A WEAK TYPE BOUND FOR A SINGULAR INTEGRAL

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Abstract. A weak type $(1,1)$ estimate is established for the first order d-commutator introduced by Christ and Journé, in dimension $d \ge 2$.

1. Introduction

Let K be regular Calderón-Zygmund convolution kernel on $\mathbb{R}^d$, $d \ge 2$, *i.e.* $K \in \mathcal{S}'$, locally bounded in $\mathbb{R}^d \setminus \{0\}$ and satisfies

$$|K(x)| \le A|x|^{-d} \quad x \ne 0, \tag{1.1}$$

and, for some $\varepsilon \in (0,1]$,

$$|K(x+h) - K(x)| \le A|h|^{\varepsilon}|x|^{-d-\varepsilon} \text{ if } |x| > 2|h|; \tag{1.2}$$

moreover

$$\|\widehat{K}\|_\infty \le A < \infty.$$

Let $a \in L^\infty(\mathbb{R}^d)$. The so-called *$d$-commutator* $T \equiv T[a]$ of first order associated with K and a is defined for Schwartz functions f by

$$T[a]f(x) = p.v. \int K(x-y) \int_0^1 a(sx + (1-s)y) ds\, f(y) dy\,.$$

In dimensions $d \ge 2$ this definition yields a rough analog of the Calderón commutator [1] in one dimension. Christ and Journé [3] proved that T and higher order versions extend to bounded operators on $L^p(\mathbb{R}^d)$, for $1 < p < \infty$. We prove that the first order d-commutator is also of weak type $(1,1)$.

Theorem 1.1. *There is $C_d < \infty$ so that for any $f \in L^1(\mathbb{R}^d)$ and any $a \in L^\infty(\mathbb{R}^d)$,*

$$\sup_{\lambda > 0} \lambda \,\mathrm{meas}\big(\{x \in \mathbb{R}^d : |T[a]f(x)| > \lambda\}\big) \le C_d A \tfrac{1}{\varepsilon} \log(\tfrac{2}{\varepsilon}) \|a\|_\infty \|f\|_{L^1(\mathbb{R}^d)}\,.$$

In two dimensions this result has recently been established by Grafakos and Honzík [6] (assuming $\varepsilon = 1$). Their approach relies on a method developed in [2], [4] and [7] for proving a weak type $(1,1)$ bound for rough singular convolution operators. A dyadic decomposition $T[a] = \sum T_j$ is used on the

1991 *Mathematics Subject Classification.* 42B15.
Supported in part by NSF grant 1200261.

kernel side, and the argument relies on the fact that in two dimensions the kernels of the operators $T_j^*T_i$ have certain Hölder continuity properties. This argument is no longer valid in higher dimensions. It is conceivable that for $d \geq 3$ one might be able to develop the more complicated iterated T^*T arguments introduced by Christ and Rubio de Francia [4] and further extended by Tao [11], but this route would lead to substantial technical difficulties and we shall not pursue it. Our approach is different and relies on an idea introduced in [8]. An orthogonality argument for a microlocal decomposition of the operator is used. The implementation of this idea in the present setting is more complicated in the convolution case as the Christ-Journé operators can be viewed as an amalgam of operators of generalized convolution type (for which there is a suitable calculus of wavefront sets) and operators of multiplication with a rough function.

Notation. We write $\mathcal{E}_1 \lesssim \mathcal{E}_2$ to indicate that $\mathcal{E}_1 \leq C_d \mathcal{E}_2$ for some 'constant' C that may depend on d. We also use the notation $\lesssim_N$ to indicate dependence on other parameters N. We denote by $\widehat{f}$ or $\mathcal{F}f$ the Fourier transform of f, defined for Schwartz functions by $\widehat{f}(\xi) = \int f(y)e^{-i\langle y,\xi\rangle}dy$.

This paper. In §2 we outline the proof of Theorem 1.1 with three technical propositions 2.2, 2.3, 2.4 proved in §3, §4, §5, respectively. In §6 we shall mention some open problems.

2. Decompositions and auxiliary estimates

We may assume that $A \leq 1$, $\|a\|_\infty \leq 1$ and write $T = T[a]$. Fix $f \in L^1(\mathbb{R}^d)$. We use the standard Calderón-Zygmund decomposition of f at height λ (see [10]). Then

$$f = g + b = g + \sum_{Q \in \mathfrak{Q}_\lambda} b_Q$$

where $\|g\|_\infty \leq \lambda$, $\|g\|_1 \lesssim \|f\|_1$, each b_Q is supported in a dyadic cube Q with sidelength $2^{L(Q)}$ and center y_Q, and $\mathfrak{Q}_\lambda$ is a family of dyadic cubes with disjoint interiors. Moreover $\|b_Q\|_1 \lesssim \lambda|Q|$ for each $Q \in \mathfrak{Q}_\lambda$ and $\sum_{Q\in\mathfrak{Q}_\lambda} |Q| \lesssim \lambda^{-1}\|f\|_1$. For each Q let Q^* be the dilate of Q with same center and $L(Q^*) = L(Q) + 10$, and let $E = \bigcup_{Q\in\mathfrak{Q}_\lambda} Q^*$. Then also

$$\text{meas}(E) \lesssim \lambda^{-1}\|f\|_1.$$

Finally, for each Q, the mean value of b_Q vanishes:

$$\int b_Q(y)dy = 0.$$

Since T is bounded on L^2 ([3]) we have, as in standard Calderón-Zygmund theory, the estimate for the good function g

$$\|Tg\|_2^2 \le \|T\|_{L^2\to L^2}^2 \|g\|_2^2 \lesssim \|g\|_1 \|g\|_\infty \lesssim \lambda \|g\|_1$$

and by Tshebyshev's inequality,

$$\big|\{x\in\mathbb{R}^d : |Tg(x)| > \lambda/10\}\big| \;\le\; 100\lambda^{-2}\|Tg\|_2^2 \lesssim \lambda^{-1}\|g\|_1 \;\lesssim\; \lambda^{-1}\|f\|_1.$$

We use a dyadic decomposition of the kernel. Let φ be a radial C^∞ function, so that $\varphi(x)=1$ for $|x|\le 1$ and $\varphi(x)=0$ for $|x|\ge 6/5$. Let

$$K_j(x) = \big(\varphi(2^{-j}x) - \varphi(2^{-j+1}x)\big)K(x)$$

so that $K=\sum K_j$ in the sense of distributions on $\mathbb{R}^d\setminus\{0\}$ and K_j is supported in the annulus $\{x : 2^{j-1}\le |x| \le \frac{6}{5}2^j\}$. Let T_j be the integral operator with Schwartz kernel

$$K_j(x-y)\int_0^1 a(sx+(1-s)y)\,ds\,.$$

For $m\in\mathbb{Z}$ let

$$B_m = \sum_{\substack{Q\in\mathfrak{Q}_\lambda\\ L(Q)=m}} b_Q.$$

Observe that for each j, m the function T_jB_m belongs to L^1, and that

$$\mathrm{supp}(T_jB_m)\subset E, \quad m\ge j.$$

Moreover, for each n,

$$\sum_j \|T_jB_{j-n}\|_1 \lesssim \|f\|_1$$

and thus, if

$$n(\varepsilon) = 10^{10} d\varepsilon^{-1}\log_2(2\varepsilon^{-1})$$

we have by Tshebyshev's inequality

$$\text{(2.1)}\quad \mathrm{meas}\Big(\Big\{x\in\mathbb{R}^d : \sum_{0<n\le n(\varepsilon)}\sum_j |T_jB_{j-n}(x)| > \lambda/10\Big\}\Big) \lesssim \varepsilon^{-1}\log(2\varepsilon^{-1})\lambda^{-1}\|f\|_1.$$

It thus suffices to show that $\sum_{n>n(\varepsilon)}(\sum_j T_jB_{j-n})$ converges in the topology of $(L^1+L^2)(\mathbb{R}^d\setminus E)$ and satisfies the inequality

$$\text{(2.2)}\quad \mathrm{meas}\Big(\Big\{x\in\mathbb{R}^d\setminus E : \sum_{n>n(\varepsilon)}\big|\sum_j T_jB_{j-n}(x)\big| > 4\lambda/5\Big\}\Big) \lesssim \lambda^{-1}\|f\|_1$$

Finer decompositions. We first slightly modify the kernel K_j and subtract an acceptable error term which is small in L^1. In what follows assume $n > n(\varepsilon)$ as defined above. Let

$$\begin{aligned}\ell(n) &= [2\log_2(n)] + 2\\ \ell_\varepsilon(n) &= [2\varepsilon^{-1}\log_2 n] + 2.\end{aligned} \tag{2.3}$$

Let Φ be a radial C_0^∞ function supported in $\{|x| \le 1\}$, and satisfying $\int \Phi(x)dx = 1$. Let $\Phi_m(x) = 2^{-md}\Phi(2^{-m}x)$. Define

$$K_j^n = K_j * \Phi_{j-\ell_\varepsilon(n)}\,.$$

Then K_j^n is supported in $\{x : 2^{j-2} \le |x| \le 2^{j+2}\}$, and, by the regularity assumption (1.2),

$$\begin{aligned}\|K_j - K_j^n\|_1 &\lesssim 2^{-(j-\ell_\varepsilon(n))d} \iint\limits_{\substack{|h|\le 2^{-(j-1-\ell_\varepsilon(n))}\\ 2^{j-2}\le|x|\le 2^{j+2}}} |K_j(x) - K_j(x-h)|\,dx\,dh\\ &\lesssim 2^{-\ell_\varepsilon(n)\varepsilon} \lesssim n^{-2}\,.\end{aligned} \tag{2.4}$$

By differentiation and (1.1)

$$|\partial^\alpha K_j^n(x)| \le C_\alpha 2^{-jd} 2^{(\ell_\varepsilon(n)-j)|\alpha|}\,. \tag{2.5}$$

Let $\vartheta_n \in C^\infty(\mathbb{R})$ be supported in $(n^{-2}, 1-n^{-2})$, such that $\vartheta_n(s) = 1$ for $s \in [2n^{-2}, 1-2n^{-2}]$, and such that the derivatives of ϑ_n satisfy the natural estimates

$$\|\vartheta_n^{(N)}\|_\infty \le C_N n^{2N}\,. \tag{2.6}$$

We then let T_j^n be the integral operator with Schwartz kernel

$$K_j^n(x-y)\int \vartheta_n(s)a(sx+(1-s)y)\,ds\,.$$

The following lemma is an immediate consequence of estimate (2.4) and the support property of ϑ_n.

Lemma 2.1. *The operator $T_j - T_j^n$ is bounded on L^1, with operator norm*

$$\|T_j - T_j^n\|_{L^1\to L^1} \lesssim n^{-2}\,.$$

The lemma implies

$$\begin{aligned}&\text{meas}\Big(\Big\{x : \sum_{n>n(\varepsilon)} \Big|\sum_j (T_jB_{j-n}(x) - T_j^nB_{j-n}(x))\Big| > \lambda/10\Big\}\Big)\\ &\le 10\lambda^{-1}\Big\|\sum_{n>n(\varepsilon)}\sum_j |T_jB_{j-n} - T_j^nB_{j-n}|\Big\|_1\\ &\lesssim \lambda^{-1}\sum_{n\ge1} n^{-2}\sum_j \|B_{j-n}\|_1 \lesssim \lambda^{-1}\|f\|_1\end{aligned}$$

and therefore it is enough to show

$$\text{meas}\big(\big\{x: \sum_{n>n(\varepsilon)}\sum_j |T_j^n B_{j-n}(x)| > \tfrac{7}{10}\lambda\big\}\big) \lesssim \lambda^{-1}\|f\|_1 . \tag{2.7}$$

For the proof of (2.7) we subtract various regular or small terms from the operators T_j^n. Let $\ell(n)$ be as in (2.3) and denote by P_m the convolution operator with convolution kernel Φ_m (defined following (2.3)). We have

Proposition 2.2. *For $n > 1$,*

$$\|P_{j-n+\ell(n)}T_j^n B_{j-n}\|_1 \lesssim n^{-2}\log n\|B_{j-n}\|_1 .$$

The proposition will be proved in §3. It yields

$$\begin{aligned}
&\text{meas}\big(\big\{x\in\mathbb{R}^d\setminus E: \sum_{n>n(\varepsilon)}\big|\sum_j P_{j-n+\ell(n)}T_j^n B_{j-n}(x))\big| > \lambda/10\big\}\big)\\
&\lesssim 10\lambda^{-1}\sum_{n>n(\varepsilon)}\sum_j \|P_{j-n+\ell(n)}T_j^n B_{j-n}\|_1\\
&\lesssim \lambda^{-1}\sum_{n>1} n^{-2}\log n\sum_j \|B_{j-n}\|_1 \lesssim \lambda^{-1}\|f\|_1
\end{aligned}$$

and thus it remains to consider the term

$$\sum_{n>n(\varepsilon)}\sum_j (I-P_{j-n+\ell(n)})T_j^n B_{j-n}(x) \tag{2.8}$$

and to estimate the measure of the set where $|(2.8)| > 3\lambda/5$. We shall need to exploit the fact that the integral $\int_0^1 a(sx+(1-s)y)ds$ smoothes the rough function a in the direction parallel to $x-y$, and use a microlocal decomposition which we now describe.

Let $1/10 < \gamma < 9/10$ (say $\gamma = 1/2$), and let Θ_n be set of unit vectors with the property that if $\nu \neq \nu'$, $\nu, \nu' \in \Theta_n$ then $|\nu-\nu'| \geq 2^{-4-n\gamma}$, and assume that Θ_n is *maximal* with respect to this property. Note that

$$\text{card}(\Theta_n) \lesssim 2^{n\gamma(d-1)} .$$

For each ν we may choose a function $\widetilde{\chi}_{n,\nu}$ on $C^\infty(S^{d-1})$ with the property that $\widetilde{\chi}_{n,\nu}(x) \geq 0$, $\widetilde{\chi}_{n,\nu}(\theta) = 1$ if $|\theta-\nu| \leq 2^{-3-n\gamma}$, $\widetilde{\chi}_{n,\nu}(\theta) = 0$ if $|\theta-\nu| > 2^{-2-n\gamma}$, and such that for each $M \in \mathbb{N}$ the functions $2^{-n\gamma M}\widetilde{\chi}_{n,\nu}$ form a bounded family in $C^M(S^{d-1})$. For each θ there is at least one ν such that $\widetilde{\chi}_{n,\nu}(\theta) = 1$, by the maximality assumption, moreover by the separatedness assumption the number of $\nu \in \Theta_n$ for which $\widetilde{\chi}_{n,\nu}(\theta) \neq 0$ is bounded above, uniformly in θ and n. Define, for $\nu \in \Theta_n$

$$\chi_{n,\nu}(x) = \frac{\widetilde{\chi}_{n,\nu}(\frac{x}{|x|})}{\sum_{\nu'\in\Theta_n}\widetilde{\chi}_{n,\nu'}(\frac{x}{|x|})} .$$

Then $\sum_{\nu\in\Theta_n}\chi_{n,\nu}(x)=1$ for every $x\in\mathbb{R}^d\setminus\{0\}$) and by homogeneity we have the following estimates for multiindices α and $x\neq 0$,

$$|(\langle\nu,\nabla\rangle)^M\chi_{n,\nu}(x)|\le C_M|x|^{-M}\,,$$
$$|\partial^\alpha\chi_{n,\nu}(x)|\,\le\,C_\alpha 2^{n\gamma|\alpha|}|x|^{-|\alpha|}\,.$$

Let $K_j^{n,\nu}(x)=K_j^n(x)\chi_{n,\nu}(x)$ and let $T_j^{n,\nu}$ be the operator with Schwartz kernel

$$K_j^{n,\nu}(x-y)\int\vartheta_n(s)\,a(sx+(1-s)y)\,ds\,.$$

We then have

$$T_j^n=\sum_{\nu\in\Theta_n}T_j^{n,\nu}\,.$$

Let $\phi\in C^\infty(\mathbb{R})$ so that $\phi(u)=1$ for $|u|<1/2$ and $\phi(u)=0$ for $|u|\ge 1$ and define the singular convolution operator $\mathfrak{S}_{n,\nu}$ by

$$\widehat{\mathfrak{S}_{n,\nu}f}(\xi)=\phi\big(2^{n\gamma}n^{-5}\langle\nu,\tfrac{\xi}{|\xi|}\rangle\big)\widehat{f}(\xi).$$

The terms involving $(I-\mathfrak{S}_{n,\nu})T_j^{n,\nu}$ can be dealt with by L^1 estimates. In §4 we shall prove

Proposition 2.3. *For $n>n(\varepsilon)$, $\nu\in\Theta_n$,*

$$\Big\|\sum_j(I-P_{j-n+\ell(n)})(I-\mathfrak{S}_{n,\nu})T_j^{n,\nu}B_{j-n}\Big\|_1\lesssim n^{-2}2^{-n\gamma(d-1)}\|f\|_1\,.$$

For the rougher terms involving $\mathfrak{S}_{n,\nu}T_j^{n,\nu}$ we shall prove in §5 the following L^2 estimate.

Proposition 2.4. *For $n>n(\varepsilon)$,*

$$\Big\|\sum_{\nu\in\Theta_n}\sum_j(I-P_{j-n+\ell(n)})\mathfrak{S}_{n,\nu}T_j^{n,\nu}B_{j-n}\Big\|_2^2\lesssim 2^{-n\gamma}n^5\lambda\|f\|_1\,.$$

Given the propositions we can finish the outline of the proof of Theorem 1.1. Namely by Tshebyshev's inequality,

$$\begin{aligned}&\operatorname{meas}\big(\{x:\big|\sum_{n>n(\varepsilon)}\sum_j(I-P_{j-n+\ell(n)})T_j^nB_{j-n}(x)\big|>\frac{3}{5}\lambda\}\big)\\&\quad\lesssim 5\lambda^{-1}\Big\|\sum_{n>n(\varepsilon)}\sum_{\nu\in\Theta_n}\sum_j(I-P_{j-n+\ell(n)})(I-\mathfrak{S}_{n,\nu})T_j^{n,\nu}B_{j-n}\Big\|_1\\&\quad+25\lambda^{-2}\Big\|\sum_{n>n(\varepsilon)}\sum_{\nu\in\Theta_n}\sum_j(I-P_{j-n+\ell(n)})\mathfrak{S}_{n,\nu}T_j^{n,\nu}B_{j-n}\Big\|_2^2\end{aligned}$$

and by the propositions and Minkowski's inequality this is bounded by a constant times

$$\lambda^{-1}\|f\|_1\Big(\sum_n n^{-2}2^{-n\gamma(d-1)}\mathrm{card}(\Theta_n)+\sum_n 2^{-n\gamma}n^5\Big)\lesssim\lambda^{-1}\|f\|_1\,.$$

3. Proof of Proposition 2.2

Let $Q\in\mathfrak{Q}_\lambda$ with $L(Q)=j-n$. We apply Fubini's theorem and write

$$P_{j-n+\ell(n)}T_j^n b_Q(x) = \int\vartheta_n(s)\int b_Q(y)\times \Big[\int\Phi_{j-n+\ell(n)}(x-w)K_j^n(w-y)a(sw+(1-s)y)\,dw\Big]\,dy\,ds\,.$$

Changing variables $z=w+\frac{1-s}{s}y$ we get

$$P_{j-n+\ell(n)}T_j^n b_Q(x)=\int\vartheta_n(s)\int a(sz)\int\mathcal{A}_{j,n}^{x,z,s}(y)b_Q(y)\,dy\,dz\,ds$$

where

$$\mathcal{A}_{j,n}^{x,z,s}(y)=\Phi_{j-n+\ell(n)}(x-z+\tfrac{1-s}{s}y)K_j^n(z-\tfrac{y}{s}).$$

We expand $\mathcal{A}_{j,n}^{x,z,s}(y)$ about the center y_Q of Q and in view of the cancellation of b_Q we may write

$$|P_{j-n+\ell(n)}T_j^n b_Q(x)| \le \iint|\vartheta_n(s)a(sz)|\Big|\int\big(\mathcal{A}_{j,n}^{x,z,s}(y)-\mathcal{A}_{j,n}^{x,z,s}(y_Q)\big)b_Q(y)\,dy\Big|dz\,ds\,.$$

Using

$$\mathcal{A}_{j,n}^{x,z,s}(y)-\mathcal{A}_{j,n}^{x,z,s}(y_Q) = \Big\langle y-y_Q,\int_0^1\nabla\mathcal{A}_{j,n}^{x,z,s}(y_Q+\sigma(y-y_Q))\,d\sigma\Big\rangle$$

in the previous display one obtains after applying Fubini's theorem

$$\|P_{j-n+\ell(n)}T_j^n b_Q(x)\|_1\le diam(Q)\int_0^1\int|\vartheta_n(s)|\times \Big[\|\nabla\Phi_{j-n+\ell(n)}\|_1\frac{1-s}{s}\int|b_Q(y)|\int\big|K_j^n(z-\tfrac{y_Q+\sigma(y-y_Q)}{s})\big|\,dz\,dy + \|\Phi_{j-n+\ell(n)}\|_1\int|b_Q(y)|\int\frac{1}{s}\big|\nabla K_j^n(z-\tfrac{y_Q+\sigma(y-y_Q)}{s})\big|\,dz\,dy\Big]\,ds\,d\sigma\,.$$

Now use $\|\nabla K_j^n\|_1\lesssim 2^{-j+\ell_\varepsilon(n)}$ and $\int_0^1|\vartheta_n(s)|s^{-1}ds\lesssim\log n$, and since $diam(Q)\lesssim 2^{j-n}$ we obtain

$$\begin{aligned}\big\|P_{j-n+\ell(n)}T_j^n b_Q\big\|_1&\lesssim\log n\,\big[2^{-\ell(n)}+2^{\ell_\varepsilon(n)-n}\big]\|b_Q\|_1\\&\lesssim n^{-2}\log n\,\|b_Q\|_1.\end{aligned}$$

Finally we sum over all $Q \in \mathfrak{Q}_\lambda$ with $L(Q) = j - n$ to obtain the asserted bound. □

4. Proof of Proposition 2.3

Let $Q \in \mathfrak{Q}_\lambda$ with $L(Q) = j - n$, and let y_Q be the center of Q. Fix a unit vector ν, and let $\pi_\nu^\perp$ be the projection to the orthogonal complement of ν, i.e. $\pi_\nu^\perp(x) = x - \langle x, \nu\rangle\nu$. In view of the support properties of the kernel it suffices to show that for $n > n(\varepsilon)$

$$\Big\|(I - P_{j-n+\ell(n)})(I - \mathfrak{S}_{n,\nu})T_j^{n,\nu} b_Q\Big\|_1 \lesssim n^{-2}2^{-n\gamma(d-1)}\|b_Q\|_1 \,, \tag{4.1}$$

under the additional assumption that the support of a is contained in

$$\big\{y : |\langle y - y_Q, \nu\rangle| \le 2^{j+4}d \,,\, |\pi_\nu^\perp(y - y_Q)| \le 2^{j+4-n\gamma}d \,\big\} \,.$$

Note that with this hypothesis

$$\|\widehat{a}\|_\infty \lesssim 2^{jd-n\gamma(d-1)} \,. \tag{4.2}$$

We introduce a frequency decomposition of a. Let φ be a radial C^∞ function as in §2, but now defined in ξ-space, so that $\varphi(\xi) = 1$ for $|\xi| \le 1$ and $\varphi(\xi) = 0$ for $|\xi| \ge 6/5$. Define $\beta_k(\xi) = \varphi(2^k\xi) - \varphi(2^{k+1}\xi)$; then β_k is supported in $\{\xi : 2^{-k-1} \le |\xi| \le \frac{6}{5}2^{-k}\}$. Let $\widetilde{\beta}$ be a radial C^∞ function so that $\widetilde{\beta}$ is supported in $\{\xi : 1/3 \le |\xi| \le 3/2\}$ and $\widetilde{\beta}(\xi) = 1$ for $1/2 \le |\xi| \le 6/5$, and define $\widetilde{\beta}_k(\xi) = \widetilde{\beta}(2^k\xi)$. Then $\beta_k\widetilde{\beta}_k = \beta_k$. Define convolution operators V_k, Λ_k, $\widetilde{\Lambda}_k$ with Fourier multipliers $\varphi(2^k\cdot)$, β_k, $\widetilde{\beta}_k$, respectively; then $\Lambda_k\widetilde{\Lambda}_k = \Lambda_k$ and, for every $m \in \mathbb{Z}$, the identity operator is decomposed as $I = V_m + \sum_{k<m}\Lambda_k$.

For fixed $y \in Q$ we define an operator $\mathcal{K}_{j,y}^{n,\nu}$ acting on a by

$$\mathcal{K}_{j,y}^{n,\nu}[a](x) = K_j^{n,\nu}(x - y)\int \vartheta_n(s)a(sx + (1-s)y)ds$$

so that

$$T_j^{n,\nu} b_Q(x) = \int b_Q(y)\mathcal{K}_{j,y}^{n,\nu}[a](x)\,dy \,. \tag{4.3}$$

We use dyadic frequency decompositions and split

$$(I - \mathfrak{S}_{n,\nu})(I - P_{j-n+\ell(n)})T_j^{n,\nu} b_Q = \sum_{k_1}\Lambda_{k_1}(I - \mathfrak{S}_{n,\nu})\widetilde{\Lambda}_{k_1}(I - P_{j-n+\ell(n)})\int b_Q(y)\mathcal{K}_{j,y}^{n,\nu}[a]\,dy \tag{4.4}$$

and then further split in (4.4)

$$a = V_{j-n+\ell(n)}a + \sum_{k_2<j-n+\ell(n)} \Lambda_{k_2} a\,. \tag{4.5}$$

We prove three lemmata with various bounds for the terms in (4.4), (4.5).

Lemma 4.1.

$$\Big\| \int b_Q(y) \mathcal{K}^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a]\,dy \Big\|_1 \lesssim n^{-2} 2^{-n\gamma(d-1)} \|b_Q\|_1\,.$$

Proof. We use the cancellation of b_Q to estimate the left-hand side by

$$\int |b_Q(y)| \int |\mathcal{K}^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a](x) - \mathcal{K}^{n,\nu}_{j,y_Q}[V_{j-n+\ell(n)}a](x)|\,dx\,dy\,.$$

For $y \in Q$ we may estimate

$$\int |\mathcal{K}^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a](x) - \mathcal{K}^{n,\nu}_{j,y_Q}[V_{j-n+\ell(n)}a](x)|\,dx \le \mathcal{E}_1(y) + \mathcal{E}_2(y)$$

where

$$\mathcal{E}_1(y) = \|V_{j-n+\ell(n)}a\|_\infty \int |K^{n,\nu}_j(x-y) - K^{n,\nu}_j(x-y_Q)|\,dx$$

and, abbreviating

$$\Gamma^Q_{j-n+\ell(n)}(x,y,z) = \int_0^1 \big\langle y-y_Q, \nabla\mathcal{F}[\varphi(2^{j-m+\ell(n)}\cdot)](sx+(1-s)(y_Q+\sigma(y-y_Q))-z)\big\rangle\,d\sigma,$$

$\mathcal{E}_2$ is given by

$$\mathcal{E}_2(y) = \int |K^{n,\nu}_j(x-y_Q)| \int |\vartheta_n(s)| \int |a(z)|\,|\Gamma^Q_{j-n+\ell(n)}(x,y,z)|\,dz\,ds\,dx.$$

Now by (2.5), and since $|\partial_x \chi_{n,\nu}(x)| \lesssim 2^{n\gamma}|x|^{-1}$ we get

$$|\mathcal{E}_1(y)| \le |y-y_Q| \|\nabla K^{n,\nu}_j\|_1 \lesssim 2^{j-n}[2^{\ell_\varepsilon(n)-j} + 2^{n\gamma-j}]2^{-n\gamma(d-1)}.$$

Notice that for $n > n(\varepsilon)$ and $\gamma > 1/10$ we have $2^{\ell_\varepsilon(n)} \lesssim 2^{n\gamma}$ and thus we see that $|\mathcal{E}_1(y)| \lesssim 2^{-n\gamma(d-1)}n^{-2}$. Moreover, with $\chi_k := \mathcal{F}^{-1}[\varphi(2^k\cdot)]$,

$$|\mathcal{E}_2(y)| \lesssim \|K^{n,\nu}_j\|_1 |y-y_Q| \|\nabla\chi_{j-n+\ell(n)}\|_1 \lesssim 2^{-n\gamma(d-1)}2^{j-n}2^{n-j-\ell(n)}$$

which is $\lesssim 2^{-n\gamma(d-1)}n^{-2}$. Integrating in y, we get

$$\int \big(|\mathcal{E}_1(y)| + |\mathcal{E}_2(y)|\big)\,|b_Q(y)|dy \lesssim 2^{-n\gamma(d-1)}n^{-2}\|b_Q\|_1,$$

and the assertion follows. □

Lemma 4.2. *Let $y \in Q$ and a be as in* (4.2)*.*

(i) Let $k_1 > k_2 + \ell(n) + 10$. Then

$$\big\|\Lambda_{k_1}\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}a]\big\|_1 \le C_N 2^{-n\gamma(d-1)}\min\{1, n^{2d+2N}2^{n\gamma}2^{(k_2-j+n\gamma)N}\}$$

(ii) Let $k_1 < k_2 - 10$. Then

$$\big\|\Lambda_{k_1}\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}a]\big\|_1 + \big\|\Lambda_{k_1}\mathcal{K}^{n,\nu}_{j,y}[V_{k_2}a]\big\|_1 \le C_N 2^{-n\gamma(d-1)}\min\{1, 2^{n\gamma}2^{(k_1-k_2)d}2^{(k_1-j+n\gamma)N}\}\,.$$

Proof. Clearly $\|\mathcal{K}^{n,\nu}_{j,y}[a]\|_1 \lesssim 2^{-n\gamma(d-1)}\|a\|_\infty$, and since the operators Λ_k, V_k are uniformly bounded we get the bound $O(2^{-n\gamma(d-1)})$ in (i) and (ii). We seek to prove the two other bounds for $\Lambda_{k_1}\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}a]$ under the assumptions $k_1 < k_2 - 10$, and $k_1 > k_2 + \ell(n) + 10$. In (ii) the corresponding estimate for $\Lambda_{k_1}\mathcal{K}^{n,\nu}_{j,y}[V_{k_2}a]$ is entirely analogous and will be omitted.

We use the Fourier inversion formula for a and for the convolution kernel of Λ_{k_1}, write

$$\Lambda_{k_1}\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}a](x) = \frac{1}{(2\pi)^{2d}}\int \vartheta_n(s)\iint \beta_{k_1}(\xi)\beta_{k_2}(\eta)\,\widehat{a}(\eta)\,\times \Big[\int_w e^{i(\langle x-w,\xi\rangle + \langle sw+(1-s)y,\eta\rangle)}K^{n,\nu}_j(w-y)\,dw\Big]\,d\xi\,d\eta\,ds\,,$$

and integrate by parts with respect to w and ξ. The integral can then be rewritten as [1]

$$\frac{1}{(2\pi)^{2d}}\int \vartheta_n(s)\int \beta_{k_2}(\eta)\widehat{a}(\eta)\int\Big[\int e^{i(\langle x-w,\xi\rangle+\langle sw+(1-s)y,\eta\rangle)}\times \frac{(I-2^{-2k_1}\Delta_\xi)^{N_1}[\beta_{k_1}(\xi)|\xi-s\eta|^{-2N_2}](-\Delta_w)^{N_2}K^{n,\nu}_j(w-y)}{(1+2^{-2k_1}|x-w|^2)^{N_1}}\,dw\Big]\,d\xi\,d\eta\,ds\,,$$

and we choose $N_1 = [d/2]+1$. Note that for $s \in supp(\vartheta_n)$,

$$|\xi - s\eta| \gtrsim \mathcal{C}(k_1,k_2,n) := \begin{cases} 2^{-k_2-\ell(n)} & \text{if } k_1 > k_2+\ell(n)+10\,,\\ 2^{-k_1-2} & \text{if } k_1 < k_2 - 10\,.\end{cases}$$

Now $(2^{-k_1}\partial_\xi)^{N_3}\beta_{k_1} = O(1)$ and thus one computes

$$\big|(I-2^{-2k_1}\Delta_\xi)^{N_1}[\beta_{k_1}(\xi)|\xi-s\eta|^{-2N_2}]\big| \lesssim [\mathcal{C}(k_1,k_2,n)]^{-N_2}\,.$$

Moreover

$$\|(-\Delta_w)^{N_2}K^{n,\nu}_j\|_1 \lesssim 2^{-2N_2 j}(2^{2N_2 n\gamma}+2^{2N_2\ell_\varepsilon(n)})2^{-n\gamma(d-1)} \lesssim 2^{-n\gamma(d-1)}2^{2N_2(n\gamma-j)}\,;$$

[1] Thanks to Xudong Lai who pointed out an error in the original version of this formula.

We integrate in η and use that the size of the support of β_{k_2} is $2^{-k_2 d}$. Then we integrate in x, ξ and use that

$$\int_{\mathrm{supp}(\beta_{k_1})} \int (1 + 2^{-2k_1}|x-w|^2)^{-N_1}\, dx\, d\xi = O(1)\,.$$

Using (4.2) we then get

$$\begin{aligned}\big\|\Lambda_{k_1}\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}a]\big\|_1 &\lesssim_{N_2} 2^{-k_2 d}\|\widehat{a}\|_\infty \|(-\Delta)^{N_2}K^{n,\nu}_j\|_1 [\mathcal{C}(k_1,k_2,n)]^{-2N_2} \\ &\lesssim_{N_2} \begin{cases} 2^{d\ell(n)-n\gamma(d-2)}2^{(2N_2-d)(k_2-j+\ell(n)+n\gamma)} & \text{if } k_1 > k_2+\ell(n)+10\,, \\ 2^{-n\gamma(d-2)}2^{(2N_2-d)(k_1-j+n\gamma)}2^{(k_1-k_2)d} & \text{if } k_1 < k_2 - 10\,. \end{cases}\end{aligned}$$

If we put $N = 2N_2 - d$ this gives the asserted bound for $\big\|\Lambda_{k_1}\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}a]\big\|_1$. For $k_1 < k_2 - 10$ the corresponding expression with Λ_{k_2} replaced by V_{k_2} is estimated in exactly the same way. □

Lemma 4.3. *Let $k_2 - 10 \le k_1 \le k_2 + \ell(n) + 10$. Then*

$$\begin{aligned}&\|\Lambda_{k_1}(I-\mathfrak{S}^{n,\nu})\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}a]\|_1 \\ &\qquad\qquad \le C_N 2^{-n\gamma(d-1)}\min\{1, n^{2(N+d)/\varepsilon}2^{(d+3)n\gamma}2^{(k_1-j+n\gamma)N}\}\end{aligned}$$

for every $y \in Q$.

Proof. We may again assume that (4.2) holds. Define the convolution operator $S_{n,\nu}$ by

$$\widehat{S^{n,\nu}g}(\eta) = \phi(2^{n\gamma}n^{-2}\langle \nu, \tfrac{\eta}{|\eta|}\rangle)\widehat{g}(\eta)$$

and split $a = S^{n,\nu}a + (I - S^{n,\nu})a$. We shall prove the following estimates,

$$\begin{aligned}(4.6)\quad &\|\Lambda_{k_1}(I-\mathfrak{S}^{n,\nu})\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}S^{n,\nu}a]\|_1 \\ &\qquad\qquad \le C_N\, n^{(2\varepsilon^{-1}-4)(N+d)}2^{4n\gamma}2^{(k_1-k_2)d}2^{(k_1-j+n\gamma)N}\end{aligned}$$

and

$$(4.7)\quad \|\Lambda_{k_1}(I-\mathfrak{S}^{n,\nu})\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}(I-S^{n,\nu})a]\|_1 \le C_N\, n^{-5d}2^{4n\gamma}2^{(k_2-j+n\gamma)N}\,,$$

which imply the somewhat weaker estimate asserted in the lemma.

Proof of (4.6). Set

$$b_{k_1,n,\nu}(\xi) = \beta_{k_1}(\xi)\big(1 - \phi(2^{n\gamma}n^{-5}\langle \nu, \tfrac{\xi}{|\xi|}\rangle)\big)$$

and write

$$\begin{aligned}&(2\pi)^{2d}\Lambda_{k_1}(I-\mathfrak{S}^{n,\nu})\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}S^{n,\nu}a](x) = \\ &\qquad \int \vartheta_n(s) \iint b_{k_1,n,\nu}(\xi)\beta_{k_2}(\eta)\phi(2^{n\gamma}n^{-2}\langle\nu, \tfrac{\eta}{|\eta|}\rangle)\,\widehat{a}(\eta) \\ &\qquad\qquad \times \Big[\int_w e^{i(\langle x-w,\xi\rangle + \langle sw+(1-s)y,\eta\rangle)}K^{n,\nu}_j(w-y)\,dw\Big]\,d\xi\,d\eta\,ds\,.\end{aligned}$$

If (ξ,η) is in the support of the amplitude then for $n>10^{10}$

$$\begin{aligned}\big|\langle \xi-s\eta,\nu\rangle\big| &\ge |\xi|\big|\langle \tfrac{\xi}{|\xi|},\nu\rangle\big| - |\eta|\big|\langle \tfrac{\eta}{|\eta|},\nu\rangle\big|\\ &\ge |\xi|\big(2^{-n\gamma-1}n^5 - 2^{|k_1-k_2|+2}2^{-n\gamma}n^2\big)\\ &\ge |\xi|2^{-n\gamma-1}(n^5-8\cdot 2^{\ell(n)+10}n^2)\ge 2^{-k_1-n\gamma}n^5\,. \end{aligned}\tag{4.8}$$

Now we can integrate by parts as in the proof of Lemma 4.2, except we use the directional derivative $\langle \nu,\nabla_w\rangle$ instead of Δ_w. The above integral is then estimated by

$$\iiiint |\beta_{k_2}(\eta)|\,|\widehat a(\eta)|\big|\phi(2^{n\gamma}n^{-2}\langle\nu,\tfrac{\eta}{|\eta|}\rangle)\big| \\ \times\frac{\big|(I-2^{-2k_1}\Delta_\xi)^{N_1}\big[\frac{b_{k_1,n,\nu}(\xi)}{\langle \xi-s\eta,\nu\rangle^{N_2}}\big]}{(1-2^{-2k_1}|x-w|^2)^{N_1}}\big|\langle\nu,\nabla_w\rangle^{N_2}K_j^{n,\nu}(w-y)\big|\,dw\,d\xi\,d\eta\,ds\,.$$

Observe that

$$\big|\partial_\xi^{N_3}b_{k_1,n,\nu}(\xi)\big|\le C_{N_1}(2^{n\gamma}n^{-5})^{N_3}2^{k_1N_3}$$

and thus

$$\Big|(I-2^{-2k_1}\Delta_\xi)^{N_1}\Big[\frac{b_{k_1,n,\nu}(\xi)}{\langle\xi-s\eta,\nu\rangle^{N_2}}\Big]\Big|\le C_{N_1}(2^{n\gamma}n^{-5})^{2N_1}(2^{-(k_1+n\gamma)}n^5)^{-N_2}. \tag{4.9}$$

Moreover,

$$\big\|\langle\nu,\nabla_w\rangle^{N_2}K_j^{n,\nu}\big\|_1\le C_{N_2}2^{(\ell_\varepsilon(n)-j)N_2}2^{-n\gamma(d-1)}\,.$$

We assume $2N_1>d$, integrate in x and ξ, and use (4.8). Then we obtain

$$\|\Lambda_{k_1}(I-\mathfrak{S}^{n,\nu})\mathcal{K}_{j,y}^{n,\nu}[\Lambda_{k_2}S^{n,\nu}a]\|_1 \\ \lesssim_{N_1,N_2}(2^{2n\gamma}n^{-5})^{2N_1}\|\widehat a\|_\infty 2^{-k_2d}\frac{2^{(\ell_\varepsilon(n)-j)N_2}2^{-n\gamma(d-1)}}{(2^{-k_1-n\gamma}n^5)^{N_2}}.$$

We use (4.2) and that the support of $\eta\mapsto\beta_{k_2}(\eta)$ has measure $O(2^{-k_2d})$. Thus the expression in the previous display can be crudely estimated by

$$C_{N_1,N_2}n^{(2\varepsilon^{-1}-4)N_2-10N_1}2^{n\gamma(2N_1-d+2)}2^{(k_1-k_2)d}2^{(k_1-j+n\gamma)(N_2-d)}$$

and, if we chose the integer $N_1\in\{\frac{d+1}{2},\frac{d+2}{2}\}$ and $N=N_2-d$ we obtain (4.6).

Proof of (4.7). Set

$$\widetilde b_{k_2,n,\nu}(\eta)=\beta_{k_2}(\eta)(1-\phi(2^{n\gamma}n^{-2}\langle\nu,\tfrac{\eta}{|\eta|}\rangle))$$

and write

$$(2\pi)^d \Lambda_{k_1}(I-\mathfrak{S}^{n,\nu})\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}(I-S^{n,\nu})a](x) \\ = \int K_j^{n,\nu}(w-y) \iint b_{k_1,n,\nu}(\xi)\widetilde{b}_{k_2,n,\nu}(\eta)\,\widehat{a}(\eta) \\ \times \Big[\int \vartheta_n(s) e^{i(\langle x-w,\xi\rangle + \langle sw+(1-s)y,\eta\rangle)}\,ds\Big]\,d\xi\,d\eta\,dw\,.$$

Now if $w-y\in \operatorname{supp}(K_j^{n,\nu})$ then $\big|\frac{w-y}{|w-y|}-\nu\big|\le 2^{-n\gamma}$ and if $\eta\in\operatorname{supp}(\widetilde{b}_{k_2,n,\nu})$ we get

$$|\langle w-y,\eta\rangle| \ge |w-y|\,\big(\langle \nu,\eta\rangle - |\eta|2^{-n\gamma}\big) \ge |w-y|\,|\eta|2^{-n\gamma}(\tfrac12 n^2-1)$$

and hence

$$|\langle w-y,\eta\rangle| \ge 2^{j-k_2-n\gamma-4}n^2. \tag{4.10}$$

Integration by parts with respect to s yields

$$(2\pi)^d \Lambda_{k_1}(I-\mathfrak{S}^{n,\nu})\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}(I-S^{n,\nu})a](x) \;= \\ \int K_j^{n,\nu}(w-y)\iint \widehat{a}(\eta)\widetilde{b}_{k_2,n,\nu}(\eta)\frac{(I-2^{-2k_1}\Delta_\xi)^{N_1}b_{k_1,n,\nu}(\xi)}{(1+2^{-2k_1}|x-w|^2)^{N_1}} e^{i(\langle x-w,\xi\rangle+\langle y,\eta\rangle)} \\ \times \Big[\int \vartheta_n^{(N_3)}(s)\frac{i^{N_3}e^{is\langle w-y,\eta\rangle}}{\langle w-y,\eta\rangle^{N_3}}\,ds\Big]\,d\xi\,d\eta\,dw\,.$$

We apply this with $N_1>d/2$ and, using (4.2), (4.9), and (4.10), obtain

$$\|\Lambda_{k_1}(I-\mathfrak{S}^{n,\nu})\mathcal{K}^{n,\nu}_{j,y}\Lambda_{k_2}(I-S^{n,\nu})a\|_1 \\ \lesssim_{N_1,N_3} (2^{n\gamma}n^{-5})^{2N_1}\|K_j^{n,\nu}\|_1 \frac{\|\vartheta_n^{(N_3)}\|_1}{(2^{j-k_2-n\gamma-4}n^2)^{N_3}} 2^{-k_2 d}\|\widehat{a}\|_\infty \\ \lesssim_{N_1,N_3} n^{-2-10N_1}2^{n\gamma(2N_1-d+2)}2^{(k_2-j+n\gamma)(N_3-d)}\,.$$

Inequality (4.7) follows if we choose $N=N_3-d$ and $N_1\in\{\frac{d+1}{2},\frac{d+2}{2}\}$. □

Proof of Proposition 2.3, conclusion. Let, for fixed n,ν,j and for a fixed cube $Q\in\mathfrak{Q}_\lambda$ with $L(Q)=j-n$,

$$I_{k_1} = \widetilde{\Lambda}_{k_1}(I-P_{j-n+\ell(n)})\Lambda_{k_1}(I-\mathfrak{S}_{n,\nu})\Big[\int b_Q(y)\mathcal{K}^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a]dy\Big]\,,$$

and

$$II_{k_1,k_2} = \widetilde{\Lambda}_{k_1}(I-P_{j-n+\ell(n)})\Lambda_{k_1}(I-\mathfrak{S}_{n,\nu})\Big[\int b_Q(y)\mathcal{K}^{n,\nu}_{j,y}[\Lambda_{k_2}a](x)dy\Big]\,.$$

By (4.4), (4.5) it is enough to show that

$$\sum_{k_1}\|I_{k_1}\|_1 + \sum_{k_1}\sum_{k_2<j-n+\ell(n)}\|II_{k_1,k_2}\|_1 \lesssim n^{-2}2^{-\gamma n(d-1)}\,\|b_Q\|_1\,. \tag{4.11}$$

We have

$$\|\Lambda_{k_1}(I-\mathfrak{S}_{n,\nu})\|_{L^1\to L^1}\le C \tag{4.12}$$

uniformly in n,ν,k_1, and using the support and cancellation properties of the kernel of $I-P_{j-n+\ell(n)}$ we also have

$$\|\widetilde{\Lambda}_{k_1}(I-P_{j-n+\ell(n)})\|_{L^1\to L^1}\lesssim \min\{1,2^{j-n+\ell(n)-k_1}\}\,. \tag{4.13}$$

Lemma 4.1 together with (4.13), (4.12) immediately gives

$$\sum_{k_1\ge j-n+\ell(n)-10}\|I_{k_1}\|_1\lesssim n^{-2}2^{-\gamma n(d-1)}\,\|b_Q\|_1. \tag{4.14}$$

It remains to verify that the other terms satisfy better bounds, namely

$$\sum_{k_1<j-n+\ell(n)-10}\|I_{k_1}\|_1+\sum_{k_1}\sum_{k_2<j-n+\ell(n)}\|II_{k_1,k_2}\|_1 \lesssim C_N n^{A_1N}2^{A_2n}2^{n(\gamma-1)N}\,\|b_Q\|_1 \tag{4.15}$$

for all N, and suitable $A_1\le 10d/\varepsilon$, $A_2\le 10$. Choose $N=100d$. Taking into account that $\gamma\le 9/10$ one may check that the bound in (4.15) is $\lesssim n^{-2}2^{-n\gamma(d-1)}\|b_Q\|_1$ for all n with $n^{-1}\log n\le 10^{-4}\varepsilon/d$, which is satisfied for $n>n(\varepsilon)$.

For the terms involving I_{k_1}, with $k_1\ge j-n+\ell(n)+10$ we get by the second estimate in part (ii) of Lemma 4.2, with $k_2=j-n+\ell(n)$,

$$\begin{aligned}&\sum_{k_1<j-n+\ell(n)-10}\|I_{k_1}\|_1\\ &\lesssim_N 2^{-n\gamma(d-2)}\sum_{k_1<j-n+\ell(n)-10}2^{(k_1-j+n-\ell(n))d}2^{(k_1-j+n\gamma)N}\,\|b_Q\|_1\\ &\lesssim_N 2^{-n\gamma(d-2)}(2^{n(\gamma-1)}n^2)^N\,\|b_Q\|_1.\end{aligned}$$

Next consider $\sum_{k_1,k_2}\|II_{k_1,k_2}\|_1$ where the k_2-summation is extended over $k_2<j-n+\ell(n)$. For $k_1\ge j-n+\ell-10$ we can sum a geometric series in k_1, with a uniform bound, due to (4.13). By Lemma 4.2, part (i)

$$\begin{aligned}&\sum_{\substack{(k_1,k_2):\,k_1\ge j-n+\ell(n)-10\\ k_2<\min\{k_1-\ell(n)-10,\,j-n+\ell(n)\}}}\|II_{k_1,k_2}\|_1\\ &\lesssim 2^{-n\gamma(d-2)}n^{2d+2N}\sum_{k_2<j-n+\ell(n)}2^{(k_2-j+n\gamma)N}\|b_Q\|_1\\ &\lesssim 2^{-n\gamma(d-2)}n^{2d+4N}2^{n(\gamma-1)N}\|b_Q\|_1\,,\end{aligned}$$

and by Lemma 4.3

$$\begin{aligned}
&\sum_{\substack{k_1\ge j-n+\ell(n)-10\\ k_1-\ell(n)-10\le k_2\le k_1+10\\ k_2<j-n+\ell(n)}} \|II_{k_1,k_2}\|_1\\
&\quad\lesssim \|b_Q\|_1 \ell(n) n^{2(N+d)/\varepsilon} 2^{4n\gamma} \sum_{k_1\le j-n+2\ell(n)+10} 2^{(k_1-j+n\gamma)N}\\
&\quad\lesssim \|b_Q\|_1 \log(n) n^{2(N+d)(\varepsilon^{-1}+2)} 2^{n(\gamma-1)N}.
\end{aligned}$$

The case $k_2 > k_1 + 10$ does not occur when $k_1 \ge j - n + \ell(n) - 10$ because of the restriction $k_2 < j - n + \ell(n)$. Thus in all cases of (4.15) which involve the restriction $k_1 \ge j - n + \ell(n) - 10$ we obtain the required estimate.

Now sum the terms $\|II_{k_1,k_2}\|_1$ with $k_1 < j - n + \ell - 10$. By Lemma 4.2, part (i)

$$\begin{aligned}
&\sum_{(k_1,k_2):\substack{k_1<j-n+\ell(n)-10\\ k_2<k_1-\ell(n)-10}} \|II_{k_1,k_2}\|_1\\
&\lesssim n^{2d+2N} 2^{-n\gamma(d-2)} \sum_{(k_1,k_2):\substack{k_1<j-n+\ell(n)-10\\ k_2<k_1-\ell(n)-10}} 2^{(k_2-j+n\gamma)N} \|b_Q\|_1\,,\\
&\lesssim n^{2d+2N} 2^{-n\gamma(d-2)} 2^{n(\gamma-1)N} \|b_Q\|_1\,,
\end{aligned}$$

by Lemma 4.2, part (ii)

$$\begin{aligned}
&\sum_{(k_1,k_2):\substack{k_1<j-n+\ell(n)-10\\ k_1+10<k_2<j-n+\ell(n)-10}} \|II_{k_1,k_2}\|_1\\
&\lesssim 2^{-n\gamma(d-2)} \sum_{k_1<j-n+\ell(n)-10} 2^{(k_1-j+n\gamma)N} \sum_{k_2>k_1+10} 2^{(k_1-k_2)d} \|b_Q\|_1\,,\\
&\lesssim n^{2N} 2^{-n\gamma(d-2)} 2^{n(\gamma-1)N} \|b_Q\|_1\,,
\end{aligned}$$

and finally, by Lemma 4.3,

$$\begin{aligned}
&\sum_{(k_1,k_2):\substack{k_1<j-n+\ell(n)-10\\ k_1-\ell(n)-10\le k_2\le k_1+10}} \|II_{k_1,k_2}\|_1\\
&\lesssim \log(n) n^{2(N+d)/\varepsilon} 2^{4n\gamma} \sum_{k_1\le j-n+\ell(n)} 2^{(k_1-j+n\gamma)N} \|b_Q\|_1\\
&\lesssim n^{2(N+d)(\varepsilon^{-1}+1)} 2^{4n\gamma} 2^{n(\gamma-1)N} \|b_Q\|_1\,.
\end{aligned}$$

This finishes the proof of (4.15). □

5. Proof of Proposition 2.4

We use a slightly modified version of an argument in [8]. The main observation is that, for fixed $n > 0$, we have

$$\sup_{\xi \neq 0} \sum_{\nu \in \Theta_n} |\phi(2^{n\gamma} n^{-5} \langle \nu, \tfrac{\xi}{|\xi|} \rangle)| \lesssim 2^{n\gamma(d-2)} n^5. \tag{5.1}$$

To see this it suffices, by homogeneity, to take the supremum over all $\xi \in S^{d-1}$. Now if $|\xi| = 1$ and $\phi(2^{n\gamma} n^{-5} \langle \theta, \xi \rangle) \neq 0$ then the distance of ν to the hyperplane ξ is at most $Cn^5 2^{-n\gamma}$ and since the vectors in Θ_n are $c2^{-n\gamma}$-separated there are $O(2^{n\gamma(d-2)} n^5)$ such vectors, hence (5.1) holds.

From (5.1) it follows that

$$\Big\| \sum_{\nu \in \Theta_n} \mathfrak{S}_{n,\nu} \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \Big\|_2^2 \\ \lesssim 2^{n\gamma(d-2)} n^5 \sum_{\nu \in \Theta_n} \Big\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \Big\|_2^2$$

and since $\#\Theta_n \lesssim 2^{n\gamma(d-1)}$ the asserted inequality is a consequence of

$$\Big\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \Big\|_2^2 \lesssim 2^{-2n\gamma(d-1)} \lambda \|f\|_1 \tag{5.2}$$

for each $\nu \in \Theta_n$.

For the proof of (5.2) the cancellation of B_{j-n} plays no role. Let

$$H_j^{n,\nu}(x) = 2^{-jd} \chi_{\tau_j^{n,\nu}}(x).$$

where

$$\tau_j^{n,\nu} = \{x : |\langle x, \nu \rangle| \leq 2^{j+2}, \ |x - \langle x, \nu \rangle| \leq 2^{j+2-\gamma n}\}.$$

Then from (1.1) we get

$$\big|(I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n}(x)\big| \lesssim H_j^{n,\nu} * |B_{j-n}|(x).$$

Therefore

$$\Big\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \Big\|_2^2 \\ \lesssim 2 \sum_j \int |B_{j-n}(x)| \sum_{i \leq j} H_j^{n,\nu} * H_i^{n,\nu} * |B_{i-n}(x)| \, dx \, .$$

Observe that $\|H_i^{n,\nu}\|_1 \lesssim 2^{-id} \text{meas}(\tau_i^{n,\nu}) \lesssim 2^{-n\gamma(d-1)}$ and thus

$$H_j^{n,\nu} * H_i^{n,\nu}(x) \lesssim 2^{-n\gamma(d-1)} \, 2^{-jd} \chi_{\tilde{\tau}_j^{n,\nu}}(x)$$

where $\widetilde{\tau}_j^{n,\nu}$ is the double of $\tau_j^{n,\nu}$. Hence, for each $x \in \mathbb{R}^d$, $j \in \mathbb{Z}$,

$$\sum_{i\le j} H_j^{n,\nu} * H_i^{n,\nu} * |B_{i-n}|(x)$$
$$\lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i\le j} \int_{x+\widetilde{\tau}_j^{n,\nu}} |B_{i-n}(y)|\, dy$$
$$\lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i\le j} \sum_{\substack{Q\in\mathfrak{Q}_\lambda:\\ L(Q)=i-n \\ Q\cap(x+\widetilde{\tau}_j^{n,\nu})\neq\emptyset}} \int |b_Q(x)|\, dx$$
$$\lesssim 2^{-n\gamma(d-1)} 2^{-jd} \lambda \operatorname{meas}(\widetilde{\tau}_j^{n,\nu}) \lesssim 2^{-2n\gamma(d-1)} \lambda\,;$$

here we have used $\|b_Q\|_1 \lesssim \lambda |Q|$, and the disjointness of the interiors of the cubes Q in $\mathfrak{Q}_\lambda$. Thus we get the estimate

$$\Big\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \Big\|_2^2 \lesssim 2^{-2n\gamma(d-1)} \lambda \sum_j \|B_{j-n}\|_1$$

which yields (5.2). □

6. Open problems

6.1. *Principal value integrals.* Let

$$\mathcal{T}_r f(x) = \int_{|x-y|>r} K(x-y) \int_0^1 a(sx + (1-s)y)\, ds\, f(y)\, dy\,.$$

Our proof shows that the operators $\mathcal{T}_r$ are of weak type $(1,1)$, with uniform bounds; moreover, for $f \in L^1$, $\mathcal{T}_r f$ converges in measure to Tf where T is weak type $(1,1)$. However it is currently open whether the principal value $\lim_{r\to 0} \mathcal{T}_r f(x)$ exists for almost every $x \in \mathbb{R}^d$. By Stein's theorem [9] this is equivalent to the open question whether the maximal singular integral $\sup_{r>0} |\mathcal{T}_r f|$ defines an operator of weak type $(1,1)$.

6.2. *Principal value integrals for rough singular convolution operators.* The question analogous to 6.1 is open for classical singular integral operators with rough convolution kernel $\Omega(y/|y|)|y|^{-d}$ where $\Omega \in L\log L(S^{d-1})$, $d \ge 2$ and $\int_{S^{d-1}} \Omega(\theta) d\sigma = 0$. These operators are known to be of weak type $(1,1)$, [8], but the a.e. existence of the principal value integrals is open even for $\Omega \in L^\infty(S^{d-1})$.

6.3. *Christ-Journé operators.* Let $F \in C^\infty(\mathbb{R})$, let K be a Calderón-Zygmund convolution kernel, and let $a \in L^\infty(\mathbb{R}^d)$. Christ and Journé [3] showed that the operator defined for $f \in C_0^\infty(\mathbb{R}^d)$ by

$$\mathcal{T}f(x) = p.v. \int F\Big(\int_0^1 a(sx + (1-s)y)dt\Big)\, K(x-y)f(y)dy$$

extends to a bounded operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. It would be interesting to get the weak type $(1,1)$ inequality for nonlinear F, in dimension $d \geq 2$.

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