perp} \left(\int_{\theta^\perp} \sup_{\tau \in \mathbb{R}} |\varsigma_R^i(s, v^\perp + \tau\theta)|^2 dv_{\theta^\perp} \right)^{\frac{1}{2}} ds \lesssim R^{\frac{d+1}{2} - \varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}$$

and

$$\int \sup_{\theta} \left(\int_{\theta^\perp} \sup_{\tau \in \mathbb{R}} |\partial_{s_i} \varsigma_R^i(s, v^\perp + \tau\theta)|^2 dv_{\theta^\perp} \right)^{\frac{1}{2}} ds \lesssim R^{\frac{d+3}{2} - \varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}.$$

Proof. Let $\beta_0 \in \mathcal{S}(\mathbb{R}^d)$ so that $\widehat{\beta}_0(\xi) = 1$ for $|\xi| < 1/2$ and $\widehat{\beta}_0$ is supported in $\{|\xi| < 1\}$. Let $\beta_1 = \beta_0^{(2)} - \beta_0$ and $\beta_k = \beta_1^{(2^{k-1})}$ so that $\widehat{\beta}_k$ has support in an annulus $\{|\xi| \approx 2^k\}$, and $f = \sum_{k=0}^{\infty} \beta_k * f$ in the sense of distributions. Let $\widetilde{\beta}_0 \in \mathcal{S}(\mathbb{R}^d)$ be such that its Fourier transform equals 1 on the support of $\widehat{\beta}_0$. Let $\widetilde{\beta}_1$ be a Schwartz function so that its Fourier transform vanishes in a neighborhood of the origin and is compactly supported, and equal to 1 on the support of $\widehat{\beta}_1$. Let $\widetilde{\beta}_k = \widetilde{\beta}_1^{(2^{k-1})}$.

Let

$$\tilde{\varsigma}_R^i(s, v) = \iint \chi_{[-R, R]}(s - \rho e_i) \varsigma(s - \rho e_i, v) R \varphi(R \rho) d\rho$$

so that $\tilde{\varsigma}_R^i(s, \cdot) * \phi_R = \varsigma_R^i$ (the definition of φ was given right before the statement of Lemma 4.2). Then

$$\varsigma_R^i(s, \cdot) = \sum_{l=0}^{\infty} \beta_l * \tilde{\varsigma}_R^i(s, \cdot) * \phi_R * \tilde{\beta}_l.$$

By Young's inequality

$$\|\varsigma_R^i(s, v) * \tilde{\beta}_l * \beta_l\|_2 \leq \|\tilde{\varsigma}_R^i(s, \cdot) * \beta_l\|_1 \|\tilde{\beta}_l * \Phi_R\|_2$$

and it is easy to see that

$$\|\tilde{\beta}_l * \Phi_R\|_2 \leq C_M 2^{ld/2} \min\{1, (R2^{-l})^M\}.$$

Thus

$$\begin{aligned} & \int \left(\int |\varsigma_R^i(s, v)|^2 dv \right)^{\frac{1}{2}} ds \\ & \lesssim \sum_{l=0}^{\infty} 2^{ld/2} \min\{1, (R2^{-l})^M\} \iint \left| \int \beta_l(v - w) \varsigma_R^i(s, w) dw \right| dv ds \\ & \lesssim \sum_{l=0}^{\infty} 2^{ld/2} \min\{1, (R2^{-l})^M\} 2^{-l\varepsilon} \|\varsigma_R^i\|_{\mathcal{B}_\varepsilon} \lesssim R^{\frac{d}{2}-\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}. \end{aligned}$$

The first inequality in (ii) is proved similarly, except that we first use the one-dimensional version of Young's inequality in the θ -direction. Since the Fourier transform of β_l is supported on a set of diameter $O(2^l)$ we have, for fixed θ and almost every s ,

$$\left(\int_{\theta^\perp} \sup_{\tau \in \mathbb{R}} |\beta_l * \varsigma_R^i(s, v^\perp + \tau\theta)|^2 dv^\perp \right)^{\frac{1}{2}} \lesssim 2^{l/2} \left(\int_{\theta^\perp} \int_{-\infty}^{\infty} |\beta_l * \varsigma_R^i(s, v^\perp + \tau\theta)|^2 d\tau dv^\perp \right)^{\frac{1}{2}}.$$

Notice that the double integral on the right hand side is just the $L^2(\mathbb{R}^d)$ norm of $\varsigma_R^i(s, \cdot)$ and thus does not depend on θ . Take the sup over θ , then integrate in s , and sum in l . Arguing as above we obtain:

$$\begin{aligned} & \int \sup_{\theta} \left(\int_{\theta^\perp} \sup_{\tau \in \mathbb{R}} |\beta_l * \varsigma_R^i(s, v^\perp + \tau\theta)|^2 dv^\perp \right)^{\frac{1}{2}} ds \\ & \lesssim \sum_{l \geq 0} 2^{l/2} \int \left(|\beta_l * \varsigma_R^i(s, v)|^2 dv \right)^{1/2} ds \\ & \lesssim \sum_{l \geq 0} 2^{l(d+1)/2} \min\{1, (R2^{-l})^M\} \iint |\beta_l * \tilde{\varsigma}_R^i(s, v)| dv ds \\ & \lesssim \sum_{l \geq 0} 2^{l(d+1)/2} \min\{1, (R2^{-l})^M\} 2^{-l\varepsilon} \|\tilde{\varsigma}_R^i\|_{\mathcal{B}_\varepsilon} \lesssim R^{\frac{d+1}{2}-\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon}. \end{aligned}$$

The second inequality in (ii) is proved in the same way. The differentiation in s_i hitting the mollifier $R\varphi(R\cdot)$ produces an additional factor of R . $\square$

By the support assumptions on ς and u , we have

$$\text{supp}(F_{k,i}[\varsigma_R^i]) \subseteq \{(x, y) : |x - y| < 1\}.$$

Thus by a trivial orthogonality argument we can reduce the estimation for $T_{k,i}[\varsigma_R^i]$ to a Hilbert-Schmidt estimate, namely

$$(65) \quad \|T_{k,i}[\varsigma_R^i]\|_{L^2 \rightarrow L^2} \lesssim \left(\sup_z \iint_{\substack{|x-z| < 1 \\ |y-z| < 1}} |F_{k,i}[\varsigma_R^i](x, y)|^2 dx dy \right)^{1/2}.$$

Proposition 4.4. *For $k \geq 0$*

$$(66) \quad \left(\sup_{y_0} \iint_{\substack{|x-y_0| < 1 \\ |y-y_0| < 1}} |F_{k,i}[\varsigma_R^i](x, y)|^2 dx dy \right)^{\frac{1}{2}} \lesssim 2^{-k/3} R^{d+1} \prod_{i=1}^n \|b_i\|_{\infty}.$$

4.1. Proof of Proposition 4.4. Note that the class of operators is invariant under translations. I.e. if $\tau_a f := f(x - a)$, then the kernel of $\tau_a T_{k,i}[\varsigma_R^i] \tau_{-a}$, i.e. $F_{k,i}[\varsigma_R^i](x - a, y - a)$, is of the same form of $F_{k,i}$, with the functions b_j replaced by $\tau_a b_j$. Therefore we may take $y_0 = 0$ in Proposition 4.4. We may also assume

$$(67) \quad \|b_j\|_{\infty} \leq 1, \quad 1 \leq j \leq n.$$

We separate s as $s = s_i e_i + s_i^{\perp}$ where $s_i^{\perp} = (\dots, s_{i-1}, s_{i+1}, \dots) \in \mathbb{R}^{n-1}$. Use the Cauchy-Schwarz inequality in the z -variable, and then Minkowski's inequality in the $s_i^{\perp}$ variables, as well as (67) for $j \neq i$, to estimate

$$\begin{aligned} & \left(\iint_{\substack{|x| < 1 \\ |y| < 1}} |F_{k,i}[\varsigma_R^i](x, y)|^2 dx dy \right)^{\frac{1}{2}} \\ & \lesssim \int \left(\iiint_{\substack{|x|, |y| \leq 1 \\ |y-z| \leq 2^{-k}}} 2^{kd} \left| \int \varsigma_R^i(s, x - y) [b_i(x - s_i(x - z)) - b_i(x - s_i(x - y))] ds_i \right|^2 dz dx dy \right)^{1/2} ds_i^{\perp} \\ & \lesssim \iiint_{\substack{|x|, |v|, |w| \leq 2 \\ |v-w| \leq 2^{-k}}} \left| \int \varsigma_R^i(s, v) [b_i(x - s_i v) - b_i(x - s_i w)] \right|^2 dv dw dx \right)^{1/2} ds_i^{\perp}. \end{aligned}$$

The proof of Proposition 4.4 will be complete after the following lemma is proved.

Lemma 4.5. *Let ς_R^i be as in Proposition 4.4. Then for $g \in L^{\infty}(\mathbb{R}^d)$ and $k > 0$,*

$$\left(2^{kd} \iiint_{\substack{|x| < 2 \\ |v|, |w| < 2 \\ |v-w| < 2^{-k}}} \left| \int \varsigma_R^i(s, v) (g(x - s_i v) - g(x - s_i w)) ds \right|^2 dx dv dw \right)^{\frac{1}{2}} \lesssim R^{d+1-\varepsilon} 2^{-k\varepsilon/3} \|\varsigma\|_{\mathcal{B}_{\varepsilon}} \|g\|_{\infty}.$$

Proof. We may and shall assume $\|g\|_{L^{\infty}} = 1$. Let $g_R(x) = g(x)$ if $|x| \leq 2R + 2$ and $g_R(x) = 0$ if $|x| > 2R + 2$. We first observe that since $\varsigma_R^i(s, v) = 0$ for $|s_i| \geq R + 1$ we may replace g by g_R in the above expression. Note that

$$(68) \quad \|g_R\|_2 \lesssim R^{d/2}.$$

We interchange the (v, w) and x integrations and then apply Plancherel's theorem, and interchange integrals again to get

$$\begin{aligned} & \int \left(2^{kd} \int \int \int_{\substack{|x| < 2 \\ |v|, |w| < 2 \\ |v-w| < 2^{-k}}} \left| \int \varsigma_R^i(s, v) (g_R(x - s_i v) - g_R(x - s_i w)) ds \right|^2 dx dv dw \right)^{\frac{1}{2}} ds_i^\perp \\ &= \int \left(\int |\widehat{g}_R(\xi)|^2 2^{kd} \int \int_{\substack{|v|, |w| < 2 \\ |v-w| < 2^{-k}}} \left| \int \varsigma_R^i(s, v) (e^{2\pi i s_i \langle v, \xi \rangle} - e^{2\pi i s_i \langle w, \xi \rangle}) ds_i \right|^2 dv dw d\xi \right)^{\frac{1}{2}} ds_i^\perp. \end{aligned}$$

For a constant $U \geq 1$ (to be determined) we split the ξ -integration into the parts for $|\xi| \leq U$ and $|\xi| \geq 1$.

For $|\xi| \leq U$ we bound $|e^{2\pi i s_i \langle v, \xi \rangle} - e^{2\pi i s_i \langle w, \xi \rangle}| \lesssim RU2^{-k}$ since $|s_i| \leq (R+1)$ and $|v-w| \leq 2^{-k}$. Hence we obtain

$$\begin{aligned} (69) \quad & \int \left(\int_{|\xi| \leq U} |\widehat{g}_R(\xi)|^2 2^{kd} \int \int_{\substack{|v|, |w| < 2 \\ |v-w| < 2^{-k}}} \left| \int \varsigma_R^i(s, v) (e^{2\pi i s_i \langle v, \xi \rangle} - e^{2\pi i s_i \langle w, \xi \rangle}) ds_i \right|^2 dv dw d\xi \right)^{\frac{1}{2}} ds_i^\perp \\ & \lesssim RU2^{-k} \|\widehat{g}_R\|_2 \int \left(\int |\varsigma_R^i(s, v)|^2 dv \right)^{1/2} ds \\ & \lesssim R^{\frac{d+2}{2}-\varepsilon} U 2^{-k} \|g_R\|_2 \|\varsigma\|_{\mathcal{B}_\varepsilon} \end{aligned}$$

where in the last inequality we have used part (i) of Lemma 4.3.

Next we consider the part when $|\xi| > U$. Using the symmetry in v, w we may estimate

$$\begin{aligned} & \int \left(\int_{|\xi| \geq U} |\widehat{g}_R(\xi)|^2 2^{kd} \int \int_{\substack{|v|, |w| < 2 \\ |v-w| < 2^{-k}}} \left| \int \varsigma_R^i(s, v) (e^{2\pi i \langle v, \xi \rangle s_i} - e^{2\pi i \langle w, \xi \rangle s_i}) ds_i \right|^2 dv dw d\xi \right)^{\frac{1}{2}} ds_i^\perp \\ & \leq 2 \int \left(\int_{|\xi| \geq U} |\widehat{g}_R(\xi)|^2 2^{kd} \int \int_{\substack{|v|, |w| < 2 \\ |v-w| < 2^{-k}}} \left| \int \varsigma_R^i(s, v) e^{2\pi i \langle v, \xi \rangle s_i} ds_i \right|^2 dv dw d\xi \right)^{\frac{1}{2}} ds_i^\perp \\ & \lesssim \int \left(\int_{|\xi| \geq U} |\widehat{g}_R(\xi)|^2 \int \left| \int \varsigma_R^i(s, v) e^{2\pi i \langle v, \xi \rangle s_i} ds_i \right|^2 dv d\xi \right)^{\frac{1}{2}} ds_i^\perp. \end{aligned}$$

For fixed $\xi = |\xi|\theta$ ($\theta \in S^{d-1}$) we separate the v -integral into two parts. Let $0 < b < 1$ (which will be optimally chosen later). For fixed $\theta = \xi/|\xi|$, $s_i^\perp$ we have $v = \pi_{\theta^\perp} v + s\theta$ where $\pi_{\theta^\perp} v$ is the projection of v to the orthogonal complement of $\mathbb{R}\theta$ and $\tau = \langle \theta, v \rangle$. We split

$$\begin{aligned} \int \left| \int \varsigma_R^i(s, v) e^{2\pi i \langle v, \xi \rangle s_i} ds_i \right|^2 dv &= \int \int \left| \int \varsigma_R^i(s, \pi_{\theta^\perp} v + \tau\theta) e^{2\pi i \tau |\xi| s_i} ds_i \right|^2 d\tau dv_{\theta^\perp} \\ &=: I_b(s_i^\perp, |\xi|\theta) + II_b(s_i^\perp, |\xi|\theta) \end{aligned}$$

where

$$\begin{aligned} I_b(s_i^\perp, |\xi|\theta) &:= \int \int_{[-b, b]} \left| \int \varsigma_R^i(s, \pi_{\theta^\perp} v + \tau\theta) e^{2\pi i \tau |\xi| s_i} ds_i \right|^2 d\tau dv_{\theta^\perp} \\ II_b(s_i^\perp, |\xi|\theta) &:= \int \int_{[-b, b]^c} \left| \int \varsigma_R^i(s, \pi_{\theta^\perp} v + \tau\theta) e^{2\pi i \tau |\xi| s_i} ds_i \right|^2 d\tau dv_{\theta^\perp} \end{aligned}$$

so that

$$\begin{aligned} & \int \left(\int_{|\xi| \geq U} |\widehat{g}_R(\xi)|^2 \int \left| \int \varsigma_R^i(s, v) e^{2\pi i \langle v, \xi \rangle s_i} ds_i \right|^2 dv d\xi \right)^{\frac{1}{2}} ds_i^\perp \\ & \lesssim \int \left(\int_{|\xi| \geq U} |\widehat{g}_R(\xi)|^2 [I_b(s_i^\perp, \xi) + II_b(s_i^\perp, \xi)] d\xi \right)^{\frac{1}{2}} ds_i^\perp. \end{aligned}$$

The expression I_b is estimated as

$$I_b(s_i^\perp, |\xi|\theta) \leq 2b \int \sup_{|\tau| \leq b} \left[\int |\varsigma_R^i(s, \pi_{\theta^\perp} v + \tau\theta)| ds_i \right]^2 dv_{\theta^\perp}$$

and we get using part (ii) of Lemma 4.3

$$\begin{aligned} & \int \left(\int_{|\xi| \geq U} |\widehat{g}_R(\xi)|^2 I_b(s_i^\perp, \xi) d\xi \right)^{\frac{1}{2}} ds_i^\perp \\ & \lesssim b^{1/2} \|g_R\|_2 \int \left(\sup_{\theta} \int \sup_{\tau} \left[\int |\varsigma_R^i(s, \pi_{\theta^\perp} v + \tau\theta)| ds_i \right]^2 dv_{\theta^\perp} \right)^{1/2} ds_i \\ (70) \quad & \lesssim b^{1/2} R^{\frac{d+1}{2}-\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon} \|g_R\|_2. \end{aligned}$$

To estimate $II_b(s_i^\perp, \xi)$ we observe that the function $s_i \mapsto \varsigma_R^i(s, v)$ is smooth and compactly supported. We use integration by parts to write

$$\int \varsigma_R^i(s, \pi_{\theta^\perp} v + \tau\theta) e^{2\pi i \tau |\xi| s_i} ds_i = - \int \partial_{s_i} \varsigma_R^i(s, \pi_{\theta^\perp} v + \tau\theta) (2\pi i |\xi|)^{-1} \tau^{-1} e^{2\pi i \tau s_i} ds_i$$

and thus for $|\xi| \geq U$

$$\begin{aligned} II_b(s_i^\perp, |\xi|\theta) & \leq \int_b^\infty |\xi|^{-2} |\tau|^{-2} d\tau \int \sup_t \left[\int |\partial_{s_i} \varsigma_R^i(s, \pi_{\theta^\perp} v + t\theta)| ds_i \right]^2 dv_{\theta^\perp} \\ & \lesssim U^{-2} b^{-1} \int \sup_t \left[\int |\partial_{s_i} \varsigma_R^i(s, \pi_{\theta^\perp} v + t\theta)| ds_i \right]^2 dv_{\theta^\perp}. \end{aligned}$$

Hence, by the second inequality in part (ii) of Lemma 4.3,

$$\begin{aligned} & \int \left(\int_{|\xi| \geq U} |\widehat{g}_R(\xi)|^2 II_b(s_i^\perp, \xi) d\xi \right)^{\frac{1}{2}} ds_i^\perp \\ & \lesssim U^{-1} b^{-1/2} \|g_R\|_2 \int \left(\sup_{\theta} \int \sup_t \left[\int |\partial_{s_i} \varsigma_R^i(s, \pi_{\theta^\perp} v + t\theta)| ds_i \right]^2 dv_{\theta^\perp} \right)^{1/2} ds_i \\ (71) \quad & \lesssim U^{-1} b^{-1/2} R^{\frac{d+3}{2}-\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon} \|g_R\|_2. \end{aligned}$$

We combine (69), (70), (71) to deduce

$$\begin{aligned} & \int \left(2^{kd} \int \int \int_{\substack{|x| < 2 \\ |v|, |w| < 2 \\ |v-w| < 2^{-k}}} \left| \int \varsigma_R^i(s, v) (g_R(x - s_i v) - g_R(x - s_i w)) ds \right|^2 dx dv dw \right)^{\frac{1}{2}} ds_i^\perp \\ (72) \quad & \lesssim \left(R^{\frac{d+2}{2}-\varepsilon} U 2^{-k} + R^{\frac{d+1}{2}-\varepsilon} b^{1/2} + R^{\frac{d+3}{2}-\varepsilon} U^{-1} b^{-1/2} \right) \|\varsigma\|_{\mathcal{B}_\varepsilon} \|g_R\|_2. \end{aligned}$$

We choose b, U so that the three terms are comparable, i.e. $b = RU^{-1}$, $U = 2^{2k/3}$. The result is that the left hand side of (72) is bounded by a constant times

$$R^{\frac{d+2}{2}-\varepsilon} 2^{-k/3} \|\varsigma\|_{\mathcal{B}_\varepsilon} \|g_R\|_2 \lesssim R^{d+1-\varepsilon} 2^{-k/3} \|\varsigma\|_{\mathcal{B}_\varepsilon},$$

by (68), and the proof is complete. $\square$

Proof of Theorem 4.1. By (63),

$$\|T_{k,0}[\varsigma]\|_{L^2 \rightarrow L^2} \lesssim 2^{-k\varepsilon} \|\varsigma\|_{\mathcal{B}_\varepsilon} \prod_{l=1}^n \|b_l\|_\infty.$$

By Lemma 4.2 and Proposition 4.4 we have for $i = 1, \dots, n$,

$$\begin{aligned} \|T_{k,i}[\varsigma]\|_{L^2 \rightarrow L^2} &\leq \|T_{k,i}[\varsigma - \varsigma_R^i]\|_{L^2 \rightarrow L^2} + \|T_{k,i}[\varsigma_R^i]\|_{L^2 \rightarrow L^2} \\ &\lesssim R^{-\varepsilon} (1 + 2^{-k/3} R^{d+1}) \|\varsigma\|_{\mathcal{B}_\varepsilon} \prod_{l=1}^n \|b_l\|_\infty. \end{aligned}$$

Choosing $R = 2^{k/(3d+3)}$ yields the bound

$$\sum_{i=0}^n \|T_{k,i}[\varsigma]\|_{L^2 \rightarrow L^2} \lesssim (n+1) 2^{-k\varepsilon/(3d+3)} \|\varsigma\|_{\mathcal{B}_\varepsilon} \prod_{l=1}^n \|b_l\|_\infty$$

and thus the estimates for the multilinear forms claimed in Theorem 4.1. $\square$

5. ON WEAK TYPE $(1, 1)$ BOUNDS

5.1. $L \log L \rightarrow L^1$ from a weak type $(1, 1)$ inequality. The $L \log L(\mathbb{T}^d) \rightarrow L^1(\mathbb{T}^d)$ bound in (16) follows from a weak type inequality for Christ-Journé operators. We recall the standard argument. Let the $L \log L$ norm be defined as in (5). Clearly $\|f\|_1 \leq \|f\|_{L \log L}$.

Lemma 5.1. *Let $q > 1$ and let T be an operator bounded on $L^q(\mathbb{R}^d)$, with $\|T\|_{L^q \rightarrow L^q} \leq A$. Let T be of weak type $(1, 1)$, i.e.*

$$\text{meas}(\{x : |Tf(x)| > \lambda\}) \leq B\lambda^{-1} \|f\|_1.$$

Then, for any compact set T maps $L \log L$ to $L^1(K)$ such that

$$(73) \quad \|Tf\|_{L^1(K)} \leq \left(\frac{2A}{q-1} \|f\|_1 + (2A \text{meas}(K) + B) \|f\|_{L \log L} \right)$$

Proof. We assume $q < \infty$ and leave the proof for $q = \infty$ to the reader. Let

$$b := 2A \|f\|_{L \log L}.$$

Write

$$\|Tf\|_{L^1(K)} = \int_0^b + \int_b^\infty \text{meas}(\{x \in K : |Tf(x)| > \lambda\}) d\lambda =: I + II.$$

Obviously

$$|I| \leq b \text{meas}(K).$$

Now let $g_\lambda(x) = f(x)$ if $|f(x)| \leq 2A\lambda$ and $g_\lambda(x) = 0$ otherwise. Let $h_\lambda(x) = f(x) - g_\lambda(x)$. Then

$$II \leq \int_b^\infty \text{meas}(\{x \in K : |Tg_\lambda(x)| > \lambda/2\}) d\lambda + \int_b^\infty \text{meas}(\{x \in K : |Th_\lambda(x)| > \lambda/2\}) d\lambda.$$

By Tshebyshev's inequality and the assumed L^q boundedness we have

$$\text{meas}(\{x \in K : |Tg_\lambda(x)| > \lambda/2\}) \leq (2A)^q \lambda^{-q} \|g_\lambda\|_q^q$$

and thus

$$\begin{aligned} \int_b^\infty \text{meas}(\{x \in K : |Tg_\lambda(x)| > \lambda/2\}) d\lambda &\leq (2A)^q \int_b^\infty \lambda^{-q} \|g_\lambda\|_q^q d\lambda \\ &\leq (2A)^q \int |f(x)|^q \int_{\lambda \geq 2A|f(x)|} \lambda^{-q} d\lambda dx \leq \frac{2A}{q-1} \int |f(x)| dx. \end{aligned}$$

Also

$$\begin{aligned}
& \int_b^\infty \text{meas}(\{x \in K : |Th_\lambda(x)| > \lambda/2\}) d\lambda \leq \int_b^\infty 2\lambda^{-1} B \|h_\lambda\|_1 d\lambda \\
& \leq 2B \int_{|f(x)| \geq 2b} |f(x)| \int_b^{2A|f(x)|} \frac{d\lambda}{\lambda} dx \leq 2B \int |f(x)| \log_+ \left(\frac{2A|f(x)|}{b} \right) dx \\
& \leq \frac{Bb}{2A} \int \frac{2A}{b} |f(x)| \log_+ \left(\frac{2A|f(x)|}{b} \right) dx \leq B \|f\|_{L \log L}
\end{aligned}$$

by the definitions of b and $\|f\|_{L \log L}$. Now combine the various estimates. $\square$

5.2. Weak type inequality. By the reduction in §1.6 the $L \log L$ bound follows from the above lemma and

Theorem 5.2. *Let T_a be as in (26), and let $a \in L^\infty$. Then T_a is of weak type $(1, 1)$.*

The proof of this result is analogous to an estimate for the original Christ-Journé operators with $n = 1$. It was shown in [36] that for $a \in L^\infty$ the operator $f \mapsto \mathcal{C}_{K,1}[f, a]$ is of weak type $(1, 1)$ (in two dimensions that was already proved by Grafakos and Honzík [21]). These arguments can be extended to cover the operators discussed in §2 when $n = 1$. Namely if $\kappa \in \mathcal{K}_\varepsilon$, $a \in L^\infty$ then the operator defined by

$$T_a f(x) = \iint \kappa(x, s) a((1-s)x + sy) f(y) dy$$

is of weak type $(1, 1)$.

The proof of these results uses methods that were previously developed to prove weak type $(1, 1)$ inequality for rough singular convolution operator, of the form

$$\int \frac{\Omega(y/|y|)}{|y|^d} f(x-y) dy$$

where Ω is homogeneous of degree 0, $\Omega \in L \log L(S^{d-1})$ (or even just $\Omega \in L^\infty(S^{d-1})$ satisfying $\int_{S^{d-1}} \Omega(w) d\sigma(w) = 0$). See Christ [8], Christ and Rubio de Francia [10], Hofmann [25], Seeger [35] and Tao [40]. The ideas in [8], [25] and especially [10] have been used by Tao [40] also for non-convolution operators, however they lead to substantial difficulties in higher dimensions. The proof in [36] is built on an approach in [35] which uses the Fourier transform in an essential way. This becomes less transparent for the non-convolution operators considered here but a main idea, an almost orthogonality argument for a microlocal decomposition of the operator, still applies.

For simplicity we will discuss only a model operator which captures the main features. Fix $a \in L^\infty(\mathbb{R}^d)$ with $\|a\|_\infty \leq 1$. Let ϑ be a C^∞ function supported in $[1/4, 3/4]$ and define

$$Tf = \int K(x-y) \int \vartheta(s) a((1-s)x + sy) ds f(y) dy$$

where $a \in L^\infty(\mathbb{R}^d)$ and K is a Calderón-Zygmund convolution kernel satisfying

$$|K(x)| + |x| |\nabla K(x)| \lesssim |x|^{-d}$$

and $\|\widehat{K}\|_\infty \leq 1$. We will outline the main steps necessary to prove the inequality

$$(74) \quad \sup_\lambda \lambda \text{meas}(\{x : |Tf(x)| > \lambda\}) \lesssim \|f\|_1.$$

5.3. *Calderón-Zygmund decomposition.* Let $f \in L^1(\mathbb{R}^d)$. We use the standard Calderón-Zygmund decomposition of f at height λ and decompose

$$f = g + b = g + \sum_{Q \in \mathfrak{Q}_\lambda} b_Q$$

where $\|g\|_\infty \leq \lambda$, $\|g\|_1 \lesssim \|f\|_1$, each b_Q is supported in a dyadic cube Q with sidelength $2^{L(Q)}$ and center y_Q , and $\mathfrak{Q}_\lambda$ is a family of dyadic cubes with disjoint interiors. Also $\|b_Q\|_1 \lesssim \lambda|Q|$ for each $Q \in \mathfrak{Q}_\lambda$ and $\sum_{Q \in \mathfrak{Q}_\lambda} |Q| \lesssim \lambda^{-1} \|f\|_1$. For each Q let Q^* be the dilate of Q with same center and $L(Q^*) = L(Q) + 10$, and let $E = \bigcup_{Q \in \mathfrak{Q}_\lambda} Q^*$. Then $\text{meas}(E) \lesssim \lambda^{-1} \|f\|_1$. Finally, for each Q , the mean value of b_Q vanishes: $\int b_Q(y) dy = 0$.

Since T is bounded on L^2 we may use the standard argument for the "good" function g

$$\|Tg\|_2^2 \leq \|T\|_{L^2 \rightarrow L^2}^2 \|g\|_2^2 \lesssim \|g\|_1 \|g\|_\infty \lesssim \lambda \|g\|_1$$

and by Tshebyshev's inequality,

$$|\{x \in \mathbb{R}^d : |Tg(x)| > \lambda/10\}| \leq 100\lambda^{-2} \|Tg\|_2^2 \lesssim \lambda^{-1} \|g\|_1 \lesssim \lambda^{-1} \|f\|_1.$$

We use a dyadic decomposition of the kernel. Let u be a radial C^∞ function, so that $u(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 6/5$. Let

$$K_j(x) = (u(2^{-j}x) - u(2^{-j+1}x))K(x)$$

so that $K = \sum K_j$ in $\mathcal{D}'(\mathbb{R}^d \setminus \{0\})$ and K_j is supported in the annulus $\{|x| \approx 2^j\}$. Let T_j be the integral operator with Schwartz kernel

$$K_j(x-y) \int_0^1 a(sx + (1-s)y) ds.$$

For $m \in \mathbb{Z}$ let

$$B_m = \sum_{\substack{Q \in \mathfrak{Q}_\lambda \\ L(Q)=m}} b_Q.$$

Then $T_j B_m \in L^1$ for each j, m , $\text{supp}(T_j B_m) \subset E$, $m \geq j$. Also, for fixed n , $\sum_j \|T_j B_{j-n}\|_1 \lesssim \|f\|_1$. It thus suffices to show that for $\tilde{\lambda} < \lambda$

$$(75) \quad \text{meas}(\{x \in \mathbb{R}^d \setminus E : \sum_{n>0} |\sum_j T_j B_{j-n}(x)| > \tilde{\lambda}\}) \lesssim \tilde{\lambda}^{-1} \|f\|_1.$$

We now need some finer decompositions of the operators T_j . Let Φ be a radial C_0^∞ function supported in $\{|x| \leq 1\}$, and satisfying $\int \Phi(x) dx = 1$, set $\Phi_m(x) = 2^{-md} \Phi(2^{-m}x)$, and denote by P_m the operator of convolution with Φ_m . Let $\ell(n) = 2 + [2 \log_2(n)]$. We have the following rather straightforward result.

Proposition 5.3. *For $n = 2, 3, \dots$,*

$$\|P_{j-n+\ell(n)} T_j B_{j-n}\|_1 \lesssim n^{-2} \log n \|B_{j-n}\|_1.$$

Proof. Fix $Q \in \mathfrak{Q}_\lambda$ with $L(Q) = j - n$. Then

$$\begin{aligned} P_{j-n+\ell(n)} T_j b_Q(x) &= \int \vartheta(s) \int b_Q(y) \times \\ &\quad \left[\int \Phi_{j-n+\ell(n)}(x-w) K_j^n(w-y) a(sw + (1-s)y) dw \right] dy ds. \end{aligned}$$

Change variables $z = w + \frac{1-s}{s}y$ to get

$$P_{j-n+\ell(n)}T_j b_Q(x) = \int \vartheta(s) \int a(sz) \int \mathcal{A}_{j,n}^{x,z,s}(y) b_Q(y) dy dz ds$$

where

$$\mathcal{A}_{j,n}^{x,z,s}(y) = \Phi_{j-n+\ell(n)}(x - z + \frac{1-s}{s}y) K_j(z - \frac{y}{s}).$$

Expand $\mathcal{A}_{j,n}^{x,z,s}(y)$ about the center y_Q of Q . Since $\int b_Q(y) dy = 0$ we have

$$|P_{j-n+\ell(n)}T_j b_Q(x)| \leq \iint |\vartheta(s)a(sz)| \left| \int (\mathcal{A}_{j,n}^{x,z,s}(y) - \mathcal{A}_{j,n}^{x,z,s}(y_Q)) b_Q(y) dy \right| dz ds.$$

Using

$$\mathcal{A}_{j,n}^{x,z,s}(y) - \mathcal{A}_{j,n}^{x,z,s}(y_Q) = \langle y - y_Q, \int_0^1 \nabla \mathcal{A}_{j,n}^{x,z,s}(y_Q + \sigma(y - y_Q)) d\sigma \rangle$$

one obtains

$$\begin{aligned} \|P_{j-n+\ell(n)}T_j^n b_Q(x)\|_1 &\leq \text{diam}(Q) \int_{1/4}^{3/4} \int |\vartheta(s)| \times \\ &\quad \left[\|\nabla \Phi_{j-n+\ell(n)}\|_1 \frac{1-s}{s} \int |b_Q(y)| \int |K_j(z - \frac{y_Q + \sigma(y - y_Q)}{s})| dz dy \right. \\ &\quad \left. + \|\Phi_{j-n+\ell(n)}\|_1 \int |b_Q(y)| \int \frac{1}{s} |\nabla K_j(z - \frac{y_Q + \sigma(y - y_Q)}{s})| dz dy \right] ds d\sigma. \end{aligned}$$

Now $\|\nabla K_j\|_1 \lesssim 2^{-j}$ and since $\text{diam}(Q) \lesssim 2^{j-n}$ we obtain $\|P_{j-n+\ell(n)}T_j^n b_Q\|_1 \lesssim n^{-2} \log n \|b_Q\|_1$ which implies the asserted bound. $\square$

As a consequence of the proposition we get by Tshebyshev's inequality It yields

$$\begin{aligned} &\text{meas}(\{x \in \mathbb{R}^d \setminus E : \sum_{n>0} \left| \sum_j P_{j-n+\ell(n)}T_j B_{j-n}(x) \right| > \lambda/10\}) \\ &\lesssim 10\lambda^{-1} \sum_{n>0} \sum_j \|P_{j-n+\ell(n)}T_j^n B_{j-n}\|_1 \\ &\lesssim \lambda^{-1} \sum_{n>0} n^{-2} \sum_j \|B_{j-n}\|_1 \lesssim \lambda^{-1} \|f\|_1. \end{aligned}$$

Thus we need to estimate

$$(76) \quad \text{meas}(\{ \sum_{n>0} \left| \sum_j (I - P_{j-n+\ell(n)})T_j^n B_{j-n}(x) \right| > \lambda/2 \}).$$

We shall need to exploit the fact that the integral $\int_0^1 a(sx + (1-s)y) ds$ has a regularizing effect for the rough function a in the direction parallel to $x - y$. Based on this insight we use a microlocal decomposition which we now describe.

Let $1/10 < \gamma < 9/10$ (say $\gamma = 1/2$), and let Θ_n be set of unit vectors with the property that if $\nu \neq \nu'$, $\nu, \nu' \in \Theta_n$ then $|\nu - \nu'| \geq 2^{-4-n\gamma}$, and assume that Θ_n is *maximal* with respect to this property. Clearly $\text{card}(\Theta_n) \lesssim 2^{n\gamma(d-1)}$. For each ν we may choose a function $\tilde{\chi}_{n,\nu}$ on $C^\infty(S^{d-1})$ with the property that $\tilde{\chi}_{n,\nu}(x) \geq 0$, $\tilde{\chi}_{n,\nu}(\theta) = 1$ if $|\theta - \nu| \leq 2^{-3-n\gamma}$, $\tilde{\chi}_{n,\nu}(\theta) = 0$ if $|\theta - \nu| > 2^{-2-n\gamma}$, and such that for each $M \in \mathbb{N}$ the functions $2^{-n\gamma M} \tilde{\chi}_{n,\nu}$ form a bounded family in $C^M(S^{d-1})$. For each θ there is at least one ν such that $\tilde{\chi}_{n,\nu}(\theta) = 1$, by the maximality assumption, moreover by the separatedness assumption the number of $\nu \in \Theta_n$

for which $\tilde{\chi}_{n,\nu}(\theta) \neq 0$ is bounded above, uniformly in θ and n . We now form a homogeneous partition of unity, defining for $\nu \in \Theta_n$

$$\chi_{n,\nu}(x) = \frac{\tilde{\chi}_{n,\nu}(\frac{x}{|x|})}{\sum_{\nu' \in \Theta_n} \tilde{\chi}_{n,\nu'}(\frac{x}{|x|})}.$$

Then $\sum_{\nu \in \Theta_n} \chi_{n,\nu}(x) = 1$ for every $x \in \mathbb{R}^d \setminus \{0\}$ and by homogeneity we have the following estimates for multiindices α and $x \neq 0$,

$$\begin{aligned} |(\langle \nu, \nabla \rangle)^M \chi_{n,\nu}(x)| &\leq C_M |x|^{-M}, \\ |\partial^\alpha \chi_{n,\nu}(x)| &\leq C_\alpha 2^{n\gamma|\alpha|} |x|^{-|\alpha|}. \end{aligned}$$

Let $K_j^{n,\nu}(x) = K_j(x) \chi_{n,\nu}(x)$ and let $T_j^{n,\nu}$ be the operator with Schwartz kernel

$$K_j^{n,\nu}(x-y) \int \vartheta(s) a(sx + (1-s)y) ds.$$

We then have $T_j = \sum_{\nu \in \Theta_n} T_j^{n,\nu}$. Let $\rho \in C^\infty(\mathbb{R})$ so that $\rho(y) = 1$ for $|y| < 1/2$ and $\rho(y) = 0$ for $|y| \geq 1$. Let

$$1/10 < \gamma_0 < \gamma < 9/10$$

and define the singular convolution operator $\mathfrak{S}_{n,\nu}$ by

$$\widehat{\mathfrak{S}_{n,\nu} f}(\xi) = \rho(2^{n\gamma_0} \langle \nu, \frac{\xi}{|\xi|} \rangle) \widehat{f}(\xi).$$

We then have the following estimates for large n .

Proposition 5.4.

$$\left\| \sum_{\nu \in \Theta_n} \sum_j (I - P_{j-n+\ell(n)}) \mathfrak{S}_{n,\nu} T_j^{n,\nu} B_{j-n} \right\|_2^2 \lesssim 2^{-n\gamma_0} \lambda \|f\|_1.$$

Proposition 5.5. *For fixed $\nu \in \Theta_n$,*

$$\left\| \sum_j (I - P_{j-n+\ell(n)}) (I - \mathfrak{S}_{n,\nu}) T_j^{n,\nu} B_{j-n} \right\|_1 \lesssim n^{-2} 2^{-n\gamma(d-1)} \|f\|_1.$$

Given the propositions it is easy to show that $|(76)| \lesssim \lambda^{-1} \|f\|_1$ and thus to finish the proof of the main inequality (75). Indeed by Tshebyshev's inequality,

$$\begin{aligned} &\text{meas} \left(\left\{ x : \left| \sum_{n > n(\varepsilon)} \sum_j (I - P_{j-n+\ell(n)}) T_j^n B_{j-n}(x) \right| > \lambda/2 \right\} \right) \\ &\leq C \lambda^{-1} \left\| \sum_{n > 0} \sum_{\nu \in \Theta_n} \sum_j (I - P_{j-n+\ell(n)}) (I - \mathfrak{S}_{n,\nu}) T_j^{n,\nu} B_{j-n} \right\|_1 \\ &\quad + C \lambda^{-2} \left\| \sum_{n > 0} \sum_{\nu \in \Theta_n} \sum_j (I - P_{j-n+\ell(n)}) \mathfrak{S}_{n,\nu} T_j^{n,\nu} B_{j-n} \right\|_2^2 \end{aligned}$$

and by the propositions and Minkowski's inequality this is bounded by a constant times

$$\lambda^{-1} \|f\|_1 \left(\sum_n n^{-2} 2^{-n\gamma(d-1)} \text{card}(\Theta_n) + \sum_n 2^{-n\gamma_1} \right) \lesssim \lambda^{-1} \|f\|_1.$$

5.4. *Proof of Proposition 5.4.* ([35], [36]). The main observation is an orthogonality argument based on the fact that for fixed $n > 0$, we have

$$(77) \quad \sup_{\xi \neq 0} \sum_{\nu \in \Theta_n} |\rho(2^{n\gamma_0} \langle \nu, \frac{\xi}{|\xi|} \rangle)| \lesssim 2^{n\gamma(d-1)-n\gamma_0}.$$

To see this let $|\xi| = 1$ and assume $\rho(2^{n\gamma} n^{-5} \langle \nu, \xi \rangle) \neq 0$ then the distance of ν to the hyperplane $\xi^\perp$ is at most $C2^{-n\gamma_0}$ and since the vectors in Θ_n are $c2^{-n\gamma}$ -separated and $\gamma_0 < \gamma$ there are $O(2^{n\gamma(d-1)-n\gamma_0})$ such vectors, hence (77) holds. This implies

$$\begin{aligned} \left\| \sum_{\nu \in \Theta_n} \mathfrak{S}_{n,\nu} \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \right\|_2^2 \\ \lesssim 2^{n\gamma(d-1)-n\gamma_0} \sum_{\nu \in \Theta_n} \left\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \right\|_2^2 \end{aligned}$$

which is a substantial improvement over what follows from Minkowski's inequality. Since $\#\Theta_n \lesssim 2^{n\gamma(d-1)}$ the asserted inequality follows from

$$(78) \quad \left\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \right\|_2^2 \lesssim 2^{-2n\gamma(d-1)} \lambda \|f\|_1$$

for each $\nu \in \Theta_n$.

For the proof of (78) the cancellation of B_{j-n} plays no role. Let

$$H_j^{n,\nu}(x) = 2^{-jd} \chi_{\tau_j^{n,\nu}}(x).$$

where

$$\tau_j^{n,\nu} = \{x : |\langle x, \nu \rangle| \leq 2^{j+2}, |x - \langle x, \nu \rangle| \leq 2^{j+2-\gamma n}\}.$$

Then

$$|(I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n}(x)| \lesssim H_j^{n,\nu} * |B_{j-n}|(x).$$

Therefore

$$\begin{aligned} \left\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \right\|_2^2 \\ \lesssim 2 \sum_j \int |B_{j-n}(x)| \sum_{i \leq j} H_j^{n,\nu} * H_i^{n,\nu} * |B_{i-n}(x)| dx. \end{aligned}$$

Observe that $\|H_i^{n,\nu}\|_1 \lesssim 2^{-id} \text{meas}(\tau_i^{n,\nu}) \lesssim 2^{-n\gamma(d-1)}$ and thus

$$H_j^{n,\nu} * H_i^{n,\nu}(x) \lesssim 2^{-n\gamma(d-1)} 2^{-jd} \chi_{\tilde{\tau}_j^{n,\nu}}(x)$$

where $\tilde{\tau}_j^{n,\nu}$ is the double of $\tau_j^{n,\nu}$. Hence, for each $x \in \mathbb{R}^d$, $j \in \mathbb{Z}$,

$$\begin{aligned} \sum_{i \leq j} H_j^{n,\nu} * H_i^{n,\nu} * |B_{i-n}|(x) \\ \lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i \leq j} \int_{x+\tilde{\tau}_j^{n,\nu}} |B_{i-n}(y)| dy \\ \lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i \leq j} \sum_{\substack{Q \in \Omega_\lambda: \\ L(Q)=i-n \\ Q \cap (x+\tilde{\tau}_j^{n,\nu}) \neq \emptyset}} \int |b_Q(x)| dx \\ \lesssim 2^{-n\gamma(d-1)} 2^{-jd} \lambda \text{meas}(\tilde{\tau}_j^{n,\nu}) \lesssim 2^{-2n\gamma(d-1)} \lambda; \end{aligned}$$

here we have used $\|b_Q\|_1 \lesssim \lambda|Q|$, and the disjointness of the interiors of the cubes Q in $\mathfrak{Q}_\lambda$. Thus we get the estimate

$$\left\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \right\|_2^2 \lesssim 2^{-2n\gamma(d-1)} \lambda \sum_j \|B_{j-n}\|_1$$

which gives (78). $\square$

5.5. Sketch of proof of Proposition 5.5. The technical proof is lengthy; we give the main ideas and refer the reader to [36] for details.

Let $Q \in \mathfrak{Q}_\lambda$ with center y_Q and with $L(Q) = j - n$. Fix a unit vector ν , and let $\pi_\nu^\perp$ be the projection to the orthogonal complement of ν . In view of the support properties of the kernel it suffices to show that for large n

$$(79) \quad \left\| (I - P_{j-n+\ell(n)}) (I - \mathfrak{S}_{n,\nu}) T_j^{n,\nu} b_Q \right\|_1 \lesssim n^{-2} 2^{-n\gamma(d-1)} \|b_Q\|_1,$$

under the additional assumption that the support of a is contained in

$$\{y : |\langle y - y_Q, \nu \rangle| \leq C2^j, |\pi_\nu^\perp(y - y_Q)| \leq C2^{j-n\gamma}\}.$$

Note that with this hypothesis

$$(80) \quad \|\widehat{a}\|_\infty \lesssim 2^{jd-n\gamma(d-1)}.$$

We need a frequency (Littlewood-Paley) decomposition of a . Let φ be a radial C^∞ function defined in ξ -space, so that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $\varphi(\xi) = 0$ for $|\xi| \geq 6/5$. Define $\beta_k(\xi) = \varphi(2^k \xi) - \varphi(2^{k+1} \xi)$; then β_k is supported where $|\xi| \approx 2^{-k}$. Let $\tilde{\beta}$ be similar but $\beta_k \tilde{\beta}_k = \beta_k$. Define convolution operators $V_k, L_k, \tilde{L}_k$ with Fourier multipliers $\varphi(2^k \cdot), \beta_k, \tilde{\beta}_k$, respectively. Then $L_k \tilde{L}_k = L_k$ and, for every $m \in \mathbb{Z}$, the identity operator is decomposed as $I = V_m + \sum_{k < m} L_k$.

For fixed $y \in Q$ we define an operator $\mathcal{K}_{j,y}^{n,\nu}$ acting on a by

$$\mathcal{K}_{j,y}^{n,\nu}[a](x) = K_j^{n,\nu}(x - y) \int \vartheta(s) a(sx + (1-s)y) ds$$

so that

$$(81) \quad T_j^{n,\nu} b_Q(x) = \int b_Q(y) \mathcal{K}_{j,y}^{n,\nu}[a](x) dy.$$

We use the above frequency decompositions to split

$$(82) \quad (I - \mathfrak{S}_{n,\nu})(I - P_{j-n+\ell(n)}) T_j^{n,\nu} b_Q = \sum_{k_1} L_{k_1} (I - \mathfrak{S}_{n,\nu}) \tilde{L}_{k_1} (I - P_{j-n+\ell(n)}) \int b_Q(y) \mathcal{K}_{j,y}^{n,\nu}[a] dy$$

and then further decompose in (82)

$$(83) \quad a = V_{j-n+\ell(n)} a + \sum_{k_2 < j-n+\ell(n)} L_{k_2} a.$$

We state three lemmas with bounds for the terms in (82), (83).

Lemma 5.6.

$$\left\| \int b_Q(y) \mathcal{K}_{j,y}^{n,\nu}[V_{j-n+\ell(n)} a] dy \right\|_1 \lesssim n^{-2} 2^{-n\gamma(d-1)} \|b_Q\|_1.$$

Based on the cancellation of b_Q one may estimate the left-hand side by

$$\int |b_Q(y)| \int |\mathcal{K}_{j,y}^{n,\nu}[V_{j-n+\ell(n)}a](x) - \mathcal{K}_{j,y_Q}^{n,\nu}[V_{j-n+\ell(n)}a](x)| dx dy.$$

From here on the estimation is straightforward; we omit the proof (cf. [36]).

Next note that $\|\mathcal{K}_{j,y}^{n,\nu}[a]\|_1 \lesssim 2^{-n\gamma(d-1)}\|a\|_\infty$ and the same bound holds for the L^1 norm of $L_{k_1}\mathcal{K}_{j,y}^{n,\nu}[L_{k_2}a]$ uniformly in k_1, k_2 . The following Lemma contains an improvement when k_1, k_2 are far apart and $k_2 \leq j - n + \ell(n)$.

Lemma 5.7. *Let $y \in Q$ and a be as in (80).*

(i) *Let $k_1 > k_2 + \ell(n) + 10$. Then*

$$\|L_{k_1}\mathcal{K}_{j,y}^{n,\nu}[L_{k_2}a]\|_1 \leq C_N 2^{-n\gamma(d-1)} \min\{1, n^{2d+2N} 2^{n\gamma(k_2-j+n\gamma)N}\}.$$

(ii) *Let $k_1 < k_2 - 10$. Then*

$$\|L_{k_1}\mathcal{K}_{j,y}^{n,\nu}[L_{k_2}a]\|_1 + \|L_{k_1}\mathcal{K}_{j,y}^{n,\nu}[V_{k_2}a]\|_1 \leq C_N 2^{-n\gamma(d-1)} \min\{1, 2^{n\gamma} 2^{(k_1-k_2)d} 2^{(k_1-j+n\gamma)N}\}.$$

We use the Fourier inversion formula for a and for the convolution kernel of L_{k_1} , and write

$$L_{k_1}\mathcal{K}_{j,y}^{n,\nu}[L_{k_2}a](x) = \frac{1}{(2\pi)^{2d}} \int \vartheta(s) \iint \beta_{k_1}(\xi) \beta_{k_2}(\eta) \widehat{a}(\eta) \times \\ \left[\int_w e^{i(\langle x-w, \xi \rangle + \langle sw + (1-s)y, \eta \rangle)} K_j^{n,\nu}(w-y) dw \right] d\xi d\eta ds.$$

We may integrate by parts with respect to w and ξ and the integral becomes

$$\frac{1}{(2\pi)^{2d}} \int \vartheta(s) \int \beta_{k_2}(\eta) \widehat{a}(\eta) \int \left[\int e^{i(\langle x-w, \xi \rangle + \langle sw + (1-s)y, \eta \rangle)} \times \right. \\ \left. \frac{(I - 2^{-2k_1} \Delta_\xi)^{N_1} [\beta_{k_1}(\xi) |\xi - s\eta|^{-2N_2}] (-\Delta_w)^{N_2} K_j^{n,\nu}(w-y)}{(1 + 2^{-2k_1} |x-w|^2)^{N_1}} dw \right] d\xi d\eta ds.$$

We use this for $N_1 = [d/2] + 1$ and N_2 large. Note that for $s \in \text{supp}(\vartheta)$,

$$|\xi - s\eta| \gtrsim \begin{cases} 2^{-k_2-\ell(n)} & \text{if } k_1 > k_2 + \ell(n) + 10, \\ 2^{-k_1-2} & \text{if } k_1 < k_2 - 10. \end{cases}$$

A straightforward evaluation leads to the assertion of Lemma 5.7.

More interesting is the analysis for $k_1 \approx k_2$.

Lemma 5.8. *Let $k_2 - 10 \leq k_1 \leq k_2 + \ell(n) + 10$. Then*

$$\|L_{k_1}(I - \mathfrak{S}^{n,\nu})\mathcal{K}_{j,y}^{n,\nu}[L_{k_2}a]\|_1 \leq C_N 2^{-n\gamma(d-1)} \min\{1, 2^{(d+3)n\gamma} 2^{(k_1-j+n\gamma)N}\}$$

for every $y \in Q$.

Proof. We may assume that (80) holds. Define the convolution operator $S_{n,\nu}$ by

$$\widehat{S^{n,\nu}g}(\eta) = \rho(2^{n\gamma} n^{-2} \langle \nu, \frac{\eta}{|\eta|} \rangle) \widehat{g}(\eta)$$

and split $a = S^{n,\nu}a + (I - S^{n,\nu})a$. One establishes the following estimates:

$$(84) \quad \|L_{k_1}(I - \mathfrak{S}^{n,\nu})\mathcal{K}_{j,y}^{n,\nu}[L_{k_2}S^{n,\nu}a]\|_1 \leq C_N n^{(2\varepsilon^{-1}-4)(N+d)} 2^{4n\gamma} 2^{(k_1-k_2)d} 2^{(k_1-j+n\gamma)N}$$

and

$$(85) \quad \|L_{k_1}(I - \mathfrak{S}^{n,\nu})\mathcal{K}_{j,y}^{n,\nu}[L_{k_2}(I - S^{n,\nu})a]\|_1 \leq C_N n^{-5d} 2^{4n\gamma} 2^{(k_2-j+n\gamma)N},$$

which imply the somewhat weaker estimate asserted in the lemma. The idea for (84) is that the supports of the multipliers for $I - \mathfrak{S}^{n,\nu}$ and $S^{n,\nu}$ and we have a situation similar to the one in the proof of Lemma 5.7. For (85) we observe that the Fourier supports are away from the hyperplane perpendicular to ν . Therefore, the s integration (which provides a microlocal regularization effect averaging in directions close to ν) can be used effectively.

For the proof of (84) set

$$b_{k_1,n,\nu}(\xi) = \beta_{k_1}(\xi)(1 - \rho(2^{n\gamma}n^{-5}\langle\nu, \frac{\xi}{|\xi|}\rangle))$$

and write

$$\begin{aligned} (2\pi)^{2d} L_{k_1}(I - \mathfrak{S}^{n,\nu}) \mathcal{K}_{j,y}^{n,\nu}[L_{k_2} S^{n,\nu} a](x) = \\ \int \vartheta(s) \iint b_{k_1,n,\nu}(\xi) \beta_{k_2}(\eta) \rho(2^{n\gamma}n^{-2}\langle\nu, \frac{\eta}{|\eta|}\rangle) \widehat{a}(\eta) \\ \times \left[\int_w e^{i(\langle x-w, \xi \rangle + \langle sw + (1-s)y, \eta \rangle)} K_j^{n,\nu}(w-y) dw \right] d\xi d\eta ds. \end{aligned}$$

If (ξ, η) is in the support of the amplitude then for large n

$$\begin{aligned} |\langle \xi - s\eta, \nu \rangle| &\geq |\xi| |\langle \frac{\xi}{|\xi|}, \nu \rangle| - |\eta| |\langle \frac{\eta}{|\eta|}, \nu \rangle| \\ &\geq |\xi| (2^{-n\gamma-1}n^5 - 2^{|k_1-k_2|+2}2^{-n\gamma}n^2) \\ &\geq |\xi| 2^{-n\gamma-1}(n^5 - 8 \cdot 2^{\ell(n)+10}n^2) \geq 2^{-k_1-n\gamma}n^5. \end{aligned}$$

Now we can integrate by parts as in the proof of Lemma 5.7, except we use the directional derivative $\langle\nu, \nabla_w\rangle$ instead of Δ_w . The above integral is then estimated by

$$\begin{aligned} \iiint |\beta_{k_2}(\eta)| |\widehat{a}(\eta)| |\rho(2^{n\gamma}n^{-2}\langle\nu, \frac{\eta}{|\eta|}\rangle)| \\ \times \frac{|(I - 2^{-2k_1}\Delta_\xi)^{N_1} [\frac{b_{k_1,n,\nu}(\xi)}{\langle \xi - s\eta, \nu \rangle^{N_2}}]|}{(1 - 2^{-2k_1}|x-w|^2)^{N_1}} |\langle\nu, \nabla_w\rangle^{N_2} K_j^{n,\nu}(w-y)| dw d\xi d\eta ds. \end{aligned}$$

The above lower bound for $\langle \xi - s\eta, \nu \rangle$ works in our favor and eventually leads to (84).

To see (85) set

$$\widetilde{b}_{k_2,n,\nu}(\eta) = \beta_{k_2}(\eta)(1 - \rho(2^{n\gamma}n^{-2}\langle\nu, \frac{\eta}{|\eta|}\rangle))$$

and write

$$\begin{aligned} (2\pi)^d L_{k_1}(I - \mathfrak{S}^{n,\nu}) \mathcal{K}_{j,y}^{n,\nu}[L_{k_2}(I - S^{n,\nu})a](x) \\ = \int K_j^{n,\nu}(w-y) \iint b_{k_1,n,\nu}(\xi) \widetilde{b}_{k_2,n,\nu}(\eta) \widehat{a}(\eta) \\ \times \left[\int \vartheta(s) e^{i(\langle x-w, \xi \rangle + \langle sw + (1-s)y, \eta \rangle)} ds \right] d\xi d\eta dw. \end{aligned}$$

Now if $w-y \in \text{supp}(K_j^{n,\nu})$ then $|\frac{w-y}{|w-y|} - \nu| \leq 2^{-n\gamma}$ and if $\eta \in \text{supp}(\widetilde{b}_{k_2,n,\nu})$ we get

$$|\langle w-y, \eta \rangle| \geq |w-y| (|\langle \nu, \eta \rangle| - |\eta| 2^{-n\gamma}) \geq |w-y| |\eta| 2^{-n\gamma} (\frac{1}{2}n^2 - 1)$$

and hence

$$(86) \quad |\langle w-y, \eta \rangle| \geq 2^{j-k_2-n\gamma-4}n^2.$$

Integration by parts with respect to s yields

$$(2\pi)^d L_{k_1}(I - \mathfrak{S}^{n,\nu}) \mathcal{K}_{j,y}^{n,\nu} [L_{k_2}(I - S^{n,\nu})a](x) = \\ \int K_j^{n,\nu}(w-y) \iint \widehat{a}(\eta) \widetilde{b}_{k_2,n,\nu}(\eta) \frac{(I - 2^{-2k_1} \Delta_\xi)^{N_1} b_{k_1,n,\nu}(\xi)}{(1 + 2^{-2k_1} |x-w|^2)^{N_1}} e^{i(\langle x-w, \xi \rangle + \langle y, \eta \rangle)} \\ \times \left[\int \vartheta^{(N_3)}(s) \frac{i^{N_3} e^{is\langle w-y, \eta \rangle}}{\langle w-y, \eta \rangle^{N_3}} ds \right] d\xi d\eta dw.$$

Now the lower bound for $\langle w-y, \eta \rangle$ works in our favor and a computation yields (85).

Finally the lemmas can be combined to prove Proposition 5.5 (cf. [36] for details). $\square$

Remark. It is currently open whether similar weak type $(1, 1)$ bounds hold for the case $n \geq 2$. Other open problems may be discussed in the lectures.

REFERENCES

- [1] G. Alberti, G. Crippa, A. Mazzucato, Exponential self-similar mixing and loss of regularity for continuity equations. arXiv:1407.2631.
- [2] J. Bergh, J. Löfström, *Interpolation spaces*. Springer-Verlag, Berlin, Heidelberg, New York, 1976.
- [3] S. Bianchini, On Bressan's conjecture on mixing properties of vector fields. Self-similar solutions of nonlinear PDE, 13–31, Banach Center Publ., 74, Polish Acad. Sci., Warsaw, 2006.
- [4] A. Bressan, A lemma and a conjecture on the cost of rearrangements. Rend. Sem. Mat. Univ. Padova 110 (2003), 97–102.
- [5] ———, Prize offered for a solution of a problem on mixing flows. Article posted on the website <https://www.math.psu.edu/bressan/PSPDF/prize1.pdf>
- [6] A.P. Calderón, Commutators of singular integrals. Proc. Nat. Acad. Sci. U.S.A., 53 (1965), 1092–1099.
- [7] C.P. Calderón, On commutators of singular integrals. Studia Math., 53 (1975), 139–174.
- [8] M. Christ, Weak type $(1, 1)$ bounds for rough operators, Annals of Math. 128 (1988), 19–42.
- [9] M. Christ, J.L. Journé, Polynomial growth estimates for multilinear singular integral operators. Acta Math. 159 (1987), no. 1-2, 51–80.
- [10] M. Christ, J.-L. Rubio de Francia, Weak type $(1, 1)$ bounds for rough operators, II, Invent. Math., 93 (1988) 225–237.
- [11] R.R. Coifman, G. David, Y. Meyer, La solution des conjecture de Calderón. Adv. in Math. 48 (1983), no. 2, 144–148.
- [12] R. Coifman and Y. Meyer, On commutators of singular integral and bilinear singular integrals. Trans. Amer. Math. Soc., 212 (1975), 315–331.
- [13] R. Coifman, A. McIntosh, Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes. Ann. of Math. (2) 116 (1982), no. 2, 361–387.
- [14] R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, 57 (1978).
- [15] G. Crippa, C. de Lellis, Estimates and regularity results for the DiPerna-Lions flow, J. Reine Angew. Math. 616 (2008), 15–46.
- [16] G. David, J.-L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators. Ann. of Math. (2) 120 (1984), 371–397.
- [17] X.T. Duong, L. Grafakos, and L. Yan, Multilinear operators with non-smooth kernels and commutators of singular integrals. Trans. Amer. Math. Soc. 362 (2010), no. 4, 2089–2113.
- [18] C. Fefferman, Inequalities for strongly singular convolution operators, Acta Math. 124 (1970), 9–36.
- [19] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math. 129 (1972), 137–194.
- [20] L. Grafakos, Multilinear harmonic analysis. In: *Nonlinear analysis, function spaces and applications* Vol. 9, 63–104, Acad. Sci. Czech Repub. Inst. Math., Prague, 2011.
- [21] L. Grafakos, and P. Honzik, A weak type estimate for commutators. Int. Math. Res. Not. IMRN 2012, no. 20, 4785–4796.
- [22] L. Grafakos and X. Li, Uniform bounds for the bilinear Hilbert transforms, I. Ann. of Math. (2), 159 (2004), 889–933.
- [23] L. Grafakos and R.H. Torres, Multilinear Calderón-Zygmund theory. Adv. in Math., 165 (2002), 124–164.
- [24] M. Hadžić, A. Seeger, C. Smart, B. Street, in preparation.
- [25] S. Hofmann, Weak $(1, 1)$ boundedness of singular integrals with nonsmooth kernel, Proc. Amer. Math. Soc. 103 (1988), 260–264.

- [26] S. Hofmann, Boundedness criteria for rough singular integrals, *Proc. London Math. Soc.* (3) 70 (1995), no. 2, 386–410.
- [27] S. Hofmann, An off-diagonal T1 theorem and applications. With an appendix "The Mary Weiss lemma" by Loukas Grafakos and the author. *J. Funct. Anal.* 160 (1998), no. 2, 581–622.
- [28] J.-L. Journé, *Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón*, Lecture Notes in Mathematics 994, Springer-Verlag, Berlin, 1983.
- [29] C. De Lellis, ODEs with Sobolev coefficients: the Eulerian and the Lagrangian approach. *Discrete Contin. Dyn. Syst. Ser. S* 1 (2008), no. 3, 405–426.
- [30] M. Lacey and C. Thiele, L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$, *Ann. of Math.* (2), 146 (1997), 693–724.
- [31] M. Lacey and C. Thiele, On Calderón's conjecture, *Ann. of Math.* (2), 149 (1999), 475–496.
- [32] X. Li, Uniform bounds for the bilinear Hilbert transforms, II, *Rev. Mat. Iberoamericana*, 22 (2006), 1069–1126.
- [33] C. Muscalu, Calderón commutators and the Cauchy integral on Lipschitz curves revisited, II. Cauchy integral and generalizations. *Rev. Mat. Iberoam.* 30 (2014), no. 3, 1089–1122.
- [34] C. Muscalu, W. Schlag, *Classical and multilinear harmonic analysis*. Vol. II. Cambridge Studies in Advanced Mathematics, 138. Cambridge University Press, Cambridge, 2013.
- [35] A. Seeger, Singular integral operators with rough convolution kernels. *J. Amer. Math. Soc.* 9 (1996), no. 1, 95–105.
- [36] A. Seeger, A weak type bound for a singular integral. *Rev. Mat. Iberoam.* 30 (2014), no. 3, 961–978.
- [37] A. Seeger, C. Smart, B. Street, *Multilinear singular integral forms of Christ-Journé type*. *Memoirs Amer. Math. Soc.* 849, to appear.
- [38] E. M. Stein, *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J. 1970.
- [39] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.
- [40] T. Tao, The weak-type $(1, 1)$ of $L \log L$ homogeneous convolution operator. *Indiana Univ. Math. J.* 48 (1999), no. 4, 1547–1584.
- [41] C. Thiele, A uniform estimate. *Ann. of Math.* (2) 156 (2002), no. 2, 519–563.
- [42] H. Triebel, *Theory of function spaces*. Birkhäuser Verlag, Basel 1983.
- [43] Y. Yao, A. Zlatoš, Mixing and un-mixing by incompressible flows, *J. Europ. Math. Soc.*, to appear.