REGULARITY PROPERTIES OF WAVE PROPAGATION ON CONIC MANIFOLDS AND APPLICATIONS TO SPECTRAL MULTIPLIERS

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1. Introduction

Let $n \geq 2$ and $\mathcal{N}$ be a compact Riemannian manifold of dimension $n-1$, possibly with boundary. Following [9], we define the *metric cone* $\Gamma(\mathcal{N})$ over $\mathcal{N}$ to be the space $\mathbb{R}^+ \times \mathcal{N}$, together with the Riemannian metric

$$dr^2 + r^2 g_{\mathcal{N}},$$

where $g_{\mathcal{N}}$ is the Riemannian metric on $\mathcal{N}$. Let $\Delta_{\mathcal{N}}$ denote the Laplacian on $\mathcal{N}$, where we assume Dirichlet or Neumann boundary conditions for $\Delta_{\mathcal{N}}$, if the boundary $\partial\mathcal{N}$ of $\mathcal{N}$ is non-empty. The Laplacian on $\Gamma(\mathcal{N})$ is then given by

$$\Delta := \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathcal{N}} \tag{1.1}$$

(here, to conform with the standard definition in $\mathbb{R}^n$ we have taken $\Delta_{\mathcal{N}}$ and Δ to be nonpositive). It has been shown in [9] that Δ is selfadjoint on $L^2(\Gamma(\mathcal{N}))$, when endowed with the appropriate domain adapted to the Dirichlet or Neumann conditions for $\Delta_{\mathcal{N}}$, if $\partial\mathcal{N} \neq \emptyset$. Thus the squareroot $\sqrt{-\Delta}$ is well defined, as are the multiplier operators $m(\sqrt{-\Delta})$ for continuous bounded m.

An important special case arises where $\mathcal{N} = S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Then Δ is the Euclidean Laplacian on $\mathbb{R}^n$, written in polar coordinates.

In this paper we are interested in deriving smoothing properties for solutions of the wave equation $u_{tt} = \Delta u$ on $\Gamma(\mathcal{N})$; so we are concerned with the wave operator U defined by

$$Uf(t,r,\theta) := \cos(t\sqrt{-\Delta})f(r,\theta).$$

Recall that

$$u(t,r,\theta) = Uf(t,r,\theta) + \int_0^t Ug(\tau,r,\theta)d\tau \tag{1.2}$$

solves $u_{tt} = \Delta u$ with initial conditions $u|_{t=0} = f$, $u_t|_{t=0} = g$.

The main purpose of this paper is to find suitable extensions of a previously known regularity theorem for radial solutions in $\mathbb{R}^n$ to more general solutions. In [25] the authors proved that for *radial* $f \in L^p(\mathbb{R}^n)$

$$\Big(\frac{1}{T}\int_0^T \|\cos(t\sqrt{-\Delta})f\|^p_{L^p(\mathbb{R}^n)}dt\Big)^{1/p} \lesssim \|f\|_{L^p_{\text{rad}}(\mathbb{R}^n)} \qquad 2 \leq p < p_0 := \frac{2n}{n-1}. \tag{1.3}$$

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The range of exponents p for which (1.3) holds is sharp, and various endpoint results can be proved for the case $p = p_0$, see [25], [10]. In particular the weak type analogue of (1.3), for radial functions in $L^{2n/(n-1)}(\mathbb{R}^n)$, was proved by Colzani, Cominardi and Stempak [10].

As pointed out in [25], the analogue of (1.3) for general L^p functions does not hold (in view of C. Fefferman's theorem [15] concerning the ball multiplier); however it is conceivable that (1.3) holds for functions in the Sobolev space L^p_ϵ for $\epsilon > 0$. This had been stated as a conjecture by Sogge [30]. A positive answer would imply the Bochner-Riesz conjecture in n dimensions; however even in two dimensions (where the Bochner-Riesz problem is well understood, [8]) only partial results are known for the corresponding smoothing estimate ([24], [3], [36]), [41]) and most of those partial results are not known in the more general context of conic manifolds.

It seems natural to ask for an analogue of (1.3) on other spaces which are left invariant by the Laplacian, such as spaces of functions of the type $f_k(|x|)Y_k(x/|x|)$ where Y_k is any spherical harmonic of degree k, normalized in $L^2(S^{n-1})$; here one is seeking estimates which are independent of the degree. More generally one would like to replace the $L^p(\mathbb{R}^n)$ norm in (1.3) by a mixed $L^p_{\text{rad}}(L^2_{\text{sph}})$ norm. The motivation for this partially came from previous work on the corresponding problem on spherical summation of the Fourier integral ([27], [11], [23], [39], [5]) and partially from mixed norm estimates derived in [21] in connection with a problem in semilinear wave equations.

We define these spaces for general conic manifolds $\Gamma(\mathcal{N})$. Let $d\theta$ denote the Riemannian volume on $\mathcal{N}$; then $r^{n-1}drd\theta$ is the Riemannian volume on $\Gamma(\mathcal{N})$. For $1 \le p < \infty$, denote by $L^p(\mathbb{R}^+, L^2)$ the space of all measurable functions on $\Gamma(\mathcal{N})$ such that

$$\|f\|_{p,2} := \Big(\int_0^\infty \|f(r,\cdot)\|^p_{L^2(\mathcal{N})} r^{n-1} dr\Big)^{1/p}$$

is finite. Here, $\mathbb{R}^+$ *is endowed with the measure* $r^{n-1}dr$.

We first formulate an L^p inequality for $p < 2n/(n-1)$; here we need a tiny bit of regularity in the "spherical" variable but no regularity in the radial variable. To describe this we use the (pseudo-)differential operator

$$\mathcal{L}^\varepsilon_T f(\rho,\theta) := \Big(I - \tfrac{T^2}{\rho^2}\Delta_{\mathcal{N}}\Big)^{\varepsilon/2} f(\rho,\theta). \tag{1.4}$$

We also prove a restricted weak type estimate for the endpoint $p_0 = 2n/(n-1)$. This is formulated using $L^{p,1}(\mathbb{R}^+, L^2)$ which is the space of all measurable functions on $\Gamma(\mathcal{N})$ such that $r \mapsto \|f(r,\cdot)\|_{L^2(\mathcal{N})}$ belongs to the Lorentz-space $L^{p,1}$, with respect to the measure $r^{n-1}dr$. All estimates below will be uniform in $T > 0$.

Our main result is:

Theorem 1.1. *Suppose that* $\varepsilon > 0$ *and* $2 \le p < 2n/(n-1)$. *Then*

$$\Big(\frac{1}{T}\int_0^T \big\|\cos(t\sqrt{-\Delta})f\big\|^p_{p,2} dt\Big)^{1/p} \le C_{p,\varepsilon}\|\mathcal{L}^\varepsilon_T f\|_{p,2} \tag{1.5}$$

for all $T > 0$.

In the complementary range $p > 2n/(n-1)$ we can prove analogous $L^p(L^2)$ smoothing estimates for an analytic family of smooth multipliers $s^{-\alpha+1/2}J_{\alpha-1/2}(s)$ (*cf.* (2.5), (2.6) below); here $J_{\alpha-1/2}$ denotes the Bessel function of order $\alpha - 1/2$. This family can be used to derive estimates for $(1-t^2\Delta)^{-\alpha/2}\cos(t\sqrt{-\Delta})$, in view of the asymptotics [13], [33]

$$\Gamma(\alpha+1/2)\frac{J_{\alpha-1/2}(s)}{s^{\alpha-1/2}} = \begin{cases} 1+o(1), & s\to 0,\\ (2/\pi)^{1/2}s^{-\alpha}\cos(s-\pi\alpha/2), & s\to\infty.\end{cases}$$

We now state $L^p(L^2)$ estimates for the family of operators

$$U_t^\alpha := \frac{J_{\alpha-1/2}(t\sqrt{-\Delta})}{(t\sqrt{-\Delta})^{\alpha-1/2}}. \tag{1.6}$$

The necessary regularization determined by α is partly mitigated by the appearance of $\mathcal{L}_T^{-\gamma}$ on the right hand side of (1.7.1/2) below. This indicates some smoothing effect in the spherical variables. We are mainly interested in the case $\gamma = 0$, though.

Theorem 1.2. *Let $0 < \alpha < (n+1)/2$. Suppose that $\frac{2n}{n-1} \le p < \infty$, $\alpha > n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$, $\gamma \le (n-1)/p'$ and assume that $\gamma < \alpha$ if $2 \le p \le 4$, and $\gamma \le \alpha - 1/2 + 2/p$ if $4 < p < \infty$.*

Then (i)

$$\Big(\frac{1}{T}\int_T^{2T} \big\|U_t^\alpha f\big\|_{p,2}^p dt\Big)^{1/p} \le C_{p,\alpha,\gamma}\big\|\mathcal{L}_T^{-\gamma} f\big\|_{p,2} \tag{1.7.1}$$

(ii) Moreover if $\alpha \ge (n-1)/2$ and $\gamma \le \alpha - 1/2$ then

$$\sup_{T\le|t|\le 2T}\sup_{r>0}\|U_t^\alpha f(t,r,\cdot)\|_{L^2(\mathcal{N})} \le C_\alpha \sup_{\rho>0}\big\|\mathcal{L}_T^{-\gamma} f(\rho,\cdot)\big\|_{L^2(\mathcal{N})}. \tag{1.7.2}$$

When $\gamma = 0$ we obtain

Corollary 1.3. *Suppose that $2 \le p < \infty$, $\alpha > \max\{0, n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}\}$.*

Then

$$\Big(\frac{1}{2T}\int_{-T}^{T} \big\|U_t^\alpha f\big\|_{p,2}^p dt\Big)^{1/p} \le C_{p,\alpha}\big\|f\big\|_{p,2}$$

and, for $\alpha \ge (n-1)/2$

$$\sup_{|t|\le T}\sup_{r>0}\|U_t^\alpha f(t,r,\cdot)\|_{L^2(\mathcal{N})} \le C_\alpha \sup_{\rho>0}\|f(\rho,\cdot)\|_{L^2(\mathcal{N})}.$$

The corollary is a straightforward consequence of Theorem 1.2; to remove the restriction $\alpha < (n+1)/2$ one uses formula (2.14) below. We note that using (2.14) one can also obtain refined versions in the spirit of Theorem 1.2 in the range $\alpha \ge (n+1)/2$ but these are of less interest.

Corollary 1.3 can be applied to solutions of the wave equation; this is done in §6. One obtains

Corollary 1.4. *Suppose that $2 \le p < \infty$, and u and v belong to $L^p(\mathbb{R}^+, L^2(\mathcal{N}))$. Suppose that $\alpha > \max\{0, n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}\}$.*

(i) If $u_{tt} = \Delta u$ and $u(0,\cdot) = f$, $u_t(0,\cdot) = 0$ then

$$\Big(\frac{1}{2T}\int_{-T}^{T} \big\|(I - t^2\Delta)^{-\alpha/2}u(t,\cdot)\big\|_{p,2}^p dt\Big)^{1/p} \le C_{p,\alpha}\big\|f\big\|_{p,2}.$$

(ii) If $v_{tt} = \Delta v$ and $v(0,\cdot) = 0$, $v_t(0,\cdot) = g$ then

$$\Big(\frac{1}{2T}\int_{-T}^{T} \big\|(I - t^2\Delta)^{-(\alpha-2)/2}v(t,\cdot)\big\|_{p,2}^p dt\Big)^{1/p} \le C_{p,\alpha}\big\|g\big\|_{p,2}.$$

We now state some endpoint (restricted weak type and weak type) estimates for the case $\alpha = n(1/2 - 1/p) - 1/2$, $2n/(n-1) \le p < \infty$. Denote by μ_n the measure $dt\,r^{n-1}dr$ on $[0,T] \times \mathbb{R}^+$.

Theorem 1.5. *Let $2n/(n-1) \le p < \infty$ and let $\alpha(p) = n(1/2 - 1/p) - 1/2$. Let*

$$\Omega^p_{\beta,T}(f) = \{(t,r) \in [T/2,T] \times \mathbb{R}^+ : \|U_t^{\alpha(p)} f(r,\cdot)\|_{L^2(\mathcal{N})} > \beta\}.$$

(i) Suppose that either one of the following cases applies.
(a) $p = 2n/(n-1)$, $\gamma < 0$;
(b) $2n/(n-1) < p \le 4$, $\gamma < \alpha(p)$;
(c) $4 < p < \infty$, $\gamma \le \alpha(p) + 2/p - 1/2$.
Then the restricted weak type inequality

$$\sup_{\beta>0} \beta^p T^{-1} \mu_n(\Omega^p_{\beta,T}(f)) \le C^p_\gamma \|\mathcal{L}_T^{-\gamma} f\|^p_{L^{p,1}(\mathbb{R}^+,L^2)}. \tag{1.8}$$

holds.

(ii) Suppose $2n/(n-1) \le p < \infty$ and $\gamma \le \alpha(p) - 1/2$. Then the weak type inequality

$$\sup_{\beta>0} \beta^p T^{-1} \mu_n(\Omega^p_{\beta,T}(f)) \le C^p_\gamma \|\mathcal{L}_T^{-\gamma} f\|^p_{L^{p}(\mathbb{R}^+,L^2)} \tag{1.9}$$

holds.

If in (1.8) or (1.9) one assumes in addition $\gamma \le 0$ then the interval $[T/2,T]$ can be replaced by $[-T,T]$. In particular this yields endpoint versions of Corollary 1.3 where $\gamma = 0$; namely a restricted weak type inequality in the case $p > 2n/(n-1)$ and a weak type inequality in the case $n \ge 3$ and $p > 2n/(n-2)$. For the better weak type estimate in the more restrictive range $\gamma \le \alpha(p) - 1/2$ the subtle Carleson-Sjölin type estimates in §3 and §5 are not needed. We do not presently know whether the Lorentz space $L^{p,1}$ in (1.8) can be replaced by L^p if $\max\{0, 1/2 - 2/p\} < \alpha(p) - \gamma < 1/2$.

Since L^p functions are locally square integrable if $p \ge 2$, the estimate in Theorem 1.2 implies an estimate for a less singular square function; namely

$$\Big\| \Big(\frac{1}{T}\int_0^T |U_t^\alpha f|^2 dt\Big)^{1/2} \Big\|_{p,2} \lesssim \|f\|_{p,2}$$

holds for $p > \frac{2n}{n-1}$, and $\alpha > n(\frac12 - \frac1p) - \frac12$. As it is well known such square functions can be used to derive various theorems for spectral multipliers; the connection is via the Fourier inversion formula

$$m(\sqrt{-\Delta}) = (2\pi)^{-1} \int \widehat{m}(\tau) \cos(\tau\sqrt{-\Delta}) d\tau \tag{1.10}$$

for even m. The estimates here are related to work on the disc multipliers on $\mathbb{R}^n$ by Rubio de Francia [27], Mockenhaupt [23], Córdoba [11] and Carbery, Romera and Soria [5]; these authors proved sharp $L^p_{\text{rad}} L^2_{\text{sph}}$ estimates. The methods in these papers does not seem to apply to the above mixed norm space-time estimates for the wave equation in $\mathbb{R}^n$, in fact they do not seem to yield the analogous bound for the above square function. We have been informed by A. Carbery that he and F. Soria [7] have recently and independently obtained a direct proof of an $L^p_{\text{rad}} L^2_{\text{sph}}$ estimate for a related square function, without first proving the stronger $L^p_{\text{time,rad}}(L^2_{\text{sph}})$ estimate.

A variant of Corollaries 1.3 and 1.4 can be used to derive stronger inequalities for multiplier transformations than those that would follow from square function estimates. We state such a theorem for the model case $\mathcal{N} = S^{n-1}$ and refer for more general statements to §6.

Theorem 1.6. *Let $K \in \mathcal{S}'(\mathbb{R}^n)$ be a radial convolution kernel whose Fourier transform has compact support in $\{\xi : R^{-1} \le |\xi| \le R\}$ for some $R > 1$. Suppose that $1 \le p \le \frac{2n}{n+1}$ and $\varepsilon > 0$. Then the inequality*

$$\|K * f\|_{L^p_{\rm rad}(L^2_{\rm sph})} \le C_{\varepsilon,R}\Big(\int \big[(1+|x|)^\varepsilon |K(x)|\big]^p dx\Big)^{1/p} \|f\|_{L^p_{\rm rad}(L^2_{\rm sph})} \tag{1.11}$$

holds for all $L^p_{\rm rad}(L^2_{\rm sph})$ functions.

This theorem is essentially sharp. Since K is radial the $L^p(L^2)$ boundedness of the convolution operator implies as a necessary condition the $L^p_{\rm rad}$ boundedness. One can test on radial Schwartz functions whose Fourier transform equals 1 on the support of $\widehat{K}$, and obtains as a necessary condition that $K \in L^p(\mathbb{R}^n)$.

The assumption that $\widehat{K}$ is compactly supported in $\mathbb{R}^n \setminus \{0\}$ can be relaxed by combining our methods with Calderón-Zygmund techniques. We hope to return to this point in a subsequent paper.

Organization of the paper: In §2 we shall discuss the expansion of the wave operator in terms of the eigenfunctions of the Laplacian on $\mathcal{N}$ and state the relevant formulas, asymptotics and estimates for the kernels. The kernels will be split into an oscillatory and a nonoscillatory part; the precise properties are proved in the Appendix A1-4. In §2 we also reduce the statement of Theorems 1.1, 1.2 and 1.5 to $L^p(\ell^2)$ estimates (*cf.* Theorems 2.3 and 2.5). The main idea is to use a vector-valued version of the Carleson-Sjölin theorem on oscillatory integral operators; an endpoint version of this is proved in §3. In §4 we give the estimates for the nonoscillatory contributions, and also prove estimates for the oscillatory terms for which the oscillation is not essential, such as the L^∞ estimates and the weak-type estimates (1.9). In §5 we use the Carleson-Sjölin estimates together with various scaling arguments to obtain improved bounds for the oscillatory kernels. In §6 various estimates for functions of the Laplacian on a conic manifold are proved, in particular §6 contains the proofs of Corollary 1.4 and of Theorem 1.6.

Notation: In sections 1 and 2 the L^p norms in the radial variable on $\Gamma(\mathcal{N})$ will be taken with respect to the measure $\rho^{n-1}d\rho$, as in the definition of $\|f\|_{p,2}$ above. However, in the various proofs in sections 3,4 and 5 we shall work with the standard L^p norm on $\mathbb{R}$, unless indicated otherwise. Given two quantities a and b we write $a \lesssim b$ if there is a positive constant C, such that $a \le Cb$. We write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$. The notation $\lesssim_\varepsilon$ indicates that in an estimate the constants are allowed to depend on a parameter (here ε). In estimates for families of operators which depend analytically on a parameter z in a vertical strip the constants in the inequalities are allowed to be $O(e^{A|\mathrm{Im}(z)|})$ for suitable positive A. We take $\widehat{f}(\xi) = \int f(y)e^{-i\langle y,\xi\rangle}dy$ as our definition for the Fourier transform.

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2. Preliminaries on the wave operator and a related analytic family

Let $\{\varphi_j\}_{j=1}^\infty$ be an orthonormal basis of eigenfunctions of $-\Delta_{\mathcal{N}}$ on $\mathcal{N}$, with corresponding eigenvalues α_j, and set

$$\lambda_j := (\alpha_j + (\tfrac{n-2}{2})^2)^{1/2}. \tag{2.1}$$

If f is smooth and compactly supported away from the boundary we may expand f as

$$f(r,\theta) = \sum_j \varphi_j(\theta) f_j(r). \tag{2.2}$$

Thus, if m is a bounded continuous function on $\mathbb{R}^+$, we have (see e.g. [9])

$$(m(\sqrt{-\Delta})f)(r,\theta) = \sum_j \varphi_j(\theta) \int_0^\infty K_{[m]}(r,\varrho,\lambda_j) f_j(\varrho) d\varrho\ , \tag{2.3}$$

where the kernel $K_{[m]}$ is given by

$$K_{[m]}(r,\varrho,\lambda) := r^{\frac{2-n}{2}}\rho^{\frac{n}{2}}\int_0^\infty m(s)J_\lambda(sr)J_\lambda(s\varrho)s\,ds. \tag{2.4}$$

The expression for $K_{[m]}$ becomes particularly simple for the analytic family of multipliers

$$m_z(s) := s^{-z}J_z(s) \tag{2.5.1}$$

so that

$$U_t^\alpha f = m_{\alpha-\frac12}(t\sqrt{-\Delta}) \tag{2.5.2}$$

if U_t^α is as in (1.6). The cases $z=-1/2$ and $z=1/2$ are relevant for the wave equation since $s^{1/2}J_{-1/2}(s) = (2/\pi)^{1/2}\cos(s)$ and $s^{-1/2}J_{1/2}(s) = (2/\pi)^{1/2}s^{-1}\sin s$. In particular the solution (1.2) is given by

$$u(t,r,\theta) = \sqrt{\frac{\pi}{2}}\Big[m_{-1/2}(t\sqrt{-\Delta})f(r,\theta) + t m_{1/2}(t\sqrt{-\Delta})g(r,\theta)\Big]. \tag{2.6}$$

We shall use the notation

$$K_{z,\lambda}(t,r,\rho) := K_{[m_z(t\cdot)]}(r,\varrho,\lambda) = r^{\frac{2-n}{2}}\rho^{\frac{n}{2}}t^{-z}\int_0^\infty J_z(ts)J_\lambda(sr)J_\lambda(s\varrho)s^{1-z}ds. \tag{2.7}$$

Explicit formulas for the expressions (2.7) are due to H.M. Macdonald [22]. Using his results the formula for $K_{z,\lambda}$, for the values of $0<\mathrm{Re}(z)<\lambda+1$, becomes then

$$K_{z,\lambda}(t,r,\rho) = \pi^{-1}r^{z-\frac n2}\rho^{\frac n2+z-1}t^{-2z}H_{z,\lambda}(\mu(t,r,\rho)) \tag{2.8}$$

where

$$\mu(t,r,\varrho) := \frac{r^2+\varrho^2-t^2}{2r\varrho} \tag{2.9}$$

and

$$H_{z,\lambda}(\mu) := \begin{cases} 0 & \text{if } \mu>1,\\ \frac{1}{\Gamma(z)}\int_0^{\arccos\mu}(\cos\theta-\mu)^{z-1}\cos(\lambda\theta)d\theta & \text{if } -1<\mu<1,\\ \frac{\sin(z\pi-\lambda\pi)}{\Gamma(z)}\int_{\operatorname{arcosh}(-\mu)}^\infty(\cosh s+\mu)^{z-1}e^{-\lambda s}ds & \text{if } \mu<-1. \end{cases} \tag{2.10}$$

One obtains (2.10) from [22] by using the Mehler-Dirichlet integral for the Legendre function $\mathrm{P}_{\lambda-1/2}^{-z+1/2}$, and an analogous formula for $Q_{\lambda-1/2}^{-z+1/2}$, see formula (A4.5) below. We note that in the special relevant case $z=1/2$ Cheeger and Taylor [9] derive the formula (2.10) using the Lipschitz-Hankel integral and analytic continuation.

The formulas (2.8/10) are valid for the range $0<\mathrm{Re}(z)<\lambda+1$, but one can extend them to values $\mathrm{Re}(z)\le 0$ by analytic continuation. To understand in particular the relevant case $z=-1/2$ we use the recursion formulas for Bessel functions (*cf.* [13, 7.2.8, (50)-(54)]); in particular

$$J_z(s) = J'_{z+1}(s) + \frac{z+1}{s}J_{z+1}(s).$$

Since $\frac{d}{dt}[t^{z+1}m_{z+1}(ts)] = J'_{z+1}(ts)s^{-z}$ we see that $m_z(t\cdot) = t^{-z}\frac{d}{dt}[t^{z+1}m_{z+1}(t\cdot)] + (z+1)m_{z+1}(t\cdot)$ and hence

$$K_{z,\lambda} = t^{-z}\frac{\partial}{\partial t}\big[t^{z+1}K_{z+1,\lambda}\big] + (z+1)K_{z+1,\lambda}. \tag{2.11}$$

If we take into account that $\mu_t = -t/(r\rho)$ we obtain for $\mathrm{Re}(z) > -1$

$$K_{z,\lambda}(t,r,\rho) = -\pi^{-1}t^{-2z}r^{z-\frac{n}{2}}\rho^{\frac{n}{2}+z-1}H'_{z+1,\lambda}(\mu(t,r,\rho)) \tag{2.12}$$

where the derivative is taken in the sense of distributions. Note that from (2.12) and (2.8)

$$H'_{z+1,\lambda} = -H_{z,\lambda} \tag{2.13}$$

which can also be checked directly from (2.10)

We shall also have to consider the case $\mathrm{Re}(z) \geq \lambda+1$ and then (2.10) is no longer available. However we may use the subordination formula $2^\nu\Gamma(\nu+1)J_{\mu+\nu+1}(t) = t^{\nu+1}\int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu ds$, $\nu > -1$; see Stein and Weiss [33, p. 170]. Applying this for $\mu = \alpha - 1/2$ one obtains

$$U_t^{\alpha+\nu+1} = \frac{1}{2^\nu\Gamma(\nu+1)}\int_0^1 s^{2\alpha}(1-s^2)^\nu U_{st}^\alpha\, ds \tag{2.14}$$

and of course an analogous formula replacing the operator U_τ^α by the kernels $K_{\alpha-1/2,\lambda}(\tau,r,\rho)$.

In order to understand the behavior of the kernels $K_{z,\lambda}$ it is useful to observe that

$$1-\mu(t,r,\rho) = \frac{t^2-(r-\rho)^2}{2r\rho} \tag{2.15.1}$$

$$1+\mu(t,r,\rho) = \frac{(r+\rho)^2-t^2}{2r\rho} \tag{2.15.2}$$

In particular, if we assume that $t > 0$ then the various restrictions on μ translate to

$$\mu(t,r,\rho) > 1 \iff t < |r-\varrho|, \tag{2.16.1}$$

$$-1 < \mu(t,r,\rho) < 1 \iff |r-\varrho| < t < r+\varrho, \tag{2.16.2}$$

$$\mu(t,r,\rho) < -1 \iff t > r+\varrho. \tag{2.16.3}$$

The fact that $H_{z,\lambda}(\mu) = 0$ for $\mu > 1$ is a reflection of finite propagation speed (for the cases $z = \pm 1/2$). Notice that if $\mathcal{N} = S^{n-1}$ and n is odd, then $\cos\lambda_j\pi = 0$ for every j [9, Cor. 2.3.], so that in this case there is no contribution in the region where $t > r+\varrho$. This is a reflection of the strong Huyghen's principle.

In the following two theorems we will state the basic asymptotics for the kernels $K_{z,\lambda}$. For the proof of Theorem 1.1 it is crucial to understand precisely the oscillatory nature of $H_{z,\lambda}(\mu)$ if $-1+\lambda^{-1} \leq \mu \leq 1-\lambda^{-1}$. We remark that asymptotics for Legendre functions are given in [13]; however we shall need more precise statements on the remainder terms which are uniform in λ. We note that some estimates in this spirit, for the main terms, are proved in the paper by Lindblad and Sogge [21]. Details about estimates and asymptotics are derived in the Appendix (§A1-4) to this paper.

Our first result deals with the wave operator $\cos(t\sqrt{-\Delta})$ (which after a normalization corresponds to the case $z = -1/2$).

Theorem 2.1. *Suppose $\lambda \ge (n-2)/2$, and $1 \le t \le 2$ and let*

$$K_\lambda(t,r,\rho) = r^{\frac{2-n}{2}} \rho^{\frac{n}{2}} \int_0^\infty \cos(ts) J_\lambda(sr) J_\lambda(s\varrho) s ds$$

so that $K_\lambda = (\pi/2)^{1/2} K_{-1/2,\lambda}$. Let $\zeta \in C_0^\infty(\mathbb{R})$ be even so that $\zeta(s) = 1$ for $|s| \le 1/2$ and $\zeta(s) = 0$ for $|s| \ge 3/4$. In what follows $\mu = \mu(t,r,\rho)$ as in (2.9). Then

$$K_\lambda = O_\lambda + R_{\lambda,1} + R_{\lambda,2} + R_{\lambda,3} + R_{\lambda,4} \tag{2.17}$$

where all distributions on the right hand side vanish if $\mu(t,r,\rho) > 1$. The function O_λ is supported where $-1 + (\lambda+1)^{-1} < \mu < 1 - (\lambda+1)^{-1}$ and is defined by

$$O_\lambda(t,r,\rho) = (2\pi)^{-1/2}(1 - \zeta((\lambda+1)\sqrt{1-\mu^2}))\chi_{(-1,1)}(\mu) \frac{\rho^{(n-3)/2} t}{r^{(n+1)/2}} \frac{\lambda^{1/2}}{(1-\mu^2)^{3/4}} \cos(\lambda \arccos \mu + \tfrac{\pi}{4}). \tag{2.18}$$

The distributions $R_{\lambda,1}$ and $R_{\lambda,2}$ are measures and principal value distributions, respectively, given for $t > 0$ by

$$R_{\lambda,1}(t,\rho,r) = \frac{1}{2}\Big(\frac{\rho}{r}\Big)^{(n-1)/2} \big[\delta(\rho - r - t) + \delta(\rho - r + t) + \sin(\lambda\pi)\delta(\rho + r - t)\big] \tag{2.19}$$

and

$$R_{\lambda,2}(t,r,\rho) = \frac{\cos(\lambda\pi)}{\pi} \frac{t}{t+r+\rho} \Big(\frac{\rho}{r}\Big)^{(n-1)/2} \text{p.v.} \frac{1}{\rho - t + r} \zeta(\lambda\sqrt{|1-\mu^2|}) \chi_{(-2,0)}(\mu). \tag{2.20}$$

The remainder term $\mathcal{R}_{\lambda,3}$ is locally integrable and supported where $-2 \le \mu \le 1$; it satisfies the estimate

$$|R_{\lambda,3}(t,r,\rho)| \le C \frac{\rho^{(n-3)/2}}{r^{(n+1)/2}} \frac{(1+\lambda)^{1/2}}{|1-\mu^2|^{3/4}} \frac{1}{1 + \lambda\sqrt{|1-\mu^2|}} \tag{2.21}$$

Finally $R_{\lambda,4}$ is supported where $\mu(t,r,\rho) \le -2$; it satisfies the estimate

$$|R_{\lambda,4}(t,r,\rho)| \le C(1+\lambda)^{1/2} \rho^{(n-3)/2} r^{-(n+1)/2} |\mu(t,r,\rho)|^{-\lambda-3/2}. \tag{2.22}$$

Theorem 2.2. *Suppose that $\varepsilon > 0$, $z = b + i\tau$, $b > -1/2$, $A \ge \pi(|b| + 1/2)$ and $\lambda \ge 0$. Let $K_{z,\lambda}(t,r,\rho)$ as in (2.8/12) and $\mu = \mu(t,r,\rho)$ as in (2.9). Suppose that $1 \le t \le 2$. Then*

$$K_{z,\lambda} = O_{z,\lambda} + V_{z,\lambda} + W_{z,\lambda} \tag{2.23}$$

where $O_{z,\lambda}$ is supported in $\{(t,r,\rho) : (\lambda+1)\sqrt{1-\mu^2} \ge 1, |\mu| < 1\}$ and

$$O_{z,\lambda}(t,r,\rho) = \pi^{-1} t^{-2z} r^{z-\frac{n}{2}} \rho^{\frac{n}{2}+z-1} (1-\mu^2)^{\frac{z-1}{2}} \lambda^{-z} \cos(\lambda \arccos \mu - z\tfrac{\pi}{2}). \tag{2.24}$$

The function $V_{z,\lambda}$ is supported where $-2 \le \mu(t,r,\rho) \le 1$. If $\lambda\sqrt{|1-\mu^2|} \le 1$ it satisfies the estimates

(2.25.1)

$$|V_{z,\lambda}(t,r,\rho)| \le C_A e^{A|\tau|} \rho^{\frac{n}{2}+b-1} r^{b-\frac{n}{2}} (1-\mu^2)^{\frac{2b-1}{2}} \qquad \textit{if } 0 \le \mu \le 1,$$

(2.25.2)

$$|V_{z,\lambda}(t,r,\rho)| \le C_A e^{A|\tau|} \rho^{\frac{n}{2}+b-1} r^{b-\frac{n}{2}} \times \begin{cases} |1-\mu^2|^{\frac{2b-1}{2}} & \textit{if } -1/2 < b < 1/2,\ -2 \le \mu \le 0, \\ \log(\frac{1}{(1+\lambda)\sqrt{|1-\mu^2|}}) & \textit{if } b = 1/2,\ -2 \le \mu \le 0, \\ (1+\lambda)^{1-2b} & \textit{if } b > 1/2,\ -2 \le \mu \le 0. \end{cases}$$

If $(1+\lambda)\sqrt{|1-\mu^2|} \geq 1$ *then*

$$|V_{z,\lambda}(t,r,\rho)| \leq C_A e^{A|\tau|}\rho^{\frac{n}{2}+b-1}r^{b-\frac{n}{2}}(1+\lambda)^{-b-1}|1-\mu^2|^{(b-2)/2} \tag{2.26}$$

The function $W_{z,\lambda}$ *is supported where* $\mu(t,r,\rho) \leq -2$ *and if* $\lambda > b-1+\varepsilon$ *it satisfies the estimate*

$$|W_{z,\lambda}(t,r,\rho)| \leq C_{b,\varepsilon}e^{A|\tau|}\rho^{\frac{n}{2}+b-1}r^{b-\frac{n}{2}}(1+\lambda)^{-b}|\mu|^{-(\lambda-b+1)} \quad \text{if } \mu \leq -2. \tag{2.27}$$

We now formulate our main technical result which will be proved in sections 4 and 5.

Define

$$S_{z,j}g(t,r) := \langle K_{z,\lambda_j}(t,r,\cdot), g\rangle \tag{2.28}$$

where the pairing $\langle\cdot,\cdot\rangle$ is the standard pairing on $\mathbb{R}$, with respect to Lebesgue measure $d\rho$ and g is smooth and compactly supported in $(0,\infty)$. L^p norms on $\mathbb{R}_+$ in the following theorem are taken with respect to the measure $r^{n-1}dr$.

Theorem 2.3. *Let* I *be a compact interval contained in* $(1,2)$ *and let* $z = b + i\tau$, *with* $-1/2 < b < n/2$, $\tau \in \mathbb{R}$ *or* $b = -1/2$ *and* $\tau = 0$, *and assume* $\gamma < (n-1)/p'$. *Let* $A > \pi(|b|+1/2)$. *Then the following inequalities hold.*

(i) Suppose that $2 \leq p < \frac{2n}{n-1}$ *and* $\varepsilon > 0$. *Then*

$$\Big(\int_I \Big\|\Big(\sum_j |S_{-1/2,j}f_j|^2\Big)^{1/2}\Big\|_p^p dt\Big)^{1/p} \leq C_{p,\varepsilon}\Big(\int_0^\infty \Big(\sum_j |(1+\lambda_j/\rho)^\varepsilon f_j(\rho)|^2\Big)^{p/2}\rho^{n-1}d\rho\Big)^{1/p}. \tag{2.29}$$

(ii) Suppose that $\frac{2n}{n-1} < p \leq 4$, $n(\frac{1}{2}-\frac{1}{p}) - \frac{1}{2} < b + \frac{1}{2}$ *and* $\gamma < b + \frac{1}{2}$. *Then*

$$\Big(\int_I \Big\|\Big(\sum_j |S_{z,j}f_j|^2\Big)^{1/2}\Big\|_p^p dt\Big)^{1/p} \leq C_{p,\gamma}e^{A|\tau|}\Big(\int_0^\infty \Big(\sum_j |(1+\lambda_j/\rho)^{-\gamma}f_j|^2\Big)^{p/2}\rho^{n-1}d\rho\Big)^{1/p}. \tag{2.30}$$

(iii) Suppose that $4 < p < \infty$ *and* $n(\frac{1}{2}-\frac{1}{p}) - \frac{1}{2} < b + \frac{1}{2}$. *Then*

$$\Big(\int_I \Big\|\Big(\sum_j |S_{z,j}f_j|^2\Big)^{1/2}\Big\|_p^p dt\Big)^{1/p} \leq C_{p,\gamma}e^{A|\tau|}\Big(\int_0^\infty \Big(\sum_j |(1+\lambda_j/\rho)^{-b-\frac{2}{p}}f_j|^2\Big)^{p/2}\rho^{n-1}d\rho\Big)^{1/p}. \tag{2.31}$$

(iv) Suppose that $b \geq \frac{n-2}{2}$. *Then*

$$\sup_{t\in I}\sup_{r>0}\Big(\sum_j |S_{z,j}f_j(t,r)|^2\Big)^{1/2} \leq Ce^{A|\tau|}\sup_{\rho>0}\Big(\sum_j |(1+\lambda_j/\rho)^{-b}f_j(\rho)|^2\Big)^{1/2}. \tag{2.32}$$

We apply Parseval's identity with respect to the spectral decomposition of $-\Delta_{\mathcal{N}}$ and obtain the following consequence

Corollary 2.4. *Theorem 2.3 implies the strong type estimates of Theorem 1.1 and 1.2 for the case where* $T = 2$ *and integration over* $[T,2T]$ *is replaced by integration over a compact subinterval of* $(1,2)$.

The (restricted) weak type inequalities (1.8), (1.9) for the endpoint $b + 1/2 = n(1/2-1/p) - 1/2$ in Theorem 1.5 will follow from

Theorem 2.5. *Let I be a compact interval contained in $(1,2)$ and μ_n denote the measure $dt\, r^{n-1}dr$. Let $G_j^{-\gamma}(f_j)(\rho) = (1+\lambda_j/\rho)^{-\gamma} f_j(\rho)$ and let $z = b + i\tau$, $-1/2 < b < n/2$, $\tau \in \mathbb{R}$ or $b = -1/2$ and $\tau = 0$.*

Then the restricted weak type estimate

$$\mu_n\big(\{(t,r) : \chi_I(t)\big(\sum_j |S_{z,j}f_j(t,r)|^2\big)^{1/2} > \beta\}\big) \le C_\gamma^p \beta^{-p} \|\{G_j^{-\gamma} f_j\}\|^p_{L^{p,1}(\ell^2,\rho^{n-1}d\rho)} \tag{2.33}$$

holds in the following cases:

(a) $b = -1/2$, $\tau = 0$, $p = 2n/(n-1)$, $\gamma < 0$;

(b) $0 < b + 1/2 = n(1/2 - 1/p) - 1/2$, $2n/(n-1) < p \le 4$, $\gamma < b + 1/2$;

(c) $0 < b + 1/2 = n(1/2 - 1/p) - 1/2$, $4 < p < \infty$, $\gamma \le b + 2/p$.

The weak type estimate

$$\mu_n\big(\{(t,r) : t \in I,\ \big(\sum_j |S_{z,j}f_j(t,r)|^2\big)^{1/2} > \beta\}\big) \le C_\gamma^p \beta^{-p} \|\{G_j^{-\gamma} f_j\}\|^p_{L^{p}(\ell^2,\rho^{n-1}d\rho)} \tag{2.34}$$

holds in the following cases:

(e) $b = -1/2$, $\tau = 0$, $p = 2n/(n-1)$, $\gamma \le -1/2$;

(f) $0 < b + 1/2 = n(1/2 - 1/p) - 1/2$, $2n/(n-1) < p < \infty$, $\gamma \le b$.

To obtain the results of Theorems 1.1, 1.2 and 1.5 from Theorems 2.3 and 2.5 one can use the conical structure and employ

Scaling arguments. For a multiplier m let

$$\mathcal{M}_j g(t,r) = \langle K_{[m(t\cdot)]}(r,\cdot,\lambda_j), g\rangle$$

where the pairing $\langle\cdot,\cdot\rangle$ is the standard pairing on $\mathbb{R}$, with respect to Lebesgue measure $d\rho$ and g is smooth and compactly supported in $(0,\infty)$.

A change of variable in formula (2.4) shows that

$$K_{[m(T\cdot)]}(r,\rho,\lambda) = \frac{1}{T} K_{[m]}\big(\frac{r}{T}, \frac{\rho}{T}, \lambda\big) \tag{2.35}$$

and, consequently,

$$\mathcal{M}_j g(t,r) = \mathcal{M}_j[g(T\cdot)]\big(\frac{t}{T}, \frac{r}{T}\big). \tag{2.36}$$

Let us now assume that

$$\Big(\int_1^2 \int_0^\infty \Big(\sum_j |\mathcal{M}_j f_j(t,r)|^2\Big)^{p/2} r^{n-1}dr\, dt\Big)^{1/p} \le A\Big(\int_0^\infty \Big(\sum_j \big|(1+\tfrac{\lambda_j}{\rho})^{-\gamma} f_j(\rho)\big|^2\Big)^{p/2} \rho^{n-1}d\rho\Big)^{1/p}. \tag{2.37}$$

holds. Then

$$\begin{aligned}
&\Big(\frac{1}{T}\int_T^{2T} \int_0^\infty \Big(\sum_j |\mathcal{M}_j f_j(t,r)|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p} \\
&= T^{n/p}\Big(\int_1^2 \int_0^\infty \Big(\sum_j |\mathcal{M}_j[f_j(T\cdot)](t,r)|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p} \\
&\le AT^{n/p}\Big(\int_0^\infty \Big(\sum_j \big|(1+\tfrac{\lambda_j}{\rho})^{-\gamma} f_j(T\rho)\big|^2\Big)^{p/2} \rho^{n-1}d\rho\Big)^{1/p} \\
&= A\Big(\int_0^\infty \Big(\sum_j \big|(1+\frac{T\lambda_j}{\rho})^{-\gamma} f_j(\rho)\big|^2\Big)^{p/2} \rho^{n-1}d\rho\Big)^{1/p}. \qquad (2.38)
\end{aligned}$$

Now we wish to replace the intervals $[T/2, T]$ by $[0, T]$ to obtain the assertions in Theorem 1.1 and Corollary 1.3. We can apply the previous estimate to the time intervals $[2^{-k-1}T, 2^{-k}T]$, and summing a geometrical series we obtain

$$(2.39)\quad \Big(\frac{1}{T}\int_0^T \int_0^\infty \Big(\sum_j |\mathcal{M}_j f_j(t,r)|^2\Big)^{p/2} r^{n-1} dr dt\Big)^{1/p} \lesssim A\Big(\int_0^\infty \Big(\sum_j \big|(1+T\lambda_j/\rho)^{-\gamma} f_j(\rho)\big|^2\Big)^{p/2} \rho^{n-1} d\rho\Big)^{1/p}.$$

(2.39) and Parseval's identity with respect to the spectral decomposition of $\Delta_{\mathcal{N}}$ show that in Theorems 1.1 and 1.2 it suffices to prove the asserted statement for $T = 1$. An analogous reduction applies to weak type inequalities. Thus we have reduced the proof of the results in the introduction to the proofs of Theorems 2.3/5.

3. A variant of the Carleson-Sjölin theorem

Let $\phi \in C^\infty(\mathbb{R}^2 \times \mathbb{R})$ be a smooth, real phase function and let $a \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R})$ be a compactly supported amplitude. Define the oscillatory integral operator $T_\lambda : L^1_{\text{loc}}(\mathbb{R}) \to L^\infty(\mathbb{R}^2)$ by

$$(3.1)\qquad T_\lambda f(z) = \int e^{i\lambda\phi(z,y)} a(z,y) f(y) dy.$$

A main tool in the proof of Theorem 1.1 will be a vector-valued variant of the Carleson-Sjölin theorem ([8]). It is assumed that the *Carleson-Sjölin determinant*

$$(3.2)\qquad CS[\phi] := \det\begin{pmatrix} \phi_{z_1 y} & \phi_{z_2 y} \\ \phi_{z_1 yy} & \phi_{z_2 yy} \end{pmatrix}$$

does not vanish on the support of a; the geometric interpretation is that for fixed z the curve $y \mapsto d_z\phi(y)$ in $T_z^*\mathbb{R}^2$ has nonvanishing curvature (*cf.* [24] for a general discussion).

Theorem 3.1. *Suppose that $|CS[\phi]| \geq c_0 > 0$ on the support of a.*

(i) Suppose that $2 \leq p \leq 4$ and let

$$w_p(\lambda) = \frac{\log^{1/2-1/p}(2+|\lambda|)}{(1+|\lambda|)^{1/2}}.$$

Then

$$(3.3)\qquad \Big\|\Big(\sum_j |T_{\lambda_j} f_j|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^2)} \leq C\Big\|\Big(\sum_j w_p^2(|\lambda_j|)\,|f_j|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R})}.$$

(ii) Suppose that $4 < q \leq \infty$ and $p \geq q/(q-3)$. Then

$$(3.4)\qquad \Big\|\Big(\sum_j |T_{\lambda_j} f_j|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^2)} \leq C(q/(q-4))^{1/4}\Big\|\Big(\sum_j (1+\lambda_j)^{-\frac{4}{q}}|f_j|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R})}.$$

Moreover, there is some $N \in \mathbb{N}$ such that C in (3.3), (3.4) depends only on c_0^{-1}, the C^N-norm of ϕ on the support of a, the C^N-norm of a and the diameter of the support of a.

Remark. The scalar version of this theorem is due to Carleson and Sjölin [8]; see also Hörmander [17] for a slightly improved version and a simpler proof. We follow the idea of Fefferman-Stein and Carleson-Sjölin according to which one should examine the $L^{q/2}$ norm of $\sum_{j,j'} |T_{\lambda_j} f_j T_{\lambda_{j'}} f_{j'}|^2$. Because of the occurence of mixed terms with $j \neq j'$ the vector-valued case can apparently not be proved by a straightforward adaptation of the proofs in the scalar situation. Moreover we did not find a vector-valued analogue of Hörmander's argument to deduce the estimate for $2 \leq p \leq 4$ from the estimate for $p > 4$.

We shall first deduce Theorem 3.1 from the following Proposition 3.2 and then give the proof of the proposition.

Proposition 3.2. *Let $\lambda > 2$, let ϕ satisfy $\big|CS[\phi]\big| \ge c_0 > 0$ on the support of a and let $\lambda^{-1} \le \kappa \le 4$. Let $b \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R})$. For $g \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R})$ define*

$$S^{\lambda,\kappa} g(z) = \iint e^{i\lambda(\phi(z,y)+\kappa\phi(z,\tilde y))} b(z,y,\tilde y) g(y,\tilde y) dy d\tilde y.$$

Then

$$\|S^{\lambda,\kappa} g\|^2_{L^2(\mathbb{R}^2)} \le C\lambda^{-2}\kappa^{-1} \iint |g(y,\tilde y)|^2 \frac{\lambda\kappa}{1+\lambda\kappa|y-\tilde y|} dy\, d\tilde y \tag{3.5}$$

where the constant C is independent of λ and κ. Moreover, there is some $N \in \mathbb{N}$ such that C depends only on c_0^{-1}, the C^N-norm of ϕ on the support of b, the C^N-norm of b and the diameter of the support of b.

Proof of Theorem 3.1. The estimates for the terms with $|\lambda_j| < 2$ is trivial, so we assume that $\lambda_j \ge 2$.

We begin by deriving (3.4). Since the Schwartz kernels of all operators involved are supported in a compact set we may assume that $q = 3p'$, by Hölder's inequality. For $1 \le r \le 2$ we have the inequality

$$\Big\| \Big(\sum_{j,k} |S^{\lambda_j, \frac{\lambda_k}{\lambda_j}} g_{jk}|^2 \Big)^{\frac12} \Big\|_{r'} \lesssim \Big(\iint \Big(\sum_{j,k} \big|(\lambda_j\lambda_k)^{\frac1r - 1} g_{jk}(y,\tilde y)\big|^2 \Big)^{\frac r2} |y-\tilde y|^{-\frac{r}{r'}} dy d\tilde y \Big)^{\frac1r}.$$

Indeed for $r = 2$ this follows from Proposition 3.2 and Fubini's theorem. For $r = 1$ this inequality follows by applying Minkowski's inequality. The general case follows by analytic interpolation for weighted L^p spaces of ℓ^2 valued functions. Now let $G_r(y) = (\sum_j |\lambda_j^{-1+1/r} f_j(y)|^2)^{1/2}$. We apply the previous inequality with $g_{jk}(y,\tilde y) = f_j(y) f_k(\tilde y)$. This yields the estimate

$$\Big(\int \Big(\sum_j |T_{\lambda_j} f_j(x)|^2 \Big)^{r'} dx \Big)^{\frac{1}{r'}} \lesssim \Big(\iint |G_r(y)|^r |G_r(\tilde y)|^r |y-\tilde y|^{1-r} dy d\tilde y \Big)^{\frac1r},$$

If $q = 2r' = 3p'$ then $p = 2r/(3-r)$. Now we proceed exactly as in Hörmander [17] and an application of the standard $L^s \to L^{s'}$ inequality for the fractional integration operator, with $s = p/r$, yields the assertion.

We now assume $2 \le p \le 4$ and prove (3.3). We write the left hand side of (3.3) as a sum of two terms one involving only positive and one involving only negative λ_j's. In order to show the inequality for the case $p = 2$, we shall just use the assumption that $\nabla_z \Phi'_y \ne 0$; without loss of generality we may assume that $\phi''_{z_1 y} \ne 0$. We can then apply Hörmander's L^2-estimate ([17], *cf.* also [32, p. 377]), to see that the oscillatory integral operator T_{λ,z_2} defined by $T_{\lambda,z_2} h(z_1) = T_\lambda h(z_1,z_2)$ is bounded on $L^2(\mathbb{R})$ with norm $O(\lambda^{-1/2})$, uniformly in z_2. This implies that T_λ is bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$ with norm $O(\lambda^{-1/2})$ and the inequality

$$\Big\| \Big(\sum_{\lambda_j \ge 1} |T_{\lambda_j} f_j|^2 \Big)^{1/2} \Big\|_{L^2(\mathbb{R}^2)} \lesssim \Big\| \Big(\sum_j |\lambda_j^{-1/2} f_j|^2 \Big)^{1/2} \Big\|_{L^2(\mathbb{R})}$$

is an immediate consequence. We now show the case $p = 4$ of Theorem 3.1, the case $2 < p < 4$ follows then by interpolation.

Let for $m = 1, 2, \dots$

$$F_m(y) = \chi_I(y) 2^{-m/2} \Big(\sum_{2^m \le \lambda_j < 2^{m+1}} |f_j(y)|^2 \Big)^{1/2}$$

where χ_I is the characteristic function of the interval I and I contains the support of $a(x,\cdot)$ for every x. Then

$$\begin{aligned}
\Big\|\Big(\sum_j |T_{\lambda_j} f_j|^2\Big)^{1/2}\Big\|_4^4 &\le 2\int \sum_{\substack{j,k\\ j\le k}} |T_{\lambda_j} f_j\, T_{\lambda_k} f_k|^2 dx \\
&\lesssim \sum_{\substack{m,n\ge 0\\ m\le n}} \sum_{\substack{2^m\le\lambda_j<2^{m+1}\\ 2^n\le\lambda_k<2^{n+1}}} \big\|S^{\lambda_k,\frac{\lambda_j}{\lambda_k}}(f_j\otimes f_k)\big\|_2^2 \\
&\lesssim \sum_{\substack{m,n\ge 0\\ m\le n}} \int \frac{2^m |F_m(\tilde y) F_n(y)|^2}{1+2^m|y-\tilde y|} dy d\tilde y \qquad (3.6)
\end{aligned}$$

by Proposition 3.2. Next, (3.6) is equal to

$$\iint_{\substack{y\in I\\ |h|\le|I|}} \sum_n |F_n(y)|^2 \sum_{m\le n} \frac{2^m|F_m(y+h)|^2}{1+2^m|h|} dy\, dh \le \mathcal{E}_1+\mathcal{E}_2$$

where

$$\mathcal{E}_1 = \iint_{\substack{y\in I\\ |h|\le|I|}} \sum_n |F_n(y)|^2 \sum_{\substack{m\le n\\ 2^m\le|h|^{-1}}} 2^m |F_m(y+h)|^2 dy dh,$$

$$\mathcal{E}_2 = \iint_{\substack{y\in I\\ |h|\le|I|}} \sum_n |F_n(y)|^2 \sum_{\substack{m\le n\\ 2^m\ge|h|^{-1}}} |F_m(y+h)|^2 dy |h|^{-1} dh.$$

We first estimate $\mathcal{E}_1$ which is slightly better behaved than $\mathcal{E}_2$. Let $\alpha>1/8$. Then

$$\begin{aligned}
\mathcal{E}_1 &\le \sum_m \int_{|h|\le 2^{-m}} \int |F_m(y+h)|^2 2^m m^{-2\alpha} \sum_{n\ge m} n^{2\alpha} |F_n(y)|^2 dy\, dh \\
&\le \sum_m \int_{|h|\le 2^{-m}} 2^m m^{-2\alpha} \Big(\int |F_m(y)|^4 dy\Big)^{1/2} \Big(\int \big(\sum_n n^{2\alpha}|F_n(z)|^2\big)^2 dz\Big)^{1/2} dh \\
&\le \sum_m m^{-4\alpha} \Big(\int |m^\alpha F_m(y)|^4 dy\Big)^{1/2} \Big(\int \big(\sum_n |n^\alpha F_n(z)|^2\big)^2 dz\Big)^{1/2}
\end{aligned}$$

and, since $8\alpha>1$, we apply the Cauchy-Schwarz inequality and obtain

$$\begin{aligned}
\sum_m m^{-4\alpha}\Big(\int |m^\alpha F_m(y)|^4 dy\Big)^{1/2} &\le C_\alpha \Big(\sum_m \int |m^\alpha F_m(y)|^4 dy\Big)^{1/2} \\
&\le C_\alpha \Big(\int \big(\sum_m |m^\alpha F_m(y)|^2\big)^2 dy\Big)^{1/2}.
\end{aligned}$$

This yields the estimate

$$\mathcal{E}_1 \le C_\alpha \int \Big[\sum_n |n^\alpha F_n(y)|^2\Big]^2 dy, \qquad \alpha>1/8, \qquad (3.7)$$

which we need for the exponent $\alpha = 1/4$.

Next we estimate $\mathcal{E}_2$. Hölder's inequality yields

$$\begin{aligned}\mathcal{E}_2 &\le \int_{|h|\le|I|}\int \sum_{\substack{m\\ 2^m\ge|h|^{-1}}} |F_m(y+h)|^2 \sum_{\substack{n\\ 2^n\ge|h|^{-1}}} |F_n(y)|^2 dy\frac{dh}{|h|}\\ &\le \int_{|h|\le|I|}\Big(\int\Big[\sum_{\substack{m\\ 2^m\ge|h|^{-1}}} |F_m(z)|^2\Big]^2 dz\Big)^{1/2}\Big(\int\Big[\sum_{\substack{n\\ 2^n\ge|h|^{-1}}} |F_m(y)|^2\Big]^2 dy\Big)^{1/2}\frac{dh}{|h|}\\ &\le \int_{|h|\le|I|}\int\Big[\sum_{\substack{m\\ 2^m\ge|h|^{-1}}} |F_m(y)|^2\Big]^2 dy\frac{dh}{|h|}.\end{aligned}$$

We interchange the order of integration and apply Minkowski's inequality to obtain

$$\begin{aligned}\mathcal{E}_2 &\le \int_I\int_{|h|\le|I|}\Big[\sum_{\substack{m\\ 2^m\ge|h|^{-1}}} |F_m(y)|^2\Big]^2\frac{dh}{|h|}dy\\ &\le \int_I\Big[\sum_n\Big(\int_{2^{-n}\le|h|\le|I|}\frac{dh}{|h|}\Big)^{1/2}|F_n(y)|^2\Big]^2 dy\\ &\lesssim \int\Big[\sum_n |n^{1/4}F_n(y)|^2\Big]^2 dy \qquad (3.8)\end{aligned}$$

and (3.7), (3.8) imply the desired estimate for $p = 4$. $\square$

Proof of Proposition 3.2. We may replace λ by λc_0 and ϕ by ϕ/c_0 and assume that $c_0 \ge 1$. Introducing a partition of unity, we may assume that b is supported in a set of positive diameter, which will be chosen appropriately small. As in the standard proof of the Carleson-Sjölin theorem we shall split the integral defining S into the region where $y > \tilde{y}$ and the region where $y < \tilde{y}$. More precisely, we introduce a dyadic partition of unity $\{\chi_\ell\}_{\ell\in\mathbb{Z}}$ on $\mathbb{R}^+$ by choosing $\chi \in C_0^\infty(\mathbb{R})$ supported in $[1/2, 2]$ such that $\sum_{\ell\in\mathbb{Z}}\chi(2^\ell s) = 1$ for every $s > 0$; we then define $\chi_\ell(s) = \chi(2^\ell s)$. Let

$$S_\ell^\pm(g)(z) = \int e^{i\lambda[\phi(z,y)+\kappa\phi(z,\tilde{y})]}\chi_\ell(\pm(y-\tilde{y}))b(z,y,\tilde{y})g(y,\tilde{y})dyd\tilde{y}\ .$$

Then

$$S = \sum_{\ell:2^\ell\le\lambda\kappa} S_\ell^+ + \sum_{\ell:2^\ell\le\lambda\kappa} S_\ell^- + R.$$

Since $\nabla_z\phi_y(z,y) \ne 0$ we can use standard L^2 estimates for oscillatory integrals [17] on $\mathbb{R}$ and obtain that for $0 < \kappa \le 4$

$$\|Rg\|_2^2 \lesssim \lambda^{-1}\int_{|y-\tilde{y}|\le 4(\lambda\kappa)^{-1}} |g(y,\tilde{y})|^2 dyd\tilde{y}$$

and $\|Rg\|_2^2$ is controlled by the right hand side of (3.5).

It remains to show

$$\Big\|\sum_{\ell:2^\ell\le\lambda\kappa} S_\ell^\pm g_\ell\Big\|_2 \lesssim \Big(\sum_\ell\Big[2^{\ell/2}\lambda^{-1}\kappa^{-1/2}\|g_\ell\|_2\Big]^2\Big)^{1/2}. \qquad (3.9_\pm)$$

The inequality claimed in the statement of the proposition is an easy consequence. Namely let $\eta \in C_0^\infty([1/4,4])$ be equal to 1 on $[1/2,2]$ and let $\eta_\ell = \eta(2^\ell \cdot)$. Then $\chi_\ell = \chi_\ell \eta_\ell$ and therefore $S_\ell^+ g = S_\ell^+ g_\ell$ with $g_\ell(y,\tilde{y}) = g(y,\tilde{y})\eta_\ell(y-\tilde{y})$. (3.9$_+$) implies then

$$\Big\| \sum_{2^\ell \le \lambda\kappa} S_\ell^+ g \Big\|_2 \lesssim \lambda^{-1}\kappa^{-1/2} \Big(\int |g(y,\tilde{y})|^2 \sum_{2^\ell \le \lambda\kappa} 2^\ell |\eta^2(2^\ell(y-\tilde{y}))| dy d\tilde{y} \Big)^{1/2}$$

which is controlled by the right hand side of (3.5). The sum $\sum_\ell S_\ell^- g$ is handled similarly.

In order to finish the proof, we have to establish the main inequality (3.9$_+$). The left hand side of (3.9$_+$) is dominated by

$$C\Big(\sum_{2^\ell \le \lambda\kappa} \|S_\ell^+ g_\ell\|_2^2 \Big)^{1/2} + C\Big(\sum_{2^\ell \le \lambda\kappa} \sum_{m < \ell - 100} \|(S_\ell^+)^* S_m^+ g_m\|_2 \|g_\ell\|_2 \Big)^{1/2}$$

and therefore (3.9$_+$) follows from the inequalities

$$\|S_\ell^+ g\|_2 \lesssim 2^{\ell/2}\kappa^{-1/2}\lambda^{-1}\|g\|_2 \tag{3.10}$$

and

$$\|(S_\ell^+)^* S_m^+ g\|_2 \lesssim 2^m \kappa^{-1}\lambda^{-2}\|g\|_2 \quad \text{if } m < \ell - 20. \tag{3.11}$$

Henceforth we shall write $S_\ell := S_\ell^+$.

The inequalities (3.10), (3.11) are proved by employing a technique of Phong and Stein [26] which was used to obtain L^p-Sobolev estimates for averaging operators with folding canonical relations. Moreover we have to use almost orthogonality arguments such as in [29, §4].

For the proof of (3.10) we fix some large positive integer $M \ge 10$ and choose $\eta \in C_0^\infty(\mathbb{R}^2)$ supported in $(-1,1)^2$ such that $\sum_{k\in\mathbb{Z}^2} \eta(y-k_1, y-k_2) = 1$ everywhere, and put $\eta_{\ell,n}(y_1,y_2)) := \eta(2^{\ell+M}y_1 - n_1, 2^{\ell+M}y_2 - n_2)$. Then for each ℓ the family $\{\eta_{\ell,n}\}_{n\in\mathbb{Z}^2}$ is a partition of unity such that each $\eta_{\ell,n}$ is supported on a square with sidelength $2^{-\ell-M+1}$. We define $S_{\ell,n}$ in the same way as S_ℓ, only with b_ℓ replaced by

$$b_{\ell,n}(z,y,\tilde{y}) := \chi_\ell(y-\tilde{y})\eta_{\ell,n}(y,\tilde{y})b(z,y,\tilde{y}).$$

Then $S_\ell = \sum_n S_{\ell,n}$, hence

$$S_\ell^* S_\ell = \sum_{\nu,n} S_{\ell,\nu}^* S_{\ell,n}. \tag{3.12}$$

The integral kernel $K_{\nu,n}^\ell$ of $S_{\ell,\nu}^* S_{\ell,n}$ is given by

$$K_{\nu,n}^\ell(x,\tilde{x},y,\tilde{y}) = \chi_\ell(y-\tilde{y})\chi_\ell(x-\tilde{x})\eta_{\ell,\nu}(x,\tilde{x})\eta_{\ell,n}(y,\tilde{y}) \int e^{i\lambda\psi(z,x,\tilde{x},y,\tilde{y})}\gamma(z,x,\tilde{x},y,\tilde{y})dz, \tag{3.13}$$

where

$$\begin{aligned} \psi(z,x,\tilde{x},y,\tilde{y}) &:= \phi(z,y) + \kappa\phi(z,\tilde{y}) - \phi(z,x) - \kappa\phi(z,\tilde{x}), \\ \gamma(z,x,\tilde{x},y,\tilde{y}) &:= b(z,y,\tilde{y})\overline{b(z,x,\tilde{x})}. \end{aligned} \tag{3.14}$$

Similarly the integral kernel $K^{\ell,m}$ of $(S_\ell)^* S_m$ is given by

$$K^{\ell,m}(x,\tilde{x},y,\tilde{y}) = \chi_m(y-\tilde{y})\chi_\ell(x-\tilde{x}) \int e^{i\lambda\psi(z,x,\tilde{x},y,\tilde{y})}\gamma(z,x,\tilde{x},y,\tilde{y})dz. \tag{3.15}$$

In order to estimate K_{n_1,n_2}^ℓ, and $K^{\ell,m}$ we examine the Taylor expansion about the diagonal $y = x$, $\tilde{y} = \tilde{x}$, of ψ_z' and its higher order z-derivatives. To simplify the notation we shall consider

$$G(x,\tilde{x},y,\tilde{y}) = F(y) - F(x) + \kappa(F(\tilde{y}) - F(\tilde{x})) \tag{3.16}$$

and use the following calculus lemma.

Lemma 3.3. *Suppose that $F = (F_1, F_2) : I \to \mathbb{R}^2$ is of class C^4 on the compact interval I and suppose that $2^{-m-1} \le y - \tilde{y} \le 2^{-m+1}$, $2^{-\ell-1} \le x - \tilde{x} \le 2^{-\ell+1}$ and that $c_1 > 0$. Then there are positive ε, A, C so that the following statements hold under the assumption that $x, \tilde{x}, y, \tilde{y}$ belong to a set of diameter $< \varepsilon$.*

(i) If $|m - \ell| > 10$ then

(3.17.1)
$$|G(x, \tilde{x}, y, \tilde{y})| \le C\big(|y - x + \kappa(\tilde{y} - \tilde{x})| + |(y - x)^2 + \kappa(\tilde{y} - \tilde{x})^2 - 2\kappa(x - \tilde{x})(\tilde{y} - \tilde{x})|\big);$$

(3.17.2)
$$|G(x, \tilde{x}, y, \tilde{y})| \le C\big(|y - x + \kappa(\tilde{y} - \tilde{x})| + |(y - x)^2 + \kappa(\tilde{y} - \tilde{x})^2 + 2\kappa(y - \tilde{y})(\tilde{y} - \tilde{x})|\big);$$

moreover if also

$$\Big|\det\begin{pmatrix} F_1' & F_2' \\ F_1'' & F_2'' \end{pmatrix}\Big| \ge c_1 \tag{3.18}$$

then

(3.19.1)
$$|G(x, \tilde{x}, y, \tilde{y})| \ge A\big(|y - x + \kappa(\tilde{y} - \tilde{x})| + |(y - x)^2 + \kappa(\tilde{y} - \tilde{x})^2 - 2\kappa(x - \tilde{x})(\tilde{y} - \tilde{x})|\big)$$

(3.19.2)
$$|G(x, \tilde{x}, y, \tilde{y})| \ge A\big(|y - x + \kappa(\tilde{y} - \tilde{x})| + |(y - x)^2 + \kappa(\tilde{y} - \tilde{x})^2 + 2\kappa(y - \tilde{y})(\tilde{y} - \tilde{x})|\big).$$

(ii) Let $\ell = m$. There is a positive integer M such that (3.17.1) and (3.17.2) hold if either

(3.20) $|x - y| \le 2^{-\ell-M}$ *and* $|\tilde{x} - \tilde{y}| \le 2^{-\ell-M}$,

or

(3.21) $|x - y| \ge 2^{-\ell+M}$ *or* $|\tilde{x} - \tilde{y}| \ge 2^{-\ell+M}$.

Moreover in both cases (3.19.1) and (3.19.2) hold under the additional assumption (3.18).

Proof of Lemma 3.3. We shall only verify the inequalities (3.17.1) and (3.19.1); (3.17.2) and (3.19.2) follow then by symmetry considerations. Let

$$r_j(F; x, y) = \frac{1}{(j-1)!}\int_0^1 (1 - s)^{j-1} F^{(j)}(x + s(y - x))ds,$$

the integral occuring in the integral remainder of Taylor's formula. Consider the Taylor expansion of F about the diagonal $y = x$, $\tilde{y} = \tilde{x}$. Then

$$\begin{aligned} G(x, \tilde{x}, y, \tilde{y}) = {} & F'(x)(y - x) + \frac{1}{2}F''(x)(y - x)^2 + r_3(F; x, y)(y - x)^3 \\ & + \kappa\big[F'(\tilde{x})(\tilde{y} - \tilde{x}) + \frac{1}{2}F''(\tilde{x})(\tilde{y} - \tilde{x})^2 + r_3(F; \tilde{x}, \tilde{y})(y - x)^3\big]. \end{aligned}$$

Moreover

$$\begin{aligned} F'(\tilde{x}) &= F'(x) + F''(x)(\tilde{x} - x) + r_2(F'; x, \tilde{x})(\tilde{x} - x)^2 \\ F''(\tilde{x}) &= F''(x) + r_1(F''; x, \tilde{x})(\tilde{x} - x) \end{aligned}$$

so that

$$\begin{aligned} G(x, \tilde{x}, y, \tilde{y}) = {} & [y - x + \kappa(\tilde{y} - \tilde{x})]F'(x) \\ & + [\frac{1}{2}(y - x)^2 + \frac{\kappa}{2}(\tilde{y} - \tilde{x})^2 + \kappa(\tilde{x} - x)(\tilde{y} - \tilde{x})]F''(x) + R(x, \tilde{x}, y, \tilde{y}) \end{aligned} \tag{3.22}$$

where

$$
\begin{aligned}
R(x,\tilde{x},y,\tilde{y}) &= r_3(F;x,y)(y-x)^3 + \kappa r_2(F';x,\tilde{x})(\tilde{x}-x)^2(\tilde{y}-\tilde{x}) \\
&\quad + \frac{\kappa}{2} r_1(F'';x,\tilde{x})(\tilde{x}-x)(\tilde{y}-\tilde{x})^2 + \kappa r_3(F;\tilde{x},\tilde{y})(\tilde{y}-\tilde{x})^3.
\end{aligned} \tag{3.23}
$$

We shall now show that under our assumptions the remainder can be considered as an error term.

To this end, let us write

$$v := y - x,\ \ \tilde{v} := \tilde{y} - \tilde{x},\ \ \delta := x - \tilde{x}, \tag{3.24}$$

and $u = (u_1, u_2)$, where

$$
\begin{aligned}
u_1 &:= y - x + \kappa(\tilde{y} - \tilde{x}) = v + \kappa\tilde{v}, \\
u_2 &:= (y-x)^2 + \kappa(\tilde{y}-\tilde{x})^2 + 2\kappa(\tilde{x}-x)(\tilde{y}-\tilde{x}) = v^2 + \kappa\tilde{v}^2 - 2\kappa\delta\tilde{v}.
\end{aligned} \tag{3.25}
$$

Notice that $2^{-\ell-1} \leq \delta \leq 2^{-\ell+1}$ and consequently

$$|R| \leq C_1(|v|^3 + 2^{-2\ell}\kappa|\tilde{v}| + 2^{-\ell}\kappa|\tilde{v}|^2 + \kappa|\tilde{v}|^3). \tag{3.26}$$

Given $A \geq 2$ we shall show that the integer $M > 0$ in (3.20), (3.21) and the number ε can be chosen so that

$$|u_1| + |u_2| \geq A|R| \tag{3.27}$$

provided that $x, y, \tilde{x}, \tilde{y}$ belong to a set of diameter ε and either $|m - \ell| > 10$ or $m = \ell$ with either (3.20) or (3.21) holds. We may assume that

$$\max\{2^{-m-2}, 2^{-\ell-2}\} + |v| + |\tilde{v}| \leq \varepsilon \tag{3.28}$$

where ε is small ($\varepsilon < 10^{-100}(1 + C_1)^2 A^{-1}$ is an acceptable choice).

If either $|v| \leq \frac{\kappa}{2}|\tilde{v}|$, or $|v| \geq 2\kappa|\tilde{v}|$, then $|u_1| \gg |R|$, so (3.27) is clear. Likewise if $\frac{\kappa}{2}|\tilde{v}| < |v| < 2\kappa|\tilde{v}|$ and if v and $\tilde{v}$ have the same sign then $|u_1| = |v| + \kappa|\tilde{v}|$, and again $|u_1| \gg |R|$. Therefore we may now assume that

$$\frac{\kappa}{2}|\tilde{v}| < |v| < 2\kappa|\tilde{v}| \text{ and sign } (v) = -\text{sign } (\tilde{v}). \tag{3.29}$$

We discuss the cases $|m - \ell| > 10$ and $m = \ell$ separately. Assume first that $|m - \ell| > 10$ and set $r = \min\{m, \ell\}$. In this case $|v - \tilde{v}| = |x - \tilde{x} + \tilde{y} - y| \in (2^{-r-2}, 2^{-r+2})$. In view of the sign condition $\max\{|v|, |\tilde{v}|\} \geq 2^{-r-2}$ and since $|v|$ and $\kappa|\tilde{v}|$ are comparable (by (3.29)) we see that $|v| \leq 2^{-r+2}$ and

$$2^{-r-4} \leq |\tilde{v}| < 2^{-r+2}. \tag{3.30}$$

Therefore also

$$|R| \leq 100 C_1 2^{-2m}\kappa|\tilde{v}|. \tag{3.31}$$

Now if $|u_1| \geq 200 C_1 A 2^{-2r}\kappa|\tilde{v}|$ then (3.27) is immediate. Therefore assume $|u_1| \leq 200 C_1 A 2^{-2r}\kappa|\tilde{v}|$.

Expanding u_2 about $v = -\kappa\tilde{v}$ we find

$$
\begin{aligned}
u_2 &= (\kappa^2 + \kappa)\tilde{v}^2 - 2\kappa u_1\tilde{v} + u_1^2 - 2\kappa\delta\tilde{v} \\
&= \kappa\tilde{v}((1+\kappa)\tilde{v} - 2\delta) - 2\kappa u_1\tilde{v} + u_1^2.
\end{aligned} \tag{3.32}
$$

By (3.30)

$$|\kappa\tilde v((1+\kappa)\tilde v-2\delta)|\ge \max\{2^{-m-4}-2^{-\ell+2},2^{-\ell-2}-2^{-m+2}\}\kappa|\tilde v|\ge 2^{-r-6}\kappa|\tilde v|$$

while the last two terms in (3.32) are $O(2^{-3r}\kappa\tilde v)$, an expression which is small since $2^{-m-2}\le\varepsilon$. We obtain the lower bound $|u_2|\ge 2^{-r-8}\kappa\tilde v$ which implies (3.27), in the present case, by our choice of ε.

We finally discuss the last case $m=\ell$, still assuming (3.29). Now $|v-\tilde v|\le 2^{-\ell+2}$, and therefore $\kappa|\tilde v|\approx|v|\le 2^{-\ell+2}$ and also $|v|\le 2^{-\ell+2}$. Therefore (3.31) holds with $m=\ell$, and again we are done in the case where $|u_1|\ge 200C_1A2^{-2\ell}\kappa|\tilde v|$. If $|u_1|\le 200C_1A2^{-2\ell}\kappa|\tilde v|$ we use again (3.32). Note that since $2^{-\ell-1}\le|x-\tilde x|,|y-\tilde y|\le 2^{-\ell+1}$ the condition $|x-y|\ge 2^{-\ell+M}$ implies $|\tilde x-\tilde y|\ge 2^{-\ell+M-2}$ and, vica versa $|\tilde x-\tilde y|\ge 2^{-\ell+M}$ implies $|x-y|\ge 2^{-\ell+M-2}$. Observe that now

$$|\kappa\tilde v((1+\kappa)\tilde v-2\delta)|\ge\begin{cases}(2^{-\ell-1}-2^{-\ell-M+4})\kappa|\tilde v| & \text{if (3.20) holds,}\\ (2^{-\ell+M}-2^{-\ell+2})\kappa|\tilde v| & \text{if (3.21) holds.}\end{cases}\tag{3.33}$$

The last two terms in (3.32) are $O(2^{-3\ell}\kappa|\tilde v|)$ while $R=O(2^{-2\ell}\kappa|\tilde v|)$ which shows that after a suitable choice of M we have $|u_2|\ge 2^{-\ell-2}\kappa|\tilde v|$ and (3.27) is then proved also in this last case.

We note that (3.17.1) is in immediate consequence of (3.27). Moreover if (3.18) holds we have $|G|\ge A^{-1}(|u_1|+|u_2|)-|R|$ for suitable $A>0$. Choosing in the above proof M suitably large, depending on A, we again obtain (3.19.1) from (3.27). □

Proofs of (3.10) and (3.11). Let

$$\chi^{\ell m}(x,\tilde x,y,\tilde y)=\chi_m(y-\tilde y)\chi_\ell(x-\tilde x)$$
$$\chi^{\ell}_{\nu,n}(x,\tilde x,y,\tilde y)=\chi_\ell(y-\tilde y)\chi_\ell(x-\tilde x)\eta_{\ell,\nu}(\tilde x)\eta_{\ell,n}(\tilde y).$$

Suppose that $(x,\tilde x,y,\tilde y)$ are either in supp $\chi^{\ell m}$ for $|m-\ell|>10$ or in supp $\chi^\ell_{\nu,n}$ for either $\nu=n$ or $|\nu-n|\ge 2^{2M+10}$.

According to Lemma 3.3 we then have, with a suitable choice of C, ε, A, M, the estimates

$$\begin{aligned}|\partial_z^\alpha\psi(x,\tilde x,y,\tilde y)|&\le C\big(|y-x+\kappa(\tilde y-\tilde x)|+|(y-x)^2+\kappa(\tilde y-\tilde x)^2-2\kappa(x-\tilde x)(\tilde y-\tilde x)|\big);\\ |\partial_z^\alpha\psi(x,\tilde x,y,\tilde y)|&\le C\big(|y-x+\kappa(\tilde y-\tilde x)|+|(y-x)^2+\kappa(\tilde y-\tilde x)^2+2\kappa(y-\tilde y)(\tilde y-\tilde x)|\big);\end{aligned}\tag{3.34}$$

for, say, $|\alpha|\le 6$; moreover by the Carleson-Sjölin condition

$$|\psi_z'(x,\tilde x,y,\tilde y)|\ge A^{-1}\big(|y-x+\kappa(\tilde y-\tilde x)|+|(y-x)^2+\kappa(\tilde y-\tilde x)^2-2\kappa(x-\tilde x)(\tilde y-\tilde x)|\big);$$
$$|\psi_z'(x,\tilde x,y,\tilde y)|\ge A^{-1}\big(|y-x+\kappa(\tilde y-\tilde x)|+|(y-x)^2+\kappa(\tilde y-\tilde x)^2+2\kappa(y-\tilde y)(\tilde y-\tilde x)|\big);$$

Using integrations by parts with respect to z we therefore obtain that for $|m-\ell|\le 10$

$$|K^{\ell,m}(x,\tilde x,y,\tilde y)|\lesssim|\chi_\ell(x-\tilde x)\chi_m(y-\tilde y)|(1+\lambda(|U_1|+|U_2|))^{-4}\tag{3.35.1}$$
$$|K^{\ell,m}(x,\tilde x,y,\tilde y)|\lesssim|\chi_\ell(x-\tilde x)\chi_m(y-\tilde y)|(1+\lambda(|V_1|+|V_2|))^{-4}\tag{3.35.2}$$

where

$$\begin{aligned}U_1(x,\tilde x,y,\tilde y)&=y-x+\kappa(\tilde y-\tilde x)\\ U_2(x,\tilde x,y,\tilde y)&=(y-x)^2+\kappa(\tilde y-\tilde x)^2-2\kappa(x-\tilde x)(\tilde y-\tilde x).\end{aligned}\tag{3.36}$$

and

$$V_i(x,\tilde{x},y,\tilde{y}) = U_i(y,\tilde{y},x,\tilde{x}) \qquad i=1,2.$$

Keeping either $(x,\tilde{x})$ or $(y,\tilde{y})$ fixed we may change variables and observe that

$$\det \frac{\partial(U_1,U_2)}{\partial(y,\tilde{y})} = 2\kappa(\tilde{y}-y), \tag{3.37.1}$$

$$\det \frac{\partial(V_1,V_2)}{\partial(x,\tilde{x})} = 2\kappa(\tilde{x}-x). \tag{3.37.2}$$

We shall now check that for $m \le \ell - 20$,

$$\int |K^{\ell m}(x,\tilde{x},y,\tilde{y})|dyd\tilde{y} \lesssim 2^m\kappa^{-1}\lambda^{-2}, \tag{3.38.1}$$

$$\int |K^{\ell m}(x,\tilde{x},y,\tilde{y})|dxd\tilde{x} \lesssim 2^m\kappa^{-1}\lambda^{-2}. \tag{3.38.2}$$

By Schur's test this implies (3.11).

Inequality (3.38.1) follows quickly by changing variables and using (3.35.1). The same argument applied to the integral in (3.38.2) only gives the weaker bound $2^\ell\kappa^{-1}\lambda^{-2}$. Instead we split the region of integration as a union of

$$\Omega_1(y) = \{(x,\tilde{x}) : |U_1(x,\tilde{x},y,\tilde{y})| \le 2^{-m-10}\}$$

and the complementary region $\Omega_2(y)$.

If $(x,\tilde{x}) \in \Omega_1(y)$ and $K^{\ell m}(x,\tilde{x},y,\tilde{y}) \ne 0$ then $U_1 = (1+\kappa)(\tilde{y}-\tilde{x}) + y - \tilde{y} - (x-\tilde{x})$ and from $|y-\tilde{y}-(x-\tilde{x})| \ge 2^{-m-2}$ we see that

$$|\tilde{y}-\tilde{x}| \ge 2^{-m-5} \ge 2^5|U_1(x,\tilde{x},y,\tilde{y})|. \tag{3.39}$$

Now

$$U_2 = (\kappa+\kappa^2)(\tilde{y}-\tilde{x})^2 - 2\kappa U_1(\tilde{y}-\tilde{x}) + U_1^2 - 2\kappa(x-\tilde{x})(\tilde{y}-\tilde{x}).$$

Since $|x-\tilde{x}| \le 2^{-\ell+1} \le 2^{-m-19} \le 2^{-9}|\tilde{y}-\tilde{x}|$ the assumption on U_1 also yields

$$|U_2| \ge \frac{\kappa}{2}(\tilde{y}-\tilde{x})^2$$

and one computes

$$\int_{\Omega_1(y)} |K^{\ell m}(x,\tilde{x},y,\tilde{y})|dxd\tilde{x} \lesssim \int_{|\tilde{y}-\tilde{x}|\ge 2^{-m-10}} (1+\lambda|U_1|)^{-2}(1+\lambda\kappa(\tilde{y}-\tilde{x})^2)^{-2}dU_1d\tilde{x} \lesssim 2^m(\lambda^2\kappa)^{-1};$$

in fact one gets the better bound $C2^m(\lambda^2\kappa)^{-1}a_m$ where $a_m = 2^{2m}(\lambda\kappa)^{-1}$ if $2^m \le \sqrt{\lambda\kappa}$ and $a_m = 2^{-m}\sqrt{\lambda\kappa}$ if $2^m \ge \sqrt{\lambda\kappa}$.

In order to evaluate the integral over $\Omega_2(y)$ we change variables $(x,\tilde{x}) \to (U_1,U_2)$; this map is at most two-to-one in the regions $U_1 > 0$ and $U_1 < 0$. We compute

$$\det \frac{\partial(U_1,U_2)}{\partial(x,\tilde{x})} = 2\kappa(\tilde{x}-x-U_1). \tag{3.40}$$

Now

$$|U_1| \ge 2^{-m-10}.$$

if $(x,\tilde x)\in\Omega_2(y)$. Since $\ell-m>20$ we see that the absolute value of the determinant in (3.40) is bounded below by $c\kappa 2^{-m}$, for the relevant region of integration, and one deduces that

(3.41)
$$\int_{\Omega_2(y)}|K^{\ell m}(x,\tilde x,y,\tilde y)|dxd\tilde x\lesssim 2^m\kappa^{-1}\int(1+\lambda|U_1|+\lambda|U_2|)^{-4}dU_1dU_2\lesssim 2^m(\lambda^2\kappa)^{-1}.$$

This yields (3.38.2).

The inequalities (3.35.1) and (3.35.2) are satisfied with $K^{\ell,m}$ replaced by $K^\ell_{\nu,n}$, provided that either $|\nu-n|\ge 2^{2M+10}$ or $\nu=n$. By the change of variables $(y,\tilde y)\mapsto(U_1,U_2)$, $(x,\tilde x)\mapsto(V_1,V_2)$ (*cf.* (3.37)) one obtains the analogue of 3.38 (for $\ell-m$). Since, for fixed ℓ, the functions $\chi^\ell_{\nu,n}$ have only finite overlap, we obtain that

$$\sum_{\substack{\nu,n\\|\nu-n|>2^{2M+10}}}\int|K^\ell_{\nu,n}(x,\tilde x,y,\tilde y)|dyd\tilde y\lesssim 2^\ell\kappa^{-1}\lambda^{-2},$$
$$\sum_{\substack{\nu,n\\|\nu-n|>2^{2M+10}}}\int|K^\ell_{\nu,n}(x,\tilde x,y,\tilde y)|dxd\tilde x\lesssim 2^\ell\kappa^{-1}\lambda^{-2},$$

and similarly

$$\sum_n\int|K^\ell_{n,n}(x,\tilde x,y,\tilde y)|dyd\tilde y\lesssim 2^\ell\kappa^{-1}\lambda^{-2},$$
$$\sum_n\int|K^\ell_{n,n}(x,\tilde x,y,\tilde y)|dxd\tilde x\lesssim 2^\ell\kappa^{-1}\lambda^{-2}.$$

Therefore

(3.42)
$$\Big\|\sum_{|nu-n|>2^{2M+10}}S^*_{\ell,\nu}S_{\ell,n}\Big\|_{L^2\to L^2}\lesssim 2^\ell\kappa^{-1}\lambda^{-2}$$

and

(3.43)
$$\Big\|\sum_n S^*_{\ell,n}S_{\ell,n}\Big\|_{L^2\to L^2}\lesssim 2^\ell\kappa^{-1}\lambda^{-2};$$

hence, for $|j|\le 2^{2M+10}$,

$$\begin{aligned}\big|\langle\sum_n S^*_{\ell,n}S_{\ell,n+j}g,g\rangle\big|&\le\sum_n\|S_{\ell,n+j}g\|_2\|S_{\ell,n}g\|_2\\&\le\big(\sum_n\|S_{\ell,n+j}g\|_2^2\big)^{1/2}\big(\sum_n\|S_{\ell,n}g\|_2^2\big)^{1/2}\\&\lesssim 2^\ell\kappa^{-1}\lambda^{-2}\|g\|_2^2.\end{aligned}$$

This, together with (3.42), yields (3.10). □

Remark. Concerning sharpness, consider the example $\Phi(z_1,z_2,y)=|(z_1,z_2)-(y,c)|$, for fixed c. Then Φ is the phase function, which comes up in connection with restriction of Fourier transforms to circles. The sharp L^4 operator norm was proved by Hörmander; and from the work by Beckner, Carbery, Semmes and Soria [1] it follows that for $2<p\le 4$ and large λ, one has the lower bound

$\|T_\lambda\|_{L^p\to L^p} \gtrsim \lambda^{-1/2}[\log\lambda/\log\log\lambda]^{1/2-1/p}$. This lower bound is deduced in [1] using a construction which proves a lower bound for the Besicovich maximal operator; the latter has been recently improved by Keich [18]. It seems likely that the factor $(\log\log\lambda)^{1/2-1/p}$ can be removed. For general Carleson-Sjölin phase functions one has the lower bound $\lambda^{-1/2}[\log\lambda]^{1/4}$ for $p=4$, by an elementary argument. To the best of our knowledge the lower bounds for $\|T_\lambda\|_{L^p\to L^p}$ are presently open in the general case when $2<p<4$. It seems reasonable to expect that counterexamples in this range will involve the constructions of Kakeya sets with respect to suitable families of curves.

We shall now discuss the relevant example

$$\psi(t,r,\rho)=\arccos\mu(t,r,\rho)=\arccos\frac{r^2+\rho^2-t^2}{2r\rho} \tag{3.44}$$

which relates to the general setting by putting $z=(t,r)$ and $y=\rho$.

Proposition 3.4. *Let ψ be as in (3.44). Then the following hold:*

(i)

$$\mu_t=-\frac{t}{r\varrho};\quad \mu_r=\frac{r^2+t^2-\varrho^2}{2r^2\varrho}=-\frac{\mu}{r}+\frac{1}{\varrho};\quad \mu_\rho=\frac{t^2+\varrho^2-r^2}{2r\varrho^2}=-\frac{\mu}{\varrho}+\frac{1}{r}. \tag{3.45}$$

(ii)

$$\begin{aligned}\psi_{t\varrho}&=(1-\mu^2)^{-3/2}\frac{t}{r\varrho}(\frac{\mu}{r}-\frac{1}{\varrho}),\\ \psi_{r\varrho}&=(1-\mu^2)^{-3/2}\frac{t^2}{r^2\varrho^2}.\end{aligned} \tag{3.46}$$

(iii)

$$CS[\psi]=\det\begin{pmatrix}\psi_{t\varrho} & \psi_{r\varrho}\\ \psi_{t\varrho\varrho} & \psi_{r\varrho\varrho}\end{pmatrix}=-\frac{t^3}{r^5\varrho^3(1-\mu^2)^3}. \tag{3.47}$$

Proof. (i) is immediate from the definition $\mu=(r^2+\varrho^2-t^2)/(2r\varrho)$.

(ii) Put $h(\mu)=\arccos\mu$. Then

$$\psi_{t\varrho}=(h''\circ\mu)\mu_t\mu_\varrho+(h'\circ\mu)\mu_{t\varrho}.$$

We have

$$h'=-(1-\mu^2)^{-1/2},\ h''=-\mu(1-\mu^2)^{-3/2}. \tag{3.48}$$

Moreover, by (i), $\mu_{t\varrho}=\frac{t}{r\varrho^2}$, hence, together again with (i), we get

$$\begin{aligned}\psi_{t\varrho}&=-\mu(1-\mu^2)^{-3/2}(-\frac{t}{r\varrho})(-\frac{\mu}{\varrho}+\frac{1}{r})-(1-\mu^2)^{-1/2}\frac{t}{r\varrho^2}\\ &=(1-\mu^2)^{-3/2}\frac{t}{r\varrho}[-\frac{\mu^2}{\varrho}+\frac{\mu}{r}-\frac{1-\mu^2}{\varrho}]\\ &=(1-\mu^2)^{-3/2}\frac{t}{r\varrho}(\frac{\mu}{r}-\frac{1}{\varrho}).\end{aligned}$$

Similarly,

$$\psi_{r\varrho}=(h''\circ\mu)\mu_r\mu_\varrho+(h'\circ\mu)\mu_{r\varrho},$$

where, by (i),

$$\mu_{r\varrho} = -\frac{\mu_r}{\varrho} - \frac{1}{r^2} = \frac{\mu}{r\varrho} - \frac{1}{\varrho^2} - \frac{1}{r^2}.$$

Hence

$$\begin{aligned}\psi_{r\varrho} &= -\mu(1-\mu^2)^{-3/2}(-\frac{\mu}{r}+\frac{1}{\varrho})(-\frac{\mu}{\varrho}+\frac{1}{r}) - (1-\mu^2)^{-1/2}(\frac{\mu}{r\varrho} - \frac{1}{\varrho^2} - \frac{1}{r^2})\\ &= (1-\mu^2)^{-3/2}[\frac{-\mu^3}{r\varrho} + \mu^2(\frac{1}{r^2}+\frac{1}{\varrho^2}) - \frac{\mu}{r\varrho} + \frac{\mu^3}{r\varrho} - \mu^2(\frac{1}{r^2}+\frac{1}{\varrho^2}) - \frac{\mu}{r\varrho} + \frac{1}{r^2} + \frac{1}{\varrho^2}]\\ &= (1-\mu^2)^{-3/2}\frac{t^2}{r^2\varrho^2}.\end{aligned}$$

For (iii) write $\psi_{t\varrho} = (\varrho\mu - r)w$, $\psi_{r\varrho} = tw$ where $w := (1-\mu^2)^{-3/2}\frac{t}{r^2\varrho^2}$. Then, in combination with (i), we get

$$\begin{aligned}CS[\psi] &= \det\begin{pmatrix}(\varrho\mu - r)w & tw\\ \frac{\partial(\varrho\mu - r)}{\partial\varrho}w + (\varrho\mu - r)w_\varrho & tw_\varrho\end{pmatrix} = -tw^2\frac{\partial(\varrho\mu - r)}{\partial\varrho}\\ &= -tw^2(\mu + \varrho\mu_\varrho) = -tw^2\frac{\varrho}{r} = -(1-\mu^2)^{-3}\frac{t^3}{r^5\varrho^3}. \quad \square\end{aligned}$$

Remark 3.5. The phase function in (3.44) does not satisfy the assumption of Theorem 3.1 because of the lack of uniform C^∞ bounds. However one can use rescaling arguments to reduce matters to the situation covered in Theorem 3.1. For this we note the following property of the Carleson-Sjölin determinant under changes of variables. If $z = (z_1, z_2) = Z(w)$, $y = Y(u)$ and if $\Phi(w_1, w_2, u) = \Psi(Z(w), Y(u))$ then

$$CS[\Phi](w,u) = \det\big(\frac{\partial Z}{\partial w}\big)\Big[\frac{dY}{du}\Big]^3 CS[\Psi](Z(w), Y(u)). \tag{3.49}$$

4. Estimation of the nonoscillatory terms

In this section we shall consider the nonoscillatory terms $\mathcal{R}_{\lambda,i}$, $i = 1,2,3,4$, and $\mathcal{V}_{\lambda,z}$, $\mathcal{W}_{\lambda,z}$ in the decompositions (2.17) and (2.23), respectively. Moreover we shall provide size estimates for the oscillatory terms (2.24) to obtain the appropriate L^∞ bounds in Theorem 2.3. These terms can be controlled by positive operators (such as maximal operators) or by maximal Hilbert transforms.

Convention. The constants in all estimates are allowed to be $O(e^{A|\mathrm{Im}(z)|})$ for some fixed A. This dependence will not be explicitely indicated.

In the first Lemma we summarize elementary properties of localizations in terms of the quantities $1 \pm \mu$ where $\mu \equiv \mu(t,r,\rho)$ as in (2.9), *cf.* also (2.15.1/2).

Lemma 4.1. *Suppose that* $1 \le t \le 2$.

(a) If $0 \le \mu(t,r,\rho) \le 1$ *then* $|r - \rho| \le 2$. *Moreover, the following holds:*

- (i) *If* $r \ge 4$ *then* $\rho \ge 2$ *and* $r \le 2|1-\mu|^{-1/2}$.
- (ii) *If* $r \le 1/2$ *then* $1/2 < \rho < 3$.
- (iii) *If* $1/2 < r < 4$ *then* $\rho < 6$.

(b) If $-2 \le \mu(t,r,\rho) \le 0$ *then* $r \le 2$, $\rho \le 2$ *and* $r + \rho > 1/2$.

(c) If $\mu \le -2$ then $r \le 2$, $\rho \le 2$ and $t \ge r$.

Proof. Note that

$$0 \le \mu < 1 \iff (r-\rho)^2 < t^2 \le r^2 + \rho^2 \tag{4.1.1}$$

$$-2 \le \mu \le 0 \iff r^2 + \rho^2 < t^2 < (r+\rho)^2 + 2r\rho \tag{4.1.2}$$

$$\mu \le -2 \iff r^2 + \rho^2 + 4r\rho \le t^2. \tag{4.1.3}$$

The assertions easily follow. □

Let χ_0 be the characteristic function of $[1/2, \infty]$ and, for $\ell \ge 1$, let χ_ℓ be the characteristic function of $[2^{-\ell-1}, 2^{-\ell}]$. It will be convenient to study operators $\mathcal{A}_\ell$, $\mathcal{B}_\ell$ with kernels

$$A_\ell(t,r,\rho) = \frac{1}{2r\rho(1-\mu)}\chi_\ell(1-\mu)\chi_{[0,1]}(\mu) \tag{4.2}$$

$$B_\ell(t,r,\rho) = \frac{1}{2r\rho|1+\mu|}\chi_\ell(|1+\mu|)\chi_{[-2,0]}(\mu). \tag{4.3}$$

Let M denote the standard Hardy-Littlewood maximal operator on the real line.

Lemma 4.2. *Let $p > 2$ and suppose that $1 \le t \le 2$. Then*

$$r^{-\frac{n-1}{2}}|\mathcal{A}_\ell[\rho^{\frac{n-1}{2}} f](t,r)| \lesssim \tfrac{p}{p-2} r^{-\frac{n-1}{2}} \sum_{\pm} M\big[\rho^{\frac{n-1}{p}}\chi_{[0,6]} f\big](r \pm t), \quad \text{if } r \le 4, \tag{4.4.1}$$

$$r^{-\frac{n-1}{2}}|\mathcal{A}_\ell[\rho^{\frac{n-1}{2}} f](t,r)| \lesssim \tfrac{p}{p-2} r^{-\frac{n-1}{p}} \sum_{\pm} M\big[\rho^{\frac{n-1}{p}} f\big](r \pm t), \quad \text{if } r \ge 4, \tag{4.4.2}$$

and

$$r^{-\frac{n-1}{2}}|\mathcal{B}_\ell[\rho^{\frac{n-1}{2}} f](t,r)| \lesssim \tfrac{p}{p-2} r^{-\frac{n-1}{2}} \sum_{\pm} M\big[\rho^{\frac{n-1}{p}}\chi_{[0,2]} f\big](t-r). \tag{4.5}$$

Proof. Note that

$$\frac{1}{2r\rho(1-\mu)} = \frac{1}{t^2 - (r-\rho)^2} = \frac{1}{(t-r+\rho)(t+r-\rho)}, \tag{4.6}$$

and that

$$A_\ell(t,r,\rho) \approx \frac{2^\ell}{r\rho}.$$

Assume first that $0 \le \mu \le 1$, so that in particular $|r-\rho| \le 2$, and that $1-\mu \approx 2^{-\ell}$.

If $|r-\rho| \le 1/4$ then, by (4.6), $A_\ell(t,r,\rho) \approx 1$. Moreover, from Lemma 4.1 and (2.15.1) one derives that $r \approx \rho \approx 2^{\ell/2}$ if $r \ge 4$. It is then immediate that $\int_{|r-\rho|\le 1/4} (\rho/r)^{(n-1)/2} A_\ell(t,r,\rho)|f(\rho)|d\rho$ is controlled by the right hand side of either (4.4.1) or (4.4.2), depending on whether $r \ge 4$ or $r \le 4$, respectively.

Next, if $|r-\rho| \ge 1/4$, then $1/4 \le |r-\rho| \le 2$. By (4.6), we have $|r-t-\rho| \approx 2^{-\ell} r\rho$ in the case $1/4 \le r-\rho \le 2$, and $|r+t-\rho| \approx 2^{-\ell} r\rho$ in the case $-2 \le r-\rho \le -1/4$.

Now, if $r \le 1/8$, then $\rho \approx 1$, and $\int_{|r-\rho|\ge 1/4} (\rho/r)^{(n-1)/2} A_\ell(t,r,\rho)|f(\rho)|d\rho$ is dominated by $r^{-(n-1)/2}$ times an average of $|f(\rho)|\rho^{(n-1)/p} \approx |f(\rho)|\rho^{(n-1)/2}$ over an interval of length $C2^{-\ell} r$ centered at either $r-t$

or $r+t$, depending on the sign of $r-\rho$. Hence we obtain the bound $\sum_\pm r^{-(n-1)/2} M\big[\rho^{(n-1)/2}\chi_{[0,6]}f\big](r\pm t)$, thus (4.4.1) for the case $r \le 1/8$.

If $1/8 \le r \le 4$ then

$$\begin{aligned} r^{-\frac{n-1}{2}}|\mathcal{A}_\ell[\rho^{\frac{n-1}{2}}f](t,r)| &\lesssim \sum_\pm \sum_{k=-2}^{\infty} 2^{\ell+k} \int_{\substack{|r\pm t-\rho|\approx 2^{-\ell-k}\\ \rho\approx 2^{-k}}} |f(\rho)|\rho^{\frac{n-1}{2}}d\rho \\ &\lesssim \sum_\pm \sum_{k=-2}^{\infty} 2^{-k(n-1)(\frac12-\frac1p)} M[\rho^{\frac{n-1}{p}}\chi_{[0,6]}f](r\pm t) \end{aligned} \tag{4.7}$$

and since now $r \approx 1$ and $p > 2$ this implies the asserted estimate (4.4.1).

We still have to consider the case $1/4 \le |r-\rho| \le 2$ and $r \ge 4$. Then $r \approx \rho$, hence

$$r^{-\frac{n-1}{2}}|\mathcal{A}_\ell[\rho^{\frac{n-1}{2}}f](t,r)| \lesssim r^{-\frac{n-1}{p}}\mathcal{A}_\ell[\rho^{\frac{n-1}{p}}|f|](t,r)$$

and the latter term is estimated by $r^{-(n-1)/p}$ times an average of $|f(\rho)|\rho^{(n-1)/p}$ over an interval of length $\approx 2^{-\ell}r^2$ centered at either $r-t$ or $r+t$; this implies (4.4.2).

Finally, if $\mu \in [-2,0]$ we write

$$\big(2r\rho(1+\mu)\big)^{-1} = (\rho-t+r)^{-1}(\rho+t+r)^{-1} \approx (\rho-t+r)^{-1}. \tag{4.8}$$

and obtain in the same way (4.5). Note that there is no contribution to B_ℓ if $r > 2$ or $\rho > 2$ by Lemma 4.1 (b). □

The next two lemmas provide estimates for the remainder terms in Theorem 2.2.

Lemma 4.3. *Let $z = b + i\tau$ with $b > -1/2$ and $\mathcal{V}_{z,\lambda}$ be the integral operator with kernel $V_{z,\lambda}$ as in (2.23/26).*

For a sequence of functions $F = \{f_j\}$ and $2 \le p \le \infty$ let

$$G_{p,\gamma}F(\rho) = \rho^{(n-1)/p}\Big(\sum_j |(1+\lambda_j/\rho)^{-\gamma}f_j(\rho)|^2\Big)^{1/2}. \tag{4.9}$$

Suppose that $1 \le t \le 2$ and $0 \le \gamma \le (2b+1)/2$. Then

$$\Big(\sum_j |\mathcal{V}_{z,\lambda_j}f_j(t,r)|^2\Big)^{1/2} \lesssim r^{b+\frac12-\frac{n-1}{2}} \sum_\pm M\big[\chi_{[0,6]}G_{p,\gamma}F\big](r\pm t) \quad \textit{if} \quad r \le 4 \tag{4.10}$$

and

$$\Big(\sum_j |\mathcal{V}_{z,\lambda_j}f_j(t,r)|^2\Big)^{1/2} \lesssim r^{-\frac{n-1}{p}} \sum_\pm M\big[G_{p,\gamma}F\big](r\pm t) \quad \textit{if} \quad r \ge 4. \tag{4.11}$$

Proof. Let $J_0 = [-1/2, 1/2] \cup [-2, -3/2]$ and for $\ell \ge 1$ let

$$\begin{aligned} J_{+,\ell} &= [1-2^{-\ell}, 1-2^{-\ell-1}] \\ J_{-,\ell} &= [-1+2^{-\ell-1}, -1+2^{-\ell}] \cup [-1-2^{-\ell}, -1-2^{-\ell-1}] \end{aligned}$$

Let $V^0_{z,\lambda} = \chi_{J_0}(\mu)V_{z,\lambda}$ and $V^{\ell,\pm}_{z,\lambda} = \chi_{J_{\pm,\ell}}(\mu)V_{z,\lambda}$. Then

$$V_{z,\lambda} = V^0_{z,\lambda} + \sum_\ell \sum_\pm V^{\ell,\pm}_{z,\lambda}.$$

Now let $\ell \geq 1$. From Theorem 2.2 we derive the estimate

$$|V^{\ell,\pm}_{z,\lambda}| \lesssim \lambda^{-b-1}2^{-\ell b/2}(\rho r)^{b+\frac12}(\rho/r)^{\frac{n-1}{2}}(A_\ell + B_\ell) \quad \text{if } \lambda 2^{-\ell/2} \geq 1, \tag{4.12}$$

and

$$|V^{\ell,\pm}_{z,\lambda}| \lesssim \alpha_\pm(\ell,\lambda)(\rho r)^{b+\frac12}(\rho/r)^{\frac{n-1}{2}}(A_\ell + B_\ell) \quad \text{if } \lambda 2^{-\ell/2} \leq 1, \tag{4.13}$$

where

$$\alpha_+(\ell,\lambda) = 2^{-\ell(2b+1)/2}, \tag{4.14.1}$$

$$\alpha_-(\ell,\lambda) = \begin{cases} 2^{-\ell(2b+1)/2} & \text{if } -1/2 < b < 1/2 \\ 2^{-\ell}\log((2+\lambda)^{-1}2^{\ell/2}) & \text{if } b = 1/2 \\ 2^{-\ell}(\lambda+1)^{1-2b} & \text{if } b > 1/2. \end{cases} \tag{4.14.2}$$

For $\ell = 0$ the previous estimates remain valid with $V^{\ell,\pm}_{z,\lambda}$ replaced by $V^0_{z,\lambda}$ and $\alpha_\pm(\ell,\lambda)$ replaced by 1. In what follows we shall only discuss the case $\ell \geq 1$ and omit the obvious modifications (and simplifications) for the case $\ell = 0$.

We shall frequently use that by Minkowski's inequality and the positivity of $\mathcal{A}_\ell$ and $\mathcal{B}_\ell$, we have the pointwise inequality

$$\Big(\sum_j |(\mathcal{A}_\ell + \mathcal{B}_\ell)[g_j]|^2\Big)^{1/2} \leq (\mathcal{A}_\ell + \mathcal{B}_\ell)\big[(\sum_j |g_j|^2)^{1/2}\big].$$

Let $\varepsilon < \min\{(2b+1)/4, 1/2\}$. We note that $\alpha_\pm(\ell,\lambda) \lesssim 2^{-\ell\varepsilon}(\lambda+1)^{-(2b+1)/2}$ if $\lambda 2^{-\ell/2} \leq 1$. For $r \leq 4$ we have $\rho \leq 6$ and therefore $(1+\lambda_j/\rho) \lesssim (1+\lambda_j)/\rho$, and we derive the estimate

$$\begin{aligned}\Big(\sum_j |\mathcal{V}_{z,\lambda_j} f_j(t,r)|^2\Big)^{1/2} &\lesssim r^{b+\frac12-\frac{n-1}{2}} \sum_{\ell\geq 0} 2^{-\ell\varepsilon}\big(\mathcal{A}_\ell + \mathcal{B}_\ell\big)\big[\big(\sum_{\lambda_j \geq 2^{\ell/2}} |\rho^{\frac{n-1}{2}}(\lambda_j/\rho)^{-(b+\frac12)} f_j|^2\big)^{1/2}\big](t,r) \\ &+ r^{b+\frac12-\frac{n-1}{2}} \sum_{\ell\geq 0} 2^{-\ell\varepsilon}\big(\mathcal{A}_\ell + \mathcal{B}_\ell\big)\big[\big(\sum_{\lambda_j \leq 2^{\ell/2}} |\rho^{\frac{n-1}{2}}(1+\lambda_j/\rho)^{-(b+\frac12)} f_j|^2\big)^{1/2}\big](t,r).\end{aligned} \tag{4.15}$$

Now (4.10) follows from Lemma 4.2.

Let now $r \geq 4$; then $V^{\ell,-}_{z,\lambda}(t,r,\rho) = 0$ and $B_\ell(t,r,\rho) = 0$, by Lemma 4.1, and furthermore we may assume that $r \approx \rho$, $4 \leq r \leq 2^{(\ell+6)/2}$ and $\mu > 0$ so that the estimate (4.13) holds with $\alpha_+(\ell,\lambda) = 2^{-\ell(2b+1)/2}$.

We shall write down different estimates for the situations $\lambda_j \leq r \leq 2^{(\ell+6)/2}$, $r \leq \lambda_j \leq 2^{(\ell+6)/2}$ and $r \leq 2^{(\ell+6)/2} \leq \lambda_j$. First

$$|\mathcal{V}^{\ell,+}_{z,\lambda} f(t,r)| \lesssim r^{-(n-1)/2} r^{2b+1} 2^{-\ell(2b+1)/2}\mathcal{A}_\ell[\rho^{(n-1)/2}|f|](t,r) \quad \text{if } \lambda \leq r \leq 2^{(\ell+6)/2}. \tag{4.16}$$

We use Lemma 4.2 and Minkowski's inequality to estimate for $p > 2$

$$\begin{aligned}\Big(\sum_{\lambda_j \leq r}\Big[\sum_{2^{(\ell+6)/2} \geq r} |\mathcal{V}^{\ell,+}_{z,\lambda_j} f_j(t,r)|\Big]^2\Big)^{1/2} &\lesssim r^{-\frac{n-1}{2}} r^{2b+1} \sum_{2^{(\ell+6)/2} \geq r} 2^{-\ell(2b+1)/2}\mathcal{A}_\ell\big[\rho^{\frac{n-1}{2}}\big(\sum_{\lambda_j \leq r} |\chi_{\rho\approx r} f_j|^2\big)^{1/2}\big](t,r) \\ &\lesssim r^{-\frac{n-1}{p}} \sum_\pm M\big[\rho^{\frac{n-1}{p}}\big(\sum_{\lambda_j \leq r} |\chi_{\rho\approx r} f_j|^2\big)^{1/2}\big](r \pm t) \\ &\lesssim r^{-\frac{n-1}{p}} \sum_\pm M\big[\rho^{\frac{n-1}{p}}\big(\sum_{\lambda_j \leq r} |(1+\lambda_j/\rho)^{-\gamma} f_j|^2\big)^{1/2}\big](r \pm t),\end{aligned} \tag{4.17}$$

for any γ.

Next we estimate the terms with $4 \le r \le \lambda_j \le 2^{(\ell+6)/2}$ and we can still use the estimate (4.13) with $\alpha_+(\ell,\lambda) = 2^{-\ell(2b+1)/2}$. We use the notation $\lambda_j \sim 2^m$ if $2^m \le \lambda_j \le 2^{m+1}$ and obtain by Minkowski's inequality and summing geometric series

$$\begin{aligned}
&\Big(\sum_{\lambda_j\ge r}\Big[\sum_{2^{(\ell+6)/2}\ge\lambda_j}|\mathcal{V}^{\ell,+}_{z,\lambda_j}f_j(t,r)|\Big]^2\Big)^{1/2}\\
&\lesssim \sum_{s=0}^{\infty}\Big(\sum_{2^m\ge r}\sum_{\lambda_j\sim 2^m}|\mathcal{V}^{2m-6+s,+}_{z,\lambda_j}f_j(t,r)|^2\Big)^{1/2}\\
&\lesssim r^{-\frac{n-1}{2}}\sum_{s=0}^{\infty}2^{-s(2b+1)/2}\Big(\sum_{2^m\ge r}\sum_{\lambda_j\sim 2^m}\big|r^{2b+1}2^{-m(2b+1)}\mathcal{A}_{2m-6+s}[\rho^{\frac{n-1}{2}}|f_j|](t,r)\big|^2\Big)^{1/2}\\
&\lesssim r^{-\frac{n-1}{2}}\sum_{s=0}^{\infty}2^{-s(2b+1)/2}\sup_{2^m\ge r} r^{(2b+1)/2}2^{-m(2b+1)/2}\mathcal{A}_{2m-6+s}\big[\rho^{\frac{n-1}{2}}(\sum_{\lambda_j\sim 2^m}|f_j|^2)^{1/2}\big](t,r)\\
&\lesssim r^{-\frac{n-1}{p}}\sum_{\pm}M\big[\rho^{\frac{n-1}{p}}(\sum_{\lambda_j\ge r}|(\lambda_j/\rho)^{-(2b+1)/2}f_j|^2)^{1/2}\big](r\pm t). \qquad (4.18)
\end{aligned}$$

Finally we consider the terms with $4 \le r \le 2^{(\ell+6)/2} \le \lambda_j$ and use the estimate (4.12). We obtain in a similar way

$$\begin{aligned}
&\Big(\sum_j\Big[\sum_{r\le 2^{(\ell+6)/2}\le\lambda_j}|\mathcal{V}^{\ell,+}_{z,\lambda_j}f_j(t,r)|\Big]^2\Big)^{1/2}\\
&\lesssim r^{-\frac{n-1}{2}}r^{2b+1}\sum_{2^{\ell+6}\ge r^2}\Big(\sum_{\lambda_j\ge 2^{\ell/2}}|\lambda_j^{-b-1}2^{-\ell b/2}\mathcal{A}_\ell\big[\rho^{\frac{n-1}{2}}|f_j|\big](t,r)|^2\Big)^{1/2}\\
&\lesssim r^{-\frac{n-1}{2}}\sum_{2^\ell\ge r^2}(r2^{-\ell/2})^{\frac{2b+1}{2}}\mathcal{A}_\ell\big[(\sum_{\lambda_j\ge 2^{\ell/2}}|(2^{\ell/2}\lambda_j^{-1})^{1/2}\lambda_j^{-(2b+1)/2}\rho^{\frac{n-1}{2}+b+\frac12}f_j|^2)^{1/2}\big](t,r)\\
&\lesssim r^{-\frac{n-1}{p}}\sum_{\pm}M\big[(\sum_j|(1+\lambda_j/\rho)^{-(2b+1)/2}\rho^{\frac{n-1}{p}}f_j|^2)^{1/2}\big](r\pm t) \qquad (4.19)
\end{aligned}$$

and the estimates (4.17-19) yield (4.11). □

Lemma 4.4. *Let $\varepsilon > 0$, $z = b + i\tau$, $b > -1/2$ and assume that $\lambda > b - 1 + \varepsilon$ and $\lambda \ge 0$. Suppose also that $p > 2$ and $\gamma \le (n-1)(1/2 - 1/p) + \min\{(2b+1)/2, (n-1)/2\}$.*

Let $\mathcal{W}_{z,\lambda}$ as in (2.23/27) and assume $1 \le t \le 2$. Then

(i)

$$W_{z,\lambda}f(t,r) = 0 \quad \text{if } r \ge t. \qquad (4.20)$$

(ii) Suppose that $t^2 - r^2 \ge 1/4$. Then

$$|\mathcal{W}_{z,\lambda}f(t,r)| \lesssim_\varepsilon \Big(\int_0^{1/2}|f(\rho)|(1+\lambda/\rho)^{-\gamma}\rho^{\frac{n-1}{p}}d\rho + w_b(r)M\big[\chi_{[1/8,2]}\rho^{n-1}f(1+\lambda/\rho)^{-\gamma}\big](\sqrt{t^2-r^2})\Big) \qquad (4.21)$$

where $w_b(r) = r^{b-\frac{n-2}{2}}$ if $-1/2 \le b < (n-2)/2$, $w_b(r) = \log(4/r)$ if $b = (n-2)/2$, and $w_b(r) = 1$ if $b > (n-2)/2$.

(iii) Suppose that $t^2-r^2\le 1/4$. Then

$$|\mathcal{W}_{z,\lambda}f(t,r)|\lesssim_\varepsilon \frac{1}{t^2-r^2}\int_0^{t^2-r^2}|f(\rho)|(1+\lambda/\rho)^{-\gamma}\rho^{\frac{n-1}{p}}d\rho. \tag{4.22}$$

(iv) The case $z=-1/2$ of the preceding estimates holds if $\mathcal{W}_{-1/2,\lambda}$ is replaced by the operator $\mathcal{R}_{\lambda,4}$ with kernel $R_{\lambda,4}$ as in (2.17).

Proof. Note that (4.20) follows from Lemma 4.1 (c). Moreover if $|\mu|\ge 2$ which is currently assumed we have by (2.9)

$$\begin{aligned}\rho^{\frac{n}{2}+b-1}r^{b-\frac{n}{2}}|\mu(t,r,\rho)|^{-(\lambda-b+1)} &\le \rho^{\frac{n}{2}+b-1}r^{b-\frac{n}{2}}\Big|\frac{t^2-r^2-\rho^2}{2r\rho}\Big|^{b-n/2}2^{\frac{n-2}{2}-\lambda}\\ &=\rho^{n-1}2^{n/2-b}|t^2-r^2-\rho^2|^{b-n/2}2^{\frac{n-2}{2}-\lambda}\end{aligned}$$

Thus setting

$$\mathcal{E}_bf(t,r):=\int_{r^2+\rho^2+4r\rho<t^2}(t^2-r^2-\rho^2)^{b-n/2}\rho^{n-1}|f(\rho)|d\rho \tag{4.23}$$

we have from (4.1.3)

$$|\mathcal{W}_{z,\lambda}f(t,r)|\lesssim(1+\lambda)^{-b}2^{-(\lambda-\frac{n-2}{2})}\mathcal{E}_bf(t,r).$$

First assume that $0\le t^2-r^2\le 1/4$; then $r\ge 1/2$. One checks that

$$r^2+\rho^2+4r\rho<t^2 \iff \frac{\rho}{r}<2\Big(\sqrt{1+\frac14\frac{t^2-r^2}{r^2}}-1\Big)$$

where the second relation implies that $\rho\le(t^2-r^2)/2r\le(t^2-r^2)$ (since $r\ge 1/2$). It follows that in this case

$$\begin{aligned}2^{-\lambda}\mathcal{E}_bf(t,r)&\lesssim 2^{-\lambda}(t^2-r^2)^{b-\frac{n}{2}}\int_0^{t^2-r^2}|f(\rho)|\rho^{n-1}d\rho\\ &\lesssim(t^2-r^2)^{b-\frac{n}{2}}\int_0^{t^2-r^2}|f(\rho)|(1+\lambda/\rho)^{-\gamma}\rho^{n-1-\gamma}d\rho\\ &\lesssim(t^2-r^2)^{(n-1)(\frac12-\frac1p)+\frac{2b+1}{2}-\gamma}\frac{1}{t^2-r^2}\int_0^{t^2-r^2}|f(\rho)|(1+\lambda/\rho)^{-\gamma}\rho^{\frac{n-1}{p}}d\rho. \qquad (4.24)\end{aligned}$$

For the second inequality observe that $2^{-\lambda}\rho^\gamma\lesssim(1+\lambda/\rho)^{-\gamma}$ uniformly for $\lambda\ge 0$ and $0<\rho\le 2$. For the last inequality we use the assumption that $\gamma\le(n-1)/p'$ and the last term is controlled by the right hand side of (4.22) since we assume that $\gamma\le(n-1)(1/2-1/p)+(2b+1)/2$.

Finally assume that $t^2-r^2\ge 1/4$. Clearly

$$\big|\mathcal{E}_b[f\chi_{[0,1/8]}](t,r)\big|\lesssim\int_0^{1/8}|f(\rho)|\rho^{n-1}d\rho$$

so that $\mathcal{W}_{z,\lambda}[f\chi_{(0,1/8)}](t,r)$ can be estimated by the first term on the right hand side of (4.21).

Next, if $\rho>1/8$ then $t^2-r^2-\rho^2\approx\sqrt{t^2-r^2}-\rho$ and also on the support of the integrand $\sqrt{t^2-r^2}-\rho\ge(t^2-r^2-\rho^2)/6\ge 2r\rho/3\ge r/12$. Therefore

$$\mathcal{E}_b[f\chi_{[1/8,2]}](t,r)\lesssim\int_{\sqrt{t^2-r^2}-\rho\ge r/12}(\sqrt{t^2-r^2}-\rho)^{b-\frac{n}{2}}|f(\rho)|\rho^{n-1}d\rho,$$

and since now $\rho \approx 1$ the estimate

$$2^{-\lambda}\mathcal{E}_b[f\chi_{[1/8,2]}](t,r) \lesssim w_b(r) M\big[\chi_{[1/8,2]}(1+\lambda/\rho)^{-\gamma}\rho^{n-1}f\big](\sqrt{t^2-r^2}) \tag{4.25}$$

follows by standard arguments. □

Now we estimate the operators $\mathcal{R}_{\lambda,i}$, with kernels $R_{\lambda,i}$, $i=1,2,3$ which occur in (2.17).

Let H be the Hilbert transform on the real line; moreover define

$$\mathfrak{M}^* f := Mf + M[Hf]. \tag{4.26}$$

Lemma 4.5. *Suppose that $p>2$. There is a constant C (independent of λ) so that the following pointwise inequalities hold for $1 \le t \le 2$.*

(i) Suppose that $r \le 4$. Then

$$|\mathcal{R}_{\lambda,1} f(t,r)| \lesssim C r^{-\frac{n-1}{2}} \sum_{\pm} \big[|f(\pm r \pm t)||(\pm r \pm t)|^{\frac{n-1}{p}}\big], \tag{4.27}$$

$$|\mathcal{R}_{\lambda,3} f(t,r)| \lesssim r^{-\frac{n-1}{2}} \sum_{\pm} M\big[\chi_{[0,6]} f(\cdot)(\cdot)^{\frac{n-1}{p}}\big](\pm r \pm t). \tag{4.28}$$

Let $I_m = [2^{-m}, 2^{1-m}]$. Then

$$|\mathcal{R}_{\lambda,2} f(t,r)| \lesssim r^{-\frac{n-1}{2}} \sum_{m=-1}^{\infty} \big|\mathfrak{M}^*\big[\chi_{I_m} f(\cdot)(\cdot)^{\frac{n-1}{2}}\big](t-r)\big|. \tag{4.29}$$

(ii) Suppose that $r \ge 4$. Then $\mathcal{R}_{\lambda,2} f(t,r) = 0$ and

$$|\mathcal{R}_{\lambda,1} f(t,r)| \le C r^{-\frac{n-1}{p}} \sum_{\pm} \big[|f(r \pm t)||(r \pm t)|^{\frac{n-1}{p}}\big], \tag{4.30}$$

$$|\mathcal{R}_{\lambda,3} f(t,r)| \le C r^{-\frac{n-1}{p}} \sum_{\pm} M\big[f(\cdot)(\cdot)^{\frac{n-1}{p}}\big](r \pm t). \tag{4.31}$$

Proof. Estimates (4.27), (4.30) are immediate from the definition (2.19) of $\mathcal{R}_{\lambda,1}$. By (2.21.1/2) we obtain the pointwise inequality

$$\chi_{(-2,1)}(\mu)|R_{\lambda,3}(t,r,\rho)| \lesssim \Big(\frac{\rho}{r}\Big)^{\frac{n-1}{2}} \sum_{m=0}^{\infty} \min\{\lambda^{1/2}2^{-m/4}, \lambda^{-1/2}2^{m/4}\}(A_m+B_m)$$

so that the estimates (4.28), (4.31) follow from Lemma 4.2.

From Lemma 4.1 it follows that $\mathcal{R}_{\lambda,2} f(t,r) = 0$ if $r>2$. For $r \le 2$ we use

$$\frac{1}{t+r+\rho}\frac{1}{\rho-t+r} = \frac{1}{2t}\Big(\frac{1}{\rho-t+r} - \frac{1}{t+r+\rho}\Big).$$

Let

$$F_m f(\rho) = \chi_{I_m}(\rho)\rho^{\frac{n-1}{2}} f(\rho).$$

The integral operator with kernel

$$(\rho/r)^{(n-1)/2}(2t)^{-1}(t+r+\rho)^{-1}\zeta(\lambda\sqrt{|1-\mu^2(t,r,\rho)|})\chi_{I_m}(\rho),$$

when applied to f, is easily seen to be bounded by $r^{-(n-1)/2}\int_{I_m}|f(\rho)|\rho^{(n-1)/2}d\rho$ which in turn is trivially bounded by $r^{-(n-1)/2}M\big[F_m f\big](t-r)$.

It remains to estimate the principal value operator $\mathcal{T}$ with kernel

$$\Big(\frac{\rho}{r}\Big)^{(n-1)/2}\text{p.v.}\frac{1}{\rho-t+r}\zeta(\lambda\sqrt{|1-\mu^2(t,r,\rho)|})\chi_{(-2,0)}(\mu(t,r,\rho)).$$

Observe that for $-2\le\mu\le 0$ we have

$$1-\mu^2\approx 1+\mu\approx\frac{|r+\rho-t|}{r\rho}$$

and also $r,\rho\le 2$, $r+\rho>1/2$ by Lemma 4.1.

Introduce the dyadic cutoff $\beta_k(s)=\zeta(2^{-k-1}s)-\zeta(2^{-k}s)$ where the even function ζ is defined as in the statement of Theorems 2.1 and 2.2. Let

$$\mathcal{T}_{k,m}f(t,r)=r^{-(n-1)/2}\times$$
$$\text{p.v.}\int\frac{1}{\rho-t+r}\zeta(\lambda\sqrt{|1-\mu^2(t,r,\rho)|})\chi_{(-2,0)}(\mu(t,r,\rho))\beta_k(r+\rho-t)\chi_{I_m}(\rho)\rho^{(n-1)/2}f(\rho)d\rho$$

and

$$\widetilde{\mathcal{T}}_{k,m}f(t,r)=r^{-(n-1)/2}\text{p.v.}\int\frac{1}{\rho-t+r}\beta_k(r+\rho-t)\chi_{I_m}(\rho)\rho^{(n-1)/2}f(\rho)d\rho.$$

We observe that $\mathcal{T}[\chi_{I_m}f](t,r)=\sum_k\mathcal{T}_{k,m}f(t,r)$ where the sum is extended over those k with $2^k\lesssim\lambda^{-2}2^{-m}r$.

Suppose that $|r+\rho-t|\approx\lambda^{-2}2^{-m-s}r$ for $s\ge 0$. Then $|\zeta(\lambda\sqrt{|1-\mu^2|})-1|\lesssim\lambda\sqrt{|1-\mu^2|}\le\lambda(\frac{|r+\rho-t|}{r\rho})^{1/2}\le 2^{-s/2}$ and from this one deduces that $\sum_{2^k\lesssim\lambda^{-2}2^{-m}r}|\mathcal{T}_{k,m}f-\widetilde{\mathcal{T}}_{k,m}f|$ is dominated by $r^{-(n-1)/2}M\big[F_m f\big](t-r)$. By Cotlar's inequality $\sum_k\widetilde{\mathcal{T}}_{k,m}$ is dominated by $M[F_m f]+M(H[F_m f])$, where H is the Hilbert transform; see [32, §I.7.3]. Thus (4.29) follows. ☐

The previous estimates can also be applied to the main (oscillatory) term in (2.24) to prove L^∞ bounds; the oscillation is irrelevant here.

Lemma 4.6. *Suppose $p>2$, $1\le t\le 2$ and*

$$P_{b,\lambda}(t,r,\rho):=\begin{cases} r^{b-\frac{n}{2}}\rho^{\frac{n}{2}+b-1}\lambda^{-b}(1-\mu(t,r,\rho)^2)^{\frac{b-1}{2}} & \text{if } |\mu|\le 1 \text{ and } \lambda\sqrt{1-\mu^2}\ge 1\\ 0 & \text{otherwise.}\end{cases}\tag{4.32}$$

Let $\mathcal{P}_{b,\lambda}$ be the associated integral operator and let $G_{p,b}F$ be as in (4.9) (with $\gamma=b$).

(i) Suppose that $b\ge -1/2$. Then

$$\Big(\sum_j|\mathcal{P}_{b,\lambda_j}f(t,r)|^2\Big)^{1/2}\lesssim r^{b+\frac12-\frac{n-1}{2}}\sum_\pm M[\chi_{[0,6]}G_{p,b}F](r\pm t)\quad \text{if}\quad r\le 4,\tag{4.33}$$

and

$$\Big(\sum_j|\mathcal{P}_{b,\lambda_j}f_j(t,r)|^2\Big)^{1/2}\lesssim r^{-\frac{n-1}{p}}\sum_\pm M\big[G_{p,b}F\big](r\pm t)\quad \text{if}\quad r\ge 4.\tag{4.34}$$

(ii) Suppose that $b \ge (n-2)/2$. Then

$$\sup_{1\le t\le 2}\sup_{r>0}\Big(\sum_j |\mathcal{P}_{b,\lambda_j} f_j(t,r)|^2\Big)^{1/2} \lesssim \sup_{\rho>0}\Big(\sum_j |(1+\lambda_j/\rho)^{-b} f_j(\rho)|^2\Big)^{1/2}. \tag{4.35}$$

Proof. This follows from the pointwise estimate

$$|P_{b,\lambda}| \lesssim \lambda^{-b}\sum_{\ell\ge 0}\rho^{\frac{n}{2}+b} r^{b+1-\frac{n}{2}} 2^{-\frac{b+1}{2}\ell}(A_\ell+B_\ell), \quad 1\le t\le 2; \tag{4.36}$$

the proof is analogous to the proof of Lemma 4.3. (ii) follows from the obvious modification of (i) for the case $p=\infty$. □

In order to derive endpoint bounds for $p = 2n/(n-1)$ we need the following lemma, a variant of which has been used already by Colzani, Cominardi and Stempak [10].

Lemma 4.7. *Let $n\in\mathbb{R}$, $n>0$ and $r\to u(r)$ be a measurable real-valued function on $\mathbb{R}^+$. Assume $1\le p<\infty$. For $g\in L^p(\mathbb{R},dx)$ define*

$$Sg(t,r) = r^{-n/p} g(t+u(r)).$$

Consider the measure $d\mu_n = dt\, r^{n-1}dr$ on $\mathbb{R}\times\mathbb{R}^+$ and let

$$E_\alpha(f) := \{(t,r) : |Sg(t,r)| > \alpha\}.$$

Then for $\alpha>0$

$$\mu_n\big(E_\alpha(f)\big) = n^{-1}\alpha^{-p}\int |g(x)|^p dx \tag{4.37}$$

Proof. We perform the change of variable $(x,r) = (t+u(r),r)$. Then

$$\begin{aligned}\mu_n(E_\alpha) &= \int r^{n-1}\int_{x:|g(x)|>\alpha r^{n/p}} dx\, dr\\ &= \int\int_{r:r<[|g(x)|/\alpha]^{p/n}} r^{n-1}dr\, dx = n^{-1}\alpha^{-p}\int |g(x)|^p dx. \quad \square\end{aligned}$$

Remark. In our context $u(r)=\pm r$. Colzani, Cominardi and Stempak [10] used a similar lemma to prove the weak type (p_0,p_0) space time estimate for the wave operator on radial functions; here $p_0 = 2n/(n-1)$. To relate this to the estimates in [25] we remark that the weak type inequality could also be obtained by applying Lemma 4.7 to the terms arising in formula (3.4) in [25].

We are now ready to give the

Applications to the operators arising in Theorems 2.1 and 2.2.

Denote by $\mathcal{R}_{\lambda,i}$, $i=1,\dots,4$ the operators with kernels $R_{\lambda,i}$ in (2.17), (2.19-22). We begin by verifying the analogue of the estimate (2.29) for these operators, as well as the corresponding weak type and restricted weak type estimates in Theorem 2.5; in fact for the operators $\{\mathcal{R}_{\lambda_j,i}\}$ this distinction is irrelevant as we prove weak type inequalities. We remark that none of the estimates involves the factor $(1+\lambda_j/\rho)^\varepsilon$ in (2.29); again this is only needed for the estimation of the oscillatory terms.

Proposition 4.8.

(i) Let $2 < p < 2n/(n-1)$ *and* $i \in \{1,2,3,4\}$. *Then*

$$\Big(\int_1^2\int_0^\infty \Big(\sum_j |\mathcal{R}_{\lambda_j,i} f_j(t,r)|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p} \lesssim \Big(\int_0^\infty \Big(\sum_j |f_j(\rho)|^2\Big)^{p/2}\rho^{n-1}d\rho\Big)^{1/p}. \tag{4.38}$$

(ii) Denote by μ_n *the measure* $dt\, r^{n-1}dr$ *on* $[1/2]\times\mathbb{R}^+$. *Let* E^i_α *be the set of all* $(t,r)\in[1,2]\times\mathbb{R}^+$ *such that* $\big(\sum_j |\mathcal{R}_{\lambda_j,i}f_j(t,r)|^2\big)^{1/2} > \alpha$. *Let* $p_0 = 2n/(n-1)$. *Then for* $i = 1,2,3,4$ *and for all* $\alpha > 0$

$$\mu_n\big(E^i_\alpha\big) \le C\alpha^{-p_0}\Big(\int_0^\infty \Big(\sum_j |f_j(\rho)|^2\Big)^{p_0/2}\rho^{n-1}d\rho\Big)^{1/p_0}. \tag{4.39}$$

(iii) Let $P_{-1/2,\lambda}$ *be as in (4.32). Then statement (ii) holds with with* $\mathcal{R}_{\lambda_j,i}$ *replaced by the operator* $(1+\lambda_j/\rho)^{-1/2}\mathcal{P}_{-1/2,\lambda_j}$.

Proof. For $i = 1,2,3$ the statements (i) and (ii) are a consequence of Lemmas 4.5 and 4.7 and the vector valued maximal inequality of Fefferman and Stein (see [32]).

First for the region where $r \ge 4$ estimates (4.30) and (4.31) yield strong type estimates for all $p > 2$. In order to prove the estimates (4.27-29) we observe that the function $r^{(n-1)(1-p/2)}$ is integrable near the origin if $p < 2n/(n-1)$, and one considers for fixed r the maximal functions as functions of t.

The term $r^{-\frac{n-1}{2}}M\big[\chi_{[1/8,2]}f(\cdot)(\cdot)^{n-1}\big](\sqrt{t^2-r^2})$ in the bound for $\mathcal{R}_{\lambda,4}$ can be reduced to the previous situation by changing variables $t\mapsto r+\sqrt{t^2-r^2}$. Note that in the present case $\sqrt{t^2-r^2}\approx 1$, $r,\rho\le 2$ so that there is no essential contribution from the Jacobian of the change of variable. This yields the asserted estimate for $\mathcal{R}_{\lambda,4}$. The endpoint weak type estimate (ii) is deduced using Lemma 4.7. (iii) follows in the same way from Lemmas 4.6 and 4.7. □

We shall now prove the analogue of (2.30-32) for the remainder terms $\{\mathcal{V}_{z,\lambda_j}\}$, $\{\mathcal{W}_{z,\lambda_j}\}$ as well as the appropriate endpoint estimates in the case $z+1/2 = n(1/2-1/p)-1/2$. Again the estimates are slightly better than what is stated in Theorems 2.3/2.5.

Proposition 4.9. *(i) Let* $p > 2$, $z = b+i\tau$, $b > -1/2$, $0\le\gamma\le(2b+1)/2$, *and* $b+1 > n(\frac12-\frac1p)$. *Then*

$$\Big(\int_1^2\int_0^\infty \Big(\sum_j |\mathcal{V}_{z,\lambda_j} f_j(t,r)|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p} \lesssim \Big(\int_0^\infty \Big(\sum_j |(1+\lambda_j/\rho)^{-\gamma} f_j(\rho)|^2\Big)^{p/2}\rho^{n-1}d\rho\Big)^{1/p}. \tag{4.40}$$

If in addition $b+1/2 \le (n-1)/p'$ *then*

$$\Big(\int_1^2\int_0^\infty \Big(\sum_{\lambda_j>\mathrm{Re}(z)-1} |\mathcal{W}_{z,\lambda_j} f_j(t,r)|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p} \lesssim \Big(\int_0^\infty \Big(\sum_j |(1+\lambda_j/\rho)^{-\gamma} f_j(\rho)|^2\Big)^{p/2}\rho^{n-1}d\rho\Big)^{1/p} \tag{4.41}$$

(ii) Denote by μ_n *the measure* $dtr^{n-1}dr$ *and assume* $p > 2$, $z = b+i\tau$, $b+1 = n(\frac12-\frac1p)$, $0\le\gamma\le(2b+1)/2$.

Let $E_\alpha(\mathcal{V},z)$ *be the set of all* $(t,r)\in[1,2]\times\mathbb{R}^+$ *such that* $(\sum_j|\mathcal{V}_{z,\lambda_j}f_j(t,r)|^2)^{1/2} > \alpha$ *and similarly define* $E_\alpha(\mathcal{W},z)$, *replacing* $\mathcal{V}_{z,\lambda}$ *by* $\mathcal{W}_{z,\lambda}$. *Define* $p_0(b)$ *by* $b+1 = n(1/2-1/p_0(b))$. *Then for all* $\alpha > 0$

$$\mu_n\big(E_\alpha(\mathcal{V},b+i\tau)\big) \lesssim \alpha^{-p_0(b)}\Big(\int_0^\infty \Big(\sum_j |(1+\lambda_j/\rho)^{-\gamma} f_j(\rho)|^2\Big)^{p_0(b)/2}\rho^{n-1}d\rho\Big)^{1/p_0(b)}. \tag{4.42}$$

The same inequality holds with $E_\alpha(\mathcal{V}, b+i\tau)$ replaced by $E_\alpha(\mathcal{W}, b+i\tau)$.

(iii) Statement (ii) holds with $\mathcal{V}_{b+i\tau,\lambda_j}$ replaced by $P_{b,\lambda_j}(1+\lambda_j/\rho)^{-1/2}$.

Proof. Similar to the proof of Proposition 4.8; for (4.40) one uses Lemma 4.3 and for (4.41) one uses Lemma 4.4. Again for the endpoint estimates one uses Lemma 4.7. Finally for statement (iii) we use the estimate in Lemma 4.6. □

We also obtain the

Proof of the L^∞ estimate (2.32). The relevant estimates follow immediately from Theorem 2.2 and Lemmas 4.3, 4.4 and 4.6. Note that (2.1) and the assumption $b < n/2$ implies $\lambda_j > b-1$, which is needed for Lemma 4.4. □

In a similar fashion we obtain the

Proof of the weak type endpoint inequality (2.34). We estimate the kernel of $\mathcal{O}_{z,\lambda_j}$ by P_{b,λ_j} and apply Proposition 4.8 (ii), (iii) in the case $b=-1/2$, $\tau=0$ and Proposition 4.9 (ii), (iii) in the case $b>-1/2$. The additional restriction $\gamma \le n(1/2-1/p)-1$ for the weak type estimates comes from parts (iii) of these propositions. □

5. Estimation of the oscillatory terms

In this section we shall prove the vector valued $L^p(\ell^2)$ estimates for the oscillatory integral operator $\mathcal{O}_{z,\lambda}$ with kernel $\{O_{z,\lambda}\}$; i.e. $\mathcal{O}_{z,\lambda} f = \langle O_{z,\lambda}(t,r,\cdot), f\rangle$.

We keep the convention about the dependence of the constants on $\mathrm{Im}(z)$ stated in the previous section. $\|\cdot\|_p$ will always denote an L^p norm on $\mathbb{R}$ with respect to the measure $d\rho$.

We first introduce a decomposition in terms of the quantity $\sqrt{1-\mu^2}$. Let $\eta \in C^\infty(\mathbb{R})$ so that $\eta(s)=0$ for $|s|\le 1/2$ and $\eta(s)=1$ for $|s|\ge 9/16$. Define

$$\begin{aligned}\eta_0(\mu) &:= \eta(1-\mu^2)\chi_{(-1,1)}(\mu)\\ \eta_\ell(\mu) &:= \big[\eta(2^\ell(1-\mu^2)) - \eta(2^{\ell-1}(1-\mu^2))\big]\chi_{(-1,1)}(\mu) \qquad \text{if } \ell\ge 1.\end{aligned}$$

Note that for $\ell \ge 1$ the function η_ℓ is supported where $(1-\mu^2)\in(2^{-\ell-1}, \frac{9}{8}2^{-\ell})$. Now $\mu^2\ge 7/16$ on its support and from this one easily sees that η_ℓ is supported in the union of the intervals $(1-2^{-\ell}, 1-2^{-\ell-2})$ and $(-1+2^{-\ell-2}, -1+2^{-\ell})$. Set

$$O_{z,\lambda,\ell}(t,r,\rho) := O_{z,\lambda}(t,r,\rho)\eta_\ell(\mu(t,r,\rho)).$$

We thus localize for $\ell>0$ where $1\pm\mu(t,r,\rho)$ is of the order of $2^{-\ell}$. For $\ell>0$ we also set

$$\begin{aligned}O^+_{z,\lambda,\ell}(t,r,\rho) &:= O_{z,\lambda,\ell}(t,r,\rho)\chi_{(0,1)}(\mu)\\ O^-_{z,\lambda,\ell}(t,r,\rho) &:= O_{z,\lambda,\ell}(t,r,\rho)\chi_{(-1,0)}(\mu).\end{aligned}$$

We denote by $\mathcal{O}_{z,\lambda,\ell}$ and $\mathcal{O}^\pm_{z,\lambda,\ell}$ the integral operators with kernels $O_{z,\lambda,\ell}$ and $O^\pm_{z,\lambda,\ell}$, respectively.

The main purpose of this section is to prove the following

Proposition 5.0. *Suppose $2\le p<\infty$, $b=Re(z)\ge -1/2$ and $0\le\gamma\le\min\{b+1/2, b+2/p\}$. Let I be a compact subinterval of $(1,2)$, and let*

$$\omega_{\gamma,p}(\lambda,\rho) := \begin{cases} \big[\log(2+\lambda/\rho)\big]^{1/2-1/p}(1+\lambda/\rho)^{-\gamma} & \text{if } 2\le p\le 4\\ \big(\frac{p}{p-4}\big)^{1/4}(1+\lambda/\rho)^{-\gamma} & \text{if } p>4\end{cases} \tag{5.0.1}$$

and, for $F = \{f_j\}$,

$$\mathcal{G}_{\gamma,p}(F,\rho) := \Big(\sum_j \big|\omega_{\gamma,p}(\lambda_j,\rho)f_j(\rho)\big|^2\Big)^{1/2}.$$

Then the following conclusions hold.

(i) $\mathcal{O}_{z,\lambda_j,\ell}f_j(t,r) = 0$ *if* $r \geq 2^{(\ell+6)/2}$, *and* $\mathcal{O}^-_{z,\lambda_j,\ell}f_j(t,r) = 0$ *if* $r \geq 4$.

(ii) For $3 \leq L \leq (\ell+6)/2$ *let* χ_L *be the characteristic function of the interval* $[2^{L-1}, 2^{L+1}]$. *Then*

$$(5.0.2)\quad \Big(\int_I\int_{2^L}^{2^{L+1}}\Big(\sum_{\lambda_j\geq 2^{\ell/2}}\big|\mathcal{O}_{z,\lambda_j,\ell}f_j(t,r)\big|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p}$$
$$\lesssim \Big(\int \big[\mathcal{G}_{\gamma,p}(F\chi_L,\rho)\big]^p(2^{-\ell}\rho^2)^{p\frac{2b+1}{4}}\min\{1,(2^{-\ell}\rho^2)^{\frac14-\frac1p}\}^p\rho^{n-1}d\rho\Big)^{1/p}.$$

(iii) Let $\chi \in C^\infty(\mathbb{R}^+)$ *be supported in* $(10^{-4},\infty)$. *Then*

$$(5.0.3)\quad \Big(\int_I\int_{10^{-3}}^{8}\Big(\sum_{\lambda_j\geq 2^{\ell/2}}\big|\mathcal{O}_{z,\lambda_j,\ell}[\chi f_j](t,r)\big|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p}$$
$$\lesssim 2^{-\ell\frac{2b+1}{4}}\min\{1,2^{-\ell(\frac14-\frac1p)}\}\Big(\int_{10^{-4}}^{10}\big[\mathcal{G}_{\gamma,p}(F,\rho)\big]^p\rho^{n-1}d\rho\Big)^{1/p}.$$

(iv) Let $R \leq 10^{-3}$. *Then*

$$(5.0.4)\quad \Big(\int_I\int_{R/2}^{R}\Big(\sum_{\lambda_j\geq 2^{\ell/2}}\big|\mathcal{O}_{z,\lambda_j,\ell}f_j(t,r)\big|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p}$$
$$\lesssim 2^{-\ell\frac{2b+1}{4}}\min\{1,2^{-\ell(\frac14-\frac1p)}\}R^{\frac np-\frac n2+b+1}\Big(\int_0^\infty\big[\mathcal{G}_{\gamma,p}(F,\rho)\big]^p\rho^{n-1}d\rho\Big)^{1/p}.$$

(v) Let $R \leq 10^{-3}$ *and let* χ_R *be the characteristic function of the interval* $[R/2,R]$. *Then*

$$(5.0.5)\quad \Big(\int_I\int_0^\infty\Big(\sum_{\lambda_j\geq 2^{\ell/2}}\big|\mathcal{O}_{z,\lambda_j,\ell}[\chi_R f_j](t,r)\big|^2\Big)^{p/2} r^{n-1}drdt\Big)^{1/p}$$
$$\lesssim 2^{-\ell\frac{2b+1}{4}}\min\{1,2^{-\ell(\frac14-\frac1p)}\}\min\{R^{(n-2)(\frac12-\frac1p)},R^{n(\frac12-\frac1p)-\frac2p}\}\Big(\int_0^\infty\big[\mathcal{G}_{\gamma,p}(F,\rho)\big]^p\rho^{n-1}d\rho\Big)^{1/p}.$$

We shall split $O_{z,\lambda,\ell}$ into a finite number of pieces, where each one of them possesses a certain localization property which then suggests an appropriate rescaling.

5.1 Localization. We shall now prove localization properties refining those in Lemma 4.1; the finer localizations depend on the size of the quantities $1 \mp \mu(t,r,\rho)$, *cf.* (2.14.1/2). The results are stated separately for the cases $\mu \geq 0$ and $\mu < 0$ in Lemma 5.1.1 and Lemma 5.1.2, respectively.

Lemma 5.1.1. *Suppose that* $1 \leq t \leq 2$ *and that* $0 < \delta/4 < 1-\mu(t,r,\rho) < \delta < 2$.

If $(t,r,\rho) \in \operatorname{supp} O_{z,\lambda}$, *then the following statements hold.*

(i) $|r-\rho| \leq 2$.

(ii) $\delta r^2 \leq 32$.

(iii) Let $R\ge 8$ and $\delta R^2\le 1/16$. If $R/2\le r,\rho\le 2R$, then we have $|r-\rho|\ge 1/2$ and

$$\begin{aligned} &\rho-8\delta R^2\le r-t\le \rho-\frac{\delta R^2}{32}, &&\text{if}\quad r-\rho\ge 0,\\ &\rho+\delta R^2/32< r+t<\rho+8\delta R^2, &&\text{if}\quad r-\rho\le 0.\end{aligned}$$

(iv) If $r\le 1/4$ then $1/2<\rho<3$. Moreover if also $R/2\le r\le 2R$ then

$$r+t-12\delta R\le \rho<r+t-\delta R/32.$$

(v) If $1/8\le r\le 4$ and if $R/2\le \rho\le 2R$, $R\le 1/4$ then

$$\delta R/200\le t-r+\rho\le 32\delta R.$$

(vi) If $1/8\le r\le 4$ and if $\rho\le 10^{-3}$ then $|t-r|\le 1/5$.
(vii) Let $10^{-4}\le r,\rho\le 10$ and $\delta\le 10^{-7}$. Then

$$\begin{aligned} &\rho-200\delta\le r-t\le \rho-10^{-7}\delta &&\text{if}\quad r-\rho\ge 0\\ &\rho+10^{-7}\delta< r+t<\rho+200\delta &&\text{if}\quad r-\rho\le 0.\end{aligned}$$

Proof. The assumption $\delta/4<1-\mu<\delta$ is equivalent with

$$t^2-2\delta r\rho<(r-\rho)^2<t^2-\frac{\delta}{2}r\rho. \tag{5.1.1}$$

(i) follows immediately since $t\le 2$. We also have

$$\delta r\rho\le 8$$

and (ii) follows since $\delta<2$ and since $\rho\ge r/2$ if $r\ge 4$.

If $R/2\le r,\rho\le 2R$, then $\sqrt{t^2-8\delta R^2}\le |r-\rho|\le \sqrt{t^2-\delta R^2/8}$ by (5.1.1). We use the inequalities $1-x\le\sqrt{1-x}\le 1-x/2$ for $0\le x\le 1$ and the assumption $1\le t\le 2$ and arrive at

$$t-8\delta R^2\le t\sqrt{1-8t^{-2}\delta R^2}\le |r-\rho|\le t\sqrt{1-t^{-2}\delta R^2/8}\le t-\delta R^2/32;$$

hence (iii).

From (5.1.1) we see that $t^2-r^2\le \rho^2+2r\rho$ and if $r\le 1/4$ it follows that $15/16\le t^2-r^2\le\rho^2+\rho/2$ which implies $\rho>1/2$. Clearly also $\rho<3$ by (i), and therefore if $r/R\in[1/2,2]$ then by (5.1.1)

$$t^2-12\delta R\le (\rho-r)^2\le t^2-\delta R/8.$$

Arguing as for (iii) we derive $t-12\delta R\le \rho-r\le t-\delta R/32$; hence (iv).

Next, (5.1.1) implies

$$\tfrac12\delta r\rho<(t-r+\rho)(t+r-\rho)<2\delta r\rho.$$

Now if $1/8\le r\le 4$ and if $R/2\le\rho\le 2R$, $R\le 1/4$ then $t+r-\rho\in[\frac12,6]$ and (v) is a consequence.

(vi) follows from (v). For (vii) we have $r\rho\ge 10^{-5}$ by (iv) and therefore $t^2-200\delta\le (r-\rho)^2\le t^2-\delta 10^{-5}/2$. Arguing as for (iii) we obtain (vii). □

Lemma 5.1.2. *Suppose that* $1 \le t \le 2$ *and that* $0 < \delta/4 < 1 + \mu(t, r, \rho) < \delta \le 1/2$.

If $(t, r, \rho) \in \operatorname{supp} O_{z,\lambda}$, *then the following statements hold.*

(i) $1 \le r + \rho \le 3$.

(ii) If $r < 3$, $10^{-4} \le \rho \le 3$, *and* $R/2 \le r \le 2R$, *then*

$$t - r + 10^{-6}\delta R \le \rho \le t - r + 6\delta R.$$

(iii) If $1/2 \le r \le 3$ *and* $\rho < 10^{-2}$, $R/2 \le \rho \le R$ *then*

$$10^{-2}\delta R \le r - t + \rho \le 3\delta R;$$

in particular $|t - r| \le 10^{-1}$.

Remark. Under the hypotheses of Lemmas 5.1.1 and 5.1.2, the cases (iii)-(vii) in Lemma 5.1.1 and (ii)-(iii) in Lemma 5.1.2 exhaust all possibilities, see also Lemma 4.1.

Proof. The assumption $\delta/4 < 1 + \mu < \delta$ is equivalent with

$$t^2 + \tfrac{1}{2}\delta r\rho < (r + \rho)^2 < t^2 + 2\delta r\rho. \tag{5.1.2}$$

(i) follows since we assume $1 \le t \le 2$ and $\delta \le 1/2$.

If $r < 3$, $10^{-4} \le \rho \le 3$, and $R/2 \le r \le 2R$, then

$$t^2 + 10^{-4}\delta R/4 < (r + \rho)^2 < t^2 + 12\delta R.$$

Since $1 + x/4 \le (1 + x)^{1/2}$ for $0 \le x \le 3$ and $(1 + x)^{1/2} \le 1 + x/2$ for $x \ge 0$ we derive

$$t + 10^{-4}\delta R/64 \le r + \rho \le t + 6\delta R$$

and therefore (ii).

Next it follows from (5.1.2) that

$$\tfrac{1}{2}\delta r\rho \le (r + \rho - t)(r + \rho + t) \le 2\delta r\rho \le 6\delta R.$$

Moreover $2 \le r + \rho + t \le 5$, by (i). Now one easily concludes (iii). $\square$

5.2 Estimates for localized operators. We shall now give estimates for various localizations of the operators $\mathcal{O}_{z,\lambda,\ell}$; the formal decomposition of $\mathcal{O}_{z,\lambda,\ell}$ is then discussed in §5.8. below. The localized operators can be estimated using Theorem 3.1 once some rescaling, and, in some cases, some nonlinear change of variables is performed. We shall first describe the general argument; it is applied in the subsequent sections 5.3-7 to specific situations.

We fix

$$0 < \delta < 3/2$$

and consider operators

$$\begin{aligned} \mathcal{A}_\lambda f(t, r) &= \int A_\lambda(t, r, \rho) f(\rho) d\rho \\ &= \int a_\lambda(t, r, \rho) e^{i\lambda \arccos \mu(t,r,\rho)} f(\rho) d\rho \end{aligned} \tag{5.2.1}$$

where the symbols a_λ have the property that

$$\mu \in (1-\delta, 1-\delta/4) \cup (-1+\delta/4, -1+\delta) \qquad \text{if } (t,r,\rho) \in \text{supp } A_\lambda. \tag{5.2.2}$$

In all cases we shall use changes of variables

$$(t,r) = (t(x)), r(x)), \quad x = (x_1, x_2), \tag{5.2.3}$$

and

$$\rho = \rho(y) = \rho_0 + \gamma y \tag{5.2.4}$$

which depend on the particular case considered. In each case we shall have

$$C_1 M \le \Big| \det \frac{\partial(t,r)}{\partial(x_1,x_2)} \Big| \le C_2 M \tag{5.2.5}$$

for some positive M and some absolute positive constants C_1, C_2. The changes of variables will also have the property that the C_N norms of

$$\tilde\mu(x,y) = \mu(t(x), r(x), \rho(y)) \tag{5.2.6}$$

$$\phi(x,y) = \delta^{-1/2} \arccos \tilde\mu(x,y) \tag{5.2.7}$$

$$\tilde a_\lambda(x,y) = a_\lambda(t(x), r(x), \rho(y)) \tag{5.2.8}$$

will be bounded by an absolute constant (here N is large, but fixed, as in Theorem 3.1), and $\widetilde a_\lambda$ will be supported in a fixed ball. Moreover we shall have the conditions

$$\big| CS[\phi] \big| \ge C \tag{5.2.9}$$

$$\big| \phi_{x_2 y} \big| \ge C. \tag{5.2.10}$$

for some absolute constant $C > 0$.

Using this setup one computes

$$\mathcal{A}_{\lambda_j} h_j(t(x), r(x)) = \gamma \widetilde{\mathcal{A}}_{\lambda_j\sqrt{\delta}} g_j(x) \qquad \text{where } g_j(y) = h_j(\rho_0 + \gamma y) \tag{5.2.11}$$

and where $\widetilde{\mathcal{A}}_{\lambda\sqrt{\delta}}$ is defined by

$$\widetilde{\mathcal{A}}_{\lambda\sqrt{\delta}} g(x) = \int \tilde a(x,y) e^{i\lambda\sqrt{\delta}\phi(x,y)} g(y)\, dy. \tag{5.2.12}$$

Assuming that the C^N norms of (5.2.6-8) are bounded and that (5.2.9/10) holds we can then apply Theorem 3.1 to the operators $\widetilde{A}_{\lambda_j\sqrt{\delta}}$ if $\lambda_j^2\delta \ge 1$. Let

$$\Omega_p(\lambda) = \begin{cases} \lambda^{-1/2}[\log(2+\lambda)]^{\frac12 - \frac1p} & \text{if } 2 \le p \le 4 \\ \big(\frac{p}{p-4}\big)^{1/4} \lambda^{-2/p} & \text{if } 4 < p < \infty \end{cases}; \tag{5.2.13}$$

we exclude the case $p = \infty$ just for notational reasons since it has already been dealt with in §4. By (5.2.4/5) we obtain

$$\begin{aligned}&\Big(\int\Big(\sum_{\lambda_j^2\delta\geq 1}|\mathcal{A}_{\lambda_j}h_j(t,r)|^2\Big)^{p/2}dtdr\Big)^{1/p}\lesssim M^{1/p}\gamma\Big(\int\Big(\sum_{\lambda_j^2\delta\geq 1}\big|\widetilde{\mathcal{A}}_{\lambda_j\sqrt{\delta}}g_j(x)\big|^2\Big)^{p/2}dx\Big)^{1/p}\\&\lesssim M^{1/p}\gamma\Big(\int\Big(\sum_{\lambda_j^2\delta\geq 1}\big|\Omega_p(\lambda_j\sqrt{\delta})g_j(y)\big|^2\Big)^{p/2}dy\Big)^{1/p}\\&\lesssim M^{1/p}\gamma^{1-1/p}\Big(\int\Big(\sum_{\lambda_j^2\delta\geq 1}\big|\Omega_p(\lambda_j\sqrt{\delta})h_j(\rho)\big|^2\Big)^{p/2}d\rho\Big)^{1/p}.\end{aligned}\tag{5.2.14}$$

Notation. In the following sections 5.3-7 we shall discuss different localizations. In each case we shall use the above notation, although the changes of variables (5.2.3/4) will differ. L^p norms will be taken with respect to Lebesgue measure in $\mathbb{R}^2$ or $\mathbb{R}$. N will be a fixed large number large (chosen so that Theorem 3.1 can be applied). $\mathfrak{B}_N$ will be the class of C^N functions with support in $(-1,1)$ so that $\|\chi\|_{C^N}\leq 1$. We assume that $\chi_1,\chi_2,\chi_3,\chi_4\in\mathfrak{B}_N$. The C^∞ function χ will be supported in $(1,2)$. We also assume that $\zeta\in C_0^\infty(-1,1)$ so that $\zeta(s)=1$ if $|s|\leq 1/2$ and that $\beta\in C_0^\infty$ so that supp $\beta\subset(1/4,1)$.

5.3. The case $\rho\approx r\geq 10^{-4}$, $\mu>-1/2$. Suppose that $R\geq 10^{-4}$ and

$$r_0,\rho_0\in(10^{-4},\infty),\qquad |r_0-\rho_0|\leq 4,\qquad R/2\leq r_0\leq 2R,\qquad \delta R^2\leq 10^2,\qquad |s_0-\rho_0|\leq 10^3\delta R^2.$$

We use the symbol $\epsilon=\pm1$ with a fixed choice of 1 or -1.

Let $\mu=\mu(t,r,\rho)$ and define

$$a_\lambda(t,r,\rho)=(1-\zeta((\lambda+1)\sqrt{1-\mu^2}))\beta(\frac{1-\mu}{\delta})\chi_1(10^{10}\frac{\rho-\rho_0}{\delta R^2})\chi_2(10^{10}(r-r_0))\chi_3(\frac{r-\epsilon t-s_0}{\delta R^2})\chi_4(t).\tag{5.3.1}$$

Lemma 5.3.1. *Let $p\geq 2$ and let $\mathcal{A}_\lambda$ be defined by (5.2.1), (5.3.1). Then*

$$\Big\|\Big(\sum_{\lambda_j^2\delta\geq 1}|\mathcal{A}_{\lambda_j}h_j|^2\Big)^{1/2}\Big\|_p\lesssim\delta R^2\Big\|\Big(\sum_{\lambda_j^2\delta\geq 1}\big|\Omega_p(\lambda_j\sqrt{\delta})h_j\big|^2\Big)^{1/2}\Big\|_p$$

where C does not depend on δ and R.

Proof. We introduce coordinates as in (5.2.3/4) with

$$\begin{aligned}(t(x),r(x))&=(x_1,\epsilon x_1+s_0+\delta R^2x_2),\\ \rho(y)&=\rho_0+\delta R^2y.\end{aligned}\tag{5.3.2}$$

Then $\gamma=M=\delta R^2$ in (5.2.4/5), so that the assertion follows from (5.2.14) once the bounds on $\tilde{\mu}$, ϕ and $\tilde{a}_\lambda$ are checked, as well as (5.2.9/10).

The assertion on the support of $\tilde{a}$ is easily verified. Concerning the lower bounds in (5.2.9-10) we use Proposition 3.4 observing that $\phi(x,y)=\delta^{-1/2}\psi(t,r,\rho)$ where ψ is defined in (3.44). Then $\psi_{r\rho}\approx\delta^{-3/2}R^{-4}$ if $1-\mu\approx\delta$. We compute

$$\phi_{x_2y}=\delta^{-1/2}(\delta R^2)^2\psi_{r\rho}\approx 1.$$

Using also Remark 3.5 we obtain

$$CS[\phi]=\delta^{-1}(\delta R^2)^4CS[\psi]=t^3(R^8r^{-5}\rho^{-3})(\delta/\sqrt{1-\mu^2})^3\approx 1.$$

Now let $s = r - \epsilon t$ and define

$$\nu(t, s, \varrho) := \mu(t, s + \epsilon t, \varrho)$$

so that

$$\widetilde{\mu}(x, y) = \nu(x_1, s_0 + \delta R^2 x_2, \rho_0 + \delta R^2 y). \tag{5.3.3}$$

Then, by Proposition 3.4

$$\nu_t = \mu_t + \epsilon\mu_r = \epsilon \frac{s^2 - \varrho^2}{2(s + \epsilon t)^2 \varrho},$$

hence

$$\partial_t^k \nu = c_k \frac{s^2 - \varrho^2}{(s + \epsilon t)^{k+1} \varrho}, \qquad k \geq 1.$$

Moreover,

$$\begin{aligned} \nu_s &= \frac{s^2 - \varrho^2}{2(s + \epsilon t)^2 \varrho} + \frac{\epsilon t}{(s + \epsilon t)\varrho}, \\ \nu_\varrho &= \frac{\varrho^2 - s^2}{2(s + \epsilon t)\varrho^2} - \frac{\epsilon t s}{(s + \epsilon t)\varrho^2} \end{aligned}$$

and, since $|s^2 - \varrho^2| = |(s - \varrho)(s + \varrho)| \approx \delta R^2 R$, one finds by induction

$$\begin{aligned} &|\partial_t^k \nu| \leq C_k (\delta R^2) R^{-k-1} \\ &|\partial_t^k \partial_s^i \partial_\rho^j \nu| \leq C_{k,i,j} R^{-2} \quad \text{if } i + j > 0. \end{aligned}$$

Using (5.3.3) this implies

$$|\partial_{x_1}^k \partial_{x_2}^i \partial_y^k \tilde{\mu}| \leq C_{k,i,j}\ \delta, \quad \text{if } k + i + j > 0. \tag{5.3.4}$$

Moreover, one checks that for $k = 0, 1, 2, \dots$

$$|\big(\tfrac{d}{d\mu}\big)^k \arccos(\mu)| \leq c_k \delta^{1/2-k} \qquad \text{if } 1 - \mu \approx \delta. \tag{5.3.5}$$

By the chain rule and induction one verifies that

$$D^\alpha \phi = \delta^{-1/2} \sum_{k=0}^{|\alpha|} \sum_{\substack{\beta^1,\dots,\beta^k \\ \sum |\beta^j| \leq |\alpha|}} C_{\alpha,j,\beta^1,\dots,\beta^k}\ \arccos^{(k)} \circ \tilde{\mu}\ (D^{\beta^1} \tilde{\mu}) \dots (D^{\beta^k} \tilde{\mu}), \tag{5.3.6}$$

and it easily follows that the C_N norm of ϕ is bounded independently of δ and R.

Note that the derivatives of the function $1 - \zeta((1 + \lambda)\sqrt{1 - \mu^2})$ vanish on the support of $\beta((1 - \mu)/\delta)$ if $\delta \gtrsim (1 + \lambda)^{-2}$. Moreover, for any $\kappa \in \mathbb{R}$,

$$|\big(\tfrac{d}{d\mu}\big)^k (1 - \mu^2)^\kappa| \leq C_k \delta^{\kappa-k} \tag{5.3.7}$$

on this support. From this and (5.3.4) one quickly deduces that the C^N norm of the amplitude $\tilde{a}_\lambda$ is bounded independently of λ, δ and R. $\square$

5.4. The case $r \le 1/4$, $\mu > -1/2$. We now assume that

$$1/4 \le \rho_0 \le 4, \qquad R/2 \le r_0 \le 2R \le 10^{-2}, \qquad 10^{-2}\delta R \le s_0 - \rho_0 \le 10^2 \delta R \tag{5.4.1}$$

and define

$$a_\lambda(t,r,\rho) = (1-\zeta((\lambda+1)\sqrt{1-\mu^2}))\beta(\frac{1-\mu}{\delta})\chi_1(10^4\frac{t+r-s_0}{\delta R})\chi_2(10^4\frac{\rho-\rho_0}{\delta R})\chi_3(4\frac{r-r_0}{R})\chi_4(t). \tag{5.4.2}$$

Lemma 5.4.1. *Let $p \ge 2$ and let $\mathcal{A}_\lambda$ be defined by (5.2.1) with a_λ as in (5.4.2). Then*

$$\Big\| \Big(\sum_{\lambda_j^2 \delta \ge 1} |\mathcal{A}_{\lambda_j} h_j|^2 \Big)^{1/2} \Big\|_p \lesssim \delta R^{1+1/p} \Big\| \Big(\sum_{\lambda_j^2\delta\ge 1} \big|\Omega_p(\lambda_j \sqrt{\delta}) h_j\big|^2 \Big)^{1/2} \Big\|_p$$

where C does not depend on δ and R.

Proof. We now introduce coordinates

$$t(x) = s_0 + \delta R x_1 - R x_2, \qquad r(x) = R x_2, \qquad \rho(y) = \rho_0 + \delta R y. \tag{5.4.3}$$

Then $\gamma = \delta R$, $M = \delta R^2$ in (5.2.4/5) and (5.2.14) will hold once uniform bounds for the functions in (5.2.6-8) and the lower bounds (5.2.9/10) are verified.

First observe that by Proposition 3.4

$$\psi_{t\rho} - \psi_{r\rho} = \frac{1}{(1-\mu^2)^{3/2}} \frac{t}{r^2\rho^2}(\rho\mu - r - t) = \frac{1}{(1-\mu^2)^{3/2}} \frac{t}{2r^3\rho^2}(\rho^2 - (t+r)^2).$$

On the support of a_λ we have that $(\rho^2 - (t+r)^2) \approx \delta R$. Therefore

$$\delta^{1/2}\phi_{x_2 y} = R^2 \delta(\psi_{t\rho} - \psi_{r\rho}) \approx \delta^{1/2};$$

hence (5.2.10). Next $CS[\delta^{-1/2}\psi] \approx \delta^{-1}\delta^{-3}R^{-5}$ on the support of a_λ and by Remark 3.5 we see that $|CS[\phi]| \approx 1$.

To verify the upper bounds on $\tilde{a}_\lambda$ and ϕ we put $s := t + r$ and define

$$\nu(s,r,\varrho) = \mu(s-r,r,\varrho). \tag{5.4.4}$$

Then, by Proposition 3.4

$$\nu_s = \mu_t = \frac{r-s}{r\varrho}, \quad \nu_\varrho = -\frac{\nu}{\varrho} + \frac{1}{r}, \tag{5.4.5}$$

and

$$\nu_r = \mu_r - \mu_t = \frac{s^2 - \varrho^2}{2r^2\varrho},$$

hence

$$\partial_r^k \nu = c_k \frac{s^2-\varrho^2}{r^{k+1}\varrho}, \quad k \ge 1. \tag{5.4.6}$$

Since $s \approx \varrho \approx 1$, $r \approx R$ and $|s^2 - \varrho^2| \approx \delta R$, one uses induction to deduce the following estimates from (5.4.5/6).

$$\begin{aligned} &|\partial_r^k \nu| \le C_k(\delta R) R^{-k-1}, && \text{if } k \ge 1, \\ &|\partial_s^i \partial_r^k \partial_\varrho^j \nu| \le C_{k,i,j} R^{-k-1}, && \text{if } i + j > 0. \end{aligned} \tag{5.4.7}$$

Since $\tilde{\mu}(x,y) = \nu(s_0 + \delta R x_1, R x_2, s_0 + \delta R y)$ this implies

$$|\partial_{x_1}^i \partial_{x_2}^k \partial_y^j \tilde{\mu}| \le C_{k,i,j}\ \delta, \quad \text{if } k + i + j \ge 1. \tag{5.4.8}$$

From here on, we can argue as in the proof of Lemma 5.3.1 to finish the proof. □

5.5. The case $\rho \ll 1$, $r \approx 1$, $\mu > -1/2$.

We now assume that

$$R \le 10^{-3}, \quad R/2 < \rho_0 < 2R, \quad \frac{1}{500}\delta R \le \rho_0 - s_0 \le 50\delta R \tag{5.5.1}$$

and define

$$a_\lambda(t,r,\rho) = (1-\zeta((\lambda+1)\sqrt{1-\mu^2}))\beta(\frac{1-\mu}{\delta})\chi_1(10^4\frac{\rho-\rho_0}{\delta R})\chi_2(10^4\frac{t-r+s_0}{\delta R})\chi_4(t). \tag{5.5.2}$$

Note that if $(r,t,\lambda) \in \text{supp } a_\lambda$ then $r \in (1/2,3)$.

Lemma 5.5.1. *Let $p \ge 2$ and let $\mathcal{A}_\lambda$ be defined by (5.2.1) with a_λ as in (5.5.2). Then*

$$\Big\|\Big(\sum_{\lambda_j^2\delta\ge1}|\mathcal{A}_{\lambda_j}h_j|^2\Big)^{1/2}\Big\|_p \lesssim \delta\min\{R^{1/2+1/p},R^{1-1/p}\}\Big\|\Big(\sum_{\lambda_j^2\delta\ge1}\big|\Omega_p(\lambda_j\sqrt{\delta})h_j\big|^2\Big)^{1/2}\Big\|_p \tag{5.5.3}$$

where C does not depend on δ and R.

Proof. We prove this inequality for $p \ge 4$ and $p = 2$; the general case follows by interpolation.

To settle the case $p \ge 4$ we set

$$\begin{aligned} t = t(x) &= \frac{R}{2x_1} - s_0 - \delta R x_2 \\ r = r(x) &= \frac{R}{2x_1} \end{aligned} \tag{5.5.4}$$

and

$$\rho = \rho(y) = \rho_0 + \delta R y. \tag{5.5.5}$$

Note that

$$\det\frac{\partial(t,r)}{\partial(x_1,x_2)} = \frac{\delta R^2}{2x_1^2}$$

and that $|x_1| \approx R$ if $(t(x),r(x),\rho(y)) \in \text{supp } a_\lambda$. We therefore observe that the constant M in (5.2.5) can be chosen to be equal to δ. Since $\gamma = \delta R$ the asserted estimate follows once the uniform estimate for $\{\mathcal{A}_{\lambda_j\sqrt{\delta}}g_j\}$ is checked. We have already noticed that $1/2 \le r \le 3$ for $(r,t,\lambda) \in \text{supp } a_\lambda$, moreover of course $1 \le t \le 2$ and $R/2 \le \rho \le 2R$. Note that the variables (x_1,x_2,y) live then in the region where $x_1 \in (R/6,R)$, $x_2 \in (10^{-3},50)$, $y \in (-10^{-4},10^{-4})$. We shall extend the change of variables (5.5.4/5) to the larger region where

$$(x_1,x_2) \in (0,10^{-3}) \times [10^{-3},50] =: I;$$

then for $(x_1,x_2) \in I$ we have $r(x) \in (500R,\infty)$.

Introduce auxiliary coordinates

$$\begin{aligned} w_1 &= \frac{1}{2r} = \frac{x_1}{R} \\ w_2 &= r - t = s_0 + \delta R x_2. \end{aligned}$$

Define ν such that $\mu(t,r,\rho) = \nu(w(t,r),\rho)$, *i.e.*

$$\nu(w,\rho) = w_1\rho + \frac{w_2 - w_2^2 w_1}{\rho} \tag{5.5.6}$$

and

$$\widetilde{\mu}(x,y) = \mu(t(x), r(x), \rho(y)) = \nu(\frac{x_1}{R}, s_0 + \delta R x_2, \rho_0 + \delta R y). \tag{5.5.7}$$

Since $\det \frac{\partial(w_1,w_2)}{\partial(t,r)} = (2r^2)^{-1}$ we see from Remark 3.5 that

$$CS[\delta\phi] = 2r^2 CS[\psi](\delta^4 R^3) = 2(1-\mu^2)^{-3}\frac{t^3}{r^3\rho^3}(\delta^4 R^3). \tag{5.5.8}$$

Now for $x \in I$ we have $|x_1 x_2| \le 1/20$ and therefore

$$\Big|\frac{t(x)}{r(x)} - 1\Big| = \frac{|s_0 + \delta R x_2||2x_1|}{R} \le \frac{1}{500} + 2\delta|x_1x_2| \le \frac{1}{4}; \tag{5.5.9}$$

therefore $|CS[\delta\phi]| \approx \delta$ and $|CS[\phi]| \approx 1$. Next

$$\psi_{t\rho}(t(x), r(x), \rho(y)) = (1-\tilde{\mu}^2)^{-3/2}\frac{t(x)}{r(x)\rho(y)}\Big(\frac{\tilde{\mu}}{r(x)} - \frac{1}{\rho(y)}\Big)$$

and since $r(x) \ge 500R$ we see from (5.5.1) that $-\psi_{t\rho}(t(x), r(x), \rho(y)) \approx \delta^{-3/2}R^{-2}$. Hence $\phi_{x_2y_2} = -\delta^{-1/2}(\delta R)^2\psi_{t\rho} \approx 1$.

We have still have to bound the C^∞ norms of $\tilde{a}_\lambda$ and ϕ. Note that

$$\begin{aligned}
\nu_{w_1} &= \frac{\varrho^2 - w_2^2}{\varrho} = \varrho - \frac{w_2^2}{\varrho}, \quad \nu_{w_1w_1} = 0,\\
\nu_{w_2} &= \frac{1 - 2w_1w_2}{\varrho}, \quad \nu_{w_2w_2} = -\frac{2w_1}{\varrho}, \quad \nu_{w_2w_2w_2} = 0,\\
\nu_\varrho &= w_1 - \frac{w_2 - w_2^2 w_1}{\varrho^2}.
\end{aligned}$$

Note that if $x \in I$ then

$$w \in I_R = [0, 10^{-3}R^{-1}] \times [s_0 + \delta R 10^{-3}, s_0 + 50\delta R].$$

and therefore $|\varrho^2 - w_2^2| = O(\delta R^2)$ on I_R. Now the previous formulas easily imply that

$$\begin{aligned}
&|\partial_{w_1}\nu| \le C\delta R,\\
&|\partial_{s_1}^k \nu| = 0, && \text{if } k \ge 2,\\
&|\partial_{w_1}\partial_{w_2}^\alpha\partial_\varrho^\beta \nu| \le C_{\alpha,\beta}R^{1-(\alpha+\beta)}, && \text{if } \alpha+\beta \ge 1,\\
&|\partial_{w_2}^\alpha\partial_\varrho^\beta \nu| \le C_{\alpha,\beta}R^{-(\alpha+\beta)}.
\end{aligned}$$

As a consequence, we find that

$$|\partial_{x_1}^k\partial_{x_2}^\alpha\partial_y^\beta\tilde{\mu}| \le C_{k,\alpha,\beta}\,\delta, \quad \text{if } k+\alpha+\beta \ge 1.$$

From here on one can argue as in the proof of Lemma 5.3.1 and this leads to the inequality

$$\Big\|\Big(\sum_{\lambda_j^2\delta\ge 1}|\mathcal{A}_{\lambda_j}h_j|^2\Big)^{1/2}\Big\|_p \le C\delta R^{1-1/p}\Big\|\Big(\sum_{\lambda_j^2\delta\ge1}\big|\Omega_p(\lambda_j\sqrt{\delta})h_j\big|^2\Big)^{1/2}\Big\|_p$$

which coincides with (5.5.3) if $p \ge 4$ (since $R \le 1$).

Next, we prove the case $p = 2$ of (5.5.3) and work with the changes of variables

$$\begin{aligned}
(t(x), r(x)) &= (x_1 - s_0 - \delta R x_2, x_1),\\
\rho(y) &= \rho_0 + \delta R y.
\end{aligned} \tag{5.5.10}$$

Since now $|\psi_{t\rho}| \approx R^{-2}\delta^{-3/2}$ by (3.46) we note that $\phi_{x_2y} = \delta^{-1/2}(\delta R)^2\psi_{t\rho} \approx 1$. Moreover $\gamma = \delta R$ and $M = \delta R$ in (5.2.4/5). One checks that the C^N norms of (5.2.6-8) are bounded and one can apply Hörmander's basic L^2 estimate ([17]) to the operators $\widetilde{\mathcal{A}}_{\lambda\sqrt{\delta}}$. Therefore the calculation (5.2.14) remains valid with our present change of variable and $p = 2$ with $M^{1/2}\gamma^{1/2} = \delta R$. This shows the validity of (5.5.3) in the case $p = 2$. □

5.6 The case $\rho \approx 1$, $r \le 3$, $\mu < 0$.

We now assume that

$$(5.6.1)\quad 10^{-3} \le \rho_0 \le 3, \qquad R/2 \le r_0 \le 2R, \qquad r_0 \le 3, \qquad s_0 \in [-2,2], \qquad 10^{-8}\delta R \le s_0 - \rho_0 \le 10\delta R$$

and define

$$(5.6.2)\quad a_\lambda(t,r,\rho) = (1-\zeta((\lambda+1)\sqrt{1-\mu^2}))\beta(\frac{1+\mu}{\delta})\chi_1(10^{10}\frac{t-r-s_0}{\delta R})\chi_2(10^{10}\frac{\rho-\rho_0}{\delta R})\chi_3(4\frac{r-r_0}{R})\chi_4(t).$$

Lemma 5.6.1. *Let $p \ge 2$ and let $\mathcal{A}_\lambda$ be defined by (5.2.1) with a_λ as in (5.6.2). Then*

$$\Big\|\Big(\sum_{\lambda_j^2\delta\ge 1} |\mathcal{A}_{\lambda_j} h_j|^2\Big)^{1/2}\Big\|_p \lesssim \delta R^{1+1/p}\Big\|\Big(\sum_{\lambda_j^2\delta\ge 1} \big|\Omega_p(\lambda_j\sqrt{\delta})h_j\big|^2\Big)^{1/2}\Big\|_p$$

where C does not depend on δ and R.

Sketch of Proof. Since both δ and R are bounded if $\mu < 0$ (*cf.* Lemma 5.1.2) we may assume that δR is small, say $\delta R \le 10^{-10}$. Otherwise both δ and R are comparable to 1 and the statement of Lemma 5.6.1 follows directly from Theorem 3.1.

Assuming that $\delta R \le 10^{-10}$ we use the changes of variables

$$(5.6.3)\qquad t(x) = s_0 + \delta R x_1 + R x_2, \qquad r(x) = R x_2, \qquad \rho(y) = \rho_0 + \delta R y.$$

We shall not give the details of the proof of Lemma 5.6.1 since it is analogous to the proof of Lemma 5.4.1. The only difference is that the analogue of the function arising in (5.4.4) is given by

$$\nu(s,r,\rho) = \mu(s+r,r,\rho).$$

One observes then that

$$\nu_s = \mu_t = -\frac{r+s}{r\varrho}, \qquad \nu_\varrho = -\frac{\nu}{\varrho} + \frac{1}{r},$$
$$\nu_r = \mu_r + \mu_t = \frac{s^2-\varrho^2}{2r^2\varrho},$$

so that one has an analogue of (5.4.5/6) in the present case. □

5.7. The case $\rho \ll 1$, $r \approx 1$, $\mu < 0$.

In this final case we analyze the situation which comes up in Lemma 5.1.2 (iii). We assume that

$$(5.7.1)\qquad R \le 10^{-3}, \quad R/2 < \rho_0 < 2R, \quad \frac{1}{500}\delta R \le \rho_0 - s_0 \le 50\delta R$$

and define

$$(5.7.2)\quad a_\lambda(t,r,\rho) = (1-\zeta((\lambda+1)\sqrt{1-\mu^2}))\beta(\frac{1+\mu}{\delta})\frac{\delta^{3/4}}{(1-\mu^2)^{3/4}}\chi_1(10^4\frac{\rho-\rho_0}{\delta R})\chi_2(10^4\frac{t-r-s_0}{\delta R})\chi_4(t).$$

Lemma 5.7.1. *Let $p \geq 2$ and let $\mathcal{A}_\lambda$ be defined by (5.2.1) with a_λ as in (5.7.2). Then*

$$\Big\| \Big(\sum_{\lambda_j^2\delta\geq 1} |\mathcal{A}_{\lambda_j} h_j|^2 \Big)^{1/2} \Big\|_p \lesssim \delta \min\{R^{1/2+1/p}, R^{1-1/p}\} \Big\| \Big(\sum_{\lambda_j^2\delta\geq 1} \big|\Omega_p(\lambda_j\sqrt{\delta})h_j\big|^2 \Big)^{1/2} \Big\|_p$$

where C does not depend on δ and R.

To prove this one now introduces coordinates by setting $t(x) = \frac{R}{2x_1} + s_0 + \delta R x_2$, $r(x) = \frac{R}{2x_1}$, $\rho(y) = \rho_0 + \delta R y$. The crucial observation is that $|t-r|$ is small and therefore one can directly adapt the proof of Lemma 5.5.1 to also prove the case $p = 4$ of Lemma 5.7.1. We omit the details.

For the case $p = 2$ we work with $t(x) = x_1 + s_0 + \delta R x_2$, $r(x) = x_1$, $\rho(y) = \rho_0 + \delta R y$. Again the argument of Lemma 5.5.1 applies.

5.8. Proof of Proposition 5.0. The first assertion (i) follows from Lemma 5.1.1 (ii) and Lemma 5.1.2 (i). We shall prove the assertions of Proposition 5.0 for large ℓ and $\mathcal{O}_{z,\lambda,\ell}$ replaced with $\mathcal{O}^+_{z,\lambda,\ell}$. The straightforward notational modifications of the easier case $r \approx \rho \approx 1$ for small ℓ, and for the operator $\mathcal{O}^-_{z,\lambda,\ell}$ are left to the reader.

In what follows we fix $z = b + i\tau$, $b \geq -1/2$ and let

(5.8.1) $$B_{\lambda,\ell}(t,r,\rho) = \pi^{-1}t^{-2z}(1-\zeta((\lambda+1)\sqrt{1-\mu^2}))\eta_\ell(\mu(t,r,\rho))\chi_{(0,1)}(\mu)(2^\ell(1-\mu^2))^{\frac{z-1}{2}}\cos(\lambda\arccos\mu(t,r,\rho) - \tfrac{\pi}{2}z).$$

Let $\mathcal{B}_{\lambda,\ell}$ be the integral operator with kernel $B_{\lambda,\ell}$. Then

$$O^+_{z,\lambda,\ell}f(t,r) = 2^{-\ell\frac{z-1}{2}} r^{z-n/2}\mathcal{B}_{\lambda,\ell}g_{z,\lambda}(t,r) \tag{5.8.2}$$

with

$$g_{z,\lambda}(\rho) = \lambda^{-z}f(\rho)\rho^{z+(n-2)/2}. \tag{5.8.3}$$

Proof of (5.0.2/3). We give the proof of (5.0.2). Let $m \geq 6$ and $I_m = [m, m+1]$, $\chi_m := \chi_{I_m}$. We note that the function $\mathcal{B}_{\lambda,\ell}[f\chi_m]$ is supported in $[m-2, m+3]$, by Lemma 5.1.1 (i).

Moreover if $2^{-\ell}m^2 \leq 2^{-10}$ and $\rho \in [m, m+1]$, $r \in [m-2, m+3]$ and $(t,r,\rho) \in \operatorname{supp} B_{\lambda,\ell}$ then $|r-\rho| \geq 1/2$, by Lemma 5.1.1 (iii), and this lemma states a finer localization property. Split the interval $I_m = [m, m+1]$ into subintervals of length $2^{-\ell-10}m^2$,

$$I_{m,\nu} = [m + 2^{-\ell-10}\nu m^2, m + 2^{-\ell-10}(\nu+1)m^2],$$

and put $\chi_{m,\nu} := \chi_{I_{m,\nu}}$. Then $\mathcal{B}_{\lambda,\ell}[f\chi_{m,\nu}]$ will be supported in sets $W^\epsilon_{m,\nu}$, $\epsilon = \pm 1$, in which $r \approx m$ and $r - \epsilon t$ is restricted to an interval of length $\approx 2^{-\ell}m^2 \ll 1$; specifically

$$W^\epsilon_{m,\nu} = \{(t,r) : r \in [m-2, m+3], m+2^{-\ell-10}\nu m^2 - \epsilon 2^{-\ell+5\epsilon}m^2 \leq r - \epsilon t \leq m + 2^{-\ell-10}(\nu+1)m^2 - \epsilon 2^{-\ell-5\epsilon}m^2\}.$$

Now it is crucial that every (t,r) is contained in at most a bounded number ($< 2^{100}$) of sets $W^\epsilon_{m,\nu}$. It therefore suffices to prove (5.0.2) under the assumption that all f_j are supported in a fixed $I_{m,\nu}$ (or I_m if $2^{-10} \leq 2^{-\ell}m^2 \leq 2^6$). We discuss the argument in the case $2^{-\ell}m^2 \leq 2^{-10}$ and leave the notational modifications in the slightly simpler case $2^{-\ell}m^2 \approx 1$ to the reader.

One checks that

$$\mathcal{B}_{\lambda,\ell}[f\chi_{I_{m,\nu}}]=\sum_{\pm}\sum_{s=1}^{N(m,\nu,\ell)}c_{s,m,\nu,\ell}\mathcal{A}_{\pm\lambda,\ell,s,m,\nu}[f\chi_{I_{m,\nu}}]$$

where each $\mathcal{A}_{\lambda,\ell,s,m,\nu}$ is of the type treated in Lemma 5.3.1, with $\delta R^2\lesssim 2^{-\ell}m^2$ and where $N(m,\nu,\ell)$ and the coefficients $c_{s,m,\nu,\ell}$ are bounded, with bounds independent of m,ν,ℓ,λ.

Therefore

$$\begin{aligned}
&\Big(\int_1^2\int_{2^L}^{2^{L+1}}\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\mathcal{O}^+_{z,\lambda_j,\ell}[f_j\chi_{I_{m,\nu}}]\big|^2\Big)^{\frac p2}r^{n-1}dr\,dt\Big)^{\frac1p}\\
&\lesssim 2^{\ell\frac{1-b}{2}}m^{\frac{n-1}{p}+b-\frac n2}\Big(\int_1^2\sum_{\epsilon=\pm1}\int_{W^\epsilon_{m,\nu}}\Big(\sum_{\lambda_j^2 2^{-\ell}\ge1}\big|\mathcal{B}_{\lambda_j,\ell}[g_{z,\lambda_j}\chi_{I_{m,\nu}}]\big|^2\Big)^{\frac p2}drdt\Big)^{\frac1p}\\
&\lesssim 2^{\ell\frac{1-b}{2}}m^{\frac{n-1}{p}+b-\frac n2}m^2 2^{-\ell}\Big(\int\Big(\sum_{\lambda_j^2 2^{-\ell}\ge1}\big|\Omega_p(\lambda_j 2^{-\ell/2})g_{z,\lambda_j}\chi_{I_{m,\nu}}\big|^2\Big)^{\frac p2}d\rho\Big)^{\frac1p}\\
&\lesssim m^{\frac{n-1}{p}}m^{2b+1}2^{-\ell\frac{b+1}{2}}\Big(\int\Big(\sum_{\lambda_j\ge 2^{\ell/2}}\big|\lambda_j^{-b}\Omega_p(\lambda_j2^{-\ell/2})f_j\chi_{I_{m,\nu}}\chi_L(\rho)\big|^2\Big)^{\frac p2}d\rho\Big)^{\frac1p}\\
&\lesssim 2^{L(2b+1)}2^{-\ell\frac{b+1}{2}}\Big(\int\Big(\sum_{\lambda_j\ge 2^{\ell/2}}\big|\lambda_j^{-b}\Omega_p(\lambda_j2^{-\ell/2})f_j\chi_{I_{m,\nu}}\chi_L(\rho)\big|^2\Big)^{\frac p2}\rho^{n-1}d\rho\Big)^{\frac1p}.
\end{aligned}$$

Now

$$\lambda_j^{-b}\Omega_p(\lambda_j2^{-\ell/2})\rho^{2b+1}2^{-\ell\frac{b+1}{2}}\lesssim\begin{cases}\big(\frac{\lambda_j}{\rho}\big)^{-b-\frac12}(2^{-\ell/2}\rho)^{\frac{2b+1}{2}}[\log(2+\lambda_j2^{-\ell/2})]^{\frac12-\frac1p} & \text{if } 2\le p\le 4\\ \big(\frac{p}{p-4}\big)^{1/4}\big(\frac{\lambda_j}{\rho}\big)^{-b-\frac2p}(2^{-\ell/2}\rho)^{b+1-\frac2p} & \text{if } p>4\end{cases}$$

and since here $2^{-\ell/2}\rho\approx 2^{L-\ell/2}\lesssim 1$, $2^{-\ell/2}\lambda_j\gtrsim 1$ (hence $\lambda_j\gtrsim\rho$) we find using the assumptions on γ that the last quantity is estimated by $\omega_{\gamma,p}(\lambda_j,\rho)\min\{(2^{-\ell/2}\rho)^{(2b+1)/2},(2^{-\ell/2}\rho)^{b+1-2/p}\}$; this implies the desired estimate with f_j replaced by $f_j\chi_{I_{m,\nu}}$. As pointed out above we obtain the full inequality (5.0.2) as a consequence.

The proof of (5.0.3) is exactly analogous. Now $r\approx\rho\approx 1$ in Lemma 5.3.1 and we note that by 5.1.1 (i) we have $\mathcal{O}_{z,\lambda_j,\ell}[\chi f_j](t,r)=\mathcal{O}_{z,\lambda_j,\ell}[\chi_{[10^{-4},10]}f_j](t,r)$ for $r\le 8$. We split the ρ-interval $[10^{-4},10]$ into intervals of length $2^{-10-\ell}$ (*cf.* the case (vii) of Lemma 5.1.1.) Then we continue as in the proof of (5.0.2). Further details are omitted. □

Proof of (5.0.4). Now we consider $\mathcal{O}^+_{z,\lambda_j,\ell}f_j(t,r)$ where $R/2\le r\le 2R$ and $R\le 10^{-3}$. We split the interval $[1/2,4]$ into subintervals

$$J_\nu=[\frac12+2^{-\ell-10}\nu R,\frac12+2^{-\ell-10}(\nu+1)R].$$

By Lemma 5.1.1, (iv), it suffices to assume that f is supported in $[1/2,4]$; moreover, if f is supported in J_ν, then $\chi_{[1,2]\times[R/2,2R]}\mathcal{O}^+_{z,\lambda_j,\ell}f$ is supported in

$$W_\nu=\{(t,r):R/2\le r\le 2R,\,1\le t\le 2,\,\frac12+\frac1{32}R2^{-\ell}+\nu R2^{-\ell-10}\le r+t\le\frac12+12R2^{-\ell}+(\nu+1)R2^{-\ell-10}\}.$$

If f is supported in J_ν then we can apply Lemma 5.4.1 to estimate the L^p norm of the square function associated to $\mathcal{B}_{\lambda,\ell} f$ (using linear combinations of terms occuring there). We obtain with $g_{\lambda_j,z}$ as in (5.8.3)

$$\Big(\iint_{W_\nu}\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\mathcal{O}^+_{z,\lambda_j,\ell}[f_j\chi_{J_\nu}]\big|^2\Big)^{p/2} r^{n-1} dr dt\Big)^{1/p}$$

$$\lesssim 2^{\ell\frac{1-b}{2}} R^{\frac{n-1}{p}+b-\frac{n}{2}}\Big(\iint_{W_\nu}\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\mathcal{B}_{\lambda_j,\ell}[g_{z,\lambda_j}\chi_{J_\nu}]\big|^2\Big)^{p/2} dr\, dt\Big)^{1/p}$$

$$\lesssim 2^{\ell\frac{1-b}{2}} R^{\frac{n-1}{p}+b-\frac{n}{2}} 2^{-\ell} R^{1+1/p}\Big(\int_{\rho\approx 1}\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\Omega_p(\lambda_j 2^{-\ell/2}) g_{\lambda_j,z}\chi_{J_\nu}(\rho)\big|^2\Big)^{p/2} d\rho\Big)^{1/p}$$

$$\lesssim 2^{-\ell\frac{b+1}{2}} R^{b+1-n(\frac12-\frac1p)}\Big(\int_{\rho\approx 1}\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\lambda_j^{-b}\Omega_p(\lambda_j 2^{-\ell/2}) f_j(\rho)\rho^{b+\frac{n-2}{2}}\chi_{J_\nu}(\rho)\big|^2\Big)^{p/2} d\rho\Big)^{1/p}$$

$$\lesssim 2^{-\ell\frac{2b+1}{4}}\min\{1, 2^{-\ell(\frac14-\frac1p)}\} R^{b+1-n(\frac12-\frac1p)}\Big(\int\Big(\sum_j\big|\omega_{\gamma,p}(\lambda_j,\rho) f_j(\rho)\chi_{J_\nu}(\rho)\big|^2\Big)^{p/2}\rho^{n-1} d\rho\Big)^{1/p}$$

and this yields the desired inequality for f_j supported in J_ν (since $\rho\approx 1$ there). Since every $(t,r)\in[1,2]\times[R/2,2R]$ is supported in only a bounded number of sets W_ν we obtain the inequality (5.0.3) for $\mathcal{O}_{z,\lambda,\ell}$ replaced by $\mathcal{O}^+_{z,\lambda,\ell}$. The corresponding inequalities for $\mathcal{O}^-_{z,\lambda,\ell}$ and for $\mathcal{O}_{z,\lambda,0}$ are derived analogously; here we have to use Lemma 5.6.1. □

Proof of (5.0.5). Let $R\le 10^{-3}$ and for $\nu\ge 0$ let

$$I_\nu=[\frac{R}{2}+2^{-\ell-10}\nu R, \frac{R}{2}+2^{-\ell-10}(\nu+1)R].$$

If f is supported in I_ν (so that $I_\nu\cap[R/2,2R]$ is not empty) then, according to Lemma 5.1.1., part (v), $\chi_{[1,2]\times[1/8,4]}\mathcal{O}^+_{z,\lambda,\ell} f$ is supported in a strip where $t-r+\rho\approx 2^{-\ell}R$, namely

$$V_\nu=\{(t,r):\frac{R}{2}-2^{-\ell+6}R+2^{-\ell-10}\nu R\le r-t\le\frac{R}{2}-2^{-\ell-8}R+2^{-\ell-10}(\nu+1)R\}\cap([1,2]\times[1/8,4]).$$

The terms $\mathcal{B}_{\lambda,\ell}[f\chi_{I_\nu}]$ can be estimated using Lemma 5.5.1. We obtain

$$\Big(\iint_{V_\nu}\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\mathcal{O}_{z,\lambda_j,\ell}[f_j\chi_{I_\nu}]\big|^2\Big)^{p/2} r^{n-1} dr\, dt\Big)^{1/p}$$

$$\lesssim 2^{-\ell\frac{b-1}{2}}\Big(\iint_{V_\nu}\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\mathcal{B}_{\lambda_j,\ell}[g_{z,\lambda_j}\chi_{I_\nu}]\big|^2\Big)^{p/2} dr\, dt\Big)^{1/p}$$

$$\lesssim 2^{-\ell\frac{b+1}{2}}\min\{R^{\frac12+\frac1p}, R^{1-\frac1p}\}\Big(\int\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\lambda_j^{-b}\Omega_p(\lambda_j 2^{-\ell/2}) f_j(\rho)\rho^{b+(n-2)/2}\chi_{I_\nu}(\rho)\big|^2\Big)^{p/2} d\rho\Big)^{1/p}$$

$$\lesssim 2^{-\ell\frac{b+1}{2}}\min\{R^{\frac12+\frac1p}, R^{1-\frac1p}\} R^{b+\frac{n-2}{2}-\frac{n-1}{p}}\Big(\int\Big(\sum_{\lambda_j^2 2^{-\ell}\ge 1}\big|\lambda_j^{-b}\Omega_p(\lambda_j 2^{-\ell/2}) f_j(\rho)\chi_{I_\nu}(\rho)\big|^2\Big)^{p/2}\rho^{n-1} d\rho\Big)^{1/p}.$$

Here we used that $\rho\approx R$ in I_ν. We can further estimate

$$\lambda_j^{-b}\Omega_p(\lambda_j 2^{-\ell/2})\lesssim\begin{cases} 2^{\frac{\ell}{4}} R^{-b-\frac12}(\lambda_j/\rho)^{-b-\frac12}[\log(2+\lambda_j/\rho)]^{1/2-1/p} & \text{if } 2\le p\le 4\\ C_p 2^{\frac{\ell}{p}} R^{-b-\frac2p}(\lambda_j/\rho)^{-b-\frac2p} & \text{if } p>4\end{cases}$$

and the asserted inequality follows after a short computation for f replaced by $f\chi_{I_\nu}$. Since every $(t,r)\in[1,2]\times[1/8,4]$ is supported in only a bounded number of the sets V_ν we obtain the full inequality (5.0.5), for $\mathcal{O}_{z,\lambda,\ell}$ replaced by $\mathcal{O}^+_{z,\lambda,\ell}$. Again the derivation of the corresponding inequalities for $\mathcal{O}^-_{z,\lambda,\ell}$ and for $\mathcal{O}_{z,\lambda,0}$ is similar, see Lemma 5.7.1. □.

5.9. Conclusion: Proofs of Theorems 2.3 and 2.5.

Proof of Theorem 2.3. The case $p=2$ follows from the spectral theorem. We show estimates (2.29-31) with $S_{z,j}$ replaced by $\mathcal{O}_{z,\lambda_j}$. The assertion of Theorem 2.3 follows then by combining these estimates with the estimates for the nonoscillatory terms in (2.17) and (2.23) which were already carried out in §4. Moreover (2.32) was already proved in §4.

We first fix $L \geq 3$ and consider $\mathcal{O}_{z,\lambda_j} f_j(t,r)$ where $t \in I$, $r \in [2^L, 2^{L+1}]$. Recall that because of the localization of the kernel the function f_j can be replaced by $f_j \chi_L$ where χ_L is the characteristic function of $[2^{L-1}, 2^{L+2}]$;

Now $\mathcal{O}_{z,\lambda_j} = \sum_\ell \mathcal{O}_{z,\lambda_j,\ell}$ where according to the localization properties of $\mathcal{O}_{\lambda,z}$ and Proposition 5.0 we sum over all ℓ with $\lambda_j \geq 2^{\ell/2}$ and $2^{(\ell+10)/2} \geq 2^{L-1}$, thus we may link $\ell = 2L-12+s$ with $s \geq 0$. We estimate with $b = \mathrm{Re}(z)$

$$\Big(\int_I \int_{2^L}^{2^{L+1}} \Big(\sum_j |\mathcal{O}_{z,\lambda_j} f_j|^2\Big)^{p/2} r^{n-1} dr dt\Big)^{1/p}$$
$$\leq \sum_{s=0}^\infty \Big(\int_I \int_{2^L}^{2^{L+1}} \Big(\sum_j |\mathcal{O}_{z,\lambda_j,2L-12+s}[f_j\chi_L]|^2\Big)^{p/2} r^{n-1} dr dt\Big)^{1/p}$$
$$\lesssim \sum_{s=0}^\infty 2^{-s\frac{2b+1}{4}} \min\{1, 2^{-s(\frac14-\frac1p)}\}\Big(\int \Big(\sum_j |\omega_{\gamma,p}(\lambda_j,\rho)[f_j\chi_L]|^2\Big)^{p/2} \rho^{n-1} d\rho\Big)^{1/p}$$

under the restriction on γ in Proposition 5.0. We can sum in s if $b > -1/2$ and also if $b = -1/2$ and $2 \leq p < 4$. Combining these estimates and summing the pth powers in L we obtain

$$\Big(\int_I \int_8^\infty \Big(\sum_j |\mathcal{O}_{z,\lambda_j} f_j|^2\Big)^{p/2} r^{n-1} dr dt\Big)^{1/p} \lesssim \Big(\int \Big(\sum_j |\omega_{\gamma,p}(\lambda_j,\rho) f_j|^2\Big)^{p/2} \rho^{n-1} d\rho\Big)^{1/p} \tag{5.9.1}$$

if $b > -1/2$ or $b = -1/2$ and $2 \leq p < 4$, and $\gamma \leq \min\{b+1/2, b+2/p\}$.

The estimate on $[0,8]$ is even more straightforward. Applying Minkowski's inequality in (5.0.3) and summing $\sum_{\ell \geq 0} 2^{-\ell(2b+1)/4} \min\{1, 2^{-\ell(1/4-1/p)}\}$ yields
(5.9.2)
$$\Big(\int_I \int_{10^{-3}}^8 \Big(\sum_j |\mathcal{O}_{z,\lambda_j}[\chi_{[10^{-4},\infty)} f_j]|^2\Big)^{p/2} r^{n-1} dr dt\Big)^{1/p} \lesssim \Big(\int \Big(\sum_j |\omega_{\gamma,p}(\lambda_j,\rho) f_j|^2\Big)^{p/2} \rho^{n-1} d\rho\Big)^{1/p}.$$

Next we apply (5.0.5) with $R = 2^{-k}$. We may sum in k if $p > 2$ and $n \geq 3$ and $p > 4$ and $n = 2$. Applying Minkowski's inequality and summing in ℓ as in the previous case yields

$$\Big(\int_I \int_{10^{-3}}^8 \Big(\sum_j |\mathcal{O}_{z,\lambda_j}[\chi_{[0,10^{-4}]} f_j]|^2\Big)^{p/2} r^{n-1} dr dt\Big)^{1/p} \lesssim \Big(\int \Big(\sum_j |\omega_{\gamma,p}(\lambda_j,\rho) f_j|^2\Big)^{p/2} \rho^{n-1} d\rho\Big)^{1/p} \tag{5.9.3}$$

if $p > 2$ and $n \geq 3$ and $p > 4$ and $n = 2$. If $n = 2$ and $p \leq 4$ we introduce an additional factor to insure convergence in k; note that for $R \approx \rho \lesssim 1$ and $\lambda_j \geq 1$ we have $1 \lesssim R^\varepsilon (1+\lambda_j/\rho)^\varepsilon$. Thus

$$\Big(\int_I \int_{10^{-3}}^8 \Big(\sum_j |\mathcal{O}_{z,\lambda_j}[\chi_{[0,10^{-4}]} f_j]|^2\Big)^{p/2} r^{n-1} dr dt\Big)^{1/p}$$
$$\leq C_\varepsilon \Big(\int \Big(\sum_j |\omega_{\gamma,p}(\lambda_j,\rho)(1+\lambda_j/\rho)^\varepsilon f_j|^2\Big)^{p/2} \rho^{n-1} d\rho\Big)^{1/p} \tag{5.9.4}$$

if $n=2$ and $2\le p\le 4$.

Finally we apply (5.0.4) with $R=10^{-3}2^{-k}$. By the assumption $b+1/2>n(1/2-1/p)-1/2$ we get convergence in k, and again we have convergence in ℓ from (5.0.4). This yields

$$\Big(\int_I\int_0^{10^{-3}}\Big(\sum_j|\mathcal{O}_{z,\lambda_j}f_j|^2\Big)^{p/2}r^{n-1}drdt\Big)^{1/p}\le C\Big(\int\Big(\sum_j|\omega_{\gamma,p}(\lambda_j,\rho)f_j|^2\Big)^{p/2}\rho^{n-1}d\rho\Big)^{1/p} \tag{5.9.5}$$

if $2\le p<2n/(n-1)$ and $b\ge -1/2$ or $p\ge 2n/(n+1)$ and $b+1/2>n(1/2-1/p)-1/2$.

Theorem 2.3 follows by combining (5.9.1-5). □

Proof of Theorem 2.5. To prove the restricted weak type inequalities in Theorem 2.5 we assume that $z=b+i\tau$ and $b+1/2\ge 0$, and define p_b by $b+1/2=n(1/2-1/p_b)-1/2$. We observe that the above arguments yield favorable *strong type* estimates for the critical exponent p_b, for the terms in (5.0.1-3) and (5.0.5). However the estimates (5.0.4) for $R\approx 2^{-k}\ll 1$ do not yield a bounded sum in k.

Now let χ_k be the characteristic function of the interval $[2^{-k-1},2^{-k}]$. We define

$$T^{z,\gamma}_{k,j}(f)=\chi_k(r)\mathcal{O}_{z,\lambda}[(1+\lambda_j/\rho)^{\gamma}f].$$

Assume first $2n/(n-1)<p_b\le 4$. Then $b+1/2=n(1/2-1/p_b)-1/2>0$. By Proposition 5.0 given $\varepsilon>0$ there is $p(\varepsilon)>p_b$ so that the inequality

(5.9.6)

$$\Big(\int_I\int_0^{\infty}\Big(\sum_j|T^{z,\gamma}_{k,j}f_j(t,r)|^2\Big)^{p/2}r^{n-1}drdt\Big)^{1/p}\lesssim 2^{k(n(\frac12-\frac1p)-b-1)}\Big(\int_0^{\infty}\Big(\sum_j|f_j(\rho)|^2\Big)^{p/2}\rho^{n-1}d\rho\Big)^{1/p}$$

holds for $2\le p\le p(\varepsilon)$ and $\gamma<b+1/2-\varepsilon$. The desired restricted weak type inequality for the vector-valued operator $\{\sum_{k\ge 0}T^{z,\gamma}_{k,j}\}_{j\ge 0}$ follows now from an interpolation lemma in §6 of [6] (see also [2] for a closely related argument).

If $p_b>4$ we have (5.9.6) with $\gamma=b+2/p$ for some open interval of p's containing p_b (by a similar argument). The same interpolation argument applies. All other operators involved are either of strong type (as a consequence of Theorem 5.0) or of weak type by the estimates of §4 (here we have to assume $\mathrm{Im}(z)=0$ if $b=-1/2$). □

6. Applications to spectral multipliers

Lemma 6.1. *Let N be the smallest integer $>(n-1)/2$. Suppose that m is even and $\kappa=\widehat{m}$ satisfies*

$$\int|t^j\kappa^{(j)}(t)|dt<\infty\qquad \text{for } 0\le j\le N \tag{6.1}$$

Then $m(\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}_+,L^2(\mathcal{N}))$, for $1\le p\le\infty$.

Proof. We use the formula

$$\frac{d}{dt}\big(t^{z+1}J_{z+1}(t\lambda)\big)=\lambda J_z(t\lambda)t^{z+1}, \tag{6.2}$$

which follows from [13, 7.2.8(50)].

Therefore integration by parts yields

$$\int\kappa(t)J_z(t\lambda)t^{z+1}dt=-\lambda^{-1}\int\frac{\kappa'(t)}{t}J_{z+1}(t\lambda)t^{z+2}dt,$$

and if we define $\Lambda\kappa(t) = t^{-1}\kappa'(t)$ and iterate we obtain

$$\begin{aligned} m(\lambda) &= \frac{1}{2\pi}\int \kappa(t)\cos(t\lambda)dt \\ &= \sqrt{\frac{\pi}{2}}\lambda^{1/2}\int \kappa(t)J_{-1/2}(t\lambda)t^{-\frac12+1}dt \\ &= (-1)^\ell\sqrt{\frac{\pi}{2}}\lambda^{1/2-\ell}\int \Lambda^\ell\kappa(t)J_{\ell-1/2}(t\lambda)t^{\ell-\frac12+1}dt. \end{aligned} \tag{6.3}$$

We use (6.3) for $\ell = N$. Since $(t\sqrt{-\Delta})^{\frac12-N}J_{N-\frac12}(t\sqrt{-\Delta})$ is uniformly bounded on $L^\infty(L^2)$ by Theorem 1.2 and therefore on all $L^p(L^2)$, $1 \le p \le \infty$, we see that the condition

$$\int |t^{2N}\Lambda^N\kappa(t)|dt < \infty \tag{6.4}$$

implies that $m(\sqrt{-\Delta})$ is bounded on $L^p(L^2)$. By induction one checks that

$$t^{2\ell}\Lambda^\ell\kappa(t) = \sum_{j=1}^{\ell} c_{j,\ell}t^j\kappa^{(j)}(t)$$

for suitable constants $c_{j,\ell}$. This completes the proof. □

Let L^2_α denote the standard L^2 Sobolev space. Applying the Cauchy-Schwarz inequality and Plancherel's theorem we see that

$$\int |t^j\kappa^{(j)}(t)| \le C_\alpha\|s^jm\|_{L^2_\alpha(\mathbb{R})}, \qquad \text{if } \alpha > j+1/2.$$

By scaling we obtain

Corollary 6.2. *Suppose that m is even and supported in $[-a,a]$. Let $\alpha > N+1/2$, where N is as in Lemma 6.1 and assume $m \in L^2_\alpha$. Then $m(\sqrt{-\Delta}/t)$ is bounded on $L^p(\mathbb{R}^+, L^2(\mathcal{N}))$, $1 \le p \le \infty$, with operator norm $\le C(a)\|m\|_{L^2_\alpha}$.*

This result is convenient but far from being sharp (compare Theorem 6.4 below).

As mentioned in the introduction one can prove local smoothing results for the wave operator in the range $p \ge 2n/(n-1)$. To deduce this from Theorem 1.2 we have to use the standard asymptotic expansion for the Bessel functions ([13, 7.13.1(3)]), namely for $x > 1$

$$(\pi/2)^{1/2}J_{\gamma-1/2}(x) = \sum_{j=0}^{M-1} c_{j,\gamma}\cos(x-\gamma\pi/2)x^{-2j-1/2} + \sum_{j=0}^{M-1} d_{j,\gamma}\sin(x-\gamma\pi/2)x^{-2j-3/2} + x^{-M}R_\gamma(x) \tag{6.5}$$

where $c_0 = 1$ and the derivatives of R_γ are bounded functions in $[1,\infty]$.

Corollary 1.4 in the introduction follows from (i) and (ii) of the following

Proposition 6.3. *Suppose $p \ge 2n/(n-1)$ and $\alpha > n(1/2-1/p)-1/2$.*

(i) Let $m_{\alpha,t}(\lambda)$ be one of the multipliers $(1+t^2\lambda^2)^{-\alpha/2}\cos(t\lambda)$ or $(1+t^2\lambda^2)^{-(\alpha-1)/2}\sin(t\lambda)/(t\lambda)$. Then

$$\Big(\frac{1}{2T}\int_{-T}^{T}\|m_{\alpha,t}(\sqrt{-\Delta})f\|_{p,2}^p dt\Big)^{1/p} \lesssim \|f\|_{p,2}.$$

(ii) Let β be an even C^∞ function with compact support in $\mathbb{R}\setminus\{0\}$. Let $R \ge 1$. Then for $\varepsilon > 0$ and $2n/(n-1) \le p \le \infty$

$$\Big(\frac{1}{2T}\int_{-T}^{T} \|\beta(\sqrt{-\Delta})\cos\tau\sqrt{-\Delta} f\|_{p,2}^p d\tau\Big)^{1/p} \le C_\varepsilon T^{n(\frac12-\frac1p)-\frac12+\varepsilon}\|f\|_{p,2}. \tag{6.6}$$

Proof. (i) Choose γ, η so that $n(1/2-1/p)-1/2<\gamma<\eta<\alpha$. We write out the asymptotic expansion for $J_{\eta-1/2}$ and $J_{\gamma-1/2}$ simultaneously. Then for large σ

$$\sqrt{\frac{\pi}{2}}\sigma^{1/2}\begin{pmatrix} J_{\eta-1/2}(\sigma)\\ J_{\gamma-1/2}(\sigma)\end{pmatrix} = \begin{pmatrix}\cos(\eta\pi/2) & \sin(\eta\pi/2)\\ \cos(\gamma\pi/2) & \sin(\gamma\pi/2)\end{pmatrix}(I-Q(\eta,\gamma,\sigma^{-1}))\begin{pmatrix}\cos\sigma\\ \sin\sigma\end{pmatrix} + \widetilde{R}(\sigma) \tag{6.7}$$

where the entries of the matrix $Q(\eta,\gamma,\sigma^{-1})$ are polynomials of σ^{-1} with no constant terms; moreover $\widetilde{R}(\sigma)=O(|\sigma|^{-N})$ for large N and the same holds for its derivatives of order $\le N$. Clearly $I-Q(\eta,\gamma,\sigma^{-1})$ is invertible for large σ and there is $\sigma_0=\sigma_0(\eta,\gamma)>1$ so that for $\sigma>\sigma_0$ each entry $a_{ij}(\sigma)$ of the inverse satisfies $|a_{ij}(\sigma)|\le C$ and $|a_{ij}^{(k)}(\sigma)|\le C_k\sigma^{-1-k}$, for $k\ge 1$.

Let ω be an even smooth function on $\mathbb{R}$, so that $\omega(\sigma)=1$ if $|\sigma|\le 2\sigma_0$ and $\omega(s)=0$ if $|\sigma|\ge 4\sigma_0$. It follows from Corollary 6.2 that the operators $\omega(\sqrt{-\Delta})(I-\Delta)^{-\alpha/2}\cos(\sqrt{-\Delta})$ and $\omega(\sqrt{-\Delta})(I-\Delta)^{-(\alpha-2)/2}\sin(\sqrt{-\Delta})/\sqrt{-\Delta}$ are bounded on all $L^p(L^2)$ and the same applies to their dilates.

Inverting (6.7) we see that for large $M>0$

$$(1-\omega(\sigma))\cos(\sigma) = (1-\omega(\sigma))\big[\Phi_\eta(\sigma)\frac{J_{\eta-1/2}(\sigma)}{\sigma^{\eta-1/2}} + \Psi_\gamma(\sigma)\frac{J_{\gamma-1/2}(\sigma)}{\sigma^{\gamma-1/2}} + r(\sigma)\big]$$

where Φ_η is a symbol of order η and Ψ_γ is a symbol of order γ, the bounds being depending on both η and γ, and r is a symbol of degree $-M$. Moreover $(1-\omega(\sqrt{-\Delta}))r(\sqrt{-\Delta})$ is bounded on all L^p, since it can be written as a converging sum of dilates of multipliers that fall under the scope of Corollary 6.2 (alternatively apply Lemma 6.1).

For $k\ge1$ let $\omega_k(\sigma)=\omega(2^{-k}\sigma)-\omega(2^{-k+1}\sigma)$. By Corollary 6.2 and scaling we see that the operators $\omega_k(t\sqrt{-\Delta})\Phi_\eta(t\sqrt{-\Delta})(I-t^2\Delta)^{-\alpha/2}$ and $\omega_k(t\sqrt{-\Delta})\Psi_\gamma(t\sqrt{-\Delta})(I-t^2\Delta)^{-\alpha/2}$ are bounded on $L^p(L^2)$ for all $p\in[1,\infty]$ with operator norm bounded $O(2^{-k(\alpha-\eta)})$ and $O(2^{-k(\alpha-\gamma)})$, respectively. From this and Theorem 1.2 the assertion for $(I-t^2\Delta)^{-\alpha/2}\cos(t\sqrt{-\Delta})$ follows immediately by summing over k.

The other assertions are proved analogously. In particular, (ii)/(6.6) follows by observing that, by Corollary 6.2, $\beta(\sqrt{(-\Delta)}(1+\tau^2(-\Delta))^{\alpha/2}$ is bounded on $L^p(L^2)$, with operator norm of order $(\tau^\alpha)\le O(T^\alpha)$, with $\alpha=n(\frac12-\frac1p)-\frac12+\varepsilon$. $\square$

We now proceed to prove a version of Theorem 1.6 on general conic manifolds.

Theorem 6.4. *Suppose that m is compactly supported in $[R^{-1},R]$, for some $R>0$ and that $\frac{2n}{n-1}\le p\le\infty$. Assume that*

$$\Big(\int\big[|\widehat{m}(r)|(1+r)^\gamma\big]^{p'}dr\Big)^{1/p'} \le A<\infty, \qquad \gamma>(n-1)(\frac12-\frac1p). \tag{6.8}$$

Then there is a constant $C_{\gamma,R}$ independent of A so that $m(\sqrt{-\Delta}/t)$ is bounded on $L^p(\mathbb{R}_+,L^2(\mathcal{N}))$ with operator norm $\le C_{\gamma,R}A$.

Proof. We extend m as an even function to $\mathbb{R}$. By decomposing the multiplier and scaling we see that the theorem follows from the special case where $t=1$ and supp $m\subset[1,2]\cup[-2,-1]$. Let β be a smooth even function with compact support so that $\beta(s)=1$ if $1/8\le|s|\le 8$ and $\beta(s)=0$ if $|s|\notin(1/16,16)$.

Let $I_0 = [-1,1]$ and $I_k = [2^k, 2^{k+1}] \cup [2^{-k-1}, 2^{-k}]$ for $k \geq 0$. Choose $\delta > 0$ so that $(n-1)(1/2-1/p)+\delta < \gamma$. Then

$$\begin{aligned}
\|m(\sqrt{-\Delta})f\|_{p,2} &= \Big\| \frac{1}{2\pi} \int \widehat{m}(\tau) \cos(\tau\sqrt{-\Delta})\beta(\sqrt{-\Delta}) f d\tau \Big\|_{p,2} \\
&\leq \frac{1}{2\pi} \int |\widehat{m}(\tau)| \| \cos(\tau\sqrt{-\Delta})\beta(\sqrt{-\Delta}) f\|_{p,2} d\tau \\
&\leq \sum_{k=0}^{\infty} \Big(\int_{I_k} |\widehat{m}(\tau)|^{p'} d\tau \Big)^{1/p'} \Big(\int_{I_k} \| \cos(\tau\sqrt{-\Delta})\beta(\sqrt{-\Delta}) f\|_{p,2}^{p} d\tau \Big)^{1/p} \\
&\lesssim \sum_{k=0}^{\infty} 2^{k(n(\frac12-\frac1p)-\frac12+\delta+\frac1p)} \Big(\int_{I_k} |\widehat{m}(\tau)|^{p'} d\tau \Big)^{1/p'} \|f\|_{p,2} \\
&\leq C_\gamma A \|f\|_{p,2} \qquad (6.9)
\end{aligned}$$

where the third inequality follows from Proposition 6.3, part (ii). □

Proof of Theorem 1.6. Assume that K is radial and $\widehat{K}$ is supported in Ω, where $\overline{\Omega}$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$. The Fourier transform of K is radial and we write $\widehat{K}(\xi) = m(|\xi|)$.

The assertion of Theorem 1.6 follows from Theorem 6.4 and duality once the inequality

$$\Big(\int_R^\infty |r^{(n-1)(\frac1p-\frac12)+\varepsilon} \widehat{m}(r)|^p dr \Big)^{1/p} \leq C_R \Big(\int |x|^{\varepsilon p} |K(x)|^p dx \Big)^{1/p} \qquad (6.13)$$

is proved for $1 \leq p \leq 2$.

We now prove (6.13). Choose $\chi \in C_0^\infty$ radial such that $\chi = 1$ on the support of $\hat{K}$ and $0 \notin \operatorname{supp} \chi$. Then

$$\begin{aligned}
\widehat{m}(r) &= \int m(s) e^{-isr} ds \\
&= c_n \int \widehat{K}(\xi)\chi(\xi)|\xi|^{1-n} e^{-i|\xi|r} d\xi \\
&= c_n \int K(x) \int \chi(\xi)|\xi|^{1-n} e^{-i(|\xi|r + \langle \xi, x\rangle)} d\xi\, dx \\
&= \int K(x) \int_0^\infty \eta(\varrho) e^{-i\varrho r} \int_{S^{n-1}} e^{-i\varrho\langle x,\theta\rangle} d\theta\, d\varrho\, dx,
\end{aligned}$$

where c_n denotes the surface measure of the unit sphere in $\mathbb{R}^n$ and $\eta \in C_0^\infty(\mathbb{R})$ is supported away from the origin.

Now, by the stationary phase method (or the asymptotics of Bessel functions),

$$\int_{S^{n-1}} e^{-i\varrho\langle x,\theta\rangle} d\theta = \sum_{\pm} a_\pm(\varrho|x|) e^{\pm i\varrho|x|},$$

where $a_\pm$ is a symbol of order $-\frac{n-1}{2}$ on $\mathbb{R}$. Thus

$$\widehat{m}(r) = \sum_{\pm} \int K(x) \int \eta(\varrho) a_\pm(\varrho|x|) e^{-i\varrho(r \mp |x|)} d\varrho\, dx.$$

Because of the support property of η and the estimates which symbols satisfy, the inner integral is bounded by

$$|x|^{-\frac{n-1}{2}} w(r \mp |x|),$$

where w is even and satisfies $w(s) = O(|s|^{-N})$ for every positive integer N. Thus, if $|K(x)| = \kappa(|x|)$, then

$$\begin{aligned} |\widehat{m}(r)| &\le \sum_{\pm} \int |K(x)|\, |x|^{-\frac{n-1}{2}} w(r \pm |x|)dx \\ &\lesssim \sum_{\pm} \int \kappa(\varrho)\varrho^{\frac{n-1}{2}} w(r \pm \varrho)d\varrho \\ &= \sum_{\pm} (\kappa \cdot \varrho^{\frac{n-1}{2}}) * w(\pm r). \end{aligned}$$

Now let $a = (n-1)(1/p - 1/2) + \varepsilon$ and estimate

$$\begin{aligned} &\Big(\int |(\kappa \cdot \varrho^{\frac{n-1}{2}}) * w(\pm r)|^p r^{ap} dr\Big)^{1/p} \\ &\lesssim \int w(\rho) \Big(\int |\kappa(r-\rho)|r-\rho|^{(n-1)/2} d\rho|^p r^{ap} dr\Big)^{1/p} \\ &\lesssim \int w(\rho) \Big[\Big(\int |\kappa(r)|^p r^{(\frac{n-1}{2}+a)p} dr\Big)^{1/p} + \rho^{\alpha}\Big(\int_0^1 |\kappa(r)|^p r^{(\frac{n-1}{2})p} dr\Big)^{1/p}\Big] d\rho \\ &\lesssim \Big(\int |\kappa(r)|^p r^{(\frac{n-1}{2}+a)p} dr\Big)^{1/p} + \Big(\int_0^1 |\kappa(r)|^p dr\Big)^{1/p} \end{aligned}$$

since w is rapidly decreasing.

Now we use that $\widehat{\kappa}(\tau)$ is compactly supported in $\{\tau : |\tau| \le R\}$ (see *e.g.*[38]) to deduce that

$$\Big(\int_0^1 |\kappa(r)|^p dr\Big)^{1/p} \lesssim C_R \Big(\int_{|r| \ge cR^{-1}} |\kappa(r)|^p dr\Big)^{1/p}$$

and the right hand side is dominated by

$$C(R)\Big(\int |\kappa(r)|^p r^{(a+\frac{n-1}{2})p} dr\Big)^{1/p} = C'(R)\Big(\int |K(x)|^p |x|^{\varepsilon p} dx\Big)^{1/p}$$

which yields (6.13). □

Remark. An alternative proof can be based on arguments in [28].

Appendix: Uniform estimates and asymptotics for Legendre functions

A1. Some asymptotics for oscillatory integrals. In this section we recall the asymptotic behavior of certain oscillatory integrals with fractional singularities. For a slight variant one may consult Erdelyi [12]. The result is based on formulas for the Fourier transforms of distributions $\chi_{\pm}^{z-1}$ defined for $\mathrm{Re}(z) > 0$ as functions by $\chi_+^{z-1}(x) = (\Gamma(z))^{-1} x_+^{z-1}$ where x_+^{z-1} equals x^{z-1} if $x > 0$ and equals 0 if $x < 0$; moreover $\chi_-^{z-1}(x) = (\Gamma(z))^{-1} x_-^{z-1}$ where $x_-^{z-1} = (-x)_+^{z-1}$. The Fourier transform of $\chi_{\pm}^{z-1}$ is given by

$$e^{\mp i\pi z/2}(\xi \mp i0)^{-z} = e^{\mp i\pi z/2}\xi_+^{-z} + e^{\pm i\pi z/2}\xi_-^{-z}, \tag{A1.1}$$

see [16]. We shall need to consider the Fourier transform of a localized version of $\chi_\pm^{z-1}$ and use its asymptotic behavior. Given a smooth function u with compact support we define

$$F_\pm^z[u,a,X] = \langle \chi_+^{z-1}(\cdot - a), ue^{\pm iX\cdot}\rangle = \frac{1}{\Gamma(z)}\int_a^\infty u(s)(s-a)^{z-1}e^{\pm isX}ds \tag{A1.2}$$

$$G_\pm^z[u,a,X] = \langle \chi_-^{z-1}(\cdot - a), ue^{\pm iX\cdot}\rangle = \frac{1}{\Gamma(z)}\int_{-\infty}^a u(s)(a-s)^{z-1}e^{\pm isX}ds. \tag{A1.3}$$

The definition of $\chi_\pm^{z-1}$ can be extended by analytic continuation to all values of z (see e.g. [16, §3]). Likewise this yields the extension of $F_\pm^z$ and $G_\pm^z$ to entire functions of z. One obtains concrete formulas for these extensions by integration by parts. In fact, for $\mathrm{Re}(z) > 0$,

$$\begin{aligned} F_\pm^z[u,a,X] &= \int_a^\infty \frac{(s-a)^{z+m-1}}{\Gamma(z+m)}(-1)^m\Big(\frac{d}{ds}\Big)^m\big[u(s)e^{\pm isX}\big]ds \\ &= \sum_{\nu=0}^m \binom{m}{\nu} e^{\mp i\pi\frac{m+\nu}{2}}X^{m-\nu}F_\pm^{z+m}[u^{(\nu)},a,X] \end{aligned} \tag{A1.4}$$

and similarly

$$\begin{aligned} G_\pm^z[u,a,X] &= \int_a^\infty \frac{(a-s)^{z+m-1}}{\Gamma(z+m)}\Big(\frac{d}{ds}\Big)^m\big[u(s)e^{\pm isX}\big]ds \\ &= \sum_{\nu=0}^m \binom{m}{\nu} e^{\pm i\pi\frac{m-\nu}{2}}X^{m-\nu}G_\pm^{z+m}[u^{(\nu)},a,X] \end{aligned} \tag{A1.5}$$

and, by analytic continuation, (A1.4), (A1.5) yield formulas valid for $\mathrm{Re}(z) > -m$.

Notation: In this appendix let ζ_0 be an even $C_0^\infty(\mathbb{R})$ function so that $\zeta_0(s) = 1$ for $|s| \le 1/4$ and $\zeta_0(s) = 0$ for $|s| \ge 1/2$, furthermore let $\zeta_k(s) = \zeta_0(2^{-k}s) - \zeta_0(2^{-k+1}s)$ for $k \ge 1$, so that $\sum_{k=0}^\infty \zeta_k \equiv 1$. The parameter $b = \mathrm{Re}(z)$ is always assumed to belong to a fixed compact interval $[-b_0, b_0]$ for large b_0, and constants may depend on b_0.

Lemma A1.1. *Let $z = b + i\tau$ and suppose that $N > |b| + 3$. Let $u \in C^{N+1}$, with compact support in an interval I of length 1. Then*

$$|F_\pm^z(u,a,X)| + |G_\pm^z(u,a,X)| \le C_N\|u\|_{C^{N+1}}(1+|\tau|)^{N+4}e^{\frac{\pi}{2}|\tau|}(1+|X|)^{-b} \tag{A1.6}$$

For $X \ge 1$,

$$F_\pm^z(u,a,X) = u(a)e^{\pm i(aX+\frac{\pi}{2}z)}X^{-z} + R_{\pm,1}(X,z,a) \tag{A1.7.1}$$

$$G_\pm^z(u,a,X) = u(a)e^{\pm i(aX-\frac{\pi}{2}z)}X^{-z} + R_{\pm,2}(X,z,a) \tag{A1.7.2}$$

where for $j = 1,2$

$$|R_{\pm,j}(X,b+i\tau,a)| \le C_N\|u\|_{C^{N+1}}(1+|\tau|)^{N-b+1/2}e^{\frac{\pi}{2}|\tau|}X^{-b-1}$$

Proof. The boundedness of $F_\pm^z$ and $G_\pm^z$ is clear for $b > 0$, we use the lower bound

$$|\Gamma(b+i\tau)| \ge Ce^{-\pi\tau/2}(|\tau|+1)^{b-1/2}. \tag{A1.8}$$

(see [19]). For $b \le 0$ the boundedness follows from formulas (A1.4), (A1.5); here we need the assumption $N > |b| + 3$ (as opposed to just $N > b + 1$ below).

We shall prove the asymptotic formulas (A1.7.1/2) under the assumption $\mathrm{Re}(z) > 0$. Again the case where $-m < b \leq -m+1$ follows by applying the formulas (A1.4), (A1.5) and the case $\mathrm{Re}(z) > 0$.

(A1.7.1) and (A1.7.2) are equivalent as one can see by performing the change of variable $s = 2a - s'$. Moreover by performing a translation it is sufficient to consider the case $a = 0$. We now examine the function F_+^z; the term F_-^z is dealt with in the same way. Let $\widetilde{u} \in C_0^\infty$ be equal to 1 on the support of u. Let $u_1(s) = \int_0^1 u'(\sigma s)d\sigma$. Since $\widetilde{u}(s) = 1$ for s in the support of $\zeta_0(X\cdot) = 1 - \sum_{k=1}^\infty \zeta_k(X\cdot)$ (with $X \geq 1$) we may split

$$\begin{aligned} u(s) &= (u(0) + su_1(s))\widetilde{u}(s) \\ &= u(0) + su_1(s)\zeta_0(Xs) - \sum_{k\geq 1}\zeta_k(Xs)u(0)(1-\widetilde{u}(s)) + \sum_{k\geq 1} su_1(s)\zeta_k(Xs)\widetilde{u}(s) \end{aligned} \tag{A1.9}$$

Replacing $u(s)$ in (A1.9) by $u(0)$ we can use (A1.1) to pick up the main term in (A1.7.1). If we replace u by any other term on the right hand side of (A1.9) we get a contribution to the remainder term. Specifically,

$$\int_0^\infty |u_1(s)s^z\zeta_0(Xs)|ds \lesssim X^{-b-1}$$

and if we integrate by parts N times we see that

$$\left|\int_0^\infty s^z u_1(s)\zeta_k(Xs)e^{\pm isX}ds\right| \leq C\|u\|_{C^{N+1}}(1+|z|)^N 2^{k(b-N+1)}X^{-b-1}.$$

We may sum in k since $N > b+1$, and also use the lower bound (A1.8). The other terms in (A1.9) are handled similarly. This finishes the proof. □

Remark. We did not attempt to optimize the bounds in τ and the dependence on N.

A2. Analytic continuation. We consider the function $H_{z,\lambda}$ as defined in (2.10) for $\mathrm{Re}(z) > 0$. We will discuss an analytic continuation, separately on the intervals $(-1,1)$ and $(-\infty,-1)$.

Assuming $\mathrm{Re}(z) > 0$ it is clear that $H_{z,\lambda}$ is a smooth function on $(-1,1)$. We use integration by parts to extend the definition of $H_{z,\lambda}$ as a function on $(-1,1)$ to all values of $z \in \mathbb{C}$. To accomplish this we rewrite the defining integral assuming $\mathrm{Re}(z) > 0$.

We set $\mu = \cos\alpha$, $0 < \alpha < \pi$, and split

$$H_{z,\lambda}(\cos\alpha) = A_{z,\lambda}(\alpha) + B_{z,\lambda}(\alpha) \tag{A2.1}$$

where

$$A_{z,\lambda}(\alpha) = \frac{1}{\Gamma(z)}\int_0^\alpha \zeta_0(\frac{\alpha-\theta}{\sin\alpha})(\cos\theta - \cos\alpha)^{z-1}\cos(\lambda\theta)d\theta \tag{A2.2}$$

$$B_{z,\lambda}(\alpha) = \frac{1}{\Gamma(z)}\int_0^\alpha (1-\zeta_0(\frac{\alpha-\theta}{\sin\alpha}))(\cos\theta - \cos\alpha)^{z-1}\cos(\lambda\theta)d\theta. \tag{A2.3}$$

Clearly $B_{z,\lambda}(\alpha)$ is an entire function in z, and for each z the function $B_{z,\lambda}$ is smooth for $\alpha \in (0,\pi)$.

Changing variables $\theta = \alpha - u\sin\alpha$ we rewrite

$$\begin{aligned} A_{z,\lambda}(\alpha) &= (\sin\alpha)^{2z-1}\frac{1}{\Gamma(z)}\int_0^\infty \zeta_0(u)\gamma(\alpha,u)^{z-1}u^{z-1}\cos(\lambda(\alpha - u\sin\alpha))du \\ &= \frac{1}{2}\sum_{\pm} e^{\pm i\lambda\alpha}(\sin\alpha)^{2z-1}\langle \chi_+^{z-1}, \zeta_0\gamma(\alpha,\cdot)^{z-1}\exp(\mp i\lambda\sin\alpha\cdot)\rangle \end{aligned} \tag{A2.4}$$

where

$$\gamma(\alpha,u)=\frac{\cos(\alpha-u\sin\alpha)-\cos\alpha}{u\sin^2\alpha}$$

$$=1-\frac{u}{2}\cos\alpha-\frac{u^2}{2}\sin\alpha\int_0^1(1-s)^2\sin(\alpha-su\sin\alpha)ds \tag{A2.5}$$

Now $\gamma(\alpha,0)^{z-1}=1$ and the C^{N+1} norm of the function $(u,\alpha)\mapsto\gamma(\alpha,u)^{z-1}$ for $(u,\alpha)\in[0,1/2]\times\mathbb{R}$ is $O(1+|\tau|)^{N+1}$. By (A2.4) and §A1 the function $A_{z,\lambda}$ can be extended to all values of z, as a smooth function on $(-1,1)$. This extends $H_{z,\lambda}$ to all values of z and yields a smooth function on $(-1,1)$ which depends analytically on the parameter z.

Similarly we may also extend $H_{z,\lambda}$ as a smooth function on $(-\infty,-1)$. We set $-\mu=\cosh a$ and repeatedly integrate by parts using the adjoint of the operator $(\sinh s)^{-1}d/ds$ to obtain

(A2.6)
$$H_{z,\lambda}(-\cosh a)=(-1)^m\frac{\sin((z-\lambda)\pi)}{\Gamma(z+m)}\int_a^\infty(\cosh s-\cosh a)^{z+m-1}\Big(\frac{1}{\sinh s}\frac{d}{ds}-\frac{\cosh s}{\sinh^2 s}\Big)^m e^{-\lambda s}ds.$$

This can be used to extend $H_{z,\lambda}$ as a smooth function on $(-\infty,-1)$ for $\mathrm{Re}(z)>-m$.

A3. Oscillatory behavior of $H_{z,\lambda}$.

In this section we examine the asymptotic behavior of $H_{z,\lambda}(\mu)$, for $\mu\in(-1,1)$, under the assumption that $\lambda\sqrt{1-\mu^2}\geq 1$.

Lemma A3.1. *Let $z=b+i\tau$ and assume $\lambda\sin\alpha\geq 1$. In the open interval $(-1,1)$ the distributions $H_{z,\lambda}$ and $H'_{z,\lambda}$, defined in (2.10), can be identified with a smooth function whose asymptotic behavior is as follows.*

(i)

$$H_{z,\lambda}(\cos\alpha)=\cos(\lambda\alpha-\tfrac{z\pi}{2})\lambda^{-z}(\sin\alpha)^{z-1}+R_{z,\lambda}(\alpha) \tag{A3.1}$$

where

$$|R_{z,\lambda}(\alpha)|\leq C_{b,\tau}\lambda^{-b-1}(\sin\alpha)^{b-2}. \tag{A3.2}$$

(ii)

$$(\sin\alpha)^{2z-1}\frac{d}{d\alpha}\big[(\sin\alpha)^{1-2z}H_{z,\lambda}(\cos\alpha)\big]=-\sin(\lambda\alpha-\tfrac{z\pi}{2})\lambda^{1-z}(\sin\alpha)^{z-1}+\widetilde{R}_{z,\lambda}(\alpha) \tag{A3.3}$$

where

$$|\widetilde{R}_{z,\lambda}(\alpha)|\leq C_{b,\tau}\lambda^{-b}(\sin\alpha)^{b-2}. \tag{A3.4}$$

Moreover

$$|H'_{z,\lambda}(\cos\alpha)-\sin(\lambda\alpha-\tfrac{z\pi}{2})\lambda^{1-z}(\sin\alpha)^{z-2}|\leq C_{b,\tau}\lambda^{-b}(\sin\alpha)^{b-3}. \tag{A3.5}$$

In the above estimates the numbers $C_{b,\tau}$ satisfy the estimates $|C_{b,\tau}|\leq A_{b,N}(1+|\tau|)^{N+4}e^{\frac{\pi}{2}\tau}$ where $N\geq|b|+3$ is a positive integer and $A_{b,N}$ stays bounded if b and $N\geq|b|+3$ are chosen in any compact interval.

Proof. We split $H_{z,\lambda}(\cos\alpha)=A_{z,\lambda}(\alpha)+B_{z,\lambda}(\alpha)$ as in (A2.2-3). We pick up the main term in (A3.1) by considering $A_{z,\lambda}$. Splitting $\cos(\lambda(\alpha-u\sin\alpha))$ as the sum of two exponentials as in (A2.4) we may apply Lemma A1.1 to obtain the desired asymptotics (A3.1) for the expression $A_{z,\lambda}(\alpha)$ in place of $H_{z,\lambda}(\cos\alpha)$.

To obtain an estimate for remainder term $B_{z,\lambda}$ we use a further splitting and integrate by parts. The argument is slightly different depending on whether $\alpha \in (0, \pi/2]$ or $\alpha \in (\pi/4, \pi)$.

We shall first assume that $\alpha \in (0, \pi/2)$. Let $\zeta_{0,\alpha}(\theta) = \zeta_0(\frac{\alpha-\theta}{\sin\alpha}))$. We use integration by parts to see that $B_{z,\lambda}(\alpha)$ is a linear combination of terms

(A3.6)
$$B_{j,\ell,\nu}(\alpha) = \begin{cases} \lambda^{-N} \int_0^\infty \sin(\lambda\theta)\partial_\theta^\nu[1-\zeta_{0,\alpha}(\theta)](\sin\theta)^j(\cos\theta)^\ell(\cos\theta-\cos\alpha)^{z-1-j-\ell}d\theta & \text{if } N \text{ is odd} \\ \lambda^{-N} \int_0^\infty \cos(\lambda\theta)\partial_\theta^\nu[1-\zeta_{0,\alpha}(\theta)](\sin\theta)^j(\cos\theta)^\ell(\cos\theta-\cos\alpha)^{z-1-j-\ell}d\theta & \text{if } N \text{ is even} \end{cases}$$

with the additional specifications that $j + 2\ell + \nu \le N$ and that $j \ge 1$ if N is odd and $\nu = 0$. The latter condition implies that no boundary terms are picked up at $\theta = 0$. In the above integrals $|\cos\theta - \cos\alpha| \approx (\sin\alpha)^2$ and therefore

$$|B_{j,\ell,\nu}(\alpha)| \lesssim (1+|z|)^N(1+\lambda)^{-N}(\sin\alpha)^{2b-1-\nu-j-2\ell}$$

where $j + 2\ell + \nu \le N$. Choosing $N \ge b+1$ shows that $B_{j,\ell,\nu} = O((1+\lambda)^{-b-1}(\sin\alpha)^{b-2})$.

Let us now assume that $\alpha > \pi/4$. Let ω be smooth so that $\omega(\theta) = 1$ if $\theta \le \pi/16$, and $\omega(\theta) = 0$ if $\theta \ge \pi/8$. For $k \ge 1$ let $\zeta_k(s) = \zeta_0(2^{-k}s) - \zeta_0(2^{1-k}s)$ and observe that $\omega(\theta)\zeta_0(\frac{\alpha-\theta}{\sin\alpha}))$ vanishes for all θ if $\alpha > \pi/4$. We may therefore split

$$B_{z,\lambda}(\alpha) = \sum_{k\ge1} I_k(\alpha) + II(\alpha) \tag{A3.7}$$

where

$$\Gamma(z)I_k(\alpha) = \int_0^\alpha (1-\omega(\theta))\zeta_k(\frac{\alpha-\theta}{\sin\alpha})(\cos\theta-\cos\alpha)^{z-1}\cos(\lambda\theta)d\theta, \tag{A3.8.1}$$

$$\Gamma(z)II(\alpha) = \int_0^\alpha \omega(\theta)(\cos\theta-\cos\alpha)^{z-1}\cos(\lambda\theta)d\theta. \tag{A3.8.2}$$

The term $II(\alpha)$ is handled by a straightforward integration by parts argument; as above one sees that no boundary terms are picked up at 0 and the result of the computation is

$$|II(\alpha)| \le C_N(1+|\tau|)^N(1+\lambda)^{-N} \tag{A3.9}$$

which is a favorable estimate since $\lambda\sin\alpha \ge 1$.

For the terms I_k we use integration by parts as well. Let

$$\zeta_{k,\alpha}(\theta) = \zeta_k(\frac{\alpha-\theta}{\sin\alpha})(1-\omega(\theta)).$$

We then see that I_k is a linear combination of terms of the form $I_{k,j,\ell,\nu}$ where

(A3.10)
$$I_{k,j,\ell,\nu}(\alpha) = \begin{cases} (1+\lambda)^{-N} \int_0^\infty \sin(\lambda\theta)\partial_\theta^\nu\zeta_{k,\alpha}(\theta)(\sin\theta)^j(\cos\theta)^\ell(\cos\theta-\cos\alpha)^{z-1-j-\ell}d\theta & \text{if } N \text{ is odd,} \\ (1+\lambda)^{-N} \int_0^\infty \cos(\lambda\theta)\partial_\theta^\nu\zeta_{k,\alpha}(\theta)(\sin\theta)^j(\cos\theta)^\ell(\cos\theta-\cos\alpha)^{z-1-j-\ell}d\theta & \text{if } N \text{ is even,} \end{cases}$$

and where $j + 2\ell + \nu \le N$.

For the integration by parts observe that the amplitudes of the integrals are supported away from the endpoints. One uses that $|\cos\theta - \cos\alpha| \approx (2^k\sin\alpha)^2$ if $|\theta - \alpha| \approx 2^k\sin\alpha$ and obtains the estimate

$$|I_{k,j,\ell,\nu}(\alpha)| \lesssim (1+|z|)^N\lambda^{-N}(2^k\sin\alpha)^{2b-1-\nu-j-2\ell}. \tag{A3.11}$$

We sum and estimate using the restriction $j + 2\ell + \nu \le N$. The sum $(1+|\tau|)^{-N}\sum_{0<2^k\le8\pi/\sin\alpha}|I_{k,j,\ell,\nu}|$ is then controlled by either $(1+\lambda)^{-N}$, or $(1+\lambda)^{-N}\log(2+(\sin\alpha)^{-1}) \lesssim (1+\lambda)^{-N}\log(2+\lambda)$, or $(1+$

$\lambda)^{-N}(\sin\alpha)^{2b-1-N}$, depending on the sign of the exponent $2b-1-\nu-j-2\ell$. If we choose $N>b$ we obtain the bound

$$\sum_{0<2^k\le 8\pi/\sin\alpha}|I_k|\lesssim(1+|z|)^N(1+\lambda)^{-b}(\sin\alpha)^{b-1}((1+\lambda)\sin\alpha)^{b-N}.$$

This finishes the proof of (A3.1/2).

We now turn to the estimates for the derivatives. The derivative of the main term in (A3.3) is given by

$$\begin{aligned}\frac{d}{d\alpha}\Big((\sin\alpha)^{1-2z}A_{z,\alpha}(\alpha)\Big)&=-\frac{1}{\Gamma(z)}\int_0^\infty\zeta_0(u)\gamma(\alpha,u)^{z-1}u^{z-1}\sin(\lambda(\alpha-u\sin\alpha))(\lambda-\lambda u\cos\alpha)du\\&+\frac{z-1}{\Gamma(z)}\int_0^\infty\zeta_0(u)\frac{\partial\gamma}{\partial\alpha}(\alpha,u)\gamma(\alpha,u)^{z-2}u^{z-1}\cos(\lambda(\alpha-u\sin\alpha))du\quad:=A_{z,\lambda,1}(\alpha)+A_{z,\lambda,2}(\alpha).\end{aligned}\tag{A3.12}$$

We apply Lemma A1.1 to the term $A_{z,\lambda,1}$ and pick up the main term in the asymptotic formula in (A3.3). Applying the same argument to the second term $A_{z,\lambda,2}$ and using (A2.5) we see that $A_{z,\lambda,2}(\alpha)$ can be subsumed under the remainder term in (A3.3/4).

The derivatives of $B_{z,\lambda}$ are estimated in the same way as $B_{z,\lambda}$ itself; for the corresponding terms I_k' the differentiation introduces factors which are all $O((\sin\alpha)^{-1})$, and this is acceptable for (A3.3). Using both the asymptotics (A3.1) and (A3.3) yields an asymptotic formula for $\partial_\alpha(H_{z,\lambda}(\cos\alpha))$ from which the asserted formula (A3.5) follows. □

A4. Estimates for the nonoscillatory terms. We begin by recalling the asymptotic behavior of the Legendre functions near the singularities.

Lemma A4.1. *(i) Suppose $z=b+i\tau$ and $b>1/2$. Then $H_{z,\lambda}$ extends to continuous function on $(-\infty,\infty)$; in particular*

$$\lim_{\mu\to-1}H_{z,\lambda}(\mu)=\frac{\sqrt{\pi}2^{z-1}\Gamma(z-\frac12)}{\Gamma(z-\lambda)\Gamma(z+\lambda)}.\tag{A4.1}$$

(ii) $H_{\frac12,\lambda}$ has a jump discontinuity at $\mu=1$ and

$$\lim_{\mu\to1-}H_{\frac12,\lambda}(\mu)=\sqrt{\frac{\pi}{2}}.\tag{A4.2}$$

(iii) $H_{\frac12,\lambda}$ has a logarithmic singularity at $\mu=-1$; moreover

$$\lim_{\mu\to-1\pm}\Big[H_{\frac12,\lambda}(\mu)+\frac{\cos(\lambda\pi)}{\sqrt{2\pi}}\log(|1+\mu|)\Big]=\gamma_\lambda^\pm\tag{A4.3}$$

where

$$\gamma_\lambda^+-\gamma_\lambda^-=-\sqrt{\frac{\pi}{2}}\sin(\lambda\pi)\tag{A4.4}$$

and $\gamma_\lambda^-=-(2\pi)^{-1/2}\cos(\lambda\pi)\big(\Psi(\lambda+\frac12)-\Psi(1)-\log2\big)$; here $\Psi=\Gamma'/\Gamma$.

Proof. We use the description of $H_{z,\lambda}$ in terms of Legendre functions of the first and second kind. Precisely, using the notation and fonts of [13], 3.7(27) and 3.7(4),

$$H_{z,\lambda}(\mu)=\begin{cases}\sqrt{\frac{\pi}{2}}(1-\mu^2)^{\frac z2-\frac14}\mathrm{P}^{\frac12-z}_{\lambda-\frac12}(\mu) & \text{if } -1<\mu<1\\ \sqrt{2\pi}\sin(z\pi-\lambda\pi)(\mu^2-1)^{\frac z2-\frac14}e^{(z-\frac12)\pi i}Q^{\frac12-z}_{\lambda-\frac12}(-\mu) & \text{if } \mu<-1.\end{cases}\tag{A4.5}$$

This is derived in [22]; alternatively one may also consult Watson's monograph ([40, §13·46, (4), (5)]) and use ([13, 3.7.4 (27), (4)]), keeping in mind that the definition of the second Legendre functions $Q^{-z+1/2}_{\lambda-1/2}$ in [40, 5·71] differs from the definition in [13, 3.3.1 (4)].

The following references concern the limiting behavior of Legendre functions as $\mu \to \pm 1$ and refer to formulas in [13]. For (i) we use 3.9.2(14) and 3.9.2(6), for (ii) we use 3.9.2(8). For (iii), for the behavior as $\mu \nearrow -1$ we use 3.9.2(7). As pointed out in [13] the behavior as $\mu \searrow -1$ can be derived from 3.4.14 and 3.9.2(8,11); the resulting formula 3.9.2(15) in [13] contains a misprint as the Euler constant there should be multiplied by 2. □

We shall need *uniform* estimates for $H_{z,\lambda}$ and its derivatives near the points ± 1.

Lemma A4.2. *Suppose that $-1 < \mu < 1$ and $z = b + i\tau$. Fix $A > 5/2$. Then the following estimates hold if $A > \pi|b| + 5/2$.*

(i) Suppose that $b > 1/2$. Then

$$|H_{z,\lambda}(\mu)| \le C_1(A,b)e^{A\tau} \times \begin{cases} (1-\mu^2)^{b-\frac{1}{2}} & \text{if } \lambda\sqrt{1-\mu^2} \le 1, \quad 0 \le \mu < 1, \\ (1+\lambda)^{-b}(1-\mu^2)^{\frac{b-1}{2}} & \text{if } \lambda\sqrt{1-\mu^2} \ge 1, \quad -1 < \mu < 1, \\ (1+\lambda)^{1-2b} & \text{if } \lambda\sqrt{1-\mu^2} \le 1, \quad -1 < \mu \le 0. \end{cases} \tag{A4.6}$$

(ii) Suppose that $b = 1/2$. Then

$$|H_{z,\lambda}(\mu)| \le C_2(A)e^{A\tau} \times \begin{cases} 1 & \text{if } \lambda\sqrt{1-\mu^2} \le 1, \quad 0 \le \mu < 1, \\ (1+\lambda)^{-1/2}(1-\mu^2)^{-\frac{1}{4}} & \text{if } \lambda\sqrt{1-\mu^2} \ge 1, \quad -1 < \mu < 1, \\ \log((2+\lambda)^{-1}(1-\mu^2)^{-1/2}) & \text{if } \lambda\sqrt{1-\mu^2} \le 1, \quad -1 < \mu \le 0. \end{cases} \tag{A4.7}$$

(iii) Suppose that $b < 1/2$ and $-1 \le \mu \le 1$. Then

$$|H_{z,\lambda}(\mu)| \le C_3(b,A)e^{A\tau} \times \begin{cases} (1-\mu^2)^{b-\frac{1}{2}} & \text{if } \lambda\sqrt{1-\mu^2} \le 1, \\ (1+\lambda)^{-b}(1-\mu^2)^{\frac{b-1}{2}} & \text{if } \lambda\sqrt{1-\mu^2} \ge 1. \end{cases} \tag{A4.8}$$

Proof. The statements for $\lambda\sqrt{1-\mu^2} \ge 1$ follow from Lemma A3.1. Therefore, in what follows we assume $\lambda\sqrt{1-\mu^2} \le 1$. We use the decomposition (A2.1).

We first consider the region $-1/8 \le \mu < 1$, in which $\alpha := \arccos\mu$ satifies $0 \le \alpha \le 3\pi/4$. Then the bound $A_{z,\lambda}(\alpha) = O(\sin\alpha)^{2b-1}$ is immediate from (A2.2) and likewise we obtain the same estimate for $B_{z,\lambda}$ from the definition.

Now assume that $\lambda \sin\alpha \le 1$ and α is near π, hence μ near -1. The bound for the term $A_{z,\lambda}$ is as above. To estimate $B_{z,\lambda}$ we have to examine the terms I_k, II in (A3.8). Notice that in the integrand of I_k we have $\cos\theta - \cos\alpha \approx 2^{2k}(\sin\alpha)^2$ so that

$$|I_k| \lesssim 2^{(2b-1)k}(\sin\alpha)^{2b-1} \tag{A4.9}$$

We use this estimate only for $2^k \le ((\lambda+1)\sin\alpha)^{-1}$ and see that the sum over these terms is bounded by $C(1+\lambda)^{1-2b}$ if $b > 1/2$, by $C\log(((2+\lambda)\sin\alpha)^{-1})$ if $b = 1/2$ and by $C(\sin\alpha)^{2b-1}$ if $b < 1/2$. The sum over the terms I_k with $2^k > ((\lambda+1)\sin\alpha)^{-1}$ is handled by integration by parts exactly as in the proof of Lemma A3.1. The same applies to the term II (if $\lambda \ge 1$).

The bounds in τ follow from the lower bound for the Γ function stated in (A1.8). □

Lemma A4.3. *Suppose that* $\varepsilon > 0$, $z = b + i\tau$, $A > 2\pi(|b| + 1/2)$, *and* $\lambda > b - 1 + \varepsilon$.

(i) If $\mu < -2$ *then*

$$|H_{z,\lambda}(\mu)| \le C(\varepsilon, b, A)e^{A\tau}(1+\lambda)^{-b}|\mu|^{b-\lambda-1}. \tag{A4.10}$$

(ii) If $-2 \le \mu \le -1 - (\lambda+1)^{-2}$ *then*

$$|H_{z,\lambda}(\mu)| \le C(\varepsilon, b, A)e^{A\tau}(1+\lambda)^{-b}(\mu^2-1)^{\frac{b-1}{2}}e^{-\lambda\sqrt{\mu^2-1}}. \tag{A4.11}$$

(iii) If $-1 - (\lambda+1)^{-2} \le \mu < -1$ *then*

$$|H_{z,\lambda}(\mu)| \le C(\varepsilon, b, A)e^{A\tau} \times \begin{cases} (1+\lambda)^{1-2b} & \text{if } b > 1/2, \\ \log((2+\lambda)\sqrt{\mu^2-1})^{-1}) & \text{if } b = 1/2, \\ (\mu^2-1)^{b-\frac{1}{2}} & \text{if } 0 < b < 1/2. \end{cases} \tag{A4.12}$$

Proof. We set $\mu = -\cosh a$, $a > 0$. Observe that for $s \ge a$

$$\cosh s - \cosh a = \sinh(\tfrac{s+a}{2})\sinh(\tfrac{s-a}{2}) \approx \begin{cases} s^2 - a^2 & \text{if } a \le 1,\ s \le 2, \\ e^{\frac{a+s}{2}}(s-a) & \text{if } a \ge 1,\ s \le a+1, \\ e^s & \text{if } \ s \ge a+1. \end{cases}$$

We assume that $\mathrm{Re}(z) > 0$. We replace z by b and then have to estimate integrals with positive integrand. After some lengthy but straightforward estimates we see that $H_{z,\lambda}(-\cosh a)$ is $O(\lambda^{-b}e^{-(\lambda-b+1)a})$ if $a \ge 1$ and $O(\lambda^{-b}a^{b-1}e^{-\lambda a})$ if $(\lambda+1)^{-1} \le a \le 1$. If $0 < a < (\lambda+1)^{-1}$ one obtains that $H_{z,\lambda}(-\cosh a)$ is $O(\lambda^{1-2b})$, $O(\log((\lambda+1)a)^{-1})$ or a^{2b-1} in the cases $b > 1/2$, $b = 1/2$ and $0 < b < 1/2$, respectively. These estimates imply (i), (ii) and (iii), for the case $b > 0$; the exponential bounds follow from (A1.8) and the obvious upper bound for the coefficient $\sin(z\pi - \lambda\pi)$ in (2.10). The same argument applies to the case $\mathrm{Re}(z) > -m$ if we use instead the formula (A2.6). □

We remark that the estimate (A4.6) for $\lambda\sqrt{1-\mu^2} \le 1$ and $\mu \to 1-$ can be slightly improved, if $b < 1/2$. Moreover the bounds for $\mu \to -1\pm$ can be replaced by asymptotic expansions. Now these improvements are only needed in this paper for the case $z = -1/2$, and in this case the corresponding statements have already been proved by Lindblad and Sogge [21]. The estimates there are stated only for integer values of λ but the analysis can be carried out for general $\lambda > 0$.

We will therefore just quote the estimates from [21]. First

$$|H'_{\frac{1}{2},\lambda}(\mu)| \lesssim (1+\lambda)^2 \quad \text{if } \lambda\sqrt{1-\mu^2} \le 1, \mu > -3/4. \tag{A4.13}$$

Next, for the asymptotic behavior at $\mu = -1$ it is natural to define

$$\widetilde{H}_\lambda(\mu) = H_{\frac{1}{2},\lambda}(\mu) + \cos(\lambda\pi)\frac{\log(|1+\mu|)}{\sqrt{2\pi}}. \tag{A4.14}$$

The function $\widetilde{H}_\lambda$ satisfies

$$\Big|\frac{d\widetilde{H}_\lambda}{d\mu}(\mu)\Big| \le C\lambda^{1/2}|1-\mu^2|^{-3/4} \qquad \text{if } \mu \in (-2,-1) \cup (-1,-1/2); \tag{A4.15}$$

see §7 of [21]. Estimate (A2.17) can be complemented by a statement of uniformity in the limit (A4.3), namely

$$|\widetilde{H}_\lambda(\mu) + \cos(\lambda\pi)(2/\pi)^{1/2}\log(\lambda+1)| \le C \quad \text{if } \lambda\sqrt{|1+\mu|} \lesssim 1, \mu \in (-2,-1) \cup (-1,-1/2); \tag{A4.16}$$

here C is independent of λ and μ. We shall omit the proof of (A4.16) as this statement does not explicitely enter in our analysis.

A5. Proof of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. By (2.12)

$$K_\lambda(t,r,\rho) = (\pi/2)^{1/2} K_{-1/2,\lambda}(t,r,\rho) = -(2\pi)^{-1/2} t \rho^{\frac{n-3}{2}} r^{-\frac{n+1}{2}} H'_\lambda(\mu)$$

where $H_\lambda := H_{1/2,\lambda}$ and H'_λ denotes the derivative in the sense of distributions. Let $\widetilde{H}_\lambda$ be as in (A4.14) We denote by $\frac{dH_\lambda}{d\mu}$, $\frac{d\widetilde{H}_\lambda}{d\mu}$ the pointwise derivative of H_λ and $\widetilde{H}_\lambda$ in $\mathbb{R}\setminus\{-1,1\}$. Moreover let $\widetilde{H}_\lambda(-1\pm)$ denote the right and left limit at -1, respectively (*cf.* (A4.3/4)).

Set $\zeta^\lambda(\mu) := \zeta((1+\lambda)\sqrt{|1-\mu^2|})$. We write with $\phi \in C_0^\infty(\mathbb{R})$

$$\langle H'_\lambda, \phi\rangle = -\Big[\int_0^1 + \int_{-1}^0 + \int_{-2}^{-1} + \int_{-\infty}^{-2}\Big]\big(H_\lambda(\mu)\zeta^\lambda(\mu) + H_\lambda(\mu)(1-\zeta^\lambda(\mu))\big)\phi'(\mu)d\mu.$$

In the integrals over $(-1,0]$ and $[-2,-1)$ we split $H_\lambda = \widetilde{H}_\lambda + c_\lambda \log|1+\mu|$ where $c_\lambda = (2\pi)^{-1/2}\cos(\lambda\pi)$. Integration by parts yields

$$\begin{aligned}
\langle H'_\lambda, \phi\rangle = &- H_\lambda(1-)\zeta^\lambda(1)\phi(1) + H_\lambda(0)\zeta^\lambda(0)\phi(0) + \int_0^1 \Big[\frac{dH_\lambda}{d\mu}\zeta^\lambda + H_\lambda\frac{d\zeta^\lambda}{d\mu}\Big]\phi(\mu)d\mu \\
&- \widetilde{H}_\lambda(0)\zeta^\lambda(0)\phi(0) + \widetilde{H}_\lambda(-1+)\zeta^\lambda(-1)\phi(-1) + \int_{-1}^0 \Big[\frac{d\widetilde{H}_\lambda}{d\mu}\zeta^\lambda + \widetilde{H}_\lambda\frac{d\zeta^\lambda}{d\mu}\Big]\phi(\mu)d\mu \\
&- \widetilde{H}_\lambda(-1-)\zeta^\lambda(-1)\phi(-1) + \widetilde{H}_\lambda(-2)\zeta^\lambda(-2)\phi(-2) + \int_{-2}^{-1} \Big[\frac{d\widetilde{H}_\lambda}{d\mu}\zeta^\lambda + \widetilde{H}_\lambda\frac{d\zeta^\lambda}{d\mu}\Big]\phi(\mu)d\mu \\
&- \lim_{\varepsilon\to 0}\Big(c_\lambda\log(|1+\mu|)\phi(\mu)\Big|_{-1+\varepsilon}^{0} - c_\lambda\log(|1+\mu|)\phi(\mu)\Big|_{-2}^{-1+\varepsilon} \\
&\qquad - \int_{\substack{[-2,0]\setminus\\ [-1-\varepsilon,-1+\varepsilon]}}\Big[\frac{c_\lambda\zeta^\lambda(\mu)}{1+\mu} + c_\lambda\log(|1+\mu|)\frac{d\zeta^\lambda}{d\mu}\Big]\phi(\mu)d\mu\Big) \\
&- H_\lambda(-2)\zeta^\lambda(-2)\phi(-2) + \int_{-\infty}^{-2}\Big[\frac{dH_\lambda}{d\mu}\zeta^\lambda + H_\lambda\frac{d\zeta^\lambda}{d\mu}\Big]\phi(\mu)d\mu \\
&+ \int\Big[\frac{dH_\lambda}{d\mu}(1-\zeta^\lambda) - H_\lambda\frac{d\zeta^\lambda}{d\mu}\Big]\phi(\mu)d\mu.
\end{aligned}$$

Observe that all terms involving a derivative of ζ^λ cancel; thus after clearing the boundary terms we obtain

$$\begin{aligned}
H'_\lambda = &\frac{dH_\lambda}{d\mu}(1-\zeta^\lambda) + \frac{dH_\lambda}{d\mu}\zeta^\lambda\chi_{(0,1)} + \frac{d\widetilde{H}_\lambda}{d\mu}\zeta^\lambda\chi_{(-2,0)} + \frac{dH_\lambda}{d\mu}\zeta^\lambda\chi_{(-\infty,-2)} \\
&+ c_\lambda\zeta^\lambda p.v.\chi_{[-2,0)}(\mu)\frac{1}{1+\mu} - H_\lambda(1-0)\delta(1-\mu) + (\gamma_\lambda^+ - \gamma_\lambda^-)\delta(1+\mu).
\end{aligned} \tag{A5.1}$$

Since $H_\lambda(1-0) = (\pi/2)^{1/2}$ and $\gamma_\lambda^+ - \gamma_\lambda^- = -(\pi/2)^{1/2}\sin(\lambda\pi)$ we pick up Dirac measures of the form

$$-\sqrt{\tfrac{1}{2\pi}}t\rho^{(n-3)/2}r^{-(n+1)/2}\sqrt{\tfrac{\pi}{2}}[-\delta(1-\mu) - \sin(\lambda\pi)\delta(1+\mu)] \tag{A5.2}$$

for the term (2.19). To express these in the form desired for (2.19) we rewrite $\delta(1\pm\mu(t,r,\rho))$ as distributions acting in the ρ variable, for fixed $r>0$, $t>0$. It is straightforward exercise in distribution theory to verify that

$$\begin{aligned}
\delta(1-\mu(t,r,\rho)) &= \delta\big(\tfrac{t^2-(r-\rho)^2}{2r\rho}\big) = 2r\rho\delta\big((t-r+\rho)(t+r-\rho)\big) \\
&= \frac{r\rho}{t}\big(\delta(t-r+\rho) + \delta(t+r-\rho)\big)
\end{aligned} \tag{A5.3}$$

and similarly

$$\begin{aligned}\delta(1+\mu(t,r,\rho)) &= \frac{r\rho}{t}\big(\delta(\rho-t+r)+\delta(\rho+r+t)\big)\\ &= \frac{r\rho}{t}\delta(\rho-t+r),\end{aligned} \tag{A5.4}$$

since we assume $t>0$. Moreover

$$\frac{1}{1+\mu} = \frac{2r\rho}{(r+\rho)^2-t^2} = \frac{2r\rho}{(r+\rho+t)(\rho-t+r)}. \tag{A5.6}$$

Using (A5.4-6) we pick up the terms in (2.19/20). For the term $\frac{dH_\lambda}{d\mu}(1-\zeta^\lambda)\chi_{(-1,1)}$ in (A5.1) we use (A3.5). The main term in the asymptotic in (A3.5) is $\sin(\lambda\arccos\mu-\pi/4)\lambda^{1/2}(1-\mu^2)^{-3/4}$ and since $\sin(\theta-\pi/4)=-\cos(\theta+\pi/4)$ this yields the term (2.18). The remainder term in this asymptotic expansion is subsumed under (2.21). The term $\frac{dH_\lambda}{d\mu}(1-\zeta^\lambda)\chi_{(-2,-1)}$ contributes to (2.21), the appropriate estimate follows from (A4.11) and (2.13). The terms $\frac{dH_\lambda}{d\mu}(1-\zeta^\lambda)\chi_{(-\infty,-2)}$ and $\frac{dH_\lambda}{d\mu}\zeta^\lambda\chi_{(-\infty,-2)}$ contribute to (2.22), here we use (A4.10) and (2.13).

For the term $\frac{dH_\lambda}{d\mu}\zeta^\lambda\chi_{(0,1)}$ we use (A4.13), and for $\frac{d\widetilde{H}_\lambda}{d\mu}\zeta^\lambda\chi_{(-2,0)}$ we use (A4.15), both terms contribute to (2.21). □

Proof of Theorem 2.2. This is analogous to the proof of Theorem 2.1; note that for the case $b>-1/2$ no singular terms occur at the boundary $\mu=\pm 1$. We use Lemma A3.1 for the contribution of $K_{z,\lambda}(1-\zeta^\lambda)\chi_{(-1,1)}$, We use Lemma A4.2 for the contribution of $K_{z,\lambda}\zeta^\lambda\chi_{(-2,-1)}$ and Lemma A4.3 for the contribution of $K_{z,\lambda}\chi_{(-\infty,-2]}$. □

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