RESTRICTION OF FOURIER TRANSFORMS TO CURVES: AN ENDPOINT ESTIMATE WITH AFFINE ARCLENGTH MEASURE

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Abstract. Consider the Fourier restriction operators associated to curves in $\mathbb{R}^d$, $d \geq 3$. We prove for various classes of curves the endpoint restricted strong type estimate with respect to affine arclength measure on the curve. An essential ingredient is an interpolation result for multilinear operators with symmetries acting on sequences of vector-valued functions.

1. Introduction

Let $t \mapsto \gamma(t)$ define a curve in $\mathbb{R}^d$, defined for t in a parameter interval I. We shall assume that γ is at least of class C^d on I.

In this paper we investigate the mapping properties of the Fourier restriction operator associated to the curve, given for Schwartz functions on $\mathbb{R}^d$ by

$$\mathcal{R}f(t) = \widehat{f}(\gamma(t));$$

here the Fourier transform is defined by $\widehat{f}(\xi) = \int f(y) e^{-\mathrm{i}\langle y,\xi\rangle} d\xi$. $\mathcal{R}f$ will be measured in Lebesgue spaces $L^q(I; d\lambda)$ where $d\lambda = w(t)dt$ is *affine arclength measure* with weight

$$w(t) = |\tau(t)|^{\frac{2}{d^2+d}} \quad \text{where } \tau(t) = \det(\gamma'(t), \dots, \gamma^{(d)}(t)). \tag{1}$$

The relevance of affine arclength measure for harmonic analysis has been discussed in [19] and [31]. There is an invariance under change of variables and reparametrizations. Fourier restriction theorems for the case of 'nondegenerate' curves (with nonvanishing τ) are supposed to extend to large classes of 'degenerate' curves when arclength measure is replaced by affine arclength measure, with uniform constants in the estimates. Finally the choice of affine arclength measure is optimal up to multiplicative constants, in a sense made precise in the next section.

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For nondegenerate curves affine arclength measure is comparable to the standard arclength measure on any compact interval. Note that for the model case $(t, t^2, \ldots, t^d)$ the weight w is constant (equal to $(d!)^{\frac{2}{d^2+d}}$). The sharp $L^p \to L^\sigma$ estimates for this case have been obtained by Zygmund [38] and Hörmander [24] in the case $d = 2$ and by Drury [18] in higher dimensions. Namely, one gets $L^p(\mathbb{R}^d) \to L^\sigma(\mathbb{R})$ boundedness for $1 < p < p_d := \frac{d^2+d+2}{d^2+d}$, $p' = \sigma \frac{d(d+1)}{2}$. A nonisotropic scaling reveals that for a global estimate this relation between p and σ is necessary in this case. Moreover, it follows from a result by Arkhipov, Chubarikov and Karatsuba [1] that the given range of p is optimal. One can ask for weaker estimates at the endpoint p_d which imply the $L^p \to L^\sigma$ estimates by interpolation. The iterative method by Drury fails to give information at the endpoint. In two dimensions, Beckner, Carbery, Semmes and Soria [8] have shown that even the restricted weak type estimate fails at the endpoint $p_2 = 4/3$. However, in [6] the authors proved for the nondegenerate model case that in dimensions $d \geq 3$ the Fourier restriction operator is of *restricted strong type* (p_d, p_d), i.e. maps the Lorentz space $L^{p_d,1}(\mathbb{R}^d)$ to $L^{p_d}(\mathbb{R}, dt)$. This result is optimal with respect to the secondary Lorentz exponents.

It is natural to ask whether for more general classes of curves the endpoint inequality

$$\Big(\int_I |\widehat{f} \circ \gamma|^{p_d} d\lambda\Big)^{1/p_d} \lesssim \|f\|_{L^{p_d,1}(\mathbb{R}^d)}, \quad p_d = \frac{d^2+d+2}{d^2+d}, \tag{2}$$

holds true with affine arclength measure $d\lambda$. This estimate of course implies the best possible $L^p(d\lambda) \to L^q$ bounds which for some classes of curves were proved in the first two papers of this series [6], [7], building on earlier work by Drury and Marshall [20], [21]. See also the very recent work by Müller and Dendrinos [16] for further extensions. In two dimensions the endpoint bound fails and sharp Lebesgue space estimate can be found in [33], [30].

Here we prove (2) for two classes of curves. We first consider the case of "monomial" curves of the form

$$t \mapsto \gamma_a(t) = (t^{a_1}, t^{a_2}, \ldots, t^{a_d}), \quad 0 < t < \infty \tag{3}$$

where $a = (a_1, \ldots, a_d)$ are arbitrary real numbers, $d \geq 3$.

Theorem 1.1. *Let $d \geq 3$ and let $w_a dt$ denote the affine arclength measure for the curve* (3)*. Then there is $C(d) < \infty$ so that for all $f \in L^{p_d,1}(\mathbb{R}^d)$*

$$\Big(\int_0^\infty |\widehat{f}(\gamma_a(t))|^{p_d} w_a(t) dt\Big)^{1/p_d} \leq C(d) \|f\|_{L^{p_d,1}(\mathbb{R}^d)}. \tag{4}$$

Note that the constant in (4) is universal in the sense that it does not depend on $a_1, \ldots, a_d$.

A similar result holds for 'simple' polynomial curves in $\mathbb{R}^d$, $d \geq 3$,

(5)
$$\Gamma_b(t) = \Big(t, \frac{t^2}{2!}, \cdots, \frac{t^{d-1}}{(d-1)!}, P_b(t)\Big), \quad t \in \mathbb{R},$$

where P_b is an arbitrary polynomial of degree $N \geq 0$, with the coefficients $(b_0, \cdots, b_N) = b \in \mathbb{R}^{N+1}$, that is, $P_b(t) = \sum_{j=0}^{N} b_j t^j$. Note that the affine arclength measure in this case is given by $W_b(t)dt$ where $W_b(t) = |P_b^{(d)}(t)|^{\frac{2}{d^2+d}}$. Then we have

Theorem 1.2. *There is $C(N) < \infty$ so that for all $f \in L^{p_d,1}(\mathbb{R}^d)$, $b \in \mathbb{R}^{N+1}$,*

(6)
$$\Big(\int_0^\infty |\widehat{f}(\Gamma_b(t))|^{p_d} W_b(t)dt\Big)^{1/p_d} \leq C(N)\|f\|_{L^{p_d,1}(\mathbb{R}^d)}.$$

It would be interesting to prove a similar theorem for general polynomial curves $(P_1(t), \dots, P_d(t))$, with a bound depending only on the highest degree. However, currently we do not even know the sharp $L^p \to L^q(w)$ bounds in the optimal range $p \in [1, \frac{d^2+d+2}{d^2+d})$. For the smaller range $1 \leq p < \frac{d^2+2d}{d^2+2d-2}$ (corresponding to the range in Christ's paper [12] for the nondegenerate case), such universal $L^p \to L^q(w)$ bounds have been recently proved by Dendrinos and Wright [17]. Their result can be slightly extended by combining an argument by Drury [19] with estimates by Stovall [37] on averaging operators, see §8.

An interpolation theorem. As in previous papers on restriction theorems for curves the results rely on the analysis of multilinear operators with a high degree of symmetry. In [6] the operators acted on n-tuples of functions in Lebesgue or Lorentz spaces, and it was important to use an interpolation procedure introduced by Christ in [12] (*cf.* also [26], [23] for related results). In the presence of weights one is led to consider interpolation results for n-linear operators acting on products of $\ell^p_s(X)$ spaces and which have values in a Lorentz space; here X is a quasi-normed space and $\ell^p_s(X)$ is the space of X valued sequences $\{f_k\}_{k\in\mathbb{Z}}$ for which $(\sum_{k\in\mathbb{Z}} 2^{ksp}\|f_k\|_X^p)^{1/p} < \infty$. For the relevance to the restriction problem see also the remarks following the statement of Theorem 1.3 below.

We recall some terminology from interpolation theory. A quasi-norm on a vector space has the same properties as a norm except that the triangle inequality is weakened to $\|x+y\| \leq C(\|x\| + \|y\|)$ for some constant C. Let $0 < r \leq 1$. The topology generated by the balls defined by this norm is called *r-convex* if there is a constant C_1 so that

(7)
$$\Big\|\sum_{i=1}^n x_i\Big\|_X \leq C_1(\sum_{i=1}^n \|x_i\|_X^r)^{1/r}$$

holds for any finite sums of elements in X. The Aoki-Rolewicz theorem states that every quasi-normed space is r-convex for some $r > 0$ (see also §3.10 in [9] for a generalization). Obviously any normed space is 1-convex. Hunt [25] showed that Lorentz spaces L^{pq} are r-convex for $r < \min\{1, p, q\}$

and they are normable for $p, q > 1$. The Lorentz space $L^{r,\infty}$ is r-convex for $0 < r < 1$; this is a result by Kalton [27] and by Stein, Taibleson and Weiss [34]. This fact plays a role in the proof of sharp endpoint theorems, in [6] as well as in the present paper.

The Lions-Peetre interpolation theory can be extended to quasi-normed spaces (see §3.11 of [9]). Here one works with couples $\overline{X} = (X_0, X_1)$ of compatible quasi-normed spaces, i.e. both X_0 and X_1 are continuously embedded in some topological vector space. We shall use both the K-functional defined on $X_0 + X_1$, given by $K(t, f; \overline{X}) = \inf_{f=f_0+f_1}[\|f_0\|_{X_0} + t\|f_1\|_{X_1}]$ and the J-functional defined on $X_0 \cap X_1$ by $J(t, f, \overline{X}) = \max\{\|f\|_{X_0}, t\|f\|_{X_1}\}$. For $0 < \theta < 1$, $0 < q < \infty$ the interpolation space $\overline{X}_{\theta,q}$ is the space of $f \in X_0 + X_1$ for which $\|f\|_{\overline{X}_{\theta,q}} = (\sum_{l\in\mathbb{Z}}[2^{-l\theta}K(2^l, f; \overline{X})]^q)^{1/q}$ is finite. Similarly one defines $\overline{X}_{\theta,\infty}$ with quasi-norm $\|f\|_{\overline{X}_{\theta,\infty}} = \sup_{l\in\mathbb{Z}} 2^{-l\theta}K(2^l, f; \overline{X})$. The space $X_0 \cap X_1$ is dense in $\overline{X}_{\theta,q}$ but not necessarily in $\overline{X}_{\theta,\infty}$; the closure of $X_0 \cap X_1$ in $\overline{X}_{\theta,\infty}$ is denoted by $\overline{X}^0_{\theta,\infty}$ and consists of all $f \in \overline{X}_{\theta,\infty}$ for which $2^{-l\theta}K(2^l, f : \overline{X})$ tends to 0 as $l \to \pm\infty$. An equivalent norm on $\overline{X}_{\theta,q}$ is given by $\|f\|_{\overline{X}_{\theta,q;J}} = \inf(\sum_{l\in\mathbb{Z}}[2^{-l\theta}J(2^l, u_l; \overline{X})]^q)^{1/q}$, where the infimum is taken over all representations $f = \sum_l u_l$, $u_l \in X_0 \cap X_1$, with convergence in $X_0 + X_1$ (see the equivalence theorem 3.11.3 in [9]).

For the formulation and proofs of interpolation results for multilinear operators with symmetries it is convenient to use the notion of a *doubly stochastic* $n \times n$ *matrix*, i.e. a matrix $A = (a_{ij})_{i,j=1,\dots,n}$ for which $a_{ij} \in [0, 1]$, $i, j = 1, \dots, n$, $\sum_{j=1}^n a_{ij} = 1$, $i = 1, \dots, n$ and $\sum_{i=1}^n a_{ij} = 1$, $j = 1, \dots, n$. Doubly stochastic matrices arise naturally in the interpolation of operators with symmetries under permutations; this is because of Birkhoff's theorem ([10], [29]) which states that the set of all doubly stochastic matrices (also called the Birkhoff polytope) is precisely the convex hull of the permutation matrices. We shall denote by $DS(n)$ the set of all doubly stochastic $n \times n$ matrices and by $DS^\circ(n)$ the subset of matrices in $DS(n)$ for which all entries lie in the open interval $(0, 1)$. In what follows given n numbers $s_1, \dots, s_n$ we let $\vec{s}$ be the column vector with entries s_i, and $\vec{e}_m$ be the mth coordinate vector.

The following interpolation theorem plays a crucial role in the proof of Theorems 1.1 and 1.2.

Theorem 1.3. *Suppose we are given* $m \in \{1, \dots, n\}$ *and* $\delta_1, \dots, \delta_n \in \mathbb{R}$ *so that the numbers* δ_i *with* $i \neq m$ *are not all equal. Let* $0 < r \leq 1$, *and let* $q_1, \dots, q_n \in [r, \infty]$ *such that* $\sum_{i=1}^n q_i^{-1} = r^{-1}$. *Let* V *be an* r*-convex Lorentz space, and let* $\overline{X} = (X_0, X_1)$ *be a couple of compatible complete quasi-normed spaces. Let* T *be a multilinear operator defined on* n*-tuples of* X_0+X_1 *valued sequences and suppose that for every permutation* π *on* n *letters we have the* *inequality*

$$\|T(f_{\pi(1)},\dots,f_{\pi(n)})\|_V \le \|f_m\|_{\ell^r_{\delta_m}(X_1)} \prod_{i\neq m} \|f_i\|_{\ell^r_{\delta_i}(X_0)}\,. \tag{8}$$

Then for every $A\in DS^\circ(n)$ *and every* $B\in DS(n)$ *such that*

$$B\vec e_m = (r/q_1,\dots,r/q_d)^T$$

there is $C=C(A,B,\vec\delta,r)$ *so that for* $\vec s = BA\vec\delta$ *and* $\vec\theta = BA\vec e_m$

$$\|T(f_1,\dots,f_n)\|_V \le C\prod_{i=1}^n \|f_i\|_{\ell^{q_i}_{s_i}(\overline{X}_{\theta_i,q_i})}\,, \tag{9}$$

for all $(f_1,\dots,f_n)\in\prod_{i=1}^n \ell^{q_i}_{s_i}(\overline{X}_{\theta_i,q_i})$.

In particular

$$\|T(f_1,\dots,f_n)\|_V \lesssim \prod_{i=1}^n \|f_i\|_{\ell^{nr}_\sigma(\overline{X}_{\frac1n,nr})}\,,\qquad \sigma=\frac1n\sum_{i=1}^n\delta_i\,. \tag{10}$$

Here, and in what follows we write $\lesssim$ if the inequality involves an implicit constant. For the proof of our restriction estimates only the special case (10) is used; it follows from (9) by choosing $a_{ij}=b_{ij}=1/n$ for all i,j.

Relevance for the adjoint restriction operator. One would like to extend the proof of the endpoint estimate for the adjoint restriction operator in [6] by using weighted Lorentz spaces, but there is the immediate difficulty that the real interpolation spaces of weighted Lebesgue or Lorentz spaces may not be weighted Lorentz spaces, and other scales of spaces have to be considered (cf. the papers by Freitag [22] and Lizorkin [28] on interpolation spaces of weighted L^p spaces).

Let X be a Lorentz space of functions on an interval I (with Lebesgue measure), and a positive measurable weight function w on I. Let $\Omega[w,k]=\{t\in I: 2^k\le w(t)<2^{k+1}\}$. We define the block Lorentz space $b^q_s(w,X)$ to be the space of measurable functions for which

$$\|f\|_{b^q_s(w,X)} := \Big(\sum_{k\in\mathbb Z}\big[2^{ks}\|\chi_{\Omega[w,k]}f\|_X\big]^q\Big)^{1/q} \tag{11}$$

is finite. These spaces arise in real interpolation of weighted Lorentz spaces with change of measure (see [2], [3]). We are not necessarily interested in the block Lorentz spaces per se, but use them as a vehicle to prove our result on $L^p(w)=b^p_{1/p}(w,L^p)$.

The connection with results on $\ell^q_s(X)$ spaces is immediate, namely $b^q_s(w,X)$ is a retract of $\ell^q_s(X)$: Define $\imath: b^q_s(w,X)\to\ell^q_s(X)$ by $[\imath(f)]_k=\chi_{\Omega[w,k]}f$ and $\varsigma:\ell^q_s(X)\to b^q_s(w,X)$ by $\varsigma(F)=\sum_k\chi_{\Omega[w,k]}F_k$ then $\imath$ and ς have operator norm 1 and $\varsigma\circ\imath$ is the identity operator on $b^q_s(w,X)$; moreover $[\imath\circ\varsigma(F)]_k=\chi_{\Omega[w,k]}F_k$. If L is a linear operator mapping $b^q_s(w,X)$ boundedly to a quasi-normed space V then $L\circ\varsigma:\ell^q_s(X)\to V$ and if $\mathcal L$ is a linear operator mapping $\ell^q_s(X)$ to V then $\mathcal L\circ\imath: b^q_s(w,X)\to V$. Analogous observations can be made for multilinear operators acting on products of such spaces. Thus Theorem 1.3 implies an immediate analog for multilinear operators acting on $\prod b^{q_i}_{s_i}(w, \overline{X}_{\vartheta_i, q_i})$ which will be used in our estimates for adjoint restriction operators.

This paper. In §2 we discuss the optimality of affine arclength measure in estimates for the Fourier restriction operators associated with curves. In §3 we prove Theorem 1.3. In §4 we formulate geometrical hypotheses for our main result on Fourier restriction from which Theorems 1.1 and 1.2 can be derived. This result is proved in §5. Theorem 1.1 is proved in §6. In §7 we make some observations on curves of simple type and prove Theorem 1.2. In §8 we give the proof of the partial result for general polynomial curves alluded to above. Some background needed for the interpolation section is provided in Appendix A.

2. Optimality of the affine arclength measure

Let $\tau(t)$ be as in (1). For $p > 1$ let $\sigma(p) = \frac{2p'}{d^2+d}$, with $p' = \frac{p}{p-1}$ (the critical σ for $L^p \to L^\sigma$ boundedness of Fourier restriction with respect to Lebesgue measure in the nondegenerate case). In particular $\sigma(p_d) = p_d$ for $p_d = \frac{d^2+d+2}{d^2+d}$.

Proposition 2.1. *Let I be an interval and $\gamma : I \to \mathbb{R}^d$ be of class C^d. Let μ be a positive Borel measure on I and suppose that the inequality*

$$\Big(\int_I |\widehat{f} \circ \gamma|^{\sigma(p)} \, d\mu \Big)^{1/\sigma(p)} \le B \|f\|_{L^{p,1}} \tag{12}$$

holds for all $f \in L^{p,1}(\mathbb{R}^d)$.

Then μ is absolutely continuous with respect to Lebesgue measure on I, so that $d\mu = \omega(t)dt$ for a nonnegative locally integrable ω, and there exists a constant C_d so that

$$\omega(t) \le C_d B^{\sigma(p)} |\tau(t)|^{\frac{2}{d^2+d}} \tag{13}$$

for almost every $t \in I$.

Proof. We argue as in the proof of Proposition 2 in [31] and use a 'Knapp example' to see that (12) implies

$$\int \chi_P(\gamma(t)) \, d\mu(t) \le C_1(d) B^{\sigma(p)} |P|^{\frac{2}{d^2+d}} \tag{14}$$

for any parallelepiped P. Indeed if $P = AQ + b$ where $Q = [0,1]^d$, $b \in \mathbb{R}^d$ and A is an invertible linear transformation then we choose f so that $\widehat{f}(\xi) = \exp(-|A^{-1}(\xi - b)|^2)$. Now $|\widehat{f}(\xi)| \ge e^{-d}$ for $\xi \in P$, and $\|f\|_{L^{p,1}} \le C_2(d) |\det(A)|^{1/p'}$, and then (14) is an immediate consequence of the relation $\sigma(p)/p' = 2/(d^2+d)$ and $|P| = |\det A|$.

We first show that μ is absolutely continuous with respect to Lebesgue measure. Let I' be a compact subinterval of I. Absolute continuity follows if we can show that $\mu(J) \le C(I')|J|$ for every subinterval J of I' with length $|J| < 1/2$. By the Radon-Nikodym theorem $d\mu = \omega(t)dt$ with locally integrable ω (in fact ω will be locally bounded by the estimate on $\mu(J)$).

Fix such a $J \subset I'$ and let t be the center of J, and let $|J| = 2h$. Consider the Taylor expansion

$$\gamma(t+u) = \sum_{j=0}^{d} \frac{u^j}{j!}\gamma^{(j)}(t) + o(u^d)\,. \tag{15}$$

Let $K = K_t$ denote the dimension of the linear span V_t of $\gamma'(t), ..., \gamma^{(d)}(t)$. Choose $1 \le j_1 < \cdots < j_K \le d$ so that the span of $\gamma^{(j_1)}(t), \dots, \gamma^{(j_K)}(t)$ is equal to V_t and so that for each $j = 1, \dots d$ the vector $\gamma^{(j)}(t)$ belongs to $\mathrm{span}(\{\gamma^{(j_k)}(t), j_k \le j\})$. Choose an orthonormal basis $\{v_k(t)\}_{1\le k\le d}$ so that $\mathrm{span}(\{v_1(t), \dots, v_l(t)\})$ is equal to $\mathrm{span}(\{\gamma^{(j_k)}(t), k = 1, \dots, l\})$, for $l = 1, \dots, K$.

Then there is a constant C (depending on I' and the C^d bounds of γ) so that $\gamma(s)$, $s \in J$, belongs to the parallelepiped

$$\mathcal{P}_C(h,t) = \gamma(t) + \big\{\sum_{j=1}^{d} Cb_j v_j(t) : 0 \le b_j \le h^j\big\}$$

which has volume $O(h^{\frac{d^2+d}{2}})$. By (14) we get

$$\mu(J) \le C_1(d)|\mathcal{P}_C(h,t)|^{\frac{2}{d^2+d}} \le C(d, I', \gamma)|h|$$

which shows the absolute continuity of μ.

In order to obtain (13) it suffices, by the Lebesgue differentiation theorem, to prove

$$\limsup_{h\to 0+} \frac{1}{h}\int_0^h \omega(t+u)du \le C_d B^{\sigma(p)}|\tau(t)|^{\frac{2}{d^2+d}} \tag{16}$$

for every t in the interior of I. In what follows fix such a t and consider the Taylor expansion (15). We distinguish the cases $\tau(t) = 0$ and $\tau(t) \neq 0$.

If $\tau(t) = 0$ then $K_t \le d-1$ and using the orthonormal basis above the Taylor expansion can be rewritten as

$$\gamma(t+u) = \gamma(t) + \sum_{l=1}^{K}(c_l(t)u^{j_l} + g_l(u,t)u^d)\, v_l(t) + u^d \sum_{l=K+1}^{d} g_l(u,t)v_l(t)$$

where $g_l(u,t) \to 0$ as $u \to 0$. Let $\rho(h,t) = \max_{K+1\le l\le d} \sup_{0\le u\le h} |g_l(u,t)|$ and

$$P(h,t,C) = \gamma(t) + \big\{\sum_{k=1}^{d} Cb_k v_k(t) :$$
$$0 \le b_k \le h^{j_k}, k = 1, \dots, K; |b_k| \le \rho(h,t)h^d, k = K+1, \dots, d\big\}.$$

If C is sufficiently large then there is $h_0(t) > 0$ so that $\gamma(t+u) \in P(h,t,C)$ whenever $h \le h_0(t)$ and $0 \le u \le h$. Also $|P(h,t,C)| \lesssim h^{(d^2+d)/2}\rho(h,t)$. Thus (14) yields

$$\int_0^h w(t+u)\,du \lesssim B^{\sigma(p)} h\rho(h,t)^{\frac{2}{d^2+d}} = o(h)$$

and we have verified (16) for the case $\tau(t) = 0$.

If $\tau(t) \neq 0$ we may replace the above orthonormal basis by the basis $\gamma'(t),...,\gamma^{(d)}(t)$ to rewrite the Taylor expansion (15) as

$$\gamma(t+u) = \sum_{j=0}^{d} \frac{u^j + u^d e_j(u,t)}{j!}\,\gamma^{(j)}(t)$$

where $\lim_{u\to 0+} |e_j(u,t)| = 0$. Let

$$P(h,t) := \gamma(t) + \Big\{ \sum_{j=1}^{d} \frac{b_j}{j!}\,\gamma^{(j)}(t) : 0 \le b_j \le 2h^j \Big\}\,.$$

Then $|P(h,t)| = C_3(d) h^{\frac{d^2+d}{2}} |\tau(t)|$ with $C_3(d) = 2^d \prod_{j=1}^{d} \frac{1}{j!}$ and there is $h_0(t) > 0$ so that for $h < h_0(t)$ we have $\gamma(t+u) \in P(h,t)$ for $0 \le u \le h$. Thus, by (14) we see that for $h \le h_0(t)$

$$\int_0^h \omega(t+u)du \;\le\; C_1(d) B^{\sigma(p)} h \big(C_3(d)|\tau(t)|\big)^{\frac{2}{d^2+d}}$$

which yields (16) in the case $\tau(t) \neq 0$. □

3. Interpolation of multilinear operators with symmetries

We shall now prove several lemmata involving real interpolation of multilinear operators with symmetry that have values in an r-convex quasi-normed space V. These will lead to the proof of Theorem 1.3. The reader may consult Appendix A for some results from interpolation theory needed here.

The following notation, for a couple $\overline{X} = (X_0, X_1)$ of compatible quasi-normed spaces, will be convenient. Set, for $0 < q \le \infty$,

$$\widetilde{X}_{\theta,q} = \begin{cases} X_0, & \text{if } \theta = 0, \\ \overline{X}_{\theta,q}, & \text{if } 0 < \theta < 1, \\ X_1, & \text{if } \theta = 1. \end{cases} \tag{17}$$

With this notation we formulate a version of Lemma A.3 for operators with symmetry.

Lemma 3.1. *Suppose $r \le 1$, and $\delta_1, \dots, \delta_n \in \mathbb{R}$. Let (X_0, X_1) be a couple of compatible complete quasi-normed spaces. Let T be a multilinear operator defined on n-tuples of $X_0 + X_1$ valued sequences, with values in an r-convex* *space V and suppose that for every permutation π on n letters we have the inequality*

$$\|T(f_{\pi(1)},\dots,f_{\pi(n)})\|_V \le \|f_1\|_{\ell^r_{\delta_1}(X_1)} \prod_{i=2}^n \|f_i\|_{\ell^r_{\delta_i}(X_0)}\,. \tag{18}$$

Then there is a constant C such that for every doubly stochastic matrix $A=(a_{ij})_{i,j=1,\dots n}$, for $s_i=\sum_{j=1}^n a_{ij}\delta_j$, $\theta_i=a_{i,1}$, $i=1,\dots,n$ and every permutation π,

$$\|T(f_{\pi(1)},\dots,f_{\pi(n)})\|_V \le C\prod_{i=1}^n \|f_i\|_{\ell^r_{s_i}(\widetilde X_{\theta_i,r})}\,. \tag{19}$$

Proof. Because of the permutation invariance of the assumption it suffices to prove (18) for $\pi=\mathrm{id}$.

The assumption says that $\|T[g_1,\dots,g_n]\|_V$ is dominated by $\|g_{\pi^{-1}(1)}\|_{\ell^r_{\delta_1}(X_1)} \times \prod_{k=2}^n \|g_{\pi^{-1}(k)}\|_{\ell^r_{\delta_k}(X_0)}$. This can be rewritten as

$$\|T[g_1,\dots,g_n]\|_V \le \prod_{i=1}^n \|g_i\|_{\ell^r_{\delta_{\pi(i)}}(\widetilde X_{\theta_i,r})} \tag{20}$$
$$\text{where } \theta_i=1 \text{ if } \pi(i)=1 \text{ and } \theta_i=0 \text{ if } \pi(i)\ne 1;$$

recall that by definition $\widetilde X_{0,r}=X_0$ and $\widetilde X_{1,r}=X_1$. Let P_π be the permutation matrix which has 1 in the positions $(i,\pi(i))$, $i=1,\dots,n$, and 0 in the other positions. Then the conditions $s_i=\delta_{\pi(i)}$ and θ_i is as in (20) can be rewritten as $\vec s=P_\pi\vec\delta$ and $\vec\theta=P_\pi\vec e_1$ (here the vectors are all understood as columns).

For a doubly stochastic matrix A let $\mathcal H(A)$ be the statement that the conclusion

$$\|T(f_1,\dots,f_n)\|_V \le \mathcal C(A)\prod_{i=1}^n \|f_i\|_{\ell^r_{s_i}(\widetilde X_{\theta_i,r})}$$

holds for the vectors $\vec s=A\vec\delta$, $\vec\theta=A\vec e_1$. Now (20) is just saying that the statement $\mathcal H(P_\pi)$ holds. By Birkhoff's theorem every $A\in DS(n)$ is a convex combination of permutation matrices and therefore the general statement in (19) follows immediately from repeated applications of a *convexity property:* Namely, if $\mathcal H(A^+)$ and $\mathcal H(A^-)$ hold for two doubly stochastic matrices A^+ and A^- then the statement $\mathcal H((1-\gamma)A^+ +\gamma A^-)$ holds for $0<\gamma<1$.

We now verify this convexity property. Let A^+ and A^- be doubly stochastic matrices for which $\mathcal H(A^+)$ and $\mathcal H(A^-)$ hold, thus we have

$$\|S(f_1,\dots,f_n)\|_V \le \mathcal C(A^+)\prod_{i=1}^n \|f_i\|_{\ell^r_{s_i^+}(\widetilde X_{\theta_i^+,r})}$$
$$\|S(f_1,\dots,f_n)\|_V \le \mathcal C(A^-)\prod_{i=1}^n \|f_i\|_{\ell^r_{s_i^-}(\widetilde X_{\theta_i^-,r})}$$

for column vectors $\vec{s}^{\pm} = A^{\pm}\vec{\delta}$, $\vec{\theta}^{\pm} = A^{\pm}\vec{e}_1$. By taking generalized geometric means we also have

$$\|S(f_1,\dots,f_n)\|_V \le \mathcal{C}(A^+)^{1-\gamma}\mathcal{C}(A^-)^{\gamma}\prod_{i=1}^{n}\big(\|f_i\|^{1-\gamma}_{\ell^r_{s_i^+}(\widetilde{X}_{\theta_i^+,r})}\|f_i\|^{\gamma}_{\ell^r_{s_i^-}(\widetilde{X}_{\theta_i^-,r})}\big)$$

for $0<\gamma<1$.

Now let temporarily $W_{i,0} = \ell^r_{s_i^+}(\widetilde{X}_{\theta_i^+,r})$, and $W_{i,1} = \ell^r_{s_i^-}(\widetilde{X}_{\theta_i^-,r})$. By the last displayed formula and Lemma A.3 we get

$$\|S(f_1,\dots,f_n)\|_V \le C\,\mathcal{C}(A^+)^{1-\gamma}\mathcal{C}(A^-)^{\gamma}\prod_{i=1}^{n}\|f_i\|_{(W_{i,0},W_{i,1})_{\gamma,r}}\,.$$

By the reiteration theorem we have

$$\big(\widetilde{X}_{\theta_i^+,r},\widetilde{X}_{\theta_i^-,r}\big)_{\gamma,r} = \widetilde{X}_{(1-\gamma)\theta_i^++\gamma\theta_i^-,r}, \quad 0<\gamma<1\,,$$

and then, by Lemma A.4 there is the continuous embedding

$$\ell^r_{(1-\gamma)s_i^++\gamma s_i^-}(\widetilde{X}_{(1-\gamma)\theta_i^++\gamma\theta_i^-,r}) \hookrightarrow (W_{i,0},W_{i,1})_{\gamma,r}\,.$$

Hence, for some $\mathcal{C}$

$$\|S(f_1,\dots,f_n)\|_V \le \mathcal{C}\prod_{i=1}^{n}\|f_i\|_{\ell^r_{(1-\gamma)s_i^++\gamma s_i^-}(\widetilde{X}_{(1-\gamma)\theta_i^++\gamma\theta_i^-,r})}\,. \tag{21}$$

Let $A^{(\gamma)} = (1-\gamma)A^+ + \gamma A^-$ then $(1-\gamma)s_i^+ + \gamma s_i^- = \sum_{j=1}^n a_{ij}^{(\gamma)}\delta_j$ and $(1-\gamma)\theta_i^+ + \gamma\theta_i^- = a_{i1}^{(\gamma)}$ and thus (21) is just $\mathcal{H}((1-\gamma)A^+ + \gamma A^-)$. □

We shall now apply an iterated version of the interpolation method by Christ [12] to upgrade $n-1$ of the n spaces $\ell^r_{s_i}(\widetilde{X}_{\theta_i,r})$ to $\ell^\infty_{s_i}(\widetilde{X}_{\theta_i,\infty})$, provided that the parameters correspond to doubly stochastic matrices in the *interior* of the Birkhoff polytope, the set $DS^\circ(n)$ of doubly stochastic $n\times n$ matrices $A=(a_{ij})$ for which all entries lie in the open interval $(0,1)$. In the following lemma we assume the conclusion of the previous lemma and also an additional assumption on $\vec{\delta}$.

Lemma 3.2. *Suppose $n\ge 3$, $0<r\le 1$, and $\delta_1,\dots,\delta_n\in\mathbb{R}$. Assume that there are two indices i_1, i_2 with $2\le i_1<i_2\le n$ so that $\delta_{i_1}\neq\delta_{i_2}$. Let X_0, X_1 be compatible, complete quasi-normed spaces and let T be an n-linear operator defined on n-tuples of X_0+X_1-valued sequences, with values in an r-convex space V. Suppose that for every $A\in DS^\circ(n)$ there is $C(A)$ such that*

$$\|T(f_1,\dots,f_n))\|_V \le C(A)\prod_{i=1}^{n}\|f_i\|_{\ell^r_{\sigma_i}(X_{\mu_i,r})} \tag{22}$$

whenever $\vec{\sigma} = A\vec{\delta}$, $\vec{\mu} = A\vec{e}_1$.

Then for every $A \in DS^\circ(n)$ there is $\widetilde{C}(A)$ such that

$$\|T(f_1,\dots,f_n)\|_V \le \widetilde{C}(A)\|f_1\|_{\ell^r_{s_1}(\overline{X}_{\theta_1,r})}\prod_{i=2}^n \|f_i\|_{\ell^\infty_{s_i}(\overline{X}_{\theta_i,\infty})} \tag{23}$$

with $\vec{s} = A\vec{\delta}$, $\vec{\theta} = A\vec{e}_1$.

Proof. Let $A \in DS^\circ(n)$, and let $2 \le k \le n+1$. Let $\mathcal{H}_{n+1}(A)$ denote the statement that (22) is true for $\vec{\sigma} = A\vec{\delta}$, $\vec{\mu} = A\vec{e}_1$.

Let $\mathcal{H}_{n+1}$ denote the hypothesis (22) for all $A \in DS^\circ(n)$. For $2 \le k \le n$ let $\mathcal{H}_k(A)$ denote the statement that there is C depending on A so that the inequality

$$\|T(f_1,\dots,f_n)\|_V \le C\Big(\prod_{i=1}^{k-1}\|f_j\|_{\ell^r_{s_i}(\overline{X}_{\theta_i,r})}\Big)\Big(\prod_{i=k}^{n}\|f_i\|_{\ell^\infty_{s_i}(\overline{X}_{\theta_i,\infty})}\Big) \tag{24}$$

holds for all $(f_1,\dots,f_{k-1}) \in \prod_{i=1}^{k-1}\ell^r_{s_i}(\overline{X}_{\theta_i,r})$, $(f_k,\dots,f_n) \in \prod_{i=k}^n \ell^\infty_{s_i}(\overline{X}_{\theta_i,\infty})$, under the condition $\vec{s} = A\vec{\delta}$, $\vec{\theta} = A\vec{e}_1$. Let $\mathcal{H}_k$ denote the statement that $\mathcal{H}_k(A)$ holds for all $A \in DS^\circ(n)$.

We seek to prove $\mathcal{H}_2$. In what follows we thus need to show for $2 \le k \le n$, $A \in DS^\circ(n)$ that $\mathcal{H}_{k+1}$ implies $\mathcal{H}_k(A)$. We assume in our writeup that $k \le n-1$ but the proof carries through to cover the initial case $k = n$ if we interpret $\prod_{i=n+1}^n \cdots$ as 1.

Assuming $\mathcal{H}_{k+1}$ we shall first prove a preliminary inequality $\mathcal{H}_k^{prel}(A)$, namely

$$\|T(f_1,\dots,f_n)\|_V \le C\Big(\prod_{i=1}^{k-1}\|f_j\|_{\ell^r_{s_i}(\overline{X}_{\theta_i,r})}\Big)\|f_k\|_{\ell^r_{s_k}(\widetilde{X}_{\theta_k,\infty})}\Big(\prod_{i=k+1}^{n}\|f_i\|_{\ell^\infty_{s_i}(\overline{X}_{\theta_i,\infty})}\Big). \tag{25}$$

We denote by $\mathcal{H}_k^{prel}$ the statement that $\mathcal{H}_k^{prel}(A)$ holds for every $A \in DS^\circ(n)$.

Proof that $\mathcal{H}_{k+1}$ implies $\mathcal{H}_k^{prel}$. Fix $A \in DS^\circ(n)$ and let $\varepsilon > 0$ with the property that all entries of A lie in the open interval $(2\varepsilon, 1-2\varepsilon)$. We define two $n \times n$ matrices $A^+ = A^+(k)$ and $A^- = A^-(k)$ by letting

$$a^\pm_{\mu\nu} = \begin{cases} a_{\mu\nu} & \text{if } (\mu,\nu) \notin \{(1,1),(1,k),(k,1),(k,k)\} \\ a_{\mu\nu} \pm \varepsilon & \text{if } (\mu,\nu) = (1,1) \quad \text{or} \quad (\mu,\nu) = (k,k), \\ a_{\mu\nu} \mp \varepsilon & \text{if } (\mu,\nu) = (1,k) \quad \text{or} \quad (\mu,\nu) = (k,1). \end{cases} \tag{26}$$

It is easy to see that $A^\pm$ belong to $DS^\circ(n)$. Also if $\vec{s} = A\vec{\delta}$, $\vec{\theta} = A\vec{e}_1$, and $\vec{s}^\pm = A^\pm\vec{\delta}$, $\vec{\theta}^\pm = A^\pm\vec{e}_1$, then $s_i^\pm = s_i$ if $i \notin \{1,k\}$, $s_1^\pm = s_1 \pm \varepsilon(\delta_1 - \delta_k)$, $s_k^\pm = s_k \pm \varepsilon(\delta_k - \delta_1)$, moreover $\theta_i = \theta_i^\pm$ if $i \notin \{1,k\}$, $\theta_1^\pm = \theta_1 \pm \varepsilon$, $\theta_k^\pm = \theta_k \mp \varepsilon$.

We interpolate the linear operator L_k given by

$$g \mapsto L_k g = T(f_1,\dots,f_{k-1},g,f_{k+1},\dots,f_n) \tag{27}$$

using the real interpolation $K_{\vartheta,q}$ method with parameters $\vartheta = 1/2$ and $q = \infty$. Since $\theta_k^+ \neq \theta_k^-$ we have by Lemma A.4 and the reiteration theorem

$$\ell^r_{s_k}(\overline{X}_{\theta_k,\infty}) = \ell^r_{s_k}\big((\overline{X}_{\theta_k^+,r}, \overline{X}_{\theta_k^-,r})_{\frac12,\infty}\big) \hookrightarrow \big(\ell^r_{s_k^+}(\overline{X}_{\theta_k^+,r}), \ell^r_{s_k^-}(\overline{X}_{\theta_k^-,r})\big)_{\frac12,\infty}.$$

Thus we obtain by interpolation of L_k (using $\mathcal{H}_{k+1}(A^{\pm})$)

$$\|T(f_1,\dots,f_n)\|_V \le C\prod_{i=1}^{k-1}\Big(\|f_j\|^{1/2}_{\ell^r_{s_i^+}(\overline{X}_{\theta_i^+,r})}\|f_i\|^{1/2}_{\ell^r_{s_i^-}(\overline{X}_{\theta_i,r})}\Big)\times \\ \|f_k\|_{\ell^r_{s_k}(\overline{X}_{\theta_k,\infty})}\Big(\prod_{i=k+1}^{n}\|f_i\|_{\ell^\infty_{s_i}(\overline{X}_{\theta_i,\infty})}\Big).$$

By Lemma A.3 and the reiteration theorem we get

$$\|T(f_1,\dots,f_n)\|_V \le \\ C'\Big(\prod_{i=1}^{k-1}\|f_i\|_{\ell^r_{s_i}(\overline{X}_{\theta_i,r})}\Big)\|f_k\|_{\ell^r_{s_k}(\widetilde{X}_{\theta_k,\infty})}\Big(\prod_{i=k+1}^{n}\|f_i\|_{\ell^\infty_{s_i}(\overline{X}_{\theta_i,\infty})}\Big).$$

This finishes the proof of the implication $\mathcal{H}_{k+1} \implies \mathcal{H}_k^{prel}$.

Proof that $\mathcal{H}_k^{prel}$ implies $\mathcal{H}_k$. Fix $A \in DS^\circ(n)$ and let $\varepsilon > 0$ so that all entries of A lie in the open interval $(2\varepsilon, 1-2\varepsilon)$. We define two $n \times n$ matrices A^+ and A^- (depending on k and different from the ones in the first step) by letting

$$a^{\pm}_{\mu\nu} = \begin{cases} a_{\mu\nu} & \text{if } (\mu,\nu) \notin \{(1,i_1),(1,i_2),(k,i_1),(k,i_2)\} \\ a_{\mu\nu} \pm \varepsilon & \text{if } (\mu,\nu) = (1,i_1) \quad \text{or} \quad (\mu,\nu) = (k,i_2), \\ a_{\mu\nu} \mp \varepsilon & \text{if } (\mu,\nu) = (k,i_1) \quad \text{or} \quad (\mu,\nu) = (1,i_2). \end{cases} \tag{28}$$

Then A^+ and A^- are in $DS^\circ(n)$. It is important for our argument that the first column of $A^\pm$ is equal to the first column of A. Let $\vec{s}^{\pm} = A^{\pm}\vec{\delta}$, $\vec{s} = A\vec{\delta}$; then $s_i^\pm = s_i$ for $i \notin \{1,k\}$ and $s_1^\pm = s_1 \pm (\delta_{i_1} - \delta_{i_2})$, $s_k^\pm = s_k \pm (\delta_{i_2} - \delta_{i_1})$, so that by the assumption $\delta_{i_1} \neq \delta_{i_2}$ we have $s_k^+ \neq s_k^-$ and s_k is the arithmetic mean of s_k^+ and s_k^-. Moreover $s_i^\pm = s_i$ if $i \notin \{1,k\}$.

We interpolate the linear operator L_k as in (27). This time we use $\mathcal{H}_k^{prel}(A^\pm)$ and the formula

$$\big(\ell^r_{s_k^+}(\overline{X}_{\theta_k,\infty}), \ell^r_{s_k^-}(\overline{X}_{\theta_k,\infty})\big)_{\frac12,\infty} = \ell^\infty_{s_k}(\overline{X}_{\theta_k,\infty}) \tag{29}$$

which is a special case of formula (89) in the appendix. This yields

$$\|T(f_1,\dots,f_n)\|_V \le \\ C\Big(\prod_{i=1}^{k-1}(\|f_i\|^{1/2}_{\ell^r_{s_i^+}(\overline{X}_{\theta_i,r})}\|f_i\|^{1/2}_{\ell^r_{s_i^-}(\overline{X}_{\theta_i,r})})\Big)\Big(\prod_{i=k}^{n}\|f_i\|_{\ell^\infty_{s_i}(\overline{X}_{\theta_i,\infty})}\Big).$$

By Lemma A.3 and the reiteration theorem we also get

$$\|T(f_1,\dots,f_n)\|_V \le C'\Big(\prod_{i=1}^{k-1}(\|f_i\|_{\ell^r_{s_i}(\overline{X}_{\theta_i,r})}\Big)\Big(\prod_{i=k}^{n}\|f_i\|_{\ell^\infty_{s_i}(\overline{X}_{\theta_i,\infty})}\Big)$$

which is $\mathcal{H}_k(A)$. □

We are now in the position to give the

Proof of Theorem 1.3. We first reduce to the case $m = 1$. If π denotes the permutation that interchanges 1 and m, with $\pi(i) = i$ for $i \notin \{1, m\}$, we define

$$T_\pi[f_1,\dots,f_n] = T[f_{\pi(1)},\dots,f_{\pi(n)}]. \tag{30}$$

If T satisfies the assumptions of Theorem 1.3 then T_π satisfies the assumptions with the choice $m = 1$ and the parameters $(\delta_1,\dots,\delta_n)$ replaced with $(\delta_{\pi(1)},\dots,\delta_{\pi(n)})$. By symmetry we get the statement for T from the statement for T_π.

After this reduction we may assume $m = 1$ in what follows. Let $A \in DS^\circ(n)$. For any $B \in DS(n)$ let $\mathcal{H}_A(B)$ denote the statement that there is $C > 0$ so that the inequality

$$\|T(f_1,\dots,f_n)\|_V \le C\prod_{i=1}^{n}\|f_i\|_{\ell^{q_i}_{s_i}(\overline{X}_{\theta_i,q_i})}$$

holds for all $(f_1,\dots,f_n) \in \prod_{i=1}^n \ell^{q_i}_{s_i}(\overline{X}_{\theta_i,q_i})$, under the condition that

$$\vec{s} = BA\vec{\delta}, \quad \vec{\theta} = BA\vec{e}_1, \quad \begin{pmatrix} r/q_1 \\ \vdots \\ r/q_n \end{pmatrix} = B\vec{e}_1\,.$$

As an immediate consequence of Lemma A.2 we see that given $A \in DS^\circ(n)$ the matrices B satisfying $\mathcal{H}_A(B)$ satisfy a convexity property, namely, if $\mathcal{H}_A(B^{(0)})$, $\mathcal{H}_A(B^{(1)})$ hold for $B^{(0)} \in DS(n)$, $B^{(1)} \in DS(n)$ then for $0 < \vartheta < 1$, $B^{(\vartheta)} = (1-\vartheta)B^{(0)} + \vartheta B^{(1)}$ the statement $\mathcal{H}_A(B^{(\vartheta)})$ also holds. By Birkhoff's theorem $\mathcal{H}_A(B)$ holds for all $B \in DS(n)$ once we have shown it for all permutation matrices.

For this we first apply Lemma 3.1, and then, by the conclusion of that lemma we can apply Lemma 3.2 to the multilinear operators T_π in (30). As a consequence there is, for every $A \in DS^\circ(n)$, a constant $C(A)$ so that for $s_j = \sum_{j=1}^n a_{ij}\delta_j$, $\theta_i = a_{i,1}$ and every permutation π

$$\|T(f_{\pi(1)},\dots,f_{\pi(n)})\|_V \le C\|f_1\|_{\ell^r_{s_1}(\widetilde{X}_{\theta_1,r})}\prod_{i=2}^{n}\|f_i\|_{\ell^\infty_{s_i}(\widetilde{X}_{\theta_i,\infty})}\,. \tag{31}$$

It is straightforward to check that this conclusion is exactly statement $\mathcal{H}_A(P)$ for all permutation matrices P. □

4. Hypotheses for the restriction theorem

Given a parameter interval I we shall consider a class $\mathfrak{C}$ of vector valued functions $\gamma : J_\gamma \to \mathbb{R}^d$ of class C^d defined on subintervals J_γ of I. For every $\gamma \in \mathfrak{C}$ the restrictions of γ to subintervals of J_γ will also be in $\mathfrak{C}$.

Definition. *(i) Let J be an interval and let $\kappa = (\kappa_1, \dots, \kappa_d)$ so that $\kappa_1 \le \kappa_2 \le \cdots \le \kappa_d$, $\kappa_d - \kappa_1 \le |J|$ and one of the coordinates κ_i is equal to 0. Let $J^\kappa = \{t : t + \kappa_1 \in J, t + \kappa_d \in J\}$. Then given a curve $t \mapsto \gamma(t) \in \mathbb{R}^d$, $t \in J$ we define the κ-offspring curve γ_κ on J^κ by*

$$\gamma_\kappa(t) = \sum_{j=1}^{d} \gamma(t + \kappa_j).$$

(ii) Let $\tau = \tau_\gamma$ be as in (1). We denote by τ_{γ_κ} the corresponding expression for the offspring curve, i.e.

$$\tau_{\gamma_\kappa} = \det\big(\sum_{j=1}^{d} \gamma'(t+\kappa_j), \dots, \sum_{j=1}^{d} \gamma^{(d)}(t+\kappa_j)\big)\,.$$

Let

$$Q = \frac{d^2 + d + 2}{2} \tag{32}$$

so that $Q = p_d'$ with p_d as in (2). For $\gamma \in \mathfrak{C}$ defined on I we shall consider the adjoint operator

$$\mathcal{E}_w f(x) = \int_I e^{-\mathrm{i}\langle x, \gamma(t)\rangle} f(t) w(t) dt$$

with $w(t) \equiv w_\gamma(t) = |\tau_\gamma(t)|^{2/(d^2+d)}$ and then examine the $L^Q(w; I) \to L^{Q,\infty}(\mathbb{R}^d)$ operator norms (here we work with a fixed equivalent norm on $L^{Q,\infty}(\mathbb{R}^d)$). We consider $L^{Q,\infty}(\mathbb{R}^d)$ as a normed space, the norm being $\|h\|^{**}_{L^{Q,\infty}} = \sup_{t>0} t^{1/Q} h^{**}(t)$ where h^{**} is as in (83) with $\rho = 1$. We also continue to use $\|h\|_{L^{Q,\infty}} = \sup \alpha(\mathrm{meas}(\{|h| > \alpha\}))^{1/Q}$, the usual equivalent quasinorm (and the constants in this equivalence are independent of the measure space).

We shall make the a priori assumption that

$$\mathcal{B} \equiv \mathcal{B}(\mathfrak{C}) := \sup_{\gamma \in \mathfrak{C}} \sup_{\|f\|_{L^Q(w_\gamma)} \le 1} \|\mathcal{E}_w f\|^{**}_{L^{Q,\infty}} \tag{33}$$

is finite and the main goal is to give a geometric bound for the constant $\mathcal{B}$. We remark that the finiteness of $\mathcal{B}$ has been shown in [6] for certain classes of smooth curves with nonvanishing torsion. Once a more effective bound for $\mathcal{B}$ is established one can prove Theorems 1.1 and 1.2 by limiting arguments.

We now formulate the hypotheses of our main estimate. Two of them were relevant already in [20], [21].

Hypotheses 4.1. *Let $\mathfrak{C}$ be a class of curves with base interval $I_\circ$. For $\gamma \in \mathfrak{C}$ defined on $I \subset I_\circ$ let*

$$E = \{(t_1, \dots, t_d) : t_1 \in I,\ t_d \in I,\ t_1 < t_2 < \cdots < t_d\}. \tag{34}$$

(i) There is $N_1 \geq 1$ so that for every $\gamma \in \mathfrak{C}$ the map $\Phi_\gamma : E \to \mathbb{R}^d$ with

$$\Phi_\gamma(t_1, \dots, t_d) = \sum_{j=1}^{d} \gamma(t_j) \tag{35}$$

is of multiplicity at most N_1.

(ii) Let $\mathcal{J}_{\Phi_\gamma}$ denote the Jacobian of Φ_γ,

$$\mathcal{J}_{\Phi_\gamma}(t_1, \dots, t_d) = \det\big(\gamma'(t_1), \dots, \gamma'(t_d)\big).$$

Then there is $\mathfrak{c}_1 > 0$ such that for every $(t_1, \dots, t_d) \in \mathcal{I}^d$ with $t_1 < \cdots < t_d$ we have the inequality

$$|\mathcal{J}_{\Phi_\gamma}(t_1, \dots, t_d)| \geq \mathfrak{c}_1 \Big(\prod_{i=1}^{d} \tau_\gamma(t_i)\Big)^{1/d} \prod_{1 \leq j < k \leq d} (t_k - t_j)\,. \tag{36}$$

(iii) Every offspring curve of a curve in $\mathfrak{C}$ is (after possible reparametrization) the affine image of a curve in $\mathfrak{C}$.

(iv) There is $\mathfrak{c}_2 > 0$ so that for every $\gamma \in \mathfrak{C}$ and every offspring curve γ_κ of γ we have the inequality

$$|\tau_{\gamma_\kappa}(t)| \geq \mathfrak{c}_2 \max_{j=1,\dots,d} |\tau_\gamma(t + \kappa_j)|\,. \tag{37}$$

Inequality (37) is a strengthening of a weaker inequality which was used in [20], [21], [6], [7], namely

$$|\tau_{\gamma_\kappa}(t)| \gtrsim \prod_{j=1}^{d} |\tau_\gamma(t + \kappa_j)|^{1/d}. \tag{38}$$

The stronger inequality allows us to replace the geometric mean on the right hand side of (38) by generalized geometric means $\prod_{j=1}^{d} |\tau_\gamma(t + \kappa_j)|^{\eta_j}$ for nonnegative η_j with $\sum_{j=1}^{d} \eta_j = 1$. Our main result is

Theorem 4.2. *Let $\mathfrak{C}$ be a class of curves satisfying Hypothesis 4.1 and $\mathcal{B}(\mathfrak{C}) < \infty$. Then*

$$\mathcal{B}(\mathfrak{C}) \leq C(d, N_1, \mathfrak{c}_1^{-1}, \mathfrak{c}_2^{-1}). \tag{39}$$

For an explicit constant see (57) below.

5. Proof of Theorem 4.2

Let $w \equiv w_\gamma$ define the affine arclength measure of γ. We start with

Observation 5.1. *If $\gamma \in \mathfrak{C}$ is defined on I and $\widetilde{w}$ is a nonnegative measurable weight satisfying $\widetilde{w}(t) \le w(t)$ then there is a constant C such that*

$$\big\|\mathcal{E}_{\widetilde{w}} f\big\|_{L^{Q,\infty}(\mathbb{R}^d)} \le C\mathcal{B}\Big(\int_I |f(t)|^Q \widetilde{w}(t)\, dt\Big)^{1/Q}.$$

Proof. One can use a duality argument (as in the submitted version) to deduce the claim from $\|\widehat{g}\circ\gamma\|_{L^{Q'}(\widetilde{w})} \le \|\widehat{g}\circ\gamma\|_{L^{Q'}(w)}$. The referee suggested a simpler and more direct argument: Write $\widetilde{w} = hw$ where $0 \le h \le 1$ and use that $\mathcal{E}_{\widetilde{w}} f = \mathcal{E}_w(hf)$ and $h^Q \le h$. □

We aim to prove estimates for the d-linear operator $\mathcal{M}$ defined by

$$\begin{aligned}\mathcal{M}[f_1,\dots,f_d](x) &= \prod_{i=1}^d \mathcal{E}_w f_i(x)\\ &= \int_{I^d} \exp(-\mathrm{i}\langle x, \sum_{j=1}^d \gamma(t_j)\rangle) \prod_{i=1}^d [f_j(t_j) w(t_j)]\, dt_1 \dots dt_d\end{aligned}$$

Denote by $V(t_1,\dots,t_d) = \prod_{1\le i<j\le d}(t_j - t_i)$ the Vandermonde determinant and let, for $l \in \mathbb{Z}$,

$$E_l = \{(t_1,\dots,t_d) \in I^d : 2^{-l-1} < |V(t)| \le 2^{-l}\}.$$

Following the reasoning in [4], [6] for the nondegenerate case we decompose $\mathcal{M} = \sum_{l\in\mathbb{Z}} \mathcal{M}_l$ where

$$\mathcal{M}_l[f_1,\dots,f_d](x) = \int_{E_l} \exp(-\mathrm{i}\langle x, \sum_{j=1}^d \gamma(t_j)\rangle) \prod_{i=1}^d [f_i(t_j) w(t_i)]\, dt_1 \dots dt_d\,. \tag{40}$$

An important ingredient is an estimate for the sublevel sets of the restriction of V to $\mathbb{R}^{d-1}$, namely

$$\mathrm{meas}\big(\{(h_1,\dots,h_{d-1}) : |h_1\cdots h_{d-1}| \prod_{1\le i<j\le d-1} |h_j - h_i| \le \alpha\}\big) \le C_d \alpha^{2/d} \tag{41}$$

where the measure is Lebesgue measure in $\mathbb{R}^{d-1}$. This was proved in [20] (*cf.* also the exposition in [6]).

Lemma 5.2. *(a) Let $\rho_i \in [2,\infty]$ be such that $\sum_{i=1}^d \rho_i^{-1} = 1/2$. Then*

$$\|\mathcal{M}_l[f_1,\dots,f_d]\|_{L^2} \lesssim (N_1/\mathfrak{c}_1)^{1/2} 2^{l\frac{d-2}{2d}} \prod_{i=1}^d \big\|f_i w^{\frac{3-d}{4}}\big\|_{\rho_i}. \tag{42}$$

(b) Let $\eta_i \in [0,1]$ so that $\sum_{i=1}^d \eta_i = 1$, and q_i such that $\sum_{i=1}^d q_i^{-1} = 1/Q$. Then

$$\|\mathcal{M}_l[f_1,\dots,f_d]\|_{L^{Q,\infty}} \lesssim \mathcal{B}\mathfrak{c}_2^{-1/Q} 2^{-2l/d} \prod_{i=1}^d \big\|f_i w^{1-\frac{\eta_i}{Q'}}\big\|_{q_i}. \tag{43}$$

Proof. For every permutation π on d letters set

$$E^\pi = \{(t_1,\dots,t_d) : t_{\pi(1)} < \dots < t_{\pi(d)}\}.$$

By assumption the map $\Phi : (t_1,\dots,t_d) \mapsto \sum_{i=1}^d \gamma(t_i)$ is of bounded multiplicity $d!N_1$ on $\cup_\pi E^\pi$. We apply the change of variable, followed by Plancherel's theorem, followed by the inverse change of variable to bound

$$\|\mathcal{M}_l[f_1,\dots,f_d]\|_2 \lesssim \Big(N_1 \int_{E_l} \big|\prod_{i=2}^d [f_i(t_i)w(t_i)\big|^2 \frac{dt_1\dots dt_d}{|J_\Phi(t_1,\dots,t_d)|}\Big)^{1/2}.$$

By our assumption (36), the right hand side is dominated by

$$\begin{aligned}&\Big(N_1 \int_{E_l} \big|\prod_{i=1}^d f_i(t_i)w(t_i)\big|^2 \frac{dt_1\dots dt_d}{\mathfrak{c}_1 |V(t)| \prod_{i=1}^d w(t_i)^{\frac{d+1}{2}}}\Big)^{1/2}\\ &\lesssim \Big(\frac{N_1}{\mathfrak{c}_1} \int_{E_l} \big|\prod_{i=1}^d f_i(t_i)w(t_i)^{\frac{3-d}{4}}\big|^2 \frac{dt_1\dots dt_d}{|V(t)|}\Big)^{1/2}\end{aligned}$$

The sublevel set estimate (41) with $\alpha = 2^{-l}$ yields

$$\Big(\int_{E_l} \big|\prod_{i=1}^d g_i(t_i)\big|^2 \frac{dt_1\dots dt_d}{|V(t)|}\Big)^{1/2} \lesssim 2^{l\frac{d-2}{2d}} \|g_d\|_2 \prod_{i=1}^{d-1} \|g_i\|_\infty .$$

By symmetry we get a similar statement with the variables permuted and then by complex interpolation also

$$\Big(\int_{E_l} \big|\prod_{i=1}^d g_i(t_i)\big|^2 \frac{dt_1\dots dt_d}{|V(t)|}\Big)^{1/2} \lesssim 2^{l\frac{d-2}{2d}} \prod_{i=1}^{d} \|g_i\|_{\rho_i} \tag{44}$$

where $\sum_{i=1}^d \rho_i^{-1} = 1/2$. If we apply this statement with $g_i = f_i w^{\frac{3-d}{4}}$ we obtain (42).

To prove (43) it suffices to show that for fixed $(\eta_1,\dots,\eta_d)$ with $\eta_i \ge 0$ and $\sum_{i=1}^d \eta_i = 1$

$$\|\mathcal{M}_l[f_1,\dots,f_d]\|_{L^{Q,\infty}} \lesssim \mathcal{B}\mathfrak{c}_2^{-1/Q} 2^{-2l/d} \big\|f_1 w^{1-\frac{\eta_1}{Q'}}\big\|_Q \prod_{i=2}^d \big\|f_i w^{1-\frac{\eta_i}{Q'}}\big\|_\infty. \tag{45}$$

Once this inequality is verified, we also get, by the symmetry of $\mathcal{M}_l$, the bound $\mathcal{B}\mathfrak{c}_2^{-1/Q} 2^{-2l/d} \|f_m w^{1-\frac{\eta_m}{Q'}}\|_Q \prod_{i\ne m} \|f_i w^{1-\frac{\eta_i}{Q'}}\|_\infty$ and then the inequality (43) follows by complex interpolation.

Let, for any permutation π on d letters

$$E_l^\pi = \{(t_1,\dots,t_d) \in I^d : t_{\pi(1)} < \dots < t_{\pi(d)},\ 2^{-l-1} < V(t_1,\dots,t_d) \le 2^{-l}\}.$$

To prove (45) we split $\mathcal{M}_l = \sum_\pi M_l^\pi$ where

$$M_l^\pi[f_1,\dots,f_d](x) = \int_{E_l^\pi} \exp(-\mathrm{i}\langle x, \sum_{j=1}^d \gamma(t_j)\rangle) \prod_{i=1}^d [f_i(t_i)w(t_i)]\, dt_1\dots dt_d\, .$$

We need to bound $\|\mathcal{M}_l^\pi[f_1,\dots,f_d]\|_{L^{Q,\infty}}$ by the right hand side of (45).

Now let $\nu=\pi^{-1}(1)$ and let H_ν be the hyperplane $\{(\kappa_1,\dots,\kappa_d):\kappa_\nu=0\}$. We change variables

$$s=t_1,\quad \kappa_j=t_{\pi(j)}-t_1,\ j\neq\nu$$

hence $t_i=s+\kappa_{\pi^{-1}(i)}$, $i=1,\dots,d$, and thus $t_1=s$. Note that with this identification

$$\begin{aligned}&|V(s,s+\kappa_{\pi^{-1}(2)},\dots,s+\kappa_{\pi^{-1}(d)})|\\ &\quad=\prod_{i=2}^d|\kappa_{\pi^{-1}(i)}|\prod_{2\le i<j\le d}|\kappa_{\pi^{-1}(j)}-\kappa_{\pi^{-1}(i)}|=\prod_{i\neq\nu}|\kappa_i|\prod_{\substack{1\le i<j\le d\\ i\neq\nu,\,j\neq\nu}}|\kappa_j-\kappa_i|\end{aligned}$$

and $\sum_{j=1}^d\gamma(s+\kappa_{\pi^{-1}(j)})=\gamma(s)+\sum_{i\neq\nu}\gamma(s+\kappa_i)$.

If $I=[a_L,a_R]$ we define $I^\kappa=[a_L-\kappa_1,a_R-\kappa_d]$ and

$$\begin{aligned}\mathfrak{D}_l^\nu=\big\{\kappa:\kappa_1<\kappa_2<\dots<\kappa_d,\ \kappa_\nu=0,\ \kappa_d-\kappa_1\le a_R-a_L,&\\ 2^{-l-1}\le\prod_{i\neq\nu}|\kappa_i|\prod_{\substack{1\le i<j\le d\\ i\neq\nu,\,j\neq\nu}}|\kappa_j-\kappa_i|<2^{-l}\big\}.&\end{aligned}$$

Let dm_{H_ν} denote Lebesgue measure on H_ν (in $d-1$ dimensions). Let, for $\kappa\in H_\nu$, denote by γ_κ the offspring curve $\gamma_\kappa(s)=\gamma(s)+\sum_{i\neq\nu}\gamma(s+\kappa_i)=\sum_{i=1}^d\gamma(s+\kappa_i)$. Then

$$\begin{aligned}(46)\quad M_l^\pi[f_1,\dots,f_d](x)=\int_{\mathfrak{D}_l^\nu}\int_{I^\kappa}\exp(-\mathrm{i}\langle x,\gamma_\kappa(s)\rangle)\times&\\ f_{\pi(\nu)}(s)w(s)\prod_{i\neq\nu}[f_{\pi(i)}(s+\kappa_i)w(s+\kappa_i)]\,ds\,dm_{H_\nu}(\kappa)\,.&\end{aligned}$$

Let

$$w_\kappa(t)=|\tau_{\gamma_\kappa}(t)|^{\frac{2}{d^2+d}},$$

the weight for the affine arclength measure associated to γ_κ. By assumption the offspring curve γ_κ is (after possible reparametrization) an affine image of a curve in the family $\mathfrak{C}$. Thus, by affine invariance and invariance under reparametrizations we have the inequality

$$\Big\|\int_{I_\kappa}\exp(-\mathrm{i}\langle\cdot,\gamma_\kappa(s)\rangle)g(s)w_\kappa(s)\,ds\Big\|_{L^{Q,\infty}}\le\mathcal{B}\|g\|_{L^Q(w_\kappa)}.$$

By hypothesis (37), and $\sum_{i=1}^d\eta_{\pi(i)}=\sum_{i=1}^d\eta_i=1$,

$$w_\kappa(s)\ge\mathfrak{c}_2^{\frac{2}{d^2+d}}\prod_{i=1}^d w(s+\kappa_i)^{\eta_{\pi(i)}},\quad s\in I^\kappa\,.$$

By Observation 5.1 we then also have

$$\Big\| \int_{I^\kappa} \exp(-\mathrm{i}\langle \cdot, \gamma_\kappa(s)\rangle) g(s) \mathfrak{c}_2^{\frac{2}{d^2+d}} \prod_{i=1}^d w(s+\kappa_i)^{\eta_{\pi(i)}} ds \Big\|_{L^{Q,\infty}} \lesssim \mathcal{B}\Big(\int_{I_\kappa} |g(s)|^Q \mathfrak{c}_2^{\frac{2}{d^2+d}} \prod_{i=1}^d w(s+\kappa_i)^{\eta_{\pi(i)}} ds \Big)^{1/Q}.$$

We apply this with

$$g(s) \equiv G^\kappa(s) := f_{\pi(\nu)}(s) w(s)^{1-\eta_{\pi(\nu)}} \prod_{i\neq\nu} [f_{\pi(i)}(s+\kappa_i) w(s+\kappa_i)^{1-\eta_{\pi(i)}}]$$

and, using the relation $\frac{2}{d^2+d}(\frac{1}{Q}-1) = -\frac{1}{Q}$, we arrive at

$$\Big\| \int_{I^\kappa} \exp(-\mathrm{i}\langle \cdot, \gamma_\kappa(s)\rangle) G^\kappa(s) \prod_{i=1}^d w(s+\kappa_i)^{\eta_{\pi(i)}} ds \Big\|_{L^{Q,\infty}} \lesssim \mathcal{B}\mathfrak{c}_2^{-1/Q} \Big(\int_{I_\kappa} |G^\kappa(s)|^Q \prod_{i=1}^d w(s+\kappa_i)^{\eta_{\pi(i)}} ds \Big)^{1/Q}. \tag{47}$$

Now use the triangle inequality for an equivalent norm in the space $L^{Q,\infty}$ and apply the analogue of the integral Minkowski inequality to get

$$\begin{aligned} &\big\| M_l^\pi [f_1, \cdots, f_d] \big\|_{L^{Q,\infty}} \\ &\lesssim \int_{\mathfrak{D}_l^\nu} \Big\| \int_{I^\kappa} e^{\mathrm{i}\langle \cdot, \gamma_\kappa(s)\rangle} G^\kappa(s) \prod_{i=1}^d w(s+\kappa_i)^{\eta_{\pi(i)}}\, ds \Big\|_{L^{Q,\infty}} dm_{H^\nu}(\kappa) \\ &\lesssim \mathfrak{c}_2^{-1/Q} \mathcal{B} \int_{\mathfrak{D}_l^\nu} \Big(\int_{I^\kappa} \big| f_{\pi(\nu)}(s) w(s)^{1-\eta_{\pi(\nu)}} \prod_{i\neq\nu} [f_{\pi(i)}(s+\kappa_i) w(s+\kappa_i)^{1-\eta_{\pi(i)}}] \big|^Q \\ &\qquad\qquad \times \prod_{i=1}^d w(s+\kappa_i)^{\eta_{\pi(i)}}\, ds \Big)^{1/Q} dm_{H^\nu}(\kappa) \end{aligned}$$

which is

$$\lesssim \mathfrak{c}_2^{-1/Q} \mathcal{B} \int_{\mathfrak{D}_l^\nu} \Big(\int | f_{\pi(\nu)}(s) w(s)^{1-\eta_{\pi(\nu)}/Q'} |^Q ds \Big)^{1/Q} \times \prod_{i\neq\nu} \| f_{\pi(i)} w^{1-\eta_{\pi(i)}/Q'} \|_\infty dm_{H^\nu}(\kappa).$$

By (41) we have $m_{H^\nu}(\mathcal{D}_l^\nu) \lesssim 2^{-2l/d}$. Thus the last displayed quantity is less than a constant times

$$\begin{aligned} &2^{-2l/d}\mathfrak{c}_2^{-1/Q}\mathcal{B}\big\|f_{\pi(\nu)}w^{1-\eta_{\pi(\nu)}/Q'}\big\|_Q \prod_{i\neq\nu} \|f_{\pi(i)}w^{1-\eta_{\pi(i)}/Q'}\|_\infty \\ =&2^{-2l/d}\mathfrak{c}_2^{-1/Q}\mathcal{B}\big\|f_1 w^{1-\eta_d/Q'}\big\|_Q \prod_{j=2}^{d} \|f_j w^{1-\eta_j/Q'}\|_\infty \end{aligned}$$

and (45) is proved. □

Lemma 5.3. *Let* $q_i \in [Q,\infty]$, $\rho_i \in [2,\infty]$, $\vartheta_i \in [0,1]$ *satisfy*

$$\sum_{i=1}^{d}\Big(\frac{1}{q_i}, \frac{1}{\rho_i}, \eta_i\Big) = \Big(\frac{1}{Q}, \frac{1}{2}, 1\Big).$$

Let

$$\frac{1}{p_i} = \frac{d-2}{d+2}\frac{1}{q_i} + \frac{4}{d+2}\frac{1}{\rho_i}\,, \tag{48}$$

$$\beta_i = \frac{d-2}{d+2}\Big(1-\frac{\eta_i}{Q'}\Big) + \frac{3-d}{d+2}\,. \tag{49}$$

Then $\sum_{i=1}^d p_i^{-1} = d/Q$, $\sum_{i=1}^d \beta_i = d/Q$, *and we have*

$$\big\|\mathcal{M}[f_1,\dots,f_d]\big\|_{L^{Q/d,\infty}} \lesssim \mathcal{C}\mathcal{B}^{\frac{d-2}{d+2}} \prod_{i=1}^{d} \|f_i\|_{b^1_{\beta_i}(w,L^{p_i,1})} \tag{50}$$

with

$$\mathcal{C} = \big(N_1/\mathfrak{c}_1\big)^{\frac{d-2}{2(d+2)}}\mathfrak{c}_2^{-\frac{4}{(d+2)Q}}. \tag{51}$$

Proof. We may interpolate the L^2 bounds and the $L^{Q,\infty}$ bounds for $\mathcal{M}_l$ to get a satisfactory estimate for the $L^{Q/d,\infty}$ norm of each $\mathcal{M}_l$ but the resulting estimates cannot be summed in l. We use a familiar trick from [11], estimating $\sum_l \mathcal{M}_l$ using the bound (42) for $2^l \le \Lambda$ and the bound (43) for $2^l > \Lambda$, for Λ to be determined. For fixed $\alpha > 0$ we need to estimate the measure of $G_\alpha = \{x : |\mathcal{M}_l[f_1,\dots,f_d]| > 2\alpha\}$. By Tshebyshev's inequality,

$$\text{meas}(G_\alpha) \le \alpha^{-2}\Big\|\sum_{2^l\le\Lambda}\mathcal{M}_l[f_1,\dots,f_d]\Big\|_2^2 + \alpha^{-Q}\Big\|\sum_{2^l>\Lambda}\mathcal{M}_l[f_1,\dots,f_d]\Big\|_{L^{Q,\infty}}^Q$$

and applying Lemma 5.2 we obtain

$$\text{meas}(G_\alpha) \le \alpha^{-2}\Lambda^{\frac{d-2}{d}}\Gamma^2 + \alpha^{-Q}\Lambda^{-2Q/d}\Delta^Q\,, \tag{52}$$

with

$$\Gamma := (N_1/\mathfrak{c}_1)^{1/2}\prod_{i=1}^{d}\big\|f_i w^{\frac{3-d}{4}}\big\|_{\rho_i}, \qquad \Delta := \mathcal{B}\mathfrak{c}_2^{-1/Q}\prod_{i=1}^{d}\big\|f_i w^{1-\frac{\eta_i}{Q'}}\big\|_{q_i}.$$

We choose Λ so that the two expressions on the right hand side of (52) balance, i.e. $\Lambda = (\alpha^{2-Q}\Gamma^Q/\Delta^2)^{\frac{d}{d-2+2Q}}$. This leads to the bound $\text{meas}(G_\alpha) \lesssim \big(\alpha^{-1}\Delta^{\frac{d-2}{d+2}}\Gamma^{\frac{4}{d+2}}\big)^{\frac{(d+2)Q}{d-2+2Q}}$ and we have $\frac{(d+2)Q}{d-2+2Q} = \frac{Q}{d}$ for $Q = \frac{d^2+d+2}{2}$. Thus

$$\|\mathcal{M}[f_1, \cdots, f_d]\|_{L^{Q/d,\infty}} \lesssim \mathcal{C}\mathcal{B}^{\frac{d-2}{d+2}}\Delta^{\frac{d-2}{d+2}}\Gamma^{\frac{4}{d+2}}$$

with $\mathcal{C}$ as in (51).

By Lemma A.3 the previous display implies that

$$\|\mathcal{M}[f_1, \cdots, f_d]\|_{L^{Q/d,\infty}} \lesssim \mathcal{C}\mathcal{B}^{\frac{d-2}{d+2}} \prod_{i=1}^{d} \|f_i\|_{\overline{Y}^i_{\frac{4}{d+2},1}} \tag{53}$$

where the interpolation space refers to the couple $\overline{Y}^i$ with

$$Y_0^i = b^1_{1-\frac{\eta_i}{Q'}}(w, L^{q_i}), \qquad Y_1^i = b^1_{\frac{3-d}{4}}(w, L^{r_i}).$$

Since the block Lorentz spaces are retracts of sequence spaces (*cf.* the discussion following (11)) the formula (90) implies the continuous embedding

$$b^1_{(1-\vartheta)(1-\frac{\eta_i}{Q'})+\vartheta\frac{3-d}{4}}((L^{q_i}, L^{\rho_i})_{\vartheta,1}) \hookrightarrow (b^1_{1-\frac{\eta_i}{Q'}}(w, L^{q_i}), b^1_{\frac{3-d}{4}}(w, L^{\rho_i}))_{\vartheta,1}$$

We apply this with $\vartheta = 4/(d+2)$. Then if p_i, β_i are in (48), (49) the usual interpolation formula for Lorentz spaces gives

$$(L^{q_i}, L^{\rho_i})_{\frac{4}{d+2},1} = L^{p_i,1}.$$

and it follows that $b^1_{\beta_i}(w, L^{p_i,1})$ is continuously embedded in $\overline{Y}^i_{4/(d+2),1}$. Now (50) follows from (53). □

As stated above the conditions (48), (49) give $\sum_{i=1}^d p_i^{-1} = \sum_{i=1}^d \beta_i = d/Q$. In particular we may choose $p_i = Q$ and $\beta_i = 1/Q$ and this choice yields an estimate for $f_i \in b^1_{1/Q}(w, L^{Q,1})$, in particular (after setting all f_i equal to f)

$$\|\mathcal{E}_w f\|_{L^{Q,\infty}} \lesssim \mathcal{C}^{1/d}\mathcal{B}^{\frac{d-2}{d^2+2d}}\|f\|_{b^1_{1/Q}(L^{Q,1})}. \tag{54}$$

However we need a better estimate for f_i in the larger space $b^Q_{1/Q}(w, L^Q) = L^Q(w)$. In what follows write $b^p_s(L^p) = b^p_s(w, L^p)$ as the weight w will be fixed.

Proof of Theorem 4.2, cont. We choose $n > Q$ and estimate the n-linear operator

$$T[f_1, \dots, f_n] = \prod_{i=1}^{n} \mathcal{E}_w f_i$$

in $L^{r,\infty}$ where $r = Q/n < 1$.

For every permutation π on n letters we may write

$$T[f_1, \dots, f_n] = \mathcal{M}[f_{\pi(1)}, \dots, f_{\pi(d)}] \prod_{i=d+1}^{n} \mathcal{E}_w f_{\pi(i)}.$$

Notice that by Hölder's inequality for Lorentz spaces
(55)
$$\|T[f_1,\dots,f_n]\|_{L^{r,\infty}} \le C\|\mathcal{M}[f_{\pi(1)},\dots,f_{\pi(d)}]\|_{L^{Q/d,\infty}} \prod_{i=d+1}^{n} \|\mathcal{E}_w f_{\pi(i)}\|_{L^{Q,\infty}}\,.$$

We apply Lemma 5.3 for special choices of the parameters q_i, ρ_i. Let μ be a small parameter (say $|\mu| \ll (10Qd^2)^{-1}$), put $\rho_i = 2d$, $i = 1,\dots,d$, let

$$\begin{aligned} \frac{1}{q_3} &= \frac{1}{Qd} + \mu\frac{d+2}{d-2}\,,\\ \frac{1}{q_2} &= \frac{1}{Qd} + \mu\frac{d+2}{n-2}\,,\\ \frac{1}{q_1} &= \frac{1}{Qd} - \mu(d+2)\frac{n-1}{n-2}\,, \end{aligned}$$

and set $q_d = \cdots = q_3$. Then $\sum_{i=1}^{d} q_i^{-1} = Q^{-1}$ and $q_3 < q_2 < Qd < q_1$ if $\mu > 0$ (for $\mu < 0$ these inequalities are reversed). Now by (48), $p_i^{-1} = \frac{d-2}{d+2}q_i^{-1} + \frac{2}{d(d+2)}$ and since $Q = \frac{d^2+d+2}{2}$ we have $\frac{d-2}{(d+2)Q} + \frac{2}{d+2} = \frac{d}{Q}$ and thus $p_i^{-1} = Q^{-1} + (q_i^{-1} - (Qd)^{-1})\frac{d-2}{d+2}$. Hence $p_3 = \cdots = p_d$ and

$$\begin{aligned} \frac{1}{p_3} &= \frac{1}{Q} + \mu\,,\\ \frac{1}{p_2} &= \frac{1}{Q} + \mu\frac{d-2}{n-2}\,,\\ \frac{1}{p_1} &= \frac{1}{Q} - \mu(d-2)\frac{n-1}{n-2}\,. \end{aligned}$$

Then $\sum_{i=1}^{d} p_i^{-1} = d/Q$, moreover $p_3 < p_2 < Q < p_1$ if $\mu > 0$. A crucial property of our choices is

$$\frac{1}{p_2} = \frac{d-2}{n-2}\frac{1}{p_3} + \frac{n-d}{n-2}\frac{1}{Q}. \tag{56}$$

Let β_i be as in (49) (with $\eta_3 = \cdots = \eta_d$ and the choice of η_2 and η_3 to be determined later). With these choices we use (55), and apply (50) for the term involving $\mathcal{M}$ and (54) for the remaining $n-d$ terms. This results in

$$\begin{aligned} \|T[f_1,\dots,f_n]\|_{L^{r,\infty}} &\lesssim \mathcal{B}^{\frac{d-2}{d+2}} \|f_{\pi(1)}\|_{b^1_{\beta_1}(L^{p_1,1})} \|f_{\pi(2)}\|_{b^1_{\beta_2}(L^{p_2,1})} \\ &\quad\times \prod_{i=3}^{d} \|f_{\pi(i)}\|_{b^1_{\beta_3}(L^{p_3,1})} \prod_{j=d+1}^{n} [\mathcal{C}^{1/d}\mathcal{B}^{\frac{d-2}{d^2+2d}} \|f_{\pi(j)}\|_{b^1_{1/Q}(L^{Q,1})}]\,. \end{aligned}$$

Now fix the first two entries and take generalized geometric means of these estimates to get

$$\big\|T[f_1,\dots,f_n]\|_{L^{r,\infty}} \lesssim \mathcal{C}\mathcal{B}^{n-d+\frac{d-2}{d+2}}\|f_{\pi(1)}\|_{b^1_{\beta_1}(L^{p_1,1})}\|f_{\pi(2)}\|_{b^1_{\beta_2}(L^{p_2,1})} \times \prod_{i=3}^{n}\big[\|f_{\pi(i)}\|^{\frac{d-2}{n-2}}_{b^1_{\beta_3}(L^{p_3,1})}\|f_{\pi(i)}\|^{\frac{n-d}{n-2}}_{b^1_{1/Q}(L^{Q,1})}\big].$$

By (56), and Lemma A.3, we get

$$\big\|T[f_1,\dots,f_n]\|_{L^{r,\infty}} \lesssim \mathcal{C}\mathcal{B}^{\frac{d-2}{d+2}}\big(\mathcal{C}^{\frac{1}{d}}\mathcal{B}^{\frac{d-2}{d^2+2d}}\big)^{n-d}\|f_{\pi(1)}\|_{b^1_{\beta_1}(L^{p_1,1})}\|f_{\pi(2)}\|_{b^1_{\beta_2}(L^{p_2,1})} \times \prod_{i=3}^{n}\|f_{\pi(i)}\|_{(b^1_{\beta_3}(L^{p_3,1}),b^1_{1/Q}(L^{Q,1}))_{\frac{n-d}{n-2},r}}.$$

The constant simplifies to $(\mathcal{C}\mathcal{B}^{\frac{d-2}{d+2}})^{n/d}$. By Lemma A.4 and a trivial embedding for the first two factors we also get

$$\big\|T[f_1,\dots,f_n]\big\|_{L^{r,\infty}} \lesssim (\mathcal{C}\mathcal{B}^{\frac{d-2}{d+2}})^{n/d}\|f_{\pi(1)}\|_{b^r_{\delta_1}(L^{p_1,r})}\|f_{\pi(2)}\|_{b^r_{\delta_2}(L^{p_2,r})}\prod_{i=3}^{n}\|f_{\pi(i)}\|_{b^r_{\delta_3}(L^{p_2,r})}$$

where $\delta_1=\beta_1$, $\delta_2=\beta_2$ and

$$\delta_3=\frac{d-2}{n-2}\beta_3+\frac{n-d}{n-2}\frac{1}{Q}\,.$$

We may choose η_2,η_3, so that $\delta_2\neq\delta_3$. This is needed for the application of Theorem 1.3. We choose $X_0=L^{p_2,1}$, $X_1=L^{p_1,1}$ and by the conclusion (10) of that theorem we obtain

$$\big\|T[f_1,\dots,f_n]\|_{L^{r,\infty}} \lesssim (\mathcal{C}\mathcal{B}^{\frac{d-2}{d+2}})^{n/d}\prod_{i=1}^{n}\|f_i\|_{b^{nr}_s((X_0,X_1)_{1/n,nr})},$$

with $s=\frac{1}{n}(\delta_1+\delta_2+(n-2)\delta_3)$. Now $r=Q/n$ and $s=1/Q$ since

$$\begin{aligned} sn=\sum_{i=1}^{n}\delta_i &= \delta_1+\delta_2+(n-2)\Big(\frac{d-2}{n-2}\beta_3+\frac{n-d}{n-2}\frac{1}{Q}\Big)\\ &=\beta_1+\beta_2+(d-2)\beta_3+\frac{n-d}{Q}\\ &=\sum_{i=1}^{d}\Big(\frac{d-2}{d+2}(1-\frac{\eta_i}{Q'})+\frac{3-d}{d+2}\Big)+\frac{n-d}{Q}\\ &=\frac{d-2}{d+2}(d-1+\frac{1}{Q})+\frac{d(3-d)}{d+2}+\frac{n-d}{Q}=\frac{d}{Q}+\frac{n-d}{Q}. \end{aligned}$$

Also $\frac{n-1}{n}\frac{1}{p_2}+\frac{1}{n}\frac{1}{p_1}=\frac{1}{Q}$ and thus $(X_0,X_1)_{1/n,nr}=(L^{p_2,r},L^{p_1,r})_{1/n,nr}=L^Q$, and therefore $b^{nr}_s(w,(X_0,X_1)_{1/n,nr})=b^Q_{1/Q}(w,L^Q)=L^Q(w)$.

Take $f_1 = \cdots = f_n = f$ and since

$$\big\|T[f,\dots,f]\big\|_{L^{r,\infty}} \approx \big\|\mathcal{E}_w f\big\|^n_{L^{Q,\infty}}$$

we get $\mathcal{B}^n \lesssim (\mathcal{C}\mathcal{B}^{\frac{d-2}{d+2}})^{n/d}$ or $\mathcal{B} \lesssim \mathcal{C}^{1/d}\mathcal{B}^{\frac{d-2}{d(d+2)}}$, and this finally implies

$$\mathcal{B} \le C(d)\mathcal{C}^{\frac{d+2}{d^2+d+2}} \tag{57}$$

where $\mathcal{C}$ is as in (51). □

6. Proof of Theorem 1.1

The crucial idea, due to Drury and Marshall [21], is to use an exponential parametrization. Fix $R > 0$, $I_R = [0, R]$ let $b = (b_1, \dots, b_d) \in \mathbb{R}^d$, $b_i \neq 0$,

$$\gamma(t) \equiv \Gamma^b(t) = (b_1^{-1}e^{b_1 t}, \dots, b_d^{-1}e^{b_n t}),$$

and let $\mathfrak{C}_{b,R}$ be the class consisting of Γ^b and restrictions of Γ^b to subintervals. The objective is to prove the bound $\|\mathcal{E}_w f\|_{L^{Q,\infty}} \le C(\int_0^R |f(t)|^Q w(t)dt)^{1/Q}$ with a constant independent of b and R. This inequality is trivial if some of the b_i coincide since then $w = 0$ and thus $\mathcal{E}_w = 0$. When the b_i are pairwise different a priori we at least know that the quantity $\mathcal{B}(\mathfrak{C}_{b,R})$ is finite, but with a bound possibly depending on b and R. To see this one may apply the result of [6] since the torsion τ is positive and Γ^b is smooth on the compact interval I_R.

We need to check Hypotheses 4.1. Most of this work has already been done in [21]. If we form the κ-offspring curves γ_κ (see the definition in §4), then $\gamma_\kappa(t) = \gamma(t)E(\kappa)$ where $E(\kappa)$ denotes the diagonal matrix with entries

$$E_{ii}(\kappa) = \sum_{j=1}^d e^{b_i\kappa_j}$$

and thus is an affine image of a curve in $\mathfrak{C}$. The bounded multiplicity hypothesis is valid by the discussion in [21], p. 549 (this goes back to a paper by Steinig [36]). The crucial inequality (36) has already been verified by Drury and Marshall who proved the relevant 'total positivity' bound in [21], p. 546; *cf.* also [16] for an alternative approach. The constant $\mathfrak{c}_1$ is independent of b and the estimate holds globally.

It remains to verify the second main assumption, inequality (37). We recall from [21], [6] formulas for the torsion $\tau(t) = \tau_b(t)$ of $\Gamma^b(t)$:

$$|\tau_b(t)| = |V(b_1,\dots,b_d)| \exp\Big(t\sum_{j=1}^d b_j\Big).$$

where $V(b_1,\dots,b_d) = \prod_{1\le i<j\le d}(b_j - b_i)$ is the Vandermonde determinant. For the torsion of the offspring curve γ_κ we have

$$|\tau_{\gamma_\kappa}(t)| = |V(b_1,\dots,b_d)| \exp\Big(t\sum_{j=1}^d b_j\Big)\prod_{i=1}^d E_{ii}(\kappa).$$

Since $E_{ii}(\kappa) \geq \exp(b_i\kappa_j)$, for $1 \leq j \leq d$, we have

$$\prod_{i=1}^{d} E_{ii}(\kappa) \geq \prod_{i=1}^{d} \exp(b_i\kappa_j) = \exp\Big(\kappa_j \sum_{i=1}^{d} b_i\Big).$$

Therefore, it follows that

$$|\tau_{\gamma_\kappa}(t)| \geq |\tau(t+\kappa_j)|, \quad 1 \leq j \leq d,$$

and (37) is proved with $\mathfrak{c}_2 = 1$. Now Theorem 4.2 gives a uniform bound for the classes $\mathfrak{C}_{b,R}$ and letting $R \to \infty$, we also get a global result. To prove the asserted result for monomial curves on $[0,\infty)$ we consider the intervals $[0,1]$ and $[1,\infty)$ separately, introduce an exponential parametrization on each interval and use the invariance of affine arclength measure under changes of parametrizations. □

Remark. The sharp $L^p \to L^q$ estimates for monomial curves in our earlier paper [6] have been recently extended by Dendrinos and Müller [16] to cover small local perturbations of monomial curves. In their setting they prove an analogue of the geometric assumption (36); moreover, Lemma 2 and Lemma 4 in [16] show that a variant of the above calculation remains true and (37) continues to hold (although no global uniformity result is proved in this setting). One can thus obtain a local analogue of Theorem 1.1 for perturbations of monomial curves. As a consequence one also gets an $L^{p_d,1} \to L^{p_d}$ endpoint result for every curve of finite type, defined on a compact interval, and the estimate is stable under small perturbations.

7. Curves of simple type and the proof of Theorem 1.2

As observed in [20] some technical issues in the restriction problem with respect to affine measure become easier for classes of curves of *simple type* on some interval I, namely $\gamma \in C^d$ is supposed to be of the form

$$\gamma(t) = \Big(t, \frac{t^2}{2!}, \dots, \frac{t^{d-1}}{(d-1)!}, \phi(t)\Big), \quad t \in I. \tag{58}$$

In this case $\tau(t) = \phi^{(d)}(t)$. Moreover, because of the triangular structure of the matrix defining the torsion, the torsion of the offspring curve is easy to compute. We get

$$\tau_{\gamma_\kappa}(t) = d^{d-1}\Big(\sum_{j=1}^{d} \phi^{(d)}(t+\kappa_j)\Big).$$

Consequently, the verification of condition (37) is often trivial:

Observation 7.1. *Let γ be as in* (58) *and assume that on an interval I the function $\phi^{(d)}$ is of constant sign. Let $t+\kappa_1$, $t+\kappa_d \in I$. Then condition* (37) *holds with $\mathfrak{c}_2 = 1$.*

Indeed, for $1 \le j \le d$,

$$|\tau_{\gamma_\kappa}(t)| = d^{d-1}\Big|\sum_{i=1}^{d} \phi^{(d)}(t+\kappa_i)\Big| \ge d^{d-1}\,|\phi^{(d)}(t+\kappa_j)| = d^{d-1}|\tau(t+\kappa_j)|$$

so that (37) holds.

In contrast, the verification of our first main hypothesis (36) can be hard. The inequality on suitable subintervals has been verified for polynomial curves $(P_1, \dots, P_d)$ by Dendrinos and Wright [17], and their argument is of great complexity. An extension to curves whose coordinate functions are rational has been worked out in [14]. Below we give a rather short argument of (36) for the case of a polynomial curve of simple type. In this case one can prove an estimate which is slightly stronger than (36).

We finally remark that both (36) and (by the observation above) (37) hold for a class of 'convex' curves of simple type considered in [7]. This class also contains nontrivial examples in which the curvature vanishes to infinite order at a point.

Jacobian estimate for polynomial curves of simple type. The strengthened version of (36) for simple polynomial curves is

Proposition 7.2. *Let* $\gamma(t) = \big(t, \frac{t^2}{2!}, \dots, \frac{t^{d-1}}{(d-1)!}, P_b(t)\big)$, $P_b(t) = \sum_{j=0}^{N} b_j t^j$. *Put*

$$J(t,\kappa) = |\det(\gamma'(t+\kappa_1), \cdots, \gamma'(t+\kappa_d))|$$

where $\kappa_1 < \cdots < \kappa_d$. *Then* $\mathbb{R}$ *is the union of* $C(N,d)$ *intervals* I_n *such that whenever* $t+\kappa_1$, $t+\kappa_d \in I_n$, *we have*

$$J(t,\kappa) \ge c(N,d)\,|V(\kappa)|\,\max\{|\phi^{(d)}(t+\kappa_j)| : 1 \le j \le d\}; \tag{59}$$

here $V(\kappa) = \prod_{1\le i<j\le d}(\kappa_j - \kappa_i)$ *and* $c(N,d) > 0$.

We begin by proving an auxiliary lemma where the polynomial assumption is not used.

Lemma 7.3. *Let* $\phi \in C^d(\mathbb{R})$ *and let* $J_d(s_1, \dots, s_d; \phi)$ *denote the determinant of the* $d \times d$ *matrix with the* j*-th column* $(1, s_j, \dots, s_j^{d-2}/(d-2)!, \phi'(s_j))^T$. *Then for* $-\infty < s_1 < \cdots < s_d < \infty$,

$$J_d(s_1, \dots, s_d; \phi) = \int_{s_1}^{s_d} \phi^{(d)}(u)\Psi(u; s_1, \dots, s_d)du \tag{60}$$

where $\Psi \equiv \Psi_d$ *satisfies*

$$0 \le \Psi(u; s_1, \dots, s_d) \le \Big|\frac{V(s_1, \dots, s_d)}{s_d - s_1}\Big| \quad \textit{for all } u \in [s_1, s_d]. \tag{61}$$

Proof. We will follow the arguments in [7]. We first show that

$$J_d(s_1, \dots, s_d; \phi) = \int_{s_1}^{s_2} \cdots \int_{s_{d-1}}^{s_d} J_{d-1}(\sigma_1, \dots, \sigma_{d-1}; \phi')\, d\sigma_{d-1} \cdots d\sigma_1\,. \tag{62}$$

To prove this we first note that since a determinant is zero if two columns are equal, we have

$$\begin{aligned} J_d(s_1,\dots,s_d;\phi) &= -\int_{s_1}^{s_2} \partial_1 J_d(\sigma_1,s_2,\dots,s_d;\phi)d\sigma_1 \\ &= (-1)^{d-1}\int_{s_1}^{s_2}\cdots\int_{s_{d-1}}^{s_d} \partial_{d-1}\cdots\partial_1 J_d(\sigma_1,\dots,\sigma_{d-1},s_d;\phi)\,d\sigma_{d-1}\cdots d\sigma_1\,. \end{aligned}$$

Now $\partial_{d-1}\cdots\partial_1 J_d(\sigma_1,\dots,\sigma_{d-1},s_d;\phi)$ is the determinant of a matrix with the first row $(0,\dots,0,1)$, and one easily checks that

$$\partial_{d-1}\cdots\partial_1 J_d(\sigma_1,\dots,\sigma_{d-1},s_d;\phi) = (-1)^{d-1} J_{d-1}(\sigma_1,\dots,\sigma_{d-1};\phi').$$

Combining the two previous displays yields (62).

We now wish to iterate this formula. It is convenient to denote by $x^m=(x^m_1,\dots,x^m_m)$ a point in $\mathbb{R}^m$ with $x^m_1\le\cdots\le x^m_m$ (i.e with increasing coordinates). We shall set $(s_1,\dots,s_d)=(x^d_1,\dots,x^d_d)=x^d$. For $1\le k\le d-2$ define

$$\mathcal{H}_{d-k}(x^{d-k+1}) = \{x^{d-k}\in\mathbb{R}^{d-k} :\, x^{d-k+1}_j \le x^{d-k}_j \le x^{d-k+1}_{j+1},\ 1\le j\le d-k\}.$$

Note that if the coordinates of x^{d-k+1}_j are increasing in j then for every $x^{d-k}\in\mathcal{H}_{d-k}$ the coordinates of x^{d-k} are increasing. The formula (62) can be rewritten as

$$J_d(x^d;\phi) = \int_{\mathcal{H}_{d-1}(x^d)} J_{d-1}(x^{d-1};\phi')\,dx^{d-1}.$$

Induction gives, for $1\le k\le d-2$

$$J_d(x^d;\phi) = \int_{\mathcal{H}_{d-1}(x^d)}\cdots\int_{\mathcal{H}_{d-k}(x^{d-k+1})} J_{d-k}(x^{d-k+1};\phi^{(k)})\,dx^{d-k}\cdots dx^{d-1}\,.$$

We also have

$$J_2(x^2;\phi^{(d-2)}) = \phi^{(d-1)}(x^2_2)-\phi^{(d-1)}(x^2_1) = \int_{x^2_1}^{x^2_2}\phi^{(d)}(u)\,du\,.$$

Hence, if

$$\begin{aligned} \mathcal{G}_u(x^d) = \{&(x^2,x^3,\cdots,x^{d-1}) : \\ &x^2_1\le u\le x^2_2,\ x^{d-k}\in\mathcal{H}_{d-k}(x^{d-k+1}),\ 1\le k\le d-2\} \end{aligned}$$

and

$$\Psi(u;x^d) := \int_{\mathcal{G}_u(x^d)} dx^2dx^3\cdots dx^{d-1}$$

we get

$$J_d(x^d;\phi) = \int_{x^d_1}^{x^d_d}\phi^{(d)}(u)\Psi(u;x^d)\,du\,.$$

If $x^d=(s_1,\dots,s_d)$ this is (60).

Observe that, for each $u \in [x_1^d, x_d^d]$, the set $\mathcal{G}_u(x^d)$ is contained in the rectangular box

$$B_2(x^d) \times \cdots \times B_{d-1}(x^d)$$

where

$$B_{d-k}(x^d) = \{x^{d-k} \in \mathbb{R}^{d-k} : \; x_j^d \le x_j^{d-k} \le x_{j+k}^d, \; 1 \le j \le d-k\}.$$

Since

$$\mathrm{vol}_{d-k}(B_{d-k}) = \prod_{j=1}^{d-k} (x_{j+k}^d - x_j^d)$$

it follows by rearranging the factors that

$$\Psi(u; x^d) = \mathrm{meas}(\mathcal{G}_u(x^d)) \le \prod_{k=2}^{d-1} \mathrm{vol}_k(B_k)$$

$$= \prod_{2\le i\le d-1} (x_d^d - x_i^d) \prod_{1\le i<j\le d-1} (x_j^d - x_i^d) = (x_d^d - x_1^d)^{-1} V(x_1^d, \cdots, x_d^d).$$

This proves (61). □

We also need the following observation on polynomials.

Lemma 7.4. *Let p be a real-valued polynomial of degree $\le N$ and $|p(t)| > 0$ on (a,b). Then, for every $\varepsilon \in (0, 2^{-N})$,*

$$\big|\{t \in (a,b) : |p(t)| < \varepsilon |p(b)|\}\big| \le 2N\varepsilon^{\frac{1}{2N}}(b-a). \tag{63}$$

Proof. To show (63) we check that for $c \in \mathbb{R}$ and $0 < \delta < 1/2$ we have

$$|\{t \in (a,b) : |t-c| < \delta|b-c|\}| \le 2\delta(b-a). \tag{64}$$

If $b < c$ then $|t-c| > |b-c|$ if $t \in [a,b]$, so $\{t \in (a,b) : |t-c| < \delta|b-c|\} = \emptyset$ since $\delta < 1/2$. If $a \le c \le b$ then

$$|\{t \in (a,b) : |t-c| < \delta|b-c|\}| \le 2\delta|b-c| \le 2\delta(b-a).$$

If $c < a < b$ and $|a-c| \le b-a$ then

$$|\{t \in (a,b) : |t-c| < \delta|b-c|\}| \le \delta|b-c| \le 2\delta(b-a).$$

And if $c < a < b$ and $|a-c| > b-a$, then if $t \in [a,b]$ we have $|t-c| \ge |a-c| \ge (|b-a| + |a-c|)/2 = |b-c|/2$, so $\{t \in (a,b) : |t-c| < \delta|b-c|\} = \emptyset$ since $\delta < 1/2$. This gives (64).

Moving towards (63), we may normalize the leading coefficient and write $p(t) = \prod_{i=1}^{N_1} p_i(t) \prod_{j=1}^{N_2} q_j(t)$ where $p_i(t) = t - c_i$, $q_j(t) = (t-d_j)^2 + e_j^2$, $c_i, d_j, e_j \in \mathbb{R}$, and $N_1 + 2N_2 \le N$. To establish (63) we show that

$$|\{t \in (a,b) : |q(t)| < \delta|q(b)|\}| \le 2\sqrt{\delta}(b-a) \tag{65}$$

if $0 < \delta < 1/2$ and $q(t) = t - c$ or $q(t) = (t-c)^2 + d^2$. The case $q(t) = t - c$ follows from (64). If $q(t) = (t-c)^2 + d^2$ then

$$\{t \in (a,b) : q(t) < \delta q(b)\} \subset \{t \in [a,b] : |t-c| \le \sqrt{\delta}|b-c|\}$$

so

$$|\{t \in (a,b) : q(t) < \delta q(b)\}| \leq 2\sqrt{\delta}(b-a)$$

by (64). This gives (65). Finally $\{t \in (a,b) : |p(t)| < \varepsilon |p(b)|\}$ is contained in the union of the N_1 sets $\{t \in (a,b) : |p_i(t)| < \varepsilon^{1/(N_1+N_2)} |p_i(b)|\}$ and the N_2 sets $\{t \in (a,b) : |q_j(t)| < \varepsilon^{1/(N_1+N_2)} |q_j(b)|\}$ and thus, if $\varepsilon < 2^{-N_1-N_2}$

$$\left|\{t \in (a,b) : |p(t)| < \varepsilon |p(b)|\}\right| \leq \left(2N_1 \varepsilon^{\frac{1}{N_1+N_2}} + 2N_2 \varepsilon^{\frac{1}{2(N_1+N_2)}}\right)(b-a)$$

This proves (63). $\square$

Proof of Proposition 7.2. With $\phi = P_b$ fixed choose the I_n such that $\phi^{(d)}$ and $\phi^{(d+1)}$ are nonzero on the interior of each I_n. We assume without loss of generality that $\phi^{(d)}$, $\phi^{(d+1)} > 0$ on the interior of I_n. If we put $s_j = t + \kappa_j$, then it follows by Lemma 7.3 that

$$J(t,\kappa) = |J_d(s_1, \cdots, s_d; \phi)| = \left| \int_{s_1}^{s_d} \phi^{(d)}(u) \Psi(u) du \right| \tag{66}$$

for some nonnegative function $\Psi(u) = \Psi(u; s_1, \ldots, s_d)$ which satisfies

$$\Psi(u) \leq V(s_1, \ldots, s_d)/(s_d - s_1). \tag{67}$$

Note that $V(s_1, \ldots, s_d) = V(\kappa_1, \ldots, \kappa_d)$. By applying (60) with $\phi(t) = t^d/(d!)$, we get

$$\int_{s_1}^{s_d} \Psi(u)\, du = c_d V(\kappa)$$

where $c_d = (2! \cdots (d-1)!)^{-1}$.

To see (59) we use this fact and (67). Thus we have

$$\int_{[s_1,s_d]\setminus E} \Psi(u)\, du = \int_{s_1}^{s_d} \Psi(u)\, du - \int_E \Psi(u)\, du \tag{68}$$

$$\geq c_d\, V(\kappa) - |E| \frac{V(\kappa)}{s_d - s_1} \tag{69}$$

if $E \subset [s_1, s_d]$. Choose $\varepsilon = \varepsilon(d, N)$ so small that $2N\varepsilon^{\frac{1}{2N}} \leq c_d/2$. Now assume that $s_1 = t + \kappa_1 \in I_n$, $s_d = t + \kappa_d \in I_n$. With

$$E = \{u \in [s_1, s_d] :\ \phi^{(d)}(u) < \varepsilon \phi^{(d)}(t + \kappa_d)\}$$

we have $|E| < (s_d - s_1)c_d/2$ by Lemma 7.4 and our choice of ε.

Hence, by (66) and (68), we have

$$\begin{aligned} J(t,\kappa) &\geq \int_{[s_1,s_d]\setminus E} \phi^{(d)}(u)\, \Psi(u)\, du \\ &\geq \varepsilon\, \phi^{(d)}(t+\kappa_d)\, V(\kappa)\, c_d/2 \\ &= \varepsilon \frac{c_d}{2}\, V(\kappa)\, \max\{|\phi^{(d)}(t+\kappa_j)| :\ 1 \leq j \leq d\}, \end{aligned}$$

giving (59) as desired. Here we put $c(N, d) = \varepsilon\, c_d/2$. $\square$

Proof of Theorem 1.2. Fix a polynomial ϕ, of degree N. If $d < N$ then the affine arclength measure is identically 0 and the assertion trivially holds with $C(N) = 0$. Now assume $N \geq d$. Let I be an interval on which the inequality (59) holds, and $\phi^{(d)}$ and $\phi^{(d+1)}$ do not change sign. Pick a subinterval I_0 on which $\phi^{(d)}$ does not vanish. Denote by $\mathfrak{C}$ the class of simple curves given by ϕ, on I_0 or on subintervals of I_0. We notice that the offspring curves are affine images of the original curves (*cf.* the proof of Lemma 3.1 in [7]). The bounded multiplicity hypothesis (when the curve is restricted to suitable subintervals) is discussed in [20], [17]. Theorem 4.2 gives the desired conclusion on the interval I_0 with no reference to a nondegeneracy assumption. A limiting argument gives the conclusion on the full interval I. Since ϕ is a polynomial and by Proposition 7.2 we have to apply this consideration to only a finite number of intervals. □

Remark. Let $\Gamma(t) = (R_1(t), \dots, R_d(t))$ where $R_i(t) = P_{1,i}(t)/P_{2,i}(t)$, $P_{1,i}, P_{2,i}$ are polynomials. It has been proved by Dendrinos, Folch-Gabayet and Wright [14] that $\mathbb{R}$ can be decomposed into a finite number of intervals (depending on d and the maximal degree of the polynomials involved) so that the crucial hypothesis (36) is satisfied on the interior of each interval. In the special case of rational curves of simple type (with $R_i(t) = t^i/i!$, $i = 1, \dots, d-1$ and $R_d(t) = P(t)/Q(t)$, P, Q polynomials) one can show that after a further decomposition Observation 7.1 applies. Thus Theorem 1.2 extends to rational curves of simple type.

8. A note on the range of the sharp $L^p \to L^q$ adjoint restriction theorem for general polynomial curves

Suppose $t \mapsto \gamma(t) = (P_1(t), \dots, P_d(t))$ is a polynomial curve in $\mathbb{R}^d$, with the P_i of degree at most n, and suppose that $d\lambda = w dt$ is the affine arclength measure on γ. Dendrinos and Wright [17] established the critical Fourier extension estimate

$$\|\widehat{f d\lambda}\|_q \leq C(n,p) \, \|f\|_{L^p(\lambda)}, \quad \frac{1}{p} + \frac{d(d+1)}{2q} = 1 \tag{70}$$

in the range $1 \leq p < d+2$ (the range obtained by Christ [12] in the non-degenerate case). Much earlier Drury [18] had proven a restriction estimate for certain curves (t, t^2, t^k) in dimension 3 that was valid for $1 \leq p < 6$ and therefore valid for some p outside of the Christ range. It turns out that by replacing two of the estimates in Drury's argument by estimates of Dendrinos and Wright and of Dendrinos, Laghi and Wright [15] one can extend Drury's result to general polynomial curves in $\mathbb{R}^3$. Moreover using an estimate of Stovall [37], one can show

Proposition 8.1. *For general polynomial curves in $\mathbb{R}^d$, $d \geq 3$, the Fourier extension estimate* (70) *holds for*

$$1 \leq p < d + 3 + \frac{2(d-3)}{d^2 - 3d + 4}. \tag{71}$$

Proof. What follows, then, is just Drury's argument run with up-to-date technology. The required result from §3 of [17] (cf. [12] for the nondegenerate case) is the d-fold convolution estimate

$$\|(fd\lambda) * \cdots * (fd\lambda)\|_r \leq C(n,t)\left(\|f\|_{L^t(\lambda)}\right)^d \quad \text{for } \frac{1}{t} + \frac{d-1}{2} = \frac{d+1}{2r}, \quad 1 \leq t < d+2. \tag{72}$$

The necessary estimate from [37] is

$$\|\lambda * g\|_{\frac{d+1}{d-1}} \leq C(n)\,\|g\|_{\frac{d^2+d}{d^2-d+2}},$$

from which it follows by the Hausdorff-Young inequality that

$$\|\widehat{fd\lambda * g}\|_{\frac{d+1}{2}} \lesssim_n \|f\|_{L^\infty(\lambda)}\,\|g\|_{\frac{d^2+d}{d^2-d+2}}. \tag{73}$$

Drury's argument is an iterative one. Begin by assuming that the $L^{p_0} \to L^{q_0}$ estimate (70) holds for some p_0 and q_0 satisfying $\frac{1}{p_0} + \frac{d(d+1)}{2q_0} = 1$. We then also have

$$\|\widehat{fd\lambda * g}\|_{s_0} \leq C(n,p_0)\|f\|_{L^{p_0}(\lambda)}\,\|g\|_2, \quad \frac{1}{s_0} = \frac{1}{q_0} + \frac{1}{2}. \tag{74}$$

To see this write $\|\widehat{fd\lambda * g}\|_{s_0} = \|\widehat{fd\lambda}\,\widehat{g}\|_{s_0}$ and estimate this by $\|\widehat{fd\lambda}\|_{q_0}\,\|\widehat{g}\|_2$, using Hölder's inequality. Now use the assumed $L^{p_0} \to L^{q_0}$ inequality and Plancherel's formula to get (74). Interpolation of (73) and (74) gives

$$\|\widehat{fd\lambda * g}\|_s \lesssim_{n,\vartheta,p_0} \|f\|_{L^a(\lambda)}\,\|g\|_r \tag{75}$$

where

$$\left(\frac{1}{s}, \frac{1}{a}, \frac{1}{r}\right) = (1-\vartheta)\left(\frac{1}{s_0}, \frac{1}{p_0}, \frac{1}{2}\right) + \vartheta\left(\frac{2}{d+1}, \frac{1}{\infty}, \frac{d^2-d+2}{d^2+d}\right) \tag{76}$$

for $0 < \vartheta < 1$. We wish to apply this inequality with g equal to the d-fold convolution in (72). This restricts the r-range to $r < \frac{d+2}{d}$ (corresponding to the range $t < d+2$). A calculation shows that this restricts the range of ϑ in (76) to

$$\frac{(d-2)(d+1)d}{(d+2)(d^2-3d+4)} =: \vartheta_{\min} < \vartheta < 1 \tag{77}$$

With $g = fd\lambda * \cdots * fd\lambda$ we obtain from (75)

$$\|\left(\widehat{fd\lambda}\right)^{d+1}\|_s \lesssim_{n,\vartheta,p_0} \|f\|_{L^a(\lambda)}\left(\|f\|_{L^t(\lambda)}\right)^d \tag{78}$$

so long as $t < d+2$, $\frac{1}{t} = \frac{d+1}{2r} - \frac{d-1}{2}$ and r, a, s are as in (76) with $\vartheta > \vartheta_{\min}$. With $f = \chi_E$ this becomes

$$\|\widehat{\chi_E d\lambda}\|_{(d+1)s} \lesssim_{n,\vartheta,p_0} \lambda(E)^{\frac{1}{d+1}(\frac{1}{a}+\frac{d}{t})},$$

which gives

$$\|\widehat{fd\lambda}\|_q \lesssim_{n,\vartheta,p_0} \|f\|_{L^{p,1}(\lambda)}, \quad \text{for } \frac{1}{q} = \frac{1}{(d+1)s} \text{ and } \frac{1}{p} = \frac{1}{d+1}\left(\frac{1}{a} + \frac{d}{t}\right),$$

where s, a, t are in

$$\Big(\frac{1}{s}, \frac{1}{a}, \frac{1}{t}\Big) = (1-\vartheta)\Big(\frac{1}{s_0}, \frac{1}{p_0}, \frac{3-d}{4}\Big) + \vartheta\Big(\frac{2}{d+1}, \frac{1}{\infty}, \frac{1}{d}\Big), \quad \vartheta_{\min} < \vartheta < 1\,.$$

A little algebra shows that p and q satisfy $1/p + d(d+1)/(2q) = 1$ (for any $\vartheta \in (\vartheta_{\min}, 1)$). Thus we get the restricted strong type version of (70) for the exponent pair (p, q). If p_1 is the exponent p corresponding to the limiting case $\vartheta_{\min}$ we obtain by real interpolation the sharp $L^p(\lambda) \to L^q$ inequality in the open range $1 \le p < p_1$. Using $\vartheta_{\min}$ in (77) we calculate that

$$\frac{1}{p_1} = \frac{8}{(d+1)(d+2)(d^2-3d+4)} \cdot \frac{1}{p_0} + \frac{d}{(d+1)(d+2)}.$$

If we define recursively $\frac{1}{p_{j+1}} = \frac{8}{(d+1)(d+2)(d^2-3d+4)p_j} + \frac{d}{(d+1)(d+2)}$ then the sequence $\{p_0, p_1, p_2, \dots\}$ converges to

$$\frac{d^3-3d+6}{d^2-3d+4} = d+3+\frac{2(d-3)}{d^2-3d+4}$$

and we can conclude that (70) holds for p in the range (71). □

Appendix A. Some results from interpolation theory

We gather various interpolation results used in the paper, especially in §3. They can be found more or less explicit in the literature and no originality is claimed. In some cases it is hard to cite exactly the precise statement that we need and the reader might find the inclusion of this appendix helpful.

On complex interpolation of multilinear operators. We use complex interpolation for multilinear operators defined for functions in a quasi-normed space with values in a Lorentz-space. We limit ourselves to the statements needed in this paper where the target space of our operator is not varied.

In the following lemma we let the measure space $\mathcal{M}$ be a finite set, with counting measure, and let V be a Lorentz space. For a positive weight on $\mathcal{M}$ the norm in $\ell^p(w)$ is given by $\|f\|_{\ell^p(w)} = (\sum_{x\in\mathcal{M}} |f(x)|^p w(x))^{1/p}$.

Lemma A.1. *Let T be a multilinear operator defined on n-tuples of functions on $\mathcal{M}$ and suppose that for some some weights $w_{i,0}$, $w_{i,1}$ on $\mathcal{M}$ and $p_{i,0}, p_{i,1} \in (0,\infty]$*

$$\|T[f_1,\dots,f_n]\|_V \le M_0 \prod_{i=1}^n \|f_i\|_{\ell^{p_{i,0}}(w_{i,0})}\,,$$
(79)
$$\|T[f_1,\dots,f_n]\|_V \le M_1 \prod_{i=1}^n \|f_i\|_{\ell^{p_{i,1}}(w_{i,1})}\,.$$

Let $0 < \vartheta < 1$ and define p_i and weight functions w_i by

$$\frac{1}{p_i} = \frac{(1-\vartheta)}{p_{i,0}} + \frac{\vartheta}{p_{i,1}}, \tag{80}$$

$$w_i = [w_{i,0}^{(1-\vartheta)/p_{i,0}} w_{i,1}^{\vartheta/p_{i,1}}]^{p_i}. \tag{81}$$

Then there is C (independent of the $f_1, \dots, f_n$ and $\mathcal{M}$) so that

$$\|T[f_1, \dots, f_n]\|_V \le C M_0^{1-\vartheta} M_1^{\vartheta} \prod_{i=1}^{n} \|f_i\|_{\ell^{p_i}(w_i)}. \tag{82}$$

Proof. This is an adaptation of the standard argument in complex interpolation ([35]) for analytic families of operators, in the setting for Lorentz spaces in [32]. We assume that V is a Lorentz space associated to the measure space Ω with measure μ, say $L^{r,q}(\mu)$. Let $\rho \in (0, 1]$ and assume in addition $\rho < \min\{q, r\}$. Define the maximal function $h_\rho^{**} \equiv h^{**}$ on $(0, \infty)$ by

(83)
$$h^{**}(t) = \begin{cases} \sup_{E:\mu(E)>t} (\mu(E)^{-1} \int_E |f(y)|^\rho d\mu(y))^{1/\rho}, & t \in (0, \mu(\Omega)) \\ (t^{-1} \int_\Omega |f(y)|^\rho d\mu(y))^{1/\rho}, & t \in [\mu(\Omega), \infty) \end{cases}.$$

The function $[h^{**}]^\rho$ is dominated by the Hardy-Littlewood maximal function of $[h^*]^\rho$, where h^* is the nonincreasing rearrangement of h.

For a function g on $(0, \infty)$ set $\|g\|_{\lambda^{r,q}} = (\frac{q}{r} \int_0^\infty t^{q/r} |g(t)|^q \frac{dt}{t})^{1/q}$ if $q < \infty$ and $\|g\|_{\lambda^{r,\infty}} = \sup_{t>0} t^{1/r} |g(t)|$. Then Hunt [25] showed that the expression $\|h^{**}\|_{\lambda^{r,q}}$ is a quasi-norm on $L^{q,r}$ which makes $L^{q,r}$ a ρ-convex space.

Let S be the strip $\{z = \vartheta + i\tau : 0 < \vartheta < 1, \tau \in \mathbb{R}\}$ and $\overline{S}$ its closure. Let $f_i \in \ell^{p_i}(w_i)$ so that $\|f_i\|_{\ell^{p_i}(w_i)} \le 1$ and define for $x \in \mathcal{M}$

$$f_{i,z}(x) = e^{i \arg(f(x))} \frac{[|f(x)|^{p_i} w_i(x)]^{\frac{1-z}{p_{i,0}} + \frac{z}{p_{i,1}}}}{w_{i,0}(x)^{\frac{1-z}{p_{i,0}}} w_{i,1}(x)^{\frac{z}{p_{i,1}}}}.$$

Then $f_{i,\vartheta} = f$. Moreover, $\|f_z\|_{\ell^{p_{i,0}}(w_{i,0})} = \|f\|_{\ell^{p_i}(w_i)}^{p_{i,0}/p_i}$ if $\mathrm{Re}\,(z) = 0$, and $\|f_z\|_{\ell^{p_{i,1}}(w_{i,1})} = \|f\|_{\ell^{p_i}(w_i)}^{p_{i,1}/p_i}$ if $\mathrm{Re}\,(z) = 1$. We define, for $y \in \Omega$,

$$H_z(y) = T[f_{1,z}, \dots, f_{1,z}](y).$$

Then $H_\vartheta = T[f_1, \dots, f_1]$ and we must show that $\|H_\vartheta\|_V \lesssim M_0^{1-\vartheta} M_1^\vartheta$. For almost every $y \in \Omega$ the function $z \mapsto H_z(y)$ is bounded and analytic in S, continuous on $\overline{S}$.

We use the standard properties of the Poisson-kernel associated with S, see Ch. V.4 in [35]. Let $P_0(\vartheta, t) = \frac{1}{2} \frac{\sin(\pi\vartheta)}{\cosh(\pi t) - \cos(\pi\vartheta)}$, $P_1(\vartheta, t) = \frac{1}{2} \frac{\sin(\pi\vartheta)}{\cosh(\pi t) + \cos(\pi\vartheta)}$. For $0 \le \vartheta \le 1$ we then have $\int_{-\infty}^\infty P_0(\vartheta, t) dt = (1 - \vartheta)$, $\int_{-\infty}^\infty P_1(\vartheta, t) dt = \vartheta$. Thus, proceeding exactly as in [32] we have

$$\log |H_{\vartheta + i\tau}(y)| \le \int_{-\infty}^\infty P_0(\vartheta, \tau) \log |H_{i\tau}(y)| d\tau + \int_{-\infty}^\infty P_1(\vartheta, \tau) \log |H_{1+i\tau}(y)| d\tau$$

and then

$$|H_{\vartheta+\mathrm{i}\tau}(y)| \le \Big(\exp\Big(\frac{1}{1-\vartheta}\int_{-\infty}^{\infty} P_0(\vartheta,\tau)\log[|H_{\mathrm{i}\tau}(y)|^\rho]d\tau\Big)\Big)^{\frac{1-\vartheta}{\rho}} \times \Big(\exp\Big(\frac{1}{\vartheta}\int_{-\infty}^{\infty} P_1(\vartheta,\tau)\log[|H_{1+\mathrm{i}\tau}(y)|^\rho]d\tau\Big)\Big)^{\frac{\vartheta}{\rho}} .$$

By Jensen's inequality,

$$|H_{\vartheta+\mathrm{i}\tau}(y)| \le A_0(y)^{1-\vartheta}A_1(y)^{\vartheta}$$

where

$$A_0(y) = \Big(\frac{1}{1-\vartheta}\int_{-\infty}^{\infty} P_0(\vartheta,\tau)|H_{\mathrm{i}\tau}(y)|^\rho d\tau\Big)^{1/\rho}$$
$$A_1(y) = \Big(\frac{1}{\vartheta}\int_{-\infty}^{\infty} P_1(\vartheta,\tau)|H_{1+\mathrm{i}\tau}(y)|^\rho d\tau\Big)^{1/\rho}.$$

By Hölder's inequality we have $(A_0^{1-\vartheta}A_1^{\vartheta})^{**}(t) \le (A_0^{**}(t))^{1-\vartheta}(A_1^{**}(t))^{\vartheta}$. By Fubini's theorem we get $A_0^{**}(t) \le B_0(t)$, $A_1^{**}(t) \le B_1(t)$ where

$$B_0(t) = \Big(\frac{1}{1-\vartheta}\int_{-\infty}^{\infty} P_0(\vartheta,\tau)|H_{\mathrm{i}\tau}^{**}(t)|^\rho d\tau\Big)^{1/\rho},$$
$$B_1(t) = \Big(\frac{1}{\vartheta}\int_{-\infty}^{\infty} P_1(\vartheta,\tau)|H_{1+\mathrm{i}\tau}^{**}(t)|^\rho d\tau\Big)^{1/\rho}.$$

Hence

$$|H_{\vartheta+\mathrm{i}\tau}^{**}(t)| \le (A_0^{**}(t))^{1-\vartheta}(A_1^{**}(t))^{\vartheta} \le B_0^{1-\vartheta}(t)B_1^{\vartheta}(t)$$

and another application of Hölder's inequality yields

$$\|H_{\vartheta+\mathrm{i}\tau}^{**}\|_{\lambda^{q,r}} \le \|B_0\|_{\lambda^{q,r}}^{1-\vartheta}\|B_1\|_{\lambda^{q,r}}^{\vartheta} .$$

Since $q > \rho$ we can apply the integral Minkowski inequality (as a version of the triangle inequality in $L^{q/\rho}(0,\infty)$)

$$\|B_0\|_{\lambda^{q,r}} \le \Big(\frac{1}{1-\vartheta}\int_{-\infty}^{\infty} P_0(\vartheta,\tau)\|H_{\mathrm{i}\tau}^{**}\|_{\lambda^{r,q}}^\rho d\tau\Big)^{1/\rho},$$
$$\|B_1\|_{\lambda^{q,r}} \le \Big(\frac{1}{\vartheta}\int_{-\infty}^{\infty} P_1(\vartheta,\tau)\|H_{1+\mathrm{i}\tau}^{**}\|_{\lambda^{r,q}}^\rho d\tau\Big)^{1/\rho}.$$

By assumption

$$\|H_{\mathrm{i}\tau}^{**}\|_{\lambda^{r,q}} \le CM_0\prod_{i=1}^n \|f_{i,\mathrm{i}\tau}\|_{\ell^{p_{i,0}}(w_{i,0})} \le CM_0 ,$$
$$\|H_{1+\mathrm{i}\tau}^{**}\|_{\lambda^{r,q}} \le CM_1\prod_{i=1}^n \|f_{i,1+\mathrm{i}\tau}\|_{\ell^{p_{i,1}}(w_{i,1})} \le CM_1 .$$

We get $\|H_{\vartheta+\mathrm{i}\tau}^{**}\|_{\lambda^{q,r}} \le CM_0^{1-\vartheta}M_1^{\vartheta}$, using the above formulas for the integrals of P_0 and P_1. □

We now use Lemma A.1 and a straightforward *transference method* to prove an interplation theorem for sequences with values in certain real interpolation spaces $\overline{X}_{\theta,q}$. When applying the complex interpolation method the case $q=\infty$ may pose some difficulties which can be avoided if ℓ^∞ is replaced by the closed subspace c_0. In particular it is convenient to replace the real interpolation space $\overline{X}_{\vartheta,\infty}$ by $\overline{X}^0_{\vartheta,\infty}$, the closure of $X_0\cap X_1$ in $\overline{X}_{\vartheta,\infty}$. To deal with this distinction we introduce some notation for the following lemma. We will work with a couple $\overline{X}=(X_0,X_1)$ of compatible complete quasi-normed spaces. If $q<\infty$ we denote by $\mathcal{Z}(q,s,\theta)=\ell^q_s(\overline{X}_{\theta,q})$ the space of $\overline{X}_{\theta,q}$-valued sequences $F=\{F_k\}_{k\in\mathbb{Z}}$ with norm

$$\|F\|_{\mathcal{Z}(q,s,\theta)}=\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\|F_k\|^q_{\overline{X}_{\theta,q}}\Big)^{1/q}.$$

For $q=\infty$ we define $\mathcal{Z}(\infty,s,\theta)=c_0(\overline{X}^0_{\theta,\infty})$, a closed subspace of $\ell^\infty_s(\overline{X}_{\theta,\infty})$, with norm $\sup_{k\in\mathbb{Z}}2^{ks}\|F_k\|_{\overline{X}_{\theta,\infty}}$. We shall say that $F=\{F_k\}\in\mathcal{Z}(q,s,\theta)$ is compactly supported if $F_k=0$ except for finitely many k.

Lemma A.2. *For $i=1,\dots,n$, let $0<\theta_{i,0},\theta_{i,1}<1$, $0<q_{i,0},q_{i,1}\le\infty$, $s_{i,1},s_{i,0}\in\mathbb{R}$. Let T be an n-linear operator defined a priori on n-tuples of compactly supported $(X_0\cap X_1)$-valued sequences, with values in a Lorentz space V, and suppose that for such sequences the inequalities*

$$\|T[F_1,\dots,F_n]\|_V\le\begin{cases}M_0\prod_{i=1}^n\|F_i\|_{\mathcal{Z}(q_{i,0},s_{i,0},\theta_{i,0})}\\ M_1\prod_{i=1}^n\|F_i\|_{\mathcal{Z}(q_{i,1},s_{i,1},\theta_{i,1})}\end{cases}\tag{84}$$

hold. Define q_i, s_i and θ_i by

$$\Big(\frac{1}{q_i},s_i,\theta_i\Big)=(1-\vartheta)\Big(\frac{1}{q_{i,0}},s_{i,0},\theta_{i,0}\Big)+\vartheta\Big(\frac{1}{q_{i,1}},s_{i,1},\theta_{i,1}\Big).$$

Then T uniquely extends to an operator bounded on $\prod_{i=1}^n\mathcal{Z}(q_i,s_i,\theta_i)$ so that

$$\|T[F_1,\dots,F_n]\|_V\lesssim M_0^{1-\vartheta}M_1^{\vartheta}\prod_{i=1}^n\|F_i\|_{\mathcal{Z}(q_i,s_i,\theta_i)}.$$

Proof. The uniqueness of the extension is clear because of the density of compactly supported $X_0\cap X_1$-valued functions in $\mathcal{Z}(q,s,\theta)$ (for $q=\infty$ this requires the modification in the definition using $c_0(\overline{X}^0_{\theta,\infty})$). In what follows we write matters out for the case that the $q_i<\infty$ and leave the obvious notational modifications in the case $q_i=\infty$ to the reader. It will be convenient to use the characterization of $\overline{X}_{\theta,q}$ by means of the J-functional.

It suffices to prove that for $\|F_i\|_{\mathcal{Z}(q_i,s_i,\theta_i)}\le1$, $i=1,\dots,d$,

$$\|T(F_1,\dots,F_n)\|_V\lesssim M_0^{1-\vartheta}M_1^{\vartheta}.\tag{85}$$

We write $F_i=\{F_{i,k}\}$ with $F_{i,k}\in\overline{X}_{\theta_i,q_i}$ and

$$\Big(\sum_k\big[2^{ks_i}\|F_{i,k}\|_{\overline{X}_{\theta_i,q_i,J}}\big]^{q_i}\Big)^{1/q_i}\le1.$$

We can decompose $F_{i,k} = \sum_l u_{i,k,l}$ with $u_{i,k,l} \in X_0 \cap X_1$ and convergence in $X_0 + X_1$ so that

$$\Big(\sum_l \big[2^{-l\theta_i} J(2^l, u_{i,k,l}; \overline{X})\big]^{q_i}\Big)^{1/q_i} \le (1 + 2^{-|k|-2}) \|F_{i,k}\|_{\overline{X}_{\theta_i,q_i;J}}$$

and thus $\big(\sum_{k,l} [2^{ks_i} 2^{-l\theta_i} J(2^l, u_{i,k,l}; \overline{X})]^{q_i}\big)^{1/q_i} \le 2$. We set $N_0 = 0$ and define numbers $N_0 < N_1 < N_2 < ...$ so that

$$\Big(\sum_{\substack{k,l \\ \max\{|k|,|l|\} \ge N_\nu}} \big[2^{ks_i} 2^{-l\theta_i} J(2^l, u_{i,k,l}; \overline{X})\big]^{q_i}\Big)^{1/q_i} \le 2^{-\nu}. \tag{86}$$

for $i = 1, \dots, n$. Let $\chi_\nu(k,l) = 1$ if $N_\nu \le \max\{|k|, |l|\} < N_{\nu+1}$ and $\chi_\nu(k,l) = 0$ otherwise, and let $F^\nu_{i,k} = \sum_l \chi_\nu(k,l) u_{i,k,l}$. Then $\sum_{\nu=1}^\infty F_i^\nu = F_i$ with convergence in $\mathcal{Z}(q_i, s_i, \theta_i)$ and, for each k, $\sum_{\nu=1}^\infty F^\nu_{i,k} = F_{i,k}$ with convergence in $\overline{X}_{\theta_i,q_i}$ and *a fortiori* with convergence in $X_0 + X_1$.

In order to prove (85) we fix the chosen vectors $u_{i,k,l}$ and define operators acting on n-tuples of functions $\mathfrak{a} = \{\mathfrak{a}_{k,l}\}$ defined on a subset of $\mathbb{Z} \times \mathbb{Z}$. Let $\mathcal{M}_{\vec\nu} = \{(k,l) : \max\{k,l\} \le \max\{N_{\nu_1+1}, \dots, N_{\nu_n+1}\}\}$. For any n-tuple $\vec\nu = (\nu_1, \dots, \nu_n)$ of nonnegative integers we let $\mathcal{F}_i^{\nu_i}(\mathfrak{a}) = \{\mathcal{F}^{\nu_i}_{i,k}(\mathfrak{a})\}_{k\in\mathbb{Z}}$, where for $i = 1, \dots, n$

$$\mathcal{F}^{\nu_i}_{i,k}(\mathfrak{a}) = \sum_l \chi_{\nu_i}(k,l) \mathfrak{a}_{k,l}\, u_{i,k,l}.$$

Now define for an n-tuple of such sequences a multilinear operator $S_{\vec\nu}$ by

$$S_{\vec\nu}(\mathfrak{a}^1, \dots, \mathfrak{a}^n) = T[\mathcal{F}_1^{\nu_1}(\mathfrak{a}^1), \dots, \mathcal{F}_n^{\nu_n}(\mathfrak{a}^n)].$$

Let

$$w_{q,s,\theta}(k,l) = \big[2^{ks} 2^{-l\theta} (\varepsilon + J(2^l, u_{i,k,l}; \overline{X}))\big]^q.$$

It is our objective to show

$$\|S_{\vec\nu}[\mathfrak{a}^1, \dots, \mathfrak{a}^n]\|_V \le C M_0^{1-\vartheta} M_1^\vartheta \prod_{i=1}^n \|\mathfrak{a}^i\|_{\ell^{q_i}(w_{q_i,s_i,\theta_i})} \tag{87}$$

where the constant C is independent of the choice of the specific $u_{i,k,l}$ and independent of $\vec\nu$. The inclusion of ε in the definition of $w_{q,s,\theta}$ guarantees the positivity of the weight. Once the bound (87) is verified we will then apply it to the sequences $\mathfrak{a}^i_{k,l} = \chi_{\nu_i}(k,l)$. For this choice of the $\mathfrak{a}^i$ an estimate for the expression $\|S_{\vec\nu}[\mathfrak{a}^1, \dots, \mathfrak{a}^n]\|_V$ becomes an estimate for $\|T[F_1^{\nu_1}, \dots, F_n^{\nu_n}]\|_V$, after letting $\varepsilon \to 0$.

Now for any admissible choice of q, s, θ

$$\|\mathcal{F}_i^{\nu_i}(\mathfrak{a}^i)\|_{\mathcal{Z}(q,s,\theta)} = \Big(\sum_k \Big[2^{ks} \Big\| \sum_l \chi_{\nu_i}(k,l) \mathfrak{a}^i_{k,l}\, u_{i,k,l} \Big\|_{\overline{X}_{\vartheta,q;J}}\Big]^q\Big)^{1/q}$$

and by definition of the $\overline{X}_{\vartheta,q;J}$ norm we have

$$\Big\| \sum_l \chi_{\nu_i}(k,l)\mathfrak{a}^i_{k,l}\, u_{i,k,l}\Big\|_{\overline{X}_{\vartheta,q;J}} \le \Big(\sum_l \big[2^{-l\vartheta} J(2^l, \mathfrak{a}^i_{k,l}\, u_{i,k,l}; \overline{X})\big]^q \Big)^{1/q}.$$

Therefore, by the homogeneity of the J-functional

$$\|\mathcal{F}_i^{\nu_i}(\mathfrak{a}^i)\|_{\mathcal{Z}(q,s,\theta)} \lesssim \Big(\sum_{k,l} 2^{(ks-l\theta)q}[\chi_{\nu_i}(k,l) J(2^l, u_{i,k,l}; \overline{X})|\mathfrak{a}^i_{k,l}|]^q \Big)^{1/q}.$$

Notice that we can consider the sequences $\mathfrak{a}^i$ as functions defined on $\mathcal{M}_{\vec{\nu}}$. By assumption

$$\begin{aligned} &\|T[\mathcal{F}_1(\mathfrak{a}^1), \dots, \mathcal{F}_n(\mathfrak{a}^n)]\|_V \\ &\quad \lesssim \min\big\{ M_0 \prod_{i=1}^n \|\mathcal{F}_i(\mathfrak{a}^i)\|_{\mathcal{Z}(q_{i,0},s_{i,0},\theta_{i,0})},\ M_1 \prod_{i=1}^n \|\mathcal{F}_i(\mathfrak{a}^i)\|_{\mathcal{Z}(q_{i,1},s_{i,1},\theta_{i,1})} \big\}, \end{aligned}$$

and by the above this implies

$$\begin{aligned} &\|S_{\vec{\nu}}[\mathfrak{a}^1, \dots, \mathfrak{a}^n]\|_V \\ &\quad \lesssim \min\big\{ M_0 \prod_{i=1}^n \|\mathfrak{a}^i\|_{\ell^{q_{i,0}}(w_{q_{i,0},s_{i,0},\theta_{i,0}})},\ M_1 \prod_{i=1}^n \|\mathfrak{a}^i\|_{\ell^{q_{i,1}}(w_{q_{i,1},s_{i,1},\theta_{i,1}})} \big\}. \end{aligned}$$

Since $w_{q_i,s_i,\theta_i} = [w^{(1-\vartheta)/q_{i,0}}_{q_{i,0},s_{i,0},\theta_{i,0}} w^{(1-\vartheta)/q_{i,1}}_{q_{i,1},s_{i,1},\theta_{i,1}}]^{q_i}$, Lemma A.1 now gives (87), with a constant independent of $\vec{\nu}$.

Finally if we apply (87) with the sequences $\mathfrak{a}^i_{k,l} = \chi_{\nu_i}(k,l)$ and let $\varepsilon \to 0$ we obtain

$$\begin{aligned} &\|T[F_1^{\nu_1}, \dots, F_n^{\nu_n}]\|_V \\ &\quad \lesssim M_0^{1-\vartheta} M_1^{\vartheta} \prod_{i=1}^n \Big(\sum_{k,l} \big[\chi_{\nu_i}(k,l) 2^{ks_i} 2^{-l\theta_i} J(2^l, u_{i,k,l}; \overline{X})\big]^{q_i} \Big)^{1/q_i} \end{aligned}$$

and, by (86) this expression is $\lesssim M_0^{1-\vartheta} M_1^{\vartheta} 2^{-(\nu_1+\dots+\nu_n)}$. Since $\sum_{\nu_i} F_i^{\nu_i} = F_i$ with convergence in $\ell^{q_i}(\overline{X}_{\theta_i,q_i})$ we see that $\sum_{\vec{\nu}} T[F_1^{\nu_1}, \dots, F_n^{\nu_n}]$ converges in V to $T[F_1, \dots, F_n]$ so that $\|T(F_1, \dots, F_n)\|_V \lesssim M_0^{1-\vartheta} M_1^{\vartheta}$. □

Means. Often one generates new estimates by taking means of given estimates. The new bounds may then be interpreted as estimates on intermediate spaces:

Lemma A.3. *Let $0 < r \le 1$ and let V be an r-convex space. For $i = 1, \dots, n$ let $\overline{X}^i = (X_0^i, X_1^i)$ be couples of compatible quasi-normed spaces and let T be an n-linear operator defined on $\prod_{i=1}^n (X_0^i \cap X_1^i)$ with values in V. Suppose that*

$$\|T(f_1, \dots, f_n)\|_V \le \prod_{i=1}^n \|f_i\|_{X_0^i}^{1-\theta_i} \|f_i\|_{X_1^i}^{\theta_i}$$

for some $0 < \theta_i < 1$. *Then there is* $C > 0$ *so that for all* $(f_1, \dots, f_n) \in \prod_{i=1}^n X_0^i \cap X_1^i$

$$\|T(f_1, \dots, f_n)\|_V \le C \prod_{i=1}^n \|f_i\|_{\overline{X}^i_{\theta_i, r}} \tag{88}$$

and T *extends to a bounded operator on* $\prod_{i=1}^n \overline{X}^i_{\theta_i, r}$.

Proof. Let $f_i \in \overline{X}_{\theta_i, r}$ and thus $f_i = \sum_{l_i \in \mathbb{N}} u_{i,l}$ with $u_{i,l} \in X_0^i \cap X_1^i$ and convergence in $X_0^i + X_1^i$. Since V is r-convex one can show

$$\|T[f_1, \dots, f_n]\|_V^r \le C^r \sum_{\vec{l} \in \mathbb{N}^n} \|T[u_{1,l_1}, \dots, u_{n,l_n}]\|_V^r\,;$$

this follows easily by considering finite sums and a limiting argument. By assumption and the definition of the J-functional the right hand side of the last display is dominated by C^r times

$$\begin{aligned} &\sum_{\vec{l} \in \mathbb{N}^n} \prod_{i=1}^n \big[\|u_{i,l_i}\|_{X_{i,0}}^{1-\theta_i} \|u_{i,l_i}\|_{X_{i,1}}^{\theta_i}\big]^r \\ &\qquad \le \sum_{\vec{l} \in \mathbb{N}^n} \prod_{i=1}^n \big[J(2^{l_i}, u_{i,l_i}; \overline{X}_i)^{1-\theta_i} (2^{-l_i} J(2^{l_i}, u_{i,l_i}; \overline{X}_i))^{\theta_i}\big]^r \\ &\qquad = \prod_{i=1}^n \Big(\sum_{l_i \in \mathbb{N}} \big[2^{-l_i \theta} J(2^{l_i}, u_{i,l_i}; \overline{X}_i)\big]^r \Big)\,. \end{aligned}$$

Taking the r-th roots and then the infimum over all decompositions $\{u_{i,l}\}$ of f_i, we get assertion (88) (by the equivalence of the J- and K-methods). The operator T extends to $\prod_{i=1}^n \overline{X}^i_{\theta_i, r}$ since $X_0^i \cap X_1^i$ is dense in $\overline{X}^i_{\theta_i, r}$. □

Spaces of vector-valued sequences. We use two results on interpolation of $\ell^p_s(X)$ spaces, for quasi-normed X and $0 < p \le \infty$, $s \in \mathbb{R}$. For fixed X the following standard formula for the real interpolation spaces can be found in §5.6 of [9].

$$\begin{aligned} &\big(\ell^{q_0}_{s_0}(X), \ell^{q_1}_{s_1}(X)\big)_{\vartheta, q} = \ell^q_s(X), \quad s = (1-\vartheta)s_0 + \vartheta s_1 \\ &\text{provided that } s_0 \ne s_1,\ 0 < q_0 \le \infty,\ 0 < q_1 \le \infty\,. \end{aligned} \tag{89}$$

Next consider the space $(\ell^r_{s_0}(X_0), \ell^r_{s_1}(X_1))_{\vartheta, q}$ for a pair of compatible quasi-normed spaces (X_0, X_1). The following lemma is essentially in Cwikel's paper [13] who considered normed spaces. We include a proof for the convenience of the reader.

Lemma A.4. *Suppose that* X_0 *and* X_1 *are compatible quasi-normed spaces. Let* $0 < r \le \infty$, s_0, $s_1 \in \mathbb{R}$, *and* $0 < \vartheta < 1$. *If* $r \le q \le \infty$, *then there is the continuous embedding*

$$\ell^r_s((X_0, X_1)_{\vartheta, q}) \hookrightarrow (\ell^r_{s_0}(X_0), \ell^r_{s_1}(X_1))_{\vartheta, q}, \quad s = (1-\vartheta)s_0 + \vartheta s_1\,. \tag{90}$$

Remark. When $0 < q \le r$, a slight modification of the argument sketched below shows that the inclusion in (90) reverses direction. In particular, equality holds when $q = r$. But this fact is not needed here. Examples disproving the equality in the cases $q \ne r$ are in [13].

Proof of Lemma A.4. Let $f = \{f_k\} \in \ell^r_{s_0}(X_0) + \ell^r_{s_1}(X_1)$. Fix $t > 0$ and $\varepsilon > 0$. For each $k \in \mathbb{Z}$, choose $f_{0,k}$ and $f_{1,k}$ with $f_k = f_{0,k} + f_{1,k}$ such that

$$\|f_{0,k}\|_{X_0} + 2^{k(s_1-s_0)}t\,\|f_{1,k}\|_{X_1} \le (1+\varepsilon)K(2^{k(s_1-s_0)}t, f_k; \overline{X}).$$

Let $W_0 = \ell^r_{s_0}(X_0)$, $W_1 = \ell^r_{s_1}(X_1)$, and let $K(t, f; \overline{W})$ be the K-functional for the pair (W_0, W_1). Then

$$\begin{aligned} K(t,f;\overline{W}) &\le \Big(\sum_k [2^{ks_0}\|f_{0,k}\|_{X_0}]^r\Big)^{1/r} + t\Big(\sum_k [2^{ks_1}\|f_{1,k}\|_{X_1}]^r\Big)^{1/r} \\ &\approx \Big(\sum_k \big[2^{ks_0}\|f_{0,k}\|_{X_0} + 2^{ks_1}t\|f_{1,k}\|_{X_1})\big]^r\Big)^{1/r} \\ &\le (1+\varepsilon)\Big(\sum_k \big[2^{ks_0}K(2^{k(s_1-s_0)}t, f_k; \overline{X})\big]^r\Big)^{1/r}. \end{aligned}$$

If $r \le q < \infty$, then it follows by Minkowski's inequality that

$$\begin{aligned} \|f\|_{\overline{W}_{\vartheta,q}} &= \Big(\int_0^\infty \big[t^{-\vartheta}K(t,f;\overline{W})\big]^q \frac{dt}{t}\Big)^{1/q} \\ &\lesssim \Big(\int_0^\infty t^{-\vartheta q}\Big(\sum_k [2^{ks_0}K(2^{k(s_1-s_0)}t, f_k; \overline{X})]^r\Big)^{q/r}\frac{dt}{t}\Big)^{1/q} \\ &\le \Big(\sum_k \Big[\int_0^\infty \Big(t^{-\vartheta}\, 2^{ks_0}K(2^{k(s_1-s_0)}t, f_k; \overline{X})\Big)^q \frac{dt}{t}\Big]^{r/q}\Big)^{1/r}. \end{aligned}$$

Let $s = (1-\vartheta)s_0 + \vartheta s_1$. By the change of variables $u = 2^{k(s_1-s_0)}t$, we see that the right hand side of the last display equals a constant multiple of

$$\begin{aligned} &\Big(\sum_k \Big[2^{ks}\Big(\int_0^\infty \Big(u^{-\vartheta}K(u, f_k; \overline{X})\Big)^q \frac{du}{u}\Big)^{1/q}\Big]^r\Big)^{1/r} \\ &= \Big(\sum_k \big[2^{ks}\|f_k\|_{\overline{X}_{\vartheta,q}}\big]^r\Big)^{1/r} = \|f\|_{\ell^r_s(\overline{X}_{\vartheta,q})}. \end{aligned}$$

The case $q = \infty$ is similar. □

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