FOURIER INTEGRAL OPERATORS WITH FOLD SINGULARITIES

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1. Introduction

Suppose that X and Y are C^∞ manifolds of dimension d_X and d_Y, respectively, and that

$$\mathcal{C} \subset T^*X \setminus 0 \times T^*Y \setminus 0$$

is a homogeneous canonical relation. By $I^\mu(X, Y; \mathcal{C}')$ we denote the class of Fourier integral operators of order μ associated to $\mathcal{C}$. Here as usual $\mathcal{C}' = \{(x, \xi; y, \eta) : (x, \xi; y, -\eta) \in \mathcal{C}\}$; if σ_X, σ_Y are the canonical two forms on T^*X and T^*Y, respectively, then $\mathcal{C}$ is Lagrangian with respect to $\sigma_x - \sigma_Y$ and $\mathcal{C}'$ contains the wavefront sets of the kernels.

We shall be concerned with $L^2_\alpha \to L^q_\beta$ mapping properties of operators in $I^\mu(X, Y; \mathcal{C}')$ (here L^q_β denotes the L^q Sobolev space). These are well known in case that $\mathcal{C}$ is locally the graph of a canonical transformation; this means that the projections $\pi_L : \mathcal{C} \to T^*X$, $\pi_R : \mathcal{C} \to T^*Y$ are locally diffeomorphisms. In particular $d_X = d_Y := d$. Then $\mathcal{F} \in I^\mu(X, Y, \mathcal{C}')$ maps $L^2_{\alpha,\mathrm{comp}}(Y)$ into $L^2_{\beta,\mathrm{loc}}(X)$ if $\beta \le \alpha - \mu$. This was shown by Hörmander as a consequence of the calculus in [7]. By composing $\mathcal{F}$ with a fractional integral operator it is easy to see that $\mathcal{F} \in I^\mu(X, Y, \mathcal{C}')$ maps $L^2_{\alpha,\mathrm{comp}}$ into $L^q_{\beta,\mathrm{loc}}$, $2 \le q < \infty$, if $\beta \le \alpha - \mu - d/2 + d/q$. More general if $d_X \le d_Y$ and $d\pi_L$ has maximal rank $2d_X$ then the same mapping properties hold for Fourier integral operators in the class $\mathcal{F} \in I^{\mu + (d_X - d_Y)/4}(X, Y, \mathcal{C}')$.

If one of the projections π_L, π_R becomes singular it follows that the other is singular as well, see [7]. However the nature of the singularities of π_L and π_R may be quite different and this is reflected in the estimates one gets. Sharp L^2 estimates are known if $\mathcal{C}$ is a folding canonical relation; one assumes that both projections are either nondegenerate or Whitney folds (again $d_X = d_Y = d$). Then there is a loss of 1/6 derivatives in the L^2 estimates; namely $\mathcal{F} \in I^\mu(X, Y, \mathcal{C}')$ maps $L^2_{\alpha,\mathrm{comp}}$ into $L^2_{\beta,\mathrm{loc}}$ if $\beta \le \alpha - \mu + 1/6$ (see [10], and [15] for a nonhomogeneous version).

In this paper we mainly consider the case of one-sided fold singularities; in the case $d_X = d_Y$ we require that one projection (say π_L) is either nondegenerate or a Whitney fold but we do not impose any condition on the other projection. If $d_X \le d_Y$ then we require that π_L is a submersion with folds. We recall the definition: Let M and N be C^∞ manifolds of dimensions m, n, respectively, where $m \ge n$. Then a C^∞ map $F : M \to N$ is a submersion with fold at $x_0 \in M$ if rank $F'(x_0) = n - 1$ (and therefore $\dim \mathrm{Ker}\, F'(x_0) = m - n + 1$ and $\dim \mathrm{Coker}\, F'(x_0) = 1$) and if the Hessian of F at x_0 is nondegenerate. The Hessian is invariantly defined as a

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quadratic form on Ker $F'(x_0)$ with values in Coker $F'(x_0)$. One can always choose local coordinates x in M vanishing at x_0 and local coordinates y in N vanishing at y_0 such that in the new coordinates

$$F(x_1, \dots, x_m) = (x_1, \dots, x_{n-1}, Q(x_n, \dots, x_m))$$

where Q is a nondegenerate quadratic form in $\mathbb{R}^{m-n+1}$ (see [4, ch. III.4]) and also [9, III, p.493]). We note that the variety $\mathcal{L}$ where F' is degenerate is a smooth surface in M of codimension $m - n + 1$. Another way of defining a submersion with folds is identifying $\mathcal{L}$ and saying that F drops rank simply by one (at least one $n \times n$ minor of dF vanishes of only first order) and that $F|_{\mathcal{L}}$ is an immersion. In particular $F(\mathcal{L})$ is a smooth hypersurface of N. In the case $m = n$ a submersion with folds is simply a Whitney fold.

Theorem 1.1. *Suppose that $d_X \le d_Y$ and that $\mathcal{C} \subset T^*X \setminus 0 \times T^*Y \setminus 0$ is a homogeneous canonical relation such that the projection $\pi_L : \mathcal{C} \to T^*X$ is a submersion with folds. Suppose that $\mathcal{F} \in I^{\mu+(d_X-d_Y)/4}(X, Y, \mathcal{C}')$. Then $\mathcal{F}$ maps $L^2_{\alpha,\mathrm{comp}}(Y)$ into $L^2_{\beta,\mathrm{loc}}(X)$ provided that*

(1) $\beta \le \alpha - \mu - 1/4$ *if* $d_Y = d_X$,
(2) $\beta \le \alpha - \mu - \epsilon$, *any* $\epsilon > 0$, *if* $d_Y = d_X + 1$,
(3) $\beta \le \alpha - \mu$ *if* $d_Y \ge d_X + 2$.

These results had been conjectured in [2], [3] where they are proved for the special case of fibered folding canonical relations (this corresponds to an assumption of maximal degeneracy on π_R). In this case there is a composition calculus which is not available in the general situation.

For averaging operators in $\mathbb{R}^2$ and some model cases in higher dimensions the L^2 estimates are already in [17]. We remark that in the case $d_X = d_Y$ Theorem 1.1 is sharp without further assumption; however it can be improved if one imposes an additional finite type condition on π_R (*cf.* [18], [19]). It is also sharp if $d_Y > d_X$; see [3] for an example in the case $d_Y = d_X + 1$ where ϵ has to be positive.

Our next result concerns $L^2_\alpha \to L^q_\beta$ estimates.

Theorem 1.2. *Suppose that $d_X \le d_Y$ and that $\mathcal{C} \subset T^*X \setminus 0 \times T^*Y \setminus 0$ is a homogeneous canonical relation such that the projection $\pi_L : \mathcal{C} \to T^*X$ is a submersion with folds. Moreover suppose that the projection $\pi^X : \mathcal{C} \to X$ is a submersion. Let $\mathcal{F} \in I^{\mu+(d_X-d_Y)/4}(X, Y; \mathcal{C}')$. Then $\mathcal{F}$ maps $L^2_{\alpha,\mathrm{comp}}(Y)$ into $L^q_{\beta,\mathrm{loc}}(X)$ provided that $\beta \le \alpha - \mu - d_X(\frac{1}{2} - \frac{1}{q})$ and*

(1) $4 \le q < \infty$ *if* $d_Y = d_X$,
(2) $2 < q < \infty$ *if* $d_Y = d_X + 1$,
(3) $2 \le q < \infty$ *if* $d_Y \ge d_X + 2$.

We note that sharp $L^2 \to L^4$ estimates for averaging operators in the plane are in [17], [19]. There is always a range of $q's$ ($4 \le q < \infty$ if $d_X = d_Y$) where the $L^2_\alpha \to L^q_\beta$ estimates are sharp and in fact the same as in the nondegenerate case. The range $[4, \infty)$ is sharp if one does not impose additional assumptions. One considers on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d-1}$ the phase function $\Phi_0(x, y, \theta) = (x_1 - y_1 + \frac{1}{2} x_d y_d^2)\theta_1 + \sum_{i=2}^{d-1} (x_i - y_i)\theta_i$ in the region $\{\theta \in \mathbb{R}^{d-1} : |\theta_1| \ge |\theta| > 0\}$. It parametrizes the canonical relation

$$\mathcal{C}_0 = \{(y_1 - x_d y_d^2/2, y'', x_d, \theta, y_d^2\theta_1/2; y, \theta, -x_d y_d \theta_1) : (y, \theta, x_d) \in \mathbb{R}^d \times \mathbb{R}^{d-1} \times \mathbb{R}, |\theta_1| \ge c|\theta|, x_d \ne 0\}$$

where we write $y = (y', y_d) = (y_1, y'', y_d)$. This is a model case for a fibered folding canonical relation, considered in [2] (here π_L is a fold and π_R is a blowdown). One can check (arguing as in [2], [19]) that Theorem 1.2 is sharp in this case.

In order to improve Theorem 1.2 one imposes additional curvature assumptions. Let us suppose that $d_X = d_Y$ and let $\mathcal{L}$ be the fold hypersurface. We assume that the projection $\pi^X : \mathcal{L} \to X$ is a submersion. Then for each $x \in X$ the image of the projection $\pi^{T_x^*X}$ of $\mathcal{L}$ to the fiber T_x^*X is a $d-1$ dimensional conic hypersurface Γ_x. For the above example these hypersurfaces are hyperplanes.

Theorem 1.3. *Let $d_X = d_Y = d$ and suppose that $\mathcal{C}$ is as in Theorem 1.2. Suppose that $\pi^X : \mathcal{L} \to X$ is a submersion and suppose that for each $x \in X$ and each $\zeta \in \Gamma_x = \pi^{T_x^*X}(\mathcal{L})$ at least ℓ principal curvatures do not vanish. Suppose $(2\ell+4)/(\ell+1) \le q < \infty$ and $\beta \le \alpha - \mu - d(\frac{1}{2} - \frac{1}{q})$. Then $\mathcal{F} \in I^\mu(X, Y; \mathcal{C}')$ maps $L^2_{\alpha,\mathrm{comp}}$ into $L^q_{\beta,\mathrm{loc}}$.*

The additional curvature condition on the fibers is close to the cone condition in [12], formulated for a class of Fourier integral operators that comes up in the study of wave equations.

We now consider averaging operators in three dimensions. Suppose that X and Y are three dimensional manifolds and suppose that $\mathcal{M} \subset X \times Y$ is a four dimensional manifold such that the projections onto X and Y are submersions; furthermore assume that

$$\mathcal{N}^*\mathcal{M} \subset T^*X \setminus 0 \times T^*Y \setminus 0$$

where $N^*\mathcal{M}$ is the normal bundle of $\mathcal{M}$. Then $\mathcal{M}_x = \{y : (x,y) \in \mathcal{M}\}$ is a curve in Y for each $x \in X$; similarly for each y define $\mathcal{M}^y$ which is a curve in X. Let $d\sigma_x$ be a smooth density on $\mathcal{M}_x$ depending smoothly on x. Then the averaging operator defined by

$$\mathcal{A}f(x) = \int_{\mathcal{M}_x} f(y) d\sigma_x(y)$$

belongs to the class $I^{-1/2}(X, Y; N^*\mathcal{M})$ (see *e.g.* [6]). Therefore Theorem 1.3 and interpolation yield

Corollary 1.4. *Suppose that $\dim X = \dim Y = 3$ and $\mathcal{M}$ is as above. Suppose that the projection $\pi_L : N^*\mathcal{M} \to T^*X$ is either nondegenerate or a Whitney fold with fold hypersurface $\mathcal{L}$, such that the projection of $\mathcal{L}$ onto X is a submersion. Suppose that for each $x \in X$ and at each $\zeta \in \Gamma_x = \pi^{T_x^*X}(\mathcal{L})$ a principal curvature does not vanish. Then $\mathcal{A}$ is bounded from $L^p_{\mathrm{comp}}(Y)$ into $L^q_{\mathrm{comp}}(X)$ if $(1/p, 1/q)$ belongs to the closed triangle with corners $(0,0)$, $(1,1)$, $(1/2, 1/3)$.*

Clearly by applying this to $\mathcal{A}^*$ we get a similar result involving assumptions on π_R. The typical example that demonstrates the sharpness of Corollary 1.4 is the X-ray transform for the family of light rays in $\mathbb{R}^3$ (considered in [2], [5], [11], [16]). The light rays are parametrized by their intersection with the (x_1, x_2)-plane and an angle α, and the averaging operator (taking the role of $\mathcal{A}^*$) is given by

$$\mathcal{R}f(x_1, x_2, \alpha) = \int f(x_1 + s\cos\alpha, x_2 + s\sin\alpha, s)\chi(s)\, ds$$

with an appropriate cutoff function χ. $(N^*\mathcal{M})'$ is a fibered canonical relation (now π_R is a fold and π_L is a blowdown) and the fold hypersurface for π_R is

$$\mathcal{L} = \{(x_1, x_2, \alpha, \mu\cos\alpha, \mu\sin\alpha, 0; x_1 + s\cos\alpha, x_2 + s\sin\alpha, s, \mu\cos\alpha, \mu\sin\alpha, \mu)\}.$$

The sharpness of Corollary 1.4 can be seen by testing $\mathcal{R}$ on characteristic functions of balls (to get the restriction $q \le 2p/(3-p)$) and on characteristic functions of rectangles with dimensions $1, \delta, \delta^2$ (to get the restriction $q \le 4p/3$); see [2]. The operator $\mathcal{R}$ is an example of a more general class of restricted X-ray transforms where one averages over lines in a well-curved hypersurface of $M_{1,d}$ (the space of lines in $\mathbb{R}^d$). This will be taken up below.

In the case of folding canonical relations one may apply Corollary 1.4 to $\mathcal{A}$ and $\mathcal{A}^*$ to get

Corollary 1.5. *Let $\mathcal{M}$ be as in Corollary 1.4 and suppose that $(N^*\mathcal{M})'$ is a folding canonical relation. Moreover suppose that the cones $\Gamma^L_x = \pi^{T^*_xX}(\mathcal{L})$ and the cones $\Gamma^R_y = \pi^{T^*_yY}(\mathcal{L})$ are curved in the sense that at every point one principal curvature does not vanish. Then $\mathcal{A}$ is bounded from $L^p_{\text{comp}}(Y)$ to $L^q_{\text{loc}}(X)$ if $(1/p, 1/q)$ belongs to the closed trapezoid with corners $(0,0)$, $(1,1)$, $(2/3, 1/2)$ and $(1/2, 2/3)$.*

In particular suppose that $t \mapsto \gamma(t)$ defines a curve in $\mathbb{R}^3$ with nonvanishing curvature $\kappa(t)$ and nonvanishing torsion $\tau(t)$. Then the translation invariant operator

$$\mathcal{A}f(x) = \int f(x - \gamma(t))\chi(t)dt$$

falls under the scope of Corollary 1.5. In this case $\mathcal{M} = \{(x, x+\gamma(t))\}$ and $(N^*\mathcal{M})'$ is a folding canonical relation with fold hypersurface

$$\mathcal{L} = \{(x, \mu B(t), y, -\mu B(t)) : x - y = \gamma(t),\ \mu \in \mathbb{R}\};$$

here $B(t)$ denotes the binormal vector. The principal curvatures of the cone $\Gamma = \{(\mu B(t))\}$ at $\mu B(t)$ are 0 and $-\mu\kappa(t)\tau(t)$. So Corollary 1.5 extends Oberlin's result [13] on translation invariant curves with nonvanishing curvature and torsion (proved in full generality by Pan [14]). It is sharp as one can see by testing $\mathcal{A}$ on characteristic functions of rectangles with dimensions $\delta, \delta^2, \delta^3$.

We shall consider more general oscillatory integral and Fourier integral operators with not necessarily homogeneous phase functions. §2 contains the main estimates for oscillatory integral operators. In §3 we apply these results to Fourier integral operators with general phase functions; the homogeneous case arises as a special case if one uses Littlewood-Paley theory. In §4 we apply our theorems to obtain new estimates for restricted X-ray transforms. Throughout the paper c, C will denote positive constants which may assume different values in different lines.

2. Estimates for oscillatory integrals

Suppose X and Z are open sets in $\mathbb{R}^d$ and $\mathbb{R}^{d+r}$, respectively. We consider oscillatory integral operators of the form

$$T_\lambda f(x) = \int e^{i\lambda\Phi(x,z)} a(x,z) f(z)\, dz \tag{2.1}$$

where the phase function $\Phi \in C^\infty(X \times Z)$ is not necessarily homogeneous and $a \in C_0^\infty(X \times Z)$. Let

$$\mathcal{C}_\Phi = \{(x, \Phi'_x; z, -\Phi'_z)\} \tag{2.2}$$

be the associated canonical relation.

It is well known ([8]) that the $L^2 \to L^q$ operator norm of T_λ is $O(\lambda^{-d/q})$ provided the differentials of the projections $\pi_L : \mathcal{C} \to T^*(X)$, $\pi_R : \mathcal{C} \to T^*(Z)$ have maximal rank d. This hypothesis is equivalent with the condition rank $\Phi''_{xz} = d$.

In this section we prove $L^2 \to L^q$ bounds for T_λ, under the assumption that the only singularities of the projection π_L are fold singularities; no assumption on π_R is made.

Theorem 2.1. *Suppose that* $\dim X = d$, $\dim Z = d + r$ *and that the projection* $\pi_L : \mathcal{C}_\Phi \to T^*X$ *is a submersion with folds. Then if* $r = 0$ *we have for* $\lambda \geq 2$

$$\begin{aligned} \|T_\lambda f\|_q &\leq C\lambda^{-\frac{d-1}{q}-\frac{1}{4}}\|f\|_2, && \text{if } 2 \leq q \leq 4 \\ \|T_\lambda f\|_q &\leq C\lambda^{-\frac{d}{q}}\|f\|_2, && \text{if } 4 \leq q \leq \infty. \end{aligned}$$

If $r = 1$ *then*

$$\begin{aligned} \|T_\lambda f\|_2 &\leq C\lambda^{-\frac{d}{2}}(\log \lambda)^{\frac{1}{2}}\|f\|_2, \\ \|T_\lambda f\|_q &\leq C\lambda^{-\frac{d}{q}}\|f\|_2, \qquad 2 < q \leq \infty. \end{aligned}$$

If $r \geq 2$ *then*

$$\|T_\lambda f\|_q \leq C\lambda^{-\frac{d}{q}}\|f\|_2, \qquad 2 \leq q \leq \infty.$$

Phong and Stein [17] noticed that the case $\dim X = \dim Z = 1$ already follows if one applies van der Corput's Lemma to the kernel of TT^*. An improvement in higher dimension may be obtained under some additional curvature assumption. Suppose $r = 0$ and denote by $\mathcal{L}$ the fold hypersurface for the projection π_L. Again if $\pi^X : \mathcal{L} \to X$ is a submersion then for each x the projection of $\mathcal{L}$ onto the fiber, $\Sigma_x = \pi^{T_x^*X}(\mathcal{L})$, is a hypersurface in T_x^*X.

Theorem 2.2. *Suppose that* $\dim X = \dim Y = d$ *and that the projection* $\pi_L : \mathcal{C}_\Phi \to T^*X$ *is either nondegenerate or a Whitney fold. Suppose in addition that for each* $x \in X$, *for each* $\zeta \in \Sigma_x$ *at least* ℓ *principal curvatures do not vanish. Then for* $\lambda \geq 1$

$$\begin{aligned} \|T_\lambda f\|_q &\leq C\lambda^{-\frac{d-1}{q}-\frac{\ell+1}{4}+\frac{\ell}{2q}}\|f\|_2, && \text{if } 2 \leq q \leq \frac{2\ell+4}{\ell+1}. \\ \|T_\lambda f\|_q &\leq C\lambda^{-\frac{d}{q}}\|f\|_2, && \text{if } \frac{2\ell+4}{\ell+1} \leq q \leq \infty. \end{aligned}$$

We shall use a general result on nondegenerate Fourier integral operators with not necessarily homogeneous phase functions. We consider operators of the form

$$S_\lambda f(x) = \iint e^{i\lambda\Psi(x,y,z)} b(x,y,z)\, dz\, f(y)\, dy \tag{2.3}$$

where $b \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N)$ and Ψ is a C^∞-function defined in a neighborhood of supp b satisfying

$$\det \begin{pmatrix} \Psi''_{xy} & \Psi''_{xz} \\ \Psi''_{zy} & \Psi''_{zz} \end{pmatrix} \neq 0. \tag{2.4}$$

This says that the associated canonical relation (see (3.2) below) is locally the graph of a canonical transformation.

The following lemma is well known; it is contained in [8] for the case $N = 0$ which corresponds to operators of type (2.1). We sketch a proof for the reader's convenience.

Lemma 2.3. *Suppose that Ψ satisfies* (2.4). *Then S_λ is a bounded operator on $L^2(\mathbb{R}^n)$ and the operator norm is $O(|\lambda|^{-(n+N)/2})$, $|\lambda| \geq 1$.*

Proof. We may assume that the support of b is small. We prove that $S_\lambda S_\lambda^*$ is a bounded on $L^2(\mathbb{R}^n)$ with norm $O(\lambda^{-n-N})$. The kernel K_λ of $S_\lambda S_\lambda^*$ is

$$K_\lambda(v,w) = \iiint e^{i\lambda[\Psi(v,y,z+h)-\Psi(w,y,z)]} b(v,y,z+h)\overline{b(w,y,z)}\, dy\, dz\, dh.$$

Observe that

$$\begin{pmatrix} \nabla_y[\Psi(v,y,z+h) - \Psi(w,y,z)] \\ \nabla_z[\Psi(v,y,z+h) - \Psi(w,y,z)] \end{pmatrix} = \begin{pmatrix} \Psi''_{yx} & \Psi''_{yz} \\ \Psi''_{zx} & \Psi''_{zz} \end{pmatrix}\Big|_{(w,y,z)} \begin{pmatrix} v-w \\ h \end{pmatrix} + O(|v-w|^2+|h|^2).$$

Therefore an integration by parts shows that

$$\begin{aligned} K_\lambda(v,w) &\leq C_M \int (1+\lambda|v-w|+\lambda|h|)^{-M} dh \\ &\leq C'_M \lambda^{-N}(1+\lambda|v-w|)^{-M+N} \end{aligned}$$

where $M > N + n$. It follows that

$$\sup_v \int |K(v,w)|dw + \sup_w \int |K(v,w)|dv \leq C\lambda^{-n-N}$$

which implies $\|S_\lambda S_\lambda^*\| = O(\lambda^{-n-N})$. □

Remark 2.4. Suppose X, Y are open sets in $\mathbb{R}^n$ and Z is an open set in $\mathbb{R}^N$. Suppose there is a family of phase functions and symbols $\{(\Psi_\nu, a_\nu)\}$ such that the Ψ_ν belong to a bounded family of $C^\infty(X \times Y \times Z)$ and the a_ν belong to a bounded family of $C_0^\infty(X \times Y \times Z)$. Suppose that the determinant (2.4) is bounded away from 0, uniformly in ν. Then the associated oscillatory integral operators S_λ are L^2-bounded with norm $O(\lambda^{-(n+N)/2})$ uniformly in ν. This is a consequence of the above proof. □

We now turn to the proofs of Theorem 2.1 and 2.2. We split coordinates $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $z = (z', z'') \in \mathbb{R}^{d-1} \times \mathbb{R}^{r+1}$ and claim that without loss of generality we can assume that $(0,0) \in X \times Y$ and

$$\det \Phi''_{x'z'}(0,0) \neq 0 \tag{2.5}$$

$$\det \Phi'''_{x_d z'' z''}(0,0) \neq 0; \tag{2.6}$$

moreover

$$\Phi''_{x'z''}(0,0) \;=\; 0 \tag{2.7}$$

$$\Phi''_{x_dz'}(0,0) \;=\; 0 \tag{2.8}$$

$$\Phi'''_{x_dz'z''}(0,0) \;=\; 0. \tag{2.9}$$

In fact if $\mathcal{C} = \{u, \phi'_u(u,v), v, \phi'_v(u,v)\}$ and π_L is a submersion with fold at $(u,v) = (x_0, y_0)$ then assume that $0 \neq a \in \text{Coker } \phi''_{uv}(x_0,y_0)$ and that $\{b_1, \dots, b_{r+1}\}$ is a basis of $\text{Ker } \phi''_{uv}(x_0,y_0)$. Set $\Phi(x,y) = \phi(x_0 + B_1x, y_0 + B_2y)$ where we require that $B_1 \in GL(d,\mathbb{R})$, $B_2 \in GL(d+r,\mathbb{R})$ with the following properties. First $B_1e_d = a$ (here $\{e_1, e_2, \dots\}$ is the standard orthonormal vectors in $\mathbb{R}^d$ or $\mathbb{R}^{d+r}$). Next $B_2e_{d-1+i} = b_i$ and for $j = 1, \dots, d-1$, B_2e_j is orthogonal to $\langle a, \phi'_u\rangle''_{vv}b_i$, for $i = 1, \dots, r+1$. The fold condition which is the nondegeneracy of the quadratic form $\eta \to \big\langle \langle a, \phi'_u\rangle''_{vv}\eta, \eta\big\rangle$ on $\text{Ker } d\pi_L$ implies that B_2 can be made invertible. Clearly $e_d \in \text{Coker } \Phi''_{xz}(0,0)$ and $e_{d-1+i} \in \text{Ker } \Phi''_{xz}(0,0)$ for $i = 1, \dots, r+1$; this is (2.7), (2.8) and the fold condition implies (2.5), (2.6). Since $\Phi'''_{x_dz_jz_{d-1+k}}\big|_{(0,0)} = (B_2e_j)^t\langle a, \phi'_u\rangle''_{vv}\big|_{(x_0,y_0)}b_k$ we get (2.9) as well.

We shall always assume that a is supported in a ball of radius ϵ and center $(0,0)$ and we shall choose ϵ small (independent of λ). Observe that

$$\Phi''_{x'z''},\ \Phi''_{x_dz'},\ \Phi'''_{x_dz'z''} \;=\; O(\epsilon)$$

in the support of a.

In order to prove our results we use an argument due to Tomas [22] according to which for $p \leq 2$

$$\|T_\lambda\|_{L^2 \to L^{p'}} \;\leq\; \|T_\lambda T^*_\lambda\|^{1/2}_{L^p \to L^{p'}}.$$

We write

$$T_\lambda T^*_\lambda f(x', x_d) \;=\; \int K_{x_dy_d}[f(\cdot, y_d)](x')\, dy_d$$

where

$$K_{x_dy_d}g(x') \;=\; \int K_\lambda(x', x_d, y', y_d)g(y')\, dy'$$

with

$$K_\lambda(x', x_d, y', y_d) \;=\; \int e^{i\lambda[\Phi(x',x_d,z) - \Phi(y',y_d,z)]}a(x,z)\overline{a(y,z)}\, dz.$$

The basic L^2 estimate is

Proposition 2.5. *For fixed x_d, y_d there is the estimate*

$$\|K_{x_dy_d}g\|_{L^2(\mathbb{R}^{d-1})} \;\leq\; C\lambda^{-(d-1)}(1 + \lambda|x_d - y_d|)^{-(r+1)/2}\|g\|_{L^2(\mathbb{R}^{d-1})}.$$

Proof. Define

$$T_{x_dz''}h(x') \;=\; \int e^{i\lambda\Phi(x',x_d,z',z'')}a(x', x_d, z', z'')\, dz'.$$

Then

$$K_{x_dy_d}g \;=\; \int T_{x_dz''}T^*_{y_dz''}g\, dz''. \tag{2.10}$$

By (2.5) it follows from Lemma 2.3 (with $N = 0$) that $T_{x_d z''}$ is bounded on $L^2(\mathbb{R}^{d-1})$ with norm $O(\lambda^{(1-d)/2})$, uniformly in x_d. Then we see from (2.10) that $K_{x_d y_d}$ is bounded on $L^2(\mathbb{R}^{d-1})$ with norm $O(\lambda^{1-d})$. This is the desired estimate in the case $|x_d - y_d| \le C\lambda^{-1}$.

Henceforth assume $|x_d - y_d| \ge \lambda^{-1}$. Note that in view of (2.5) and (2.8)

$$\big|\nabla_{z'}\big[\Phi(x', x_d, z) - \Phi(y', y_d, z)\big]\big| \ge c\big[|x' - y'| - C_0\epsilon|x_d - y_d|\big].$$

Therefore an integration by parts argument shows that

$$(2.11) \qquad |K_\lambda(x, y)| \le C_N(1 + \lambda|x' - y'|)^{-N} \qquad \text{if } |x' - y'| \ge 2C_0\epsilon|x_d - y_d|.$$

Let

$$\chi_\epsilon(x, y) = \chi(3C_0\epsilon^{-1}\frac{|x' - y'|}{|x_d - y_d|})$$

and let

$$H(x', y') \equiv H_{x_d y_d}(x', y') = \chi_\epsilon(x, y)K_\lambda(x, y)$$

and

$$R_{x_d y_d}(x', y') = (1 - \chi_\epsilon(x, y))K_\lambda(x, y).$$

From (2.11) we obtain

$$\sup_{y'}\int |R_{x_d y_d}(x', y')|dx' + \sup_{x'}\int |R_{x_d y_d}(x', y')|dy' \le C_N(1 + \lambda|x_d - y_d|)^{-N+d-1}.$$

Choosing $N > d + (r-1)/2$ we see that the operator with kernel $R_{x_d y_d}$ is bounded on L^2 with the desired bound.

In view of the support properties of the kernel H it is appropriate to introduce another localization. Let $\beta \in C_0^\infty(\mathbb{R}^{d-1})$ be supported in $[-1, 1]^{d-1}$ with $\sum_{n\in\mathbb{Z}^{d-1}} \beta(\cdot - n) \equiv 1$. We split

$$H(x', y') = \sum_{n\in\mathbb{Z}^{d-1}} H^n(x', y')$$

where

$$H^n(x', y') = \beta(|x_d - y_d|^{-1}x' - n)H(x', y').$$

Note that $H^n(x', y') = 0$ if $|x' - n|x_d - y_d|| \ge 2\sqrt{d-1}|x_d - y_d|$ or if $|y' - n|x_d - y_d|| \ge C_1|x_d - y_d|$ (with $C_1 = 2\sqrt{d-1} + (3C_0\epsilon)^{-1}$). Let $\mathcal{H}^n$ denote the operator with kernel H^n; then $\mathcal{H}^n(\mathcal{H}^{n'})^* = 0$, $(\mathcal{H}^n)^*\mathcal{H}^{n'} = 0$ if $|n - n'| \ge C$, for suitable C, and therefore it suffices to prove the required bound for an individual $\mathcal{H}^n$. We define rescaled operators $\widetilde{\mathcal{H}}^n$ with kernels

$$\widetilde{H}^n(u, v) = H^n(|x_d - y_d|(n + u), |x_d - y_d|(n + v)).$$

Then

$$(2.12) \qquad \mathcal{H}^n g(x') = |x_d - y_d|^{d-1}\widetilde{\mathcal{H}}^n[f(|x_d - y_d|\cdot + n)](\frac{x'}{|x_d - y_d|} - n).$$

Let

$$\Psi_{n,x_d,y_d}(u,v,z) \;=\; \frac{\Phi((u+n)|x_d-y_d|,x_d,z)-\Phi((v+n)|x_d-y_d|,y_d,z)}{|x_d-y_d|}.$$

Then $\Psi = \Psi_{n,x_d,y_d}$ is a C^∞ phase function which satisfies the assumptions of Lemma 2.3, uniformly in x_d, y_d and n. In fact we have

$$\begin{aligned}
\Psi''_{uv}(u,v,z) &= 0\\
\Psi''_{uz}(u,v,z) &= \Phi''_{x'z}(u+n|x_d-y_d|,x_d,z)\\
\Psi''_{zv}(u,v,z) &= -(\Phi''_{x'z})^t(v+n|x_d-y_d|,y_d,z)\\
\Psi''_{zz}(u,v,z) &= \frac{\Phi''_{zz}((u+n)|x_d-y_d|,x_d,z)-\Phi''_{zz}((v+n)|x_d-y_d|,y_d,z)}{|x_d-y_d|}.
\end{aligned}$$

In view of (2.5), (2.6) and the support properties of χ_ϵ ($|u-v|\ll|x_d-y_d|$) we see that $|\Psi''_{z''z''}|\ge c>0$. Taking also into account (2.7) and (2.9) we obtain

$$\det\begin{pmatrix}\Psi''_{uv} & \Psi''_{uz}\\ \Psi''_{zv} & \Psi''_{zz}\end{pmatrix}\Big|_{(u,v,z)} \neq 0$$

in the support of χ_ϵ if ϵ is chosen sufficiently small. Observe that

$$\widetilde{H}^n(u,v) \;=\; \int e^{i\lambda|x_d-y_d|\Psi_{n,x_d,y_d}(u,v,z)}b_{n,x_d,y_d}(u,v,z)\,dz$$

where b_{n,x_d,y_d} is a C^∞-function with bounds independent of n, x_d and y_d. Hence we may apply Lemma 2.3 and it follows from Remark 2.4 that

$$\|\widetilde{\mathcal{H}}^n\|_{L^2\to L^2} \;\le\; C(\lambda|x_d-y_d|)^{-d-r/2+1/2}$$

where C does not depend on x_d, y_d or n. Therefore by (2.12)

$$\begin{aligned}
\|\mathcal{H}^n g\|_2 &= |x_d-y_d|^{(d-1)/2}\big\|\widetilde{\mathcal{H}}^n[g(|x_d-y_d|(n+\cdot))]|x_d-y_d|^{d-1}\big\|_2\\
&\le C\lambda^{-d+1}(\lambda|x_d-y_d|)^{-(r+1)/2}\|g\|_2.
\end{aligned}$$

This is the desired estimate since we assume $|x_d-y_d|\ge\lambda^{-1}$. $\Box$

In order to complete the proof of the $L^2\to L^q$ estimates for T_λ we need an $L^1\to L^\infty$ estimate for $K_{x_dy_d}$.

Proposition 2.6. *Let $\mathcal{C}_\Phi$ be as in Theorem 2.1 and assume $r=0$. Then*

$$\|K_{x_dy_d}g\|_{L^\infty(\mathbb{R}^{d-1})} \;\le\; C(1+\lambda|x_d-y_d|)^{-1/2}\|g\|_{L^1(\mathbb{R}^{d-1})}. \tag{2.13}$$

Suppose that $\mathcal{C}_\Phi$ satisfies the additional curvature assumption of Theorem 2.2. Then

$$\|K_{x_dy_d}g\|_{L^\infty(\mathbb{R}^{d-1})} \;\le\; C(1+\lambda|x_d-y_d|)^{-(\ell+1)/2}\|g\|_{L^1(\mathbb{R}^{d-1})}. \tag{2.14}$$

Proof. We first prove (2.13). Split $K_{x_dy_d}=H_{x_dy_d}+R_{x_dy_d}$ as in the proof of Proposition 2.5. The appropriate inequality for the operator with kernel $R_{x_dy_d}$

follows at once from (2.11). An application of the method of stationary phase which uses only the fold condition (2.6) (and (2.8)) yields

$$|H_{x_d y_d}(u,v)| \le C(1+\lambda|x_d-y_d|)^{-1/2}$$

and (2.13).

If $\mathcal{C}_\Phi$ is as in Theorem 2.2 then Σ_x can be parametrized by $z' \mapsto \Phi'_x(x,z',g(z'))$ for suitable smooth g and e_d is a normal vector for Σ_0 at $z'=0$. The curvature condition on Σ_0 at $z'=0$ is

$$\operatorname{rank} \Phi'''_{x_d z' z'} = \ell.$$

In order to see this one uses (2.8) and (2.9). By (2.6) and (2.9) it follows that

$$\operatorname{rank} \Phi'''_{x_d zz} = \ell+1.$$

In this case the application of the method of stationary phase yields

$$|H_{x_d y_d}(u,v)| \le C(1+\lambda|x_d-y_d|)^{-(\ell+1)/2}$$

and therefore (2.14). □

Proof of Theorems 2.1 and 2.2. We first assume that $r=0$. Using complex interpolation we deduce from Propositions 2.5 and 2.6 that for $1 \le p \le 2$

$$\|K_{x_d y_d} g\|_{L^{p'}(\mathbb{R}^{d-1})} \le C\lambda^{-2(d-1)/p'}(1+\lambda|x_d-y_d|)^{\ell/p'-(\ell+1)/2}\|g\|_{L^p(\mathbb{R}^{d-1})}$$

where $\ell=0$ in Theorem 2.1 and $0<\ell\le d-1$ in Theorem 2.2. Of course $K_{x_d y_d}=0$ if $|x_d-y_d|\ge 1$. By the theorem on fractional integration we know that for $0<a<1$ the integral operator $|x_d-y_d|^{a-1}\chi(x_d-y_d)$ (where χ is a cutoff function) maps $L^p(\mathbb{R})$ into $L^{p'}(\mathbb{R})$ if $2/(a+1)\le p\le 2$. We want to apply this with $a-1=\ell/p'-(\ell+1)/2$ which yields the limitation $2\le p'\le(2\ell+4)/(\ell+1)$. For this range we obtain (using an idea by Oberlin [13])

$$\begin{aligned}
&\big\|T_\lambda T_\lambda^* f\big\|_{L^{p'}(\mathbb{R}^d)}\\
&\le C\Big(\int\Big[\int\big\|K_{x_d y_d}[f(\cdot,y_d)]\big\|_{L^{p'}(\mathbb{R}^{d-1})}dy_d\Big]^{p'}dx_d\Big)^{1/p'}\\
&\le C\lambda^{-2(d-1)/p'-(\ell+1)/2+\ell/p'}\Big(\int_{-1}^1\Big[\int_{-1}^1\frac{\|f(\cdot,y_d)\|_{L^p(\mathbb{R}^{d-1})}}{|x_d-y_d|^{(\ell+1)(1/2-1/p)-1/p+1}}dy_d\Big]^{p'}dx_d\Big)^{1/p'}\\
&\le C\lambda^{-2(d-1)/p'-(\ell+1)/2+\ell/p'}\|f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Consequently T_λ is bounded from L^2 into $L^{p'}$ with operator norm $O(\lambda^{-(d-1)/p'-(\ell+1)/4+\ell/(2p')}$. This settles the case $r=0$.

If $\dim X=d$, $\dim Y=d+r$ then we replace Proposition 2.6 by the trivial estimate $\|K_{x_d y_d}\|_{L^1\to L^\infty}=O(1)$ and obtain

$$\|K_{x_d y_d} g\|_{L^{p'}(\mathbb{R}^{d-1})} \le C\lambda^{-2(d-1)/p'}(1+\lambda|x_d-y_d|)^{-(r+1)/p'}\|g\|_{L^p(\mathbb{R}^{d-1})}.$$

Let $w_{\lambda,p}(t)=\lambda^{-2/p'}(1+\lambda|t|)^{-(r+1)/p'}\chi(t)$. If $r\ge 1$, $p'>2$ or if $r>1$, $p'\ge 2$ then the convolution with $w_{\lambda,p}$ defines a bounded operator from $L^p(\mathbb{R})$ into $L^{p'}(\mathbb{R})$, with norm independent of λ. If $r=1$, $p=2$ the L^2 operator norm is $O(\log\lambda)$. This together with the argument above settles the case $d_X<d_Y$. □

3. Application to Fourier integral operators

Let X, Y be open sets in $\mathbb{R}^{d_X}$ and $\mathbb{R}^{d_Y}$, respectively. We consider operators of the form

$$S_\lambda f(x) = \iint e^{i\lambda\Psi(x,y,z)} b(x,y,z)\, dz\, f(y) dy \tag{3.1}$$

where $b \in C_0^\infty(\mathbb{R}^{d_X} \times \mathbb{R}^{d_Y} \times \mathbb{R}^N)$ and Ψ is a not necessarily homogeneous nondegenerate phase function in the sense that Ψ is C^∞ in a neighborhood of supp b and the gradients $\nabla_{x,y,z}\Psi'_{z_i}$, $i = 1, \dots, N$ are linearly independent if $N > 0$. We allow $N = 0$ to include operators of type (2.1); in this case the nondegeneracy condition is void. If $N > 0$ it implies that

$$\mathcal{C}_\Psi = \{(x, \Psi'_x; y, -\Psi'_y) : \Psi'_z = 0\} \tag{3.2}$$

is an immersed Lagrangian submanifold of $T^*X \times T^*Y$, *i.e.* a canonical relation.

We shall show that L^2 estimates for operators of type (2.1) can be reduced to L^2 estimates for operators of type (3.1). The same is true for $L^2 \to L^q$ estimates if one assumes that the projection $\pi^X : \mathcal{C} \to X$ is a submersion. We note that similar arguments come up in the calculus for Fourier integral operators [7], [9, vol.IV], and in fact one can develop a similar theory for operators with nonhomogeneous phase functions of type (3.1). Since we are not attempting to develop a calculus we prefer to give more elementary arguments using only linear canonical transformations. We begin with some simple facts from symplectic linear algebra.

Lemma 3.1. *Suppose that $\mathcal{C} \subset T^*X \times T^*Y$ is a canonical relation. Then for each $\rho \in \mathcal{C}$ the subspace $d\pi_L(T_\rho\mathcal{C})$ of $T_{\pi_L\rho}T^*X$ contains a Lagrangian subspace.*

Proof. We have to show that $V = d\pi_L(T_\rho\mathcal{C})$ is coisotropic with respect to the symplectic form $\sigma_X = d\xi \wedge dx$ on T^*X. Suppose $\sigma_X((\delta x, \delta\xi), (\delta x', \delta\xi')) = 0$ for all $(\delta x', \delta\xi,) \in V$. This implies that if $t' = (\delta x', \delta\xi', \delta y', \delta\eta')$ is a tangent vector in $T_\rho\mathcal{C}$ and $\sigma = \sigma_X - \sigma_Y$ then $\sigma((\delta x, \delta\xi, 0, 0), t') = 0$. Therefore the span of $T_\rho\mathcal{C}$ and $(\delta x, \delta\xi, 0, 0)$ is isotropic and since $T_\rho\mathcal{C}$ was already Lagrangian we see that $(\delta x, \delta\xi, 0, 0) \in T_\rho\mathcal{C}$ and therefore $(\delta x, \delta\xi) \in V$. $\square$

Lemma 3.2. *Suppose that $\mathcal{C} \subset T^*X \times T^*Y$ is a canonical relation and suppose that the projection $\pi^X : \mathcal{C} \to X$ is a submersion. Then*

$$\mathcal{C}^x = \{(y, \eta) \in T^*Y : (x, \xi; y, \eta) \in \mathcal{C} \text{ for some } \xi\}$$

is an immersed Lagrangian submanifold of T^*Y.

Proof. Since rank $d\pi^X = d_X$ we see that $\mathcal{N}^x = (\{x\} \times T^*_xX \times T^*Y) \cap \mathcal{C}$ is an isotropic d_X-dimensional immersed submanifold of $T^*X \times T^*Y$. We observe that the projection of $\mathcal{N}^x$ to T^*Y at a point ρ has injective differential. Indeed suppose that $(0, \delta\xi, 0, 0) \in T_\rho\mathcal{N}^x$. By our assumption on π^X we may find d_X tangent vectors $t^{(i)} = (\delta x_i, \dots)$ (with the δx_i being a basis of the tangent space to X at $\pi^X(\rho)$). If we apply $\sigma_X - \sigma_Y$ to the tangent vectors $t^{(i)}$ and to $(0, \delta\xi, 0, 0)$ we find $\langle \delta x_i, \delta\xi\rangle = 0$ for $i = 1, \dots, d_X$ and therefore $\delta\xi = 0$. We have shown that the intersection of the tangent spaces of $\mathcal{N}^x$ with (the tangent space of) $0 \times T^*_xX \times 0$ is $\{0\}$ and therefore $\mathcal{C}^x$ is an immersed manifold of T^*Y of dimension d_Y. σ_Y vanishes on $\mathcal{C}^x$ and hence $\mathcal{C}^x$ is Lagrangian. $\square$

We now consider operators of type (3.1).

Lemma 3.3. *Suppose that $\Psi_z'(x_0,y_0,z_0)\neq 0$ and $M\in\mathbb{N}$. Then there is a neighborhood W of (x_0,y_0,z_0) such that $\|S_\lambda\|_{L^p\to L^p}=O(\lambda^{-M})$ for all M and $1\le p\le\infty$ provided that b is supported in W.*

Proof. Let K_λ be the kernel of S_λ. Since $\Psi_z'\neq 0$ near (x_0,y_0,z_0) we may use integration by parts to see that $|K_\lambda(x,y)|\le C_M\lambda^{-M}$ provided that the support of b is contained in a small neighborhood of (x_0,y_0,z_0). $\square$

Proposition 3.4. *Let Ψ be nondegenerate and suppose that $\Psi_z'(x_0,y_0,z_0)=0$. Then there is a neighborhood W of (x_0,y_0,z_0) such that if b is supported in W we can write*

$$S_\lambda \;=\; \lambda^{d_X/2}G_\lambda V_\lambda \;+\; R_\lambda \tag{3.3}$$

and G_λ, V_λ, and R_λ are as follows: G_λ is an unitary operator on $L^2(\mathbb{R}^{d_X})$. The kernel of V_λ is given by

$$K_\lambda(x,y) \;=\; \int e^{i\lambda\phi(x,y,\vartheta)}\gamma(x,y,\vartheta)\,d\vartheta \tag{3.4}$$

where $\gamma\in C_0^\infty(\mathbb{R}^{d_X}\times\mathbb{R}^{d_Y}\times\mathbb{R}^{N+d_X})$ and ϕ is nondegenerate in a neighborhood of supp γ; moreover the projection π_X to X of the associated canonical relation $\mathcal{C}_\phi$ is a submersion. $\mathcal{C}_\phi$ is given by

$$\begin{aligned}\mathcal{C}_\phi \;&=\; \{(x,\phi_x;y,-\phi_y');\phi_\vartheta'=0\}\\ &=\; \{(x,\xi;y,\eta):(x,\xi)=\chi(w,\zeta),(w,\zeta;y,\eta)\in\mathcal{C}_\Psi\};\end{aligned} \tag{3.5}$$

here χ is a linear canonical transformation. Finally R_λ is bounded on L^p, $1\le p\le\infty$ with operator norm $O(\lambda^{-M})$.

Proof. Let $\rho_0=(x_0,\xi_0;y_0,\eta_0)=(x_0,\Psi_x'(x_0,y_0,z_0);y_0,-\Psi_y'(x_0,y_0,z_0))$. By Lemma 3.1 there is a Lagrangian subspace L_0 of $d\pi_L T_{\rho_0}\mathcal{C}_\Psi$. Consider the fiber $L_1=\{(0,\delta\xi)\}$ as a Lagrangian subspace of $T_{(x_0,\xi_0)}T^*X$. Then one can choose another Lagrangian subspace L_2 which is transversal to both L_0 and L_1 (see [9, p.289]). Therefore $L_2=\{(\delta x,A\delta x)\}$ for some symmetric A. Let $B_1,B_2\in Sp(d_X,\mathbb{R})$ be defined by

$$B_1=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix},\qquad B_2=\begin{pmatrix}I & 0\\ -A & I\end{pmatrix}$$

and consider $B_i(L_j)=\{v\in T_{(x_0,\xi_0)}T^*X : B_i^{-1}v\in L_j\}$. Then $B_2(L_2)=\{(\delta x,0)\}$ and $B_1B_2L_0$ is transversal to $B_1B_2(L_2)=L_1$; hence $B_1B_2L_0=\{(\delta x,A_0\delta x)\}$ for some A_0.

Next define

$$F_\lambda g(x) \;=\; \int e^{-i\lambda[\langle x,w\rangle+\frac12\langle Aw,w\rangle]}g(w)\,dw$$

Then the operator $(\lambda/2\pi)^{d_X/2}F_\lambda$ is a unitary operator on $L^2(\mathbb{R}^{d_X})$, by Plancherel's theorem.

Let $\chi(w,\zeta)=(-Aw+\zeta,-w)$; that is $\chi=B_1B_2$ if we consider B_1, B_2 as acting on T^*X. Let

$$(\tilde{x}_0,\tilde{\xi}_0)=\chi(x_0,\xi_0)=(-Ax_0+\Psi_x'(x_0,y_0,z_0),-x_0)$$

and let $\beta \in C_0^\infty(\mathbb{R}^{d_X})$ be equal to 1 in a neighborhood U of $\tilde{x}_0$. Let $G_\lambda = (\lambda/2\pi)^{-d_X/2} F_\lambda^{-1}$ and let

$$\begin{aligned} V_\lambda f(x) &= (2\pi)^{d_X/2}\beta(x) F_\lambda S_\lambda f(x) \\ \widetilde{R}_\lambda f(x) &= (2\pi)^{d_X/2}(1-\beta(x)) F_\lambda S_\lambda f(x) \end{aligned}$$

Then we have the decomposition (3.3) with $R_\lambda = \lambda^{d_X/2} G_\lambda \widetilde{R}_\lambda$.

The kernel of V_λ is (3.4) with $\vartheta = (w,z)$ and

$$\begin{aligned} \phi(x,y,(w,z)) &= -\langle x, w\rangle - \frac{1}{2}\langle Aw, w\rangle + \Psi(w,y,z) \\ \gamma(x,y,(w,z)) &= (2\pi)^{d_X/2}\beta(x) b(w,y,z) \end{aligned}$$

Then

$$\mathcal{C}_\phi = \{(x,-w;y,\Psi'_y(w,y,z)) : -x - Aw + \Psi'_w(w,y,z) = 0,\ \Psi'_z(w,y,z) = 0\}.$$

Hence $\mathcal{C}_\phi$ is given by (3.5) with $\chi = B_1B_2$. Let $\tilde{\rho}_0 = (\tilde{x}_0, \tilde{\xi}_0, y_0, \eta_0)$. Then the space $d\pi_L(T_{\tilde{\rho}_0}\mathcal{C}_\phi) \subset T_{(\tilde{x}_0,\tilde{\xi}_0)}T^*X$ contains the Lagrangian subspace $B_1B_2L_0$ and hence the differential of the projection $\pi^X : \mathcal{C}_\phi \to X$ is surjective at $\tilde{\rho}_0$. Therefore π^X is a submersion provided the support of b and β are small.

In order to complete the proof we have to show that the L^p-operator norm of $\widetilde{R}_\lambda$ is $O(\lambda^{-M})$, provided that the support of b is sufficiently close to (x_0, y_0, z_0). In order to see this we note that

$$\begin{aligned} \phi'_w(x,y,(w,z)) &= -x - Aw + \Psi'_w(w,y,z) \\ &= -(x-\tilde{x}_0) - \tilde{x}_0 - Ax_0 + \Psi'_w(x_0,y_0,z_0) + O(|w-x_0| + |y-y_0| + |z-z_0|) \\ &= -(x-\tilde{x}_0) + O(|w-x_0| + |y-y_0| + |z-z_0|). \end{aligned}$$

In view of Lemma 3.3 and the support properties of $1-\beta$ we get the required estimate for the kernel of $\widetilde{R}_\lambda$ provided that the support of b is sufficiently close to (x_0, y_0, z_0). $\square$

Proposition 3.5. *Let Ψ be nondegenerate and suppose that $\Psi'_z(x_0,y_0,z_0) = 0$. Let $\rho_0 = (x_0,\xi_0;y_0,\eta_0) = (x_0, \Psi'_x(x_0,y_0,z_0); y_0, -\Psi'_y(x_0,y_0,z_0))$. Suppose that near ρ_0 the projection $\pi_X : \mathcal{C}_\Psi \to X$ is a submersion. Then there is a neighborhood W of (x_0,y_0,z_0) such that if b is supported in W we can write*

$$S_\lambda = \lambda^{-N/2} T_\lambda G_\lambda + R_\lambda \tag{3.6}$$

where G_λ is a unitary operator on $L^2(\mathbb{R}^{d_X})$ and R_λ is bounded on L^p, $1 \le p \le \infty$ with operator norm $O(\lambda^{-M})$. T_λ is given by

$$T_\lambda f(x) = \int e^{i\lambda\Phi(x,y)} b_\lambda(x,y) f(y)\, dy \tag{3.7}$$

where b_λ belongs to a bounded set of functions in $C_0^\infty(X \times Y)$. The canonical relation (2.2) associated to T_λ can be written as

$$\mathcal{C}_\Phi = \{(x,\xi;y,\eta) : (y,\eta) = \chi(w,\zeta), (x,\xi;w,\zeta) \in \mathcal{C}_\Psi\};$$

where χ is a linear canonical transformation.

Proof. We use exactly the same reasoning as in the proof of Proposition 3.4 (this time working in T^*Y). Again we want to choose a Lagrangian subspace L_0 of $d\pi_R T_{\rho_0}\mathcal{C}$. By our assumption on π^X and Lemma 3.2 we may choose L_0 to be the tangent space of the Lagrangian manifold $\mathcal{C}_\phi^{x_0}$. Arguing as in the proof of Proposition 3.4 we may write

$$S_\lambda = \lambda^{d_Y/2} V_\lambda G_\lambda + R_\lambda$$

where the kernel of V_λ is given by (3.4) with $\vartheta = (w,z) \in \mathbb{R}^{d_Y} \times \mathbb{R}^N$ and $\gamma \in C_0^\infty(\mathbb{R}^{d_X} \times \mathbb{R}^{d_Y} \times \mathbb{R}^{N+d_Y})$. Moreover the projection of $\mathcal{C}_\phi^x$ onto Y has surjective differential at $(\tilde{y}_0, \tilde{\eta}_0) = \chi(y_0, \eta_0) = (y_0, \phi_y'(x_0, y_0, \vartheta_0))$ for x close to x_0. This means that that the projection $\mathcal{C}_\phi \to X \times Y$ has surjective differential at $(x_0, \xi_0, \tilde{y}_0, \tilde{\eta}_0)$. Since ϕ is nondegenerate this implies

$$\det \phi''_{\vartheta\vartheta}(x_0, y_0, \vartheta_0) \neq 0; \tag{3.8}$$

cf. [7, p.137] (note that (3.8) can never happen if ϕ is a homogeneous phase function). Now if the support of b is sufficiently small we may apply the method of stationary phase with respect to the ϑ-variables (analogous to the reduction of frequency variables in [7]) and obtain

$$V_\lambda = \lambda^{-(N+d_Y)/2} T_\lambda + R'_\lambda;$$

here T_λ is as in (3.7) and R'_λ is bounded on L^p with norm $O(\lambda^{-M})$. The canonical relations associated to T_λ and V_λ coincide. $\square$

An immediate consequence of Theorems 2.1 and 2.2 and Propositions 3.4 and 3.5 is

Theorem 3.6. *Let S_λ be as in (3.1) and suppose $d_Y \geq d_X$. Suppose that the projection $\pi_L : \mathcal{C}_\Psi \to T^*X$ is a submersion with folds. Then for $\lambda \geq 2$*

$$\begin{aligned}
\|S_\lambda f\|_{L^2(X)} &\leq C\lambda^{-\frac{N+d_X}{2}+\frac14} \|f\|_{L^2(Y)} && \text{if } d_Y = d_X\\
\|S_\lambda f\|_{L^2(X)} &\leq C\lambda^{-\frac{N+d_X}{2}} (\log\lambda)^{1/2} \|f\|_{L^2(Y)} && \text{if } d_Y = d_X + 1\\
\|S_\lambda f\|_{L^2(X)} &\leq C\lambda^{-\frac{N+d_X}{2}} \|f\|_{L^2(Y)} && \text{if } d_Y \geq d_X + 2.
\end{aligned}$$

Suppose in addition that the projection of $\mathcal{C}_\Psi$ to X is a submersion. Then

$$\|S_\lambda f\|_{L^q(X)} \leq C\lambda^{-\frac{N}{2}-\frac{d_X}{q}} \|f\|_{L^2(Y)} \tag{3.9}$$

provided that $4 \leq q \leq \infty$ if $d_X = d_Y$ and $2 < q \leq \infty$ if $d_Y \geq d_X + 1$. If one imposes the additional assumption that the projection of the fold hypersurface $\mathcal{L}$ to X is a submersion and that at least ℓ principal curvatures of the surfaces $\Sigma_x = \pi^{T_x^*X}\mathcal{C}_\Psi$ do not vanish (here $1 \leq \ell \leq d_X - 1$) then (3.9) holds for $(2\ell+4)/(\ell+1) \leq q \leq \infty$.

Remark. There is a more straightforward reduction to oscillatory integral operators in the case of averaging operators, given by $\mathcal{A}f(x) = \int_{\mathcal{M}_x} f(y)d\sigma_x(y)$. If $\mathcal{M}_x$ is parametrized by $y'' = S(x,y')$, $y' \in \mathbb{R}^k$, $y'' \in \mathbb{R}^{d-k}$ then one is led to consider S_λ with $\Psi(x,y,z) = \sum_{i=k+1}^d (y_i - S_i(x,y'))z_i$, $z \in \mathbb{R}^{d-k}$, and by an application of a partial Fourier transform in the y''-variables one reduces the study of S_λ to the study of T_λ with $\Phi(x,y) = \sum_{j=k+1}^d S_j(x,y'))y_j$; see Sogge and Stein [21].

We now apply Theorem 3.6 to the homogeneous case and use the following Lemma.

Lemma 3.7. *Let Ψ be a homogeneous nondegenerate phase function defined in $X \times Y \times \Gamma$ where Γ is an open cone in $\mathbb{R}^N \setminus 0$ and suppose that $\Psi'_x \neq 0$, $\Psi'_y \neq 0$ in Γ. Let U be an open subset of $X \times Y$ with compact closure and let Γ_0 be a subcone of Γ such that $\overline{\Gamma_0} \setminus 0 \subset \Gamma$. Let $\mathcal{F}$ be the Fourier integral operator*

$$\mathcal{F}f(x) = \iint e^{i\Psi(x,y,\theta)} a(x,y,\theta)\, d\theta\, f(y) dy$$

where $a(x,y,\theta)$ is a symbol of class $S^m(X \times Y \times \mathbb{R}^N)$ supported in $U \times \Gamma_0$. Let $\beta \in C_0^\infty(\mathbb{R})$ be such that $\beta(s) > 0$ if $1/\sqrt{2} \le |s| \le \sqrt{2}$ and $\beta(s) = 0$ if $|s| \notin (1/2, 2)$. For $\lambda > 0$ let

$$a_\lambda(x,y,\theta) = \beta(|\theta|/\lambda) a(x,y,\theta)$$

and let $\mathcal{F}^\lambda$ be similarly defined as $\mathcal{F}$ with a replaced by a_λ.

Suppose that $1 < p \le 2 \le q < \infty$ and that

$$\|\mathcal{F}^\lambda f\|_{L^q(\mathbb{R}^{d_X})} \le A \|f\|_{L^p(\mathbb{R}^{d_Y})}$$

for all $f \in L^p(\mathbb{R}^{d_Y})$, for all $\lambda > C_0$ (where C_0 is a fixed positive constant). Then $\mathcal{F}$ is bounded from $L^p(Y)$ to $L^q(X)$.

The proof is a well known application of Littlewood Paley theory and easy estimates for oscillatory integrals ([7, p.177]), based on the assumptions $\Psi'_x \neq 0$, $\Psi'_y \neq 0$. For details of this standard argument see [19].

Proof of Theorems 1.1-3. By conjugating $\mathcal{F}$ with pseudodifferential operators $(I - \Delta)^{\gamma/2}$ and standard calculations we see that the estimates involving Sobolev spaces follow from the L^2 or $L^2 \to L^q$ estimates. It suffices to consider $\mathcal{F}_\lambda$ as in Lemma 3.7. A change of variable shows that $\mathcal{F}^\lambda = \lambda^{m+N} S_\lambda$ where S_λ is as in Theorem 3.6. Now the asserted estimates follow easily from Theorem 3.6.

4. Application to restricted X-ray transforms

We now show how the previous results can be applied to obtain local estimates for restricted X-ray transforms on d-dimensional Riemannian or semi-Riemannian manifolds. We shall be interested in hypersurfaces in the $(2d-2)$-dimensional space $\mathcal{M}$ of geodesics in (M,g). Recall the following description of $\mathcal{M}$ (*cf.* [1], [3]). For $(x,\xi) \in T^*M \backslash 0$, let $\xi^\sharp \in T_xM$ be the corresponding tangent vector (so that $g(\xi^\sharp, v) = \langle \xi, v \rangle$ for all $v \in T_xM$.) To (x,ξ) we associate the geodesic $s \to \gamma_{x,\xi}(s) = \exp_x(s\xi^\sharp)$. There are two redundancies in this parametrization of all geodesics: dilation in ξ and translation along the geodesic flow; we take these into account by noting the (locally defined) action of $\mathbb{R}_+ \times \mathbb{R}$ on $T^*M \backslash 0$,

$$U_{(\rho,r)} \cdot (x,\xi) = \exp(rH_g)(x, \rho\xi),$$

where H_g is the Hamiltonian vector field of the metric $g(x,\xi)$. If $\sim$ is the resulting equivalence relation, and $(x',\xi') \sim (x,\xi)$, then $\gamma_{x',\xi'} = \gamma_{x,\xi}$ as sets. Thus, the (locally defined) space of unparametrized geodesics is $\mathcal{M} = (T^*M \backslash 0)/\sim$, which is $(2d-2)$-dimensional.

We consider a hypersurface $\mathfrak{C} \subset \mathcal{M}$ with the property that for each $y \in M$ the family of all geodesics in $\mathfrak{C}$ passing through y form a $d-2$-dimensional smooth submanifold $\mathfrak{C}_y$ of $\mathcal{M}_{1,d}$. $\mathfrak{C}$ can be locally specified by a defining function $f(x,\xi)$ on T^*M, homogeneous of some degree and and invariant under the Hamiltonian flow: $f(\exp sH_g(x,\xi)) = f(x,\xi)$. We may locally make a smooth choice of representative, $\mathfrak{C} \ni \gamma \to (x,\xi)$; in the Riemannian case it is customary to normalize $g(x,\xi) = 1$, but in the semi-Riemannian case this is not possible if there are null-geodesics in $\mathfrak{C}$. In any case for suitable cutoff-functions $\chi_1 \in C_0^\infty(\mathfrak{C})$, $\chi_2 \in C_0^\infty(M)$ with small support the restricted X-ray transform,

$$\mathcal{R}_{\mathfrak{C}}\phi(\gamma) = \chi_1(\gamma)\int \chi_2\phi(\exp_x(s\xi^\sharp))ds \quad \gamma \in \mathfrak{C},$$

is well defined. $\mathcal{R}_{\mathfrak{C}}$ is a generalized Radon transform in the sense of [6]. The Schwartz kernel of $\mathcal{R}_{\mathfrak{C}}$ is supported on the point-geodesic relation

$$\mathcal{Z}_{\mathfrak{C}} = \{(\gamma, y) \in \mathfrak{C} \times M : y \in \gamma\}.$$

$\mathcal{Z}_{\mathfrak{C}}$ is a smooth, $(2d-2)$-dimensional submanifold of $\mathfrak{C} \times M$; away from the critical points of $\pi_M : \mathcal{Z}_{\mathfrak{C}} \to M$, $K_{\mathcal{R}_{\mathfrak{C}}}(\gamma, y)$ is a smooth density on $\mathcal{Z}_{\mathfrak{C}}$, and thus $\mathcal{R}_{\mathfrak{C}}$ is a Fourier integral operator, $\mathcal{R}_{\mathfrak{C}} \in I^{-(d-1)/4}(\mathfrak{C}, M; N^*\mathcal{Z}_{\mathfrak{C}})$. It is assumed henceforth that we have localized away from any critical points.

We are going to impose a curvature assumption on $\mathfrak{C}$. For each $y \in M$ let $\mathfrak{c}_y$ be the cone in T_yM consisting of all lines tangent at y to a geodesic in $\mathfrak{C}_y$. Then $\mathfrak{c}_y = \{\xi^\sharp : f(y,\xi) = 0\}$. Following [3] we say that $\mathfrak{C}$ is *well-curved* if each cone $\mathfrak{c}_y$ has $d-2$ nonvanishing principal curvatures. In terms of the defining function f the well-curvedness of $\mathfrak{C}$ means that for all (x,ξ) with $f(x,\xi) = 0$ we have

$$\text{(4.1)} \qquad \operatorname{rank}\, d^2_{\xi\xi} f(x,\xi)\Big|_{d_\xi f^\perp} = d-2.$$

Proposition 4.1. *If M has no conjugate points and $\mathfrak{C} \subset M$ is a well-curved hypersurface then $\pi : \mathcal{N}^*\mathcal{Z}_{\mathfrak{C}} \to T^*M$ is a submersion with folds. If $\mathcal{L} \subset \mathcal{N}^*\mathcal{Z}_{\mathfrak{C}}$ is the fold surface, then the projection $\mathcal{L} \to M$ is a submersion and for each $y \in M$, $\Gamma_y = \pi(\mathcal{L}) \cap T^*_yM\backslash 0$ is an immersed conic hypersurface with $d-2$ nonvanishing principal curvatures.*

Corollary 4.2. *If $\mathfrak{C} \subset \mathcal{M}_{1,d}$ is as in Proposition 4.1, then $\mathcal{R}_{\mathfrak{C}} : L^2_{\alpha,\mathrm{comp}}(M) \to L^2_{\alpha+s_0,\mathrm{loc}}(\mathfrak{C})$ with $s_0 = 1/4$ if d=3, $s_0 = 1/2-\epsilon$ if $d = 4$ (any $\epsilon > 0$) and $s_0 = 1/2$ if $d \geq 5$. Furthermore $\mathcal{R}_{\mathfrak{C}} : L^p_{\mathrm{comp}}(M) \to L^q_{\mathrm{loc}}(\mathfrak{C})$ is bounded, provided $1 \leq p \leq \frac{2d}{d+1}$ and $q \leq \frac{dp-p}{d-p}$.*

Proof. Given Proposition 4.1, the first part follows immediately from Theorem 1.1 (by duality), and the second part follows from Theorem 1.2 if $d \geq 4$ and Theorem 1.3 if $d = 3$, and an interpolation with the easy $L^1 \to L^1$ estimate. □

Remarks.

1) The first part of Corollary 4.2 was conjectured in [3] and proved for *admissible* $\mathfrak{C} \subset M_{1,d}$ (the manifold of lines in $\mathbb{R}^d$). In this case the projection $\mathcal{C} \to T^*\mathfrak{C}$ has maximal degeneracy.

2) Corollary 4.2 applies in particular when (M,g) is a non-Riemannian, semi-Riemannian manifold and $\mathfrak{C}$ is the hypersurface of null geodesics in M. In this

case we take $f(x,\xi)$ to be the metric function $g(x,\xi)$; since this is a nonsingular quadratic form in ξ, it clearly satisfies the criterion of Proposition 4.1.

3) As shown in [2], [3] the L^2 estimates are sharp. The $L^p \to L^q$ estimates are sharp for $p \le 2d/(d+1)$ as one can see by testing $\mathcal{R}$ on characteristic functions of balls of small radius. However for $p > 2d/(d+1)$ the sharp $L^p \to L^q$ estimates are not known except for $d = 3$.

Proof of Proposition 4.1. It is convenient to work not with $\mathfrak{C}$, $\mathcal{Z}_{\mathfrak{C}}$ and $\mathcal{C} = (N^*\mathcal{Z}_{\mathfrak{C}})'$, but rather $\widetilde{\mathfrak{C}}$, $\widetilde{\mathcal{Z}} = \mathcal{Z}_{\widetilde{\mathfrak{C}}}$ and $\widetilde{\mathcal{C}}$, where

$$\begin{aligned}\widetilde{\mathfrak{C}} &= \{(x,\xi) \in T^*M\backslash 0 : f(x,\xi) = 0\}\\ \widetilde{\mathcal{Z}} &= \{(x,\xi;y) \in \widetilde{\mathfrak{C}} \times M : y \in \gamma_{x,\xi}\}\\ \widetilde{\mathcal{C}} &= (N^*\widetilde{\mathfrak{C}})' \subset T^*\widetilde{\mathfrak{C}} \setminus 0 \times T^*M \setminus 0.\end{aligned}$$

As described above, $\widetilde{\mathfrak{C}}$ has two redundant variables, since $\mathfrak{C} = \widetilde{\mathfrak{C}}/\sim$, where $\sim$ is the equivalence relation induced by the action $U_{(\rho,r)}$. The projection $\widetilde{\mathfrak{C}} \to \mathfrak{C}$ is a submersion (with two-dimensional fibers), and so is the projection $\widetilde{\mathcal{Z}} \to \mathcal{Z}_{\mathfrak{C}}$. Letting $\tilde{\pi}_M$ and $\tilde{\pi}_{T^*M}$ be the projections from $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{C}}$ into M and T^*M, respectively, we have that $\tilde{\pi}_M \circ U_{(\rho,r)} = \tilde{\pi}_M$ on $\widetilde{\mathcal{Z}}$ and so $\tilde{\pi}_{T^*M} \circ dU_{(\rho,r)} = \tilde{\pi}_{T^*M}$ on $\widetilde{\mathcal{C}}$. Thus, to show that the projection $\pi_{T^*M} : \mathcal{C} \to T^*M$ is a submersion with folds, it suffices to show that $\tilde{\pi}_{T^*M}$ is a submersion off a codimension $d-2$ submanifold $\widetilde{\mathcal{L}} \subset \widetilde{\mathfrak{C}}$; that $\tilde{\pi}_{T^*M}$ drops rank simply at $\widetilde{\mathcal{L}}$ (*i.e.*, some $2d \times 2d$ minor of $d\tilde{\pi}_{T^*M}$ vanishes to first order at $\widetilde{\mathcal{L}}$); and $\tilde{\pi}_{T^*M}|_{\widetilde{\mathcal{L}}}$ is a submersion, with $\mathrm{Ker}(d\tilde{\pi}_{T^*M}) \cap T\tilde{L}$ equaling the tangent space of the fibers of $\widetilde{\mathfrak{C}} \to \mathfrak{C}$.

Now parametrize $\widetilde{\mathcal{Z}}$ by

$$\widetilde{\mathcal{Z}} = \{(x,\xi; \exp_x(s\xi^\sharp)) : (x,\xi) \in \widetilde{\mathfrak{C}}, s \in I_{x,\xi}\},$$

where $I_{x,\xi} \subset \mathbb{R}$ is an open interval depending on (x,ξ). From [3; eqn. (2.15)], we have that $\widetilde{\mathcal{C}}$ is parametrized by $(x,\xi) \in \widetilde{\mathfrak{C}}$, $s \in I_{x,\xi}$, and $\eta \in \xi^\perp \subset T_x^*M$, with

$$\tilde{\pi}_{T^*M}(x,\xi,s,\eta) = (\exp_x(s\xi^\sharp), (D_v\mathrm{exp})^{*^{-1}}(\eta)),$$

where $D_v\mathrm{exp}$ is the derivative of the exponential map in the tangent vector variable. We now calculate the kernel of $d\tilde{\pi}_{T^*M}$ at $\rho = (x,\xi,s,\eta)$. Note first that for a tangent vector $(\delta x, \delta\xi, \delta s, \delta\eta) \in T_\rho\widetilde{\mathfrak{C}}$, we have

$$\langle \eta, \delta\xi^\sharp \rangle + \langle \xi^\sharp, \delta\eta \rangle = 0 \tag{4.2}$$

since $\langle \xi, \eta \rangle = 0$. Via the metric, we convert the defining function for $\widetilde{\mathfrak{C}}$ to a function on TM, which we denote by $f^\sharp$ (since this involves a fiberwise linear change of variables, it does not affect the criterion of Proposition 4.1). Since we assume that $f^\sharp$ is invariant under the geodesic flow, *i.e.*

$$f^\sharp(\exp_x(s\xi^\sharp), D_v\mathrm{exp}_x^*(s\xi^\sharp)) = f^\sharp(x,\xi^\sharp)$$

we obtain by differentiation

$$\langle d_{\xi^\sharp} f^\sharp + D_v\mathrm{exp}^* D_x\mathrm{exp}^{-1^*} d_x f^\sharp, \xi^\sharp \rangle = 0 \tag{4.3}$$

Since $f^\sharp(x,\xi^\sharp)=0$ on $\widetilde{\mathfrak{C}}$, we have

$$\langle d_x f^\sharp, \delta x\rangle + \langle d_{\xi^\sharp} f^\sharp, \delta\xi^\sharp\rangle = 0. \tag{4.4}$$

Now, if the tangent vector also belongs to $\mathrm{Ker}(d\tilde\pi_{T^*M})$, then

$$D_x\exp(\delta x) + sD_v\exp(\delta\xi^\sharp) + D_v\exp(\xi^\sharp\delta s) = 0 \tag{4.5}$$

and
(4.6)
$$-(D_v\exp)^{*^{-1}}(D^2_{vx}\exp)^*(D_v\exp)^{*^{-1}}(\eta,\delta x) + s(D_x\exp)^{-1}(D^2_{vv}\exp)(D_x\exp)^{-1}(\eta,\delta\xi^\sharp)$$
$$+(D_v\exp)^{*^{-1}}(D^2_{vv}\exp)^*(D_v\exp)^{*^{-1}}(\eta,\xi^\sharp)\delta s + (D_v\exp)^{*^{-1}}(\delta\eta) \;=\; 0.$$

Solving for δx in (4.5) and substituting in (4.4), we obtain

$$\langle d_{\xi^\sharp} f^\sharp - sD_v\exp^* D_x\exp^{-1^*} d_x f^\sharp, \delta\xi^\sharp\rangle - \langle D_v\exp^* D_x\exp^{-1^*} d_x f^\sharp, \xi^\sharp\rangle\delta s = 0. \tag{4.7}$$

For $\eta\wedge(d_{\xi^\sharp} f^\sharp - sD_v\exp^* D_x\exp^{-1^*} d_x f^\sharp)\neq 0$, the system of equations (4.2), (4.7), (4.5), (4.6) has rank $2d+2$ (acting on the full tangent space $T_{(x,\xi,s,\eta)}(T^*M\times\mathbb{R}\times T^*_xM)$, $d\tilde\pi_{T^*M}$ has a $(d-1)$-dimensional kernel, and thus $\tilde\pi_{T^*M}$ is a submersion there. Now let $\widetilde{\mathcal{L}}$ be the submanifold of $\widetilde{\mathcal{C}}$ given by the equation

$$\eta\wedge(d_{\xi^\sharp} f^\sharp - sD_v\exp^* D_x\exp^{-1^*} d_x f^\sharp) = 0.$$

Since $f^\sharp$ is homogeneous of some degree Euler's relation yields $\langle\xi^\sharp, d_{\xi^\sharp} f^\sharp\rangle = 0$ on $\widetilde{\mathfrak{C}}$ and from (4.3) we see that $d_{\xi^\sharp} f^\sharp - sD_v\exp^* D_x\exp^{-1^*} d_x f^\sharp$ belongs to ${\xi^\sharp}^\perp$. Since also $\eta\in{\xi^\sharp}^\perp$ it follows that $\widetilde{\mathcal{L}}\subset\widetilde{\mathcal{C}}$ is a submanifold of codimension $d-2$. Using (4.1) one checks that $d\tilde\pi_{T^*M}$ drops rank simply at $\widetilde{\mathcal{L}}$.

It remains to show that $d\tilde\pi_{T^*M}|_{T\widetilde{\mathcal{L}}}$ has a two-dimensional kernel (which must equal the tangent space of the fiber of $\widetilde{\mathcal{C}}\to\mathcal{C}$ since that is two-dimensional and in the kernel.) Again, since $\tilde L$ is defined by a collineation, we have a redundant family of defining functions: for each C^∞-section Ω of $\bigwedge^2 TM$ we have

$$h_\Omega(x,\xi,s,\eta) := \Omega(\eta\wedge(d_{\xi^\sharp} f^\sharp - sD_v\exp^* D_x\exp^{-1^*} d_x f^\sharp)) = 0.$$

Then, at $\tilde L$ and $s=0$ (which we can always assume : given x, we can pick all the representatives of the geodesics through x to be of the form $\gamma_{x,\xi}$), the derivative of h_Ω is given by

$$dh_\Omega(\delta x,\delta\xi^\sharp,\delta s,\delta\eta) = d^2_{\xi^\sharp x} f^\sharp(\eta\lrcorner\Omega,\delta x) + d^2_{\xi^\sharp\xi^\sharp} f^\sharp(\eta\lrcorner\Omega,\delta\xi^\sharp)$$
$$-(\Omega(\eta\wedge D_v\exp^* D_x\exp^{-1^*} d_x f^\sharp))\delta s - \langle d_{\xi^\sharp} f^\sharp\lrcorner\Omega,\delta\eta\rangle;$$

As Ω_x ranges over $\bigwedge^2 T_xM$, $v=\eta\lrcorner\Omega$ ranges over $\eta^\perp=(d_{\xi^\sharp} f^\sharp)^\perp$ (this last equality holds since $\eta\wedge d_{\xi^\sharp} f^\sharp = 0$ at $\widetilde{\mathcal{L}}$.) Using (4.3), (4.4), the equation $dh_\Omega=0$ (all evaluated at $\rho=(x,\xi,0,\eta)$) and the H_g invariance of f, one sees after a short calculation that $\mathrm{Ker}(d\tilde\pi_{T^*M})\cap T_\rho\widetilde{\mathcal{L}}$ is given by the system of equations

$$d^2_{\xi^\sharp\xi^\sharp} f^\sharp(v,\delta\xi^\sharp) = l(\delta s),\quad \text{all } v\in(d_{\xi^\sharp} f^\sharp)^\perp, \tag{4.8}$$

where l is a linear mapping. Since $d^2_{\xi\xi} f$ has rank $d-2$ on $d_\xi f^\perp$, $d^2_{\xi^\sharp\xi^\sharp} f^\sharp$ has rank $d-2$ on $(d_{\xi^\sharp} f^\sharp)^\perp$, and thus (4.8) has a two-dimensional space of solutions, finishing the proof that π_{T^*M} is a submersion with folds.

It is clear from the definition of $\mathcal{L}$ that the projection $\mathcal{L} \to M$ is a submersion. Finally, each cone $\Gamma_{y_0} = \pi_{T^*M}(\mathcal{L}) \cap T^*_{y_0} M$ is parametrized by

$$\{\xi^\sharp : f^\sharp(y_0, \xi^\sharp) = 0\} \to \{(y_0, d_{\xi^\sharp} f^\sharp(y_0, \xi^\sharp)\},$$

and thus has $d-2$ nonvanishing principal curvatures. □

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