WEIGHTED INEQUALITIES FOR BOCHNER-RIESZ MEANS IN THE PLANE

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1. Introduction

For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^2)$ let $\widehat{f}(\xi) = \int f(y)e^{-i\langle y,\xi\rangle}dy$ denote the Fourier transform. We consider the Bochner-Riesz means of index λ defined by

$$S_t^\lambda f(x) = \frac{1}{(2\pi)^2}\int_{|\xi|\le t}(1-\frac{|\xi|^2}{t^2})^\lambda \widehat{f}(\xi)e^{i\langle x,\xi\rangle}\,d\xi$$

and the maximal operator

$$S_*^\lambda f(x) = \sup_{t>0}|S_t^\lambda f(x)|$$

which controls the pointwise behavior of S_t^λ as $t\to\infty$.

In this paper we prove weighted L^2 inequalities

$$\int \big|S_*^\lambda f(x)\big|^2 w(x)dx \le C_\lambda \int |f(x)|^2 W(x)dx \tag{1.1}$$

for appropriate weights w, W. We shall always assume that all weights w under consideration are nonnegative, locally integrable and satisfy some mild growth condition at infinity, namely

$$\int w(x)(1+|x|)^{-N_0}dx<\infty \tag{1.2}$$

for some fixed large N_0; we shall call such weights *admissible.*

Rubio de Francia [11] showed that for every $w\in L^2(\mathbb{R}^2)$ there is a nonnegative $W\in L^2(\mathbb{R}^2)$ such that $\|W\|_2\le C_\lambda\|w\|_2$, $C_\lambda<\infty$ if $\lambda>0$, and the analogous weighted norm inequality for S_t^λ holds uniformly in t. He used methods related to factorization theory of operators and the proof gave no information on how to construct w from W. In [3] the first author explicitly constructed for every $q\ge 2$ an operator $\mathcal{W}_{q,\lambda}$, bounded on $L^q(\mathbb{R}^2)$, such that (1.1) holds for $w\in L^q(\mathbb{R}^2)$ and $W=\mathcal{W}_{q,\lambda}w$; in fact given $\mathcal{W}_{2,\lambda}$ one choses $\mathcal{W}_{q,\lambda}w$ to be $(\mathcal{W}_{2,\lambda}(w^{q/2}))^{2/q}$. See also Córdoba [8] for a related result concerning S_t^λ. In [3] it was observed that the operator $\mathcal{W}_{q,\lambda}$ was bounded on $L^r(\mathbb{R}^2)$ for $q\le r\le 2q$ and the question arose whether $\mathcal{W}_{q,\lambda}$ can be chosen to be independent of q. We shall show that this is indeed the case; for each $\lambda>0$ we construct an operator W_λ such that (1.1) holds with $W=W_\lambda$ and W_λ is bounded on L^r if $2\le r\le\infty$. Moreover this operator is pointwise bounded by a positive operator (involving a Besicovich-type maximal function acting on w^2) which itself is bounded on L^s for $4\le s\le\infty$.

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Let $\mathcal{B}_N$ be the family of all rectangles centered at the origin, with the property that the eccentricity (the ratio of the larger and the smaller sidelength) is equal to 2^N. Define

$$\mathfrak{M}_N g(x) = \sup_{\substack{R\in\mathcal{B}_N\\ x\in R}} \frac{1}{|R|}\int_R |g(x+y)|dy.$$

This maximal function is known to be bounded on $L^2(\mathbb{R}^2)$ with norm $O(N)$, moreover for $q>2$ it is bounded with norm $O(\frac{q}{q-2}N^{1/2-1/q})$ (see [7], [17]). Denote by M the standard Hardy-Littlewood maximal operator and let $M_s g = M(|g|^s)^{1/s}$.

Theorem 1. *Given $\lambda>0$ there is $\delta_\lambda>0$ and an operator W_λ, bounded on $L^q(\mathbb{R}^2)$, $2-\delta_\lambda\le q\le\infty$, such that for all admissible weights the inequality*

$$\sup_{t>0}\int \left|S_t^\lambda f(x)\right|^2 w(x)\,dx \le C_\lambda \int |f(x)|^2 W_\lambda w(x)dx \tag{1.3}$$

holds. Moreover if $s>1$ then

$$\int \left|S_*^\lambda f(x)\right|^2 w(x)\,dx \le C_{\lambda,s}\int |f(x)|^2 M_s(W_\lambda w)(x)dx. \tag{1.4}$$

The operator W_λ satisfies the pointwise estimate

$$W_\lambda w(x) \le C_\varepsilon \sum_{j\ge1} 2^{-j\varepsilon}\big(\mathfrak{M}_{j/2}[w^2](x)\big)^{1/2}, \qquad \varepsilon<2\lambda. \tag{1.5}$$

A definition of W_λ and somewhat sharper results are given in §2. Stein [14, p.7] posed the question whether W_λ can be essentially realized as $\sum_{l>0}2^{-l\varepsilon}\mathfrak{M}_{l/2}w$, $\varepsilon<2\lambda$. An affirmative answer seems to be known only for radial weights (see Carbery, Romera and Soria [4]), and then only for the operator S_t^λ. Since by (1.4) the L^p operator norm of S_*^λ is controlled by the square root of the $L^{(p/2)'}$ operator norm of M_sW_λ, and since this operator is bounded for all $q\ge2$, Theorem 1.1 implies Carbery's theorem [2] saying that S_*^λ is bounded on L^p if $\lambda>0$ and $2\le p\le4$. The weaker weighted norm inequality

$$\int \left|S_*^\lambda f(x)\right|^2 w(x)\,dx \le C_{\varepsilon,s}\sum_{l>0}2^{-l\varepsilon}\int |f(x)|^2 M_s[(\mathfrak{M}_{l/2}|w|^2)^{1/2}]dx \tag{1.6}$$

which by (1.5) also holds true implies the known L^p result for all $\lambda>0$ only for the range $2\le p\le 8/3$. Moreover our estimate is interesting only for small values of λ. In fact for $\lambda>1/6$ M. Christ [6] showed that (1.1) holds with $W(x)=M_s[M_rw](x)$ where $r>\max\{2/(2\lambda+1);1\}$, $s>1$.

The proof of Theorem 1.1 relies on the method used in [3]; the improvement is achieved by using arguments along the lines of [13].

In what follows c and C will always be positive numbers which may assume different values in different formulas.

2. Weighted estimates for square-functions

Let $j > 10^5$ and let I be a fixed interval of length 2^{-j} contained in $[1/2, 2]$. Let $\psi_I \in C_0^\infty$ be supported in I and satisfy the estimates

$$\Big|\Big(\frac{d}{ds}\Big)^n \psi_I(s)\Big| \le C_n 2^{nj}. \tag{2.1}$$

Let $\eta \in C^\infty(\mathbb{R}^2)$ supported in $\{\xi \in \mathbb{R}^2 : |\xi_1| \le 10^{-2}\xi_2\}$ and define

$$\Psi(\xi) \;=\; \psi_I(|\xi|)\eta(\xi)$$

and an operator T_t by

$$\widehat{T_t f}(\xi) \;=\; \Psi(t^{-1}\xi)\widehat{f}(\xi).$$

We are going to derive weighted L^2 inequalities for the square-functions

$$\mathcal{G}^k f(x) \;=\; \Big(\int_{2^k}^{2^{k+1}} |T_t f(x)|^2 \frac{dt}{t}\Big)^{1/2}$$

and

$$\mathcal{G} f(x) \;=\; \Big(\int_0^\infty |T_t f(x)|^2 \frac{dt}{t}\Big)^{1/2}.$$

with suitable bounds depending on j.

Let $\gamma \in C_0^\infty(\mathbb{R})$ be supported in $(2^{-11}, 2^{11})$ such that $\gamma(t) = 1$ for $t \in (2^{-10}, 2^{10})$. Let $\phi \in C_0^\infty(\mathbb{R})$ be supported in $(-1, 1)$ such that

$$\sum_{L\in\mathbb{Z}} \phi(s - L) = 1, \qquad s \in \mathbb{R}.$$

For $l \le j/2$, $\tau \in \mathbb{Z}$, $|\tau| \le 2^{j/2-l}$ let

$$q_\tau^{jl}(\xi) \;=\; \gamma(2^{-l+j/2}|\xi|)\phi(2^{-l+j/2}\frac{\xi_1}{|\xi|} - \tau)$$

and define for $k \in \mathbb{Z}$ an operator $Q_\tau^{jl,k}$ by

$$\widehat{Q_\tau^{jl,k} g}(\xi) \;=\; q_\tau^{jl}(2^{-k}\xi)\widehat{g}(\xi).$$

In section 2 below we use the notation Q_τ^{jl} for $Q_\tau^{jl,0}$.

The multipliers $q_\tau^{jl}(2^{-k}\cdot)$ are supported in rectangles with a longer side of length $C2^{k+l-j/2}$ and a shorter side of length $C2^{k+2l-j}$; the longer side is parallel to the radial direction $\theta(\tau)$ where $\theta(\tau) = e_\tau/|e_\tau|$ with $e_\tau = (2^{l-j/2}, 1)$. The distance of these rectangles to the origin is $\approx 2^{k+l-j/2}$.

For $\theta \in S^1$ let $\theta^\perp$ be the unit vector the vector perpendicular to θ such that $\det(\theta, \theta^\perp) = 1$. Define

$$H_{\theta,N}^{jl}(x) = 2^{-l-3j/2}(1 + 2^{-l-j/2}|\langle x, \theta\rangle|)^{-N}(1 + 2^{-j}|\langle x, \theta^\perp\rangle|)^{-N}.$$

and for $k \in \mathbb{Z}$ the dilates

$$H_{\theta,N}^{jl,k}(x) = 2^{2k} H_{\theta,N}^{jl}(2^k x).$$

We shall always assume that $N \ge 100 + N_0$ where N_0 is the number in the definition of admissibility.

Now let

$$\mathcal{W}_j^k g(x) \;=\; \sup_\theta H_{\theta,N}^{j0,k} * |g|(x)$$

and for $l > 0$

$$\mathcal{W}_j^{l,k} g(x) \;=\; \sup_\tau \big(H_{\theta(\tau),N}^{jl,k} * |(Q_\tau^{jl,k})^* g|^2(x)\big)^{1/2}$$

Theorem 2.1. *For all $k \in \mathbb{Z}$, for all Schwartz-functions f, for all admissible weights w*

$$\int |\mathcal{G}^k f(x)|^2 w(x)\, dx \le C 2^{-j} \int |f(x)|^2 W_j^k w(x) dx \tag{2.2}$$

where

$$W_j^k w(x) = \mathcal{W}_j^k w(x) + \sqrt{j}\Big(\sum_{0<l<j/2} |\mathcal{W}_j^{l,k} w(x)|^2\Big)^{1/2}.$$

Moreover if $s > 1$ then

$$\int |\mathcal{G} f(x)|^2 w(x)\, dx \le C 2^{-j} \int |f(x)|^2 M_s[\sup_k W_j^k w](x) dx. \tag{2.3}$$

The mapping properties of W_j^k are contained in

Proposition 2.2. *The inequalities*

$$\begin{aligned} \|W_j^k g\|_q &\le C(1+j)^{1-1/q}\|g\|_q \\ \|\sup_k W_j^k g\|_q &\le C(1+j)\|g\|_q \end{aligned} \tag{2.4}$$

hold for $2 \le q \le \infty$; here C does not depend on q, j or k. Moreover there is the pointwise estimate

$$\sup_k |W_j^k g(x)| \le Cj(\mathfrak{M}_{j/2}|g|^2)^{1/2}. \tag{2.5}$$

We note that (2.2), (2.4) and a duality argument imply the sharp L^4 estimate for the square-function $\mathcal{G}^k$, namely

$$\|\mathcal{G}^k f\|_4 \le C(1+j)^{1/4} 2^{-j/2}\|f\|_4.$$

This estimate implies the known bound $\|T_t\|_{L^4 \to L^4} = O(j^{1/4})$, obtained by Córdoba [7]. The sharpness of the L^4 estimate for $\mathcal{G}^k$ follows from the sharpness of Córdoba's estimate. For earlier related results on $\mathcal{G}^k$ see [2], [3].

Remark 2.3. The estimates (2.2) and (2.3) remain true if $W_j^k w$ is replaced by

$$W_{j,p}^k w = \mathcal{W}_j^k w + \sqrt{j}\Big(\sum_{0<l<j/2} |2^{4l(1/p-1/2)}\mathcal{W}_{j,p}^{l,k} w|^2\Big)^{1/2}$$

where

$$\mathcal{W}_{j,p}^{l,k} g(x) = \sup_\tau \big(H_{\theta(\tau),N}^{jl,k} * |(Q_\tau^{jl,k})^* g|^p(x)\big)^{1/p}$$

and $1 \le p \le 2$. The proof of this assertion will be given below.

Standard arguments ([15], [3], [16]) relating maximal operators to square functions can be used to deduce Theorem 1 from the above results. Namely let $\widetilde{\Psi}(\xi) = \langle \xi, \nabla\Psi(\xi)\rangle$ and and let $\widetilde{T}_t$ be the convolution operator with Fourier multiplier $\widetilde{\Psi}(t^{-1}\cdot)$; then $2^{-j}\widetilde{T}_t$ satisfies the same quantitative properties as T_t. Using [16, p.499] one obtains the estimate

$$\begin{aligned} \sup_{t>0} |T_t f(x)| &\le \Big(\sum_{k\in\mathbb{Z}} \sup_{1\le s\le 2} |T_{2^k s} f|^2\Big)^{1/2} \\ &\le \Big(\sum_{k\in\mathbb{Z}} \Big[2^{j/2}\Big(\int_1^2 |T_{2^k s} f|^2 ds\Big)^{1/2} + 2^{-j/2}\Big(\int_1^2 |\frac{d}{ds} T_{2^k s} f|^2 ds\Big)^{1/2}\Big]^2\Big)^{1/2} \\ &\le C\Big[2^{j/2}\Big(\int_0^\infty |T_t f|^2 \frac{dt}{t}\Big)^{1/2} + 2^{-j/2}\Big(\int_0^\infty |\widetilde{T}_t f|^2 \frac{dt}{t}\Big)^{1/2}\Big] \end{aligned}$$

and therefore

$$\int \sup_{t>0} |T_t f(x)|^2 w(x)dx \le C \int |f(x)|^2 M_s(\sup_k W_j^k w)(x)dx.$$

Now $S_t^\lambda = \sum_{j=0}^\infty 2^{-j\lambda} S_{j,t}$ where $\sup_{t>0} |S_{j,t}f|$ is pointwise bounded by $2^j Mf$ and where for $j \ge 10^5$ the Fourier multiplier for $S_{j,t}$ is $C\psi_I(t^{-1}\cdot)$ with a suitable ψ_I satisfying the bounds (2.1). Therefore Theorem 2.1, Remark 2.3 and a weighted inequality for M due to Fefferman and Stein ([16, p. 53]) imply that (1.1) holds with $W = M_s(W_{\varepsilon,p}w)$, $s > 1$, where

$$W_{\varepsilon,p}w(x) = Mw(x) + \sum_{j\ge 10^5} 2^{-j\varepsilon} \sup_k \widetilde{W}_{j,p}^k w(x).$$

Here $1 \le p \le 2$, $\varepsilon < 2\lambda$ and $\widetilde{W}_{j,p}^k w$ is a sum of less than 10^6 operators satisfying the same quantitative estimates as the operator $W_{j,p}^k$ in Remark 2.3 (they are essentially rotates of this operator). By Proposition 2.2 the operator $W_{\varepsilon,2}$ is bounded on L^q, for $2 < q \le \infty$. An examination of the operators $\widetilde{W}_{j,p}^k$ and an interpolation argument show that for $1 < p < 2$

$$\| \sup_k \widetilde{W}_{j,p}^k w\|_p \le C 2^{ja(1-2/p)} \|w\|_p$$

for some $a > 0$. This implies that given $\varepsilon > 0$ there is $p < 2$ and $\delta(\varepsilon, p) > 0$ such that $W_{\varepsilon,p}$ is bounded on L^q for $2 - \delta(\varepsilon, p) \le q \le \infty$. Theorem 1 follows by choosing $0 < \varepsilon < 2\lambda$.

Before proceeding with the proof of Theorem 2.1 we state without proof a lemma which is closely related to a theorem of Carleson concerning square-functions with equally spaced decompositions. For published proofs see [7], [12] (and also [13] for a simple proof based on Bernstein's Theorem).

Lemma 2.4. *Let $\{Q_l\}$ be a sequence of disjoint unit cubes and let m_l be supported in Q_l; moreover assume that the estimates*

$$\int |\partial_\xi^\alpha m_l(\xi)|^2 d\xi \le B^2$$

hold for all multiindices $|\alpha| \le N$, uniformly in l. Let $A \in GL(2, \mathbb{R})$. Then

$$\sum_l \big|\mathcal{F}^{-1}[m_l(A\cdot)\mathcal{F}f](x)]\big|^2 \le C_N B^2 \int |f(y)|^2 \frac{|\det A|^{-1}}{1 + |{}^tA^{-1}(x-y)|^{2N}}\, dy.$$

We now fix $j \ge 10$. In what follows we shall introduce various decompositions depending on j without always indicating the dependence on j. Consequently we shall also omit the index j in $H_{\tau,N}^{jkl}$ or Q_τ^{jkl}. Various constants C in inequalities may depend on N.

Proof of Theorem 2.1. Denote by δ_k the dilation operator given by $\delta_k f(x) = f(2^{-k}x)$. Then $\mathcal{G}_k = \delta_{-k}\mathcal{G}_0\delta_k$ and $W_j^k = \delta_{-k}W_j^0\delta_k$. A scaling argument shows that in order to prove (2.2) it suffices to prove (2.2) for $k = 0$ which is henceforth assumed.

For $m \in \mathbb{Z}$, define operators P^m by

$$\widehat{P^m f}(\xi) = \phi(2^j|\xi| - m)\widehat{f}(\xi)$$

so that the $\widehat{P^m f}$ are supported in thin annuli of width 2^{-m+1} and $\sum P^m f = f$. Observe that for fixed t there are at most three m such that $P^m T_t \ne 0$. Therefore

$$\int |\mathcal{G}^0 f(x)|^2 w(x)\, dx = \iint_{\mathbb{R}^2\times[1,2]} \Big|\sum_m P^m T_t f(x)\Big|^2 w(x) dx \frac{dt}{t}$$

$$\le C \iint_{\mathbb{R}^2\times[1,2]} \sum_m \big|P^m T_t f(x)\big|^2 w(x) dx \frac{dt}{t}. \tag{2.6}$$

Define for $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$, $|\nu_1|, |\nu_2| \le 2^{j/2}$

$$\widehat{P_\nu f}(\xi) \;=\; \phi(2^{j/2}\xi_1 - \nu_1)\phi(2^{j/2}\xi_2 - \nu_2)\widehat{f}(\xi)$$
$$\widehat{P_\nu^m f}(\xi) \;=\; \phi(2^{j/2}\xi_1 - \nu_1)\phi(2^{j/2}\xi_2 - \nu_2)\widehat{P^m f}(\xi).$$

Thus $\widehat{P_\nu f}$ is supported in a square of sidelength $2^{1-j/2}$, and $\widehat{P_\nu^m f}$ is supported in the intersection of such a square with a thin annulus of width 2^{1-j}; therefore it is supported in a rectangle of dimensions $C2^{-j/2} \times C2^{-j}$.

Moreover for fixed l and for $\mu \in \mathbb{Z}$ we define operators $P_{\nu\mu}^{ml}$ by

$$\widehat{P_{\nu\mu}^{ml} f}(\xi) \;=\; \phi(2^{l+j/2}\xi_1 - \mu)\widehat{P_\nu^m f}(\xi).$$

so that $\widehat{P_{\nu\mu}^{ml} f}$ is supported in a smaller rectangle of dimensions $C2^{-l-j/2} \times C2^{-j}$.

Finally define operators $B_{\tau\sigma\rho}^l$ by

$$\widehat{B_{\tau\sigma\rho}^l f}(\xi) \;=\; b_{\tau\sigma\rho}^l(\xi)\widehat{f}(\xi)$$

where

$$b_{\tau\sigma\rho}^l(\xi) \;=\; q_\tau^{jl}(\xi)\phi(2^{l+j/2}\langle \xi, \theta(\tau)\rangle - \rho)\phi(2^j\langle \xi, \theta^\perp(\tau)\rangle - \sigma)$$

here q_τ^{jl} and the corresponding operator Q_τ^{jl} were defined above.

Let

$$\mathcal{Z}_l = \{(\nu, \nu') \in \mathbb{Z}^2 \times \mathbb{Z}^2 : 2^l \le |\nu_1 - \nu_1'| < 2^{l+1}\}$$

and for $\kappa \in \mathbb{Z}$, $|\kappa| \le 2^{j/2-\ell}$ let

$$\mathfrak{Z}_{l\kappa} \;=\; \{\nu : 2^l(\kappa - 1) < \nu \le 2^l\kappa\}.$$
$$\mathfrak{A}_{l\kappa} \;=\; \{\mu : 2^{2l}(\kappa - 4) \le \mu \le 2^{2l}(\kappa + 4)\}.$$

We shall use the following elementary geometrical facts (2.7-2.15), assuming $l \ge 10$ in what follows.

(2.7) For each m, ν_1 there are at most three ν_2 such that $P_\nu^m \ne 0$ (here $\nu = (\nu_1, \nu_2)$).

(2.8) For each μ there are at most nine ν such that $P_{\nu\mu}^{ml} \ne 0$.

(2.9) If $\nu \in \mathfrak{Z}_{l\kappa}$, $(\nu, \nu') \in \mathcal{Z}_l$ then $\nu' \in \mathfrak{Z}_{l\kappa'}$ with $|\kappa - \kappa'| \le 1$.

(2.10) If $\nu \in \mathfrak{Z}_{l\kappa'}$, with $|\kappa - \kappa'| \le 1$ and if $P_{\nu\mu}^{ml} \ne 0$ then $\mu \in \mathfrak{A}_{l\kappa}$.

(2.11) For each $\mu \in \mathfrak{A}_{l\kappa}$ the support of $\widehat{P_{\nu\mu}^{ml} f}$ is contained in a rectangle R_μ^l with sidelengths $C_1 2^{-l-j/2}$ and $C_1 2^{-j}$ where the orientation of R_μ^l only depends on κ; the longer side is parallel to $u_\kappa = (-1, 2^{l-j/2}\kappa)$. The rectangle is contained in the annulus $\{\xi : \big||\xi| - 2^{-j}m\big| \le C2^{-j}\}$ (here $2^{-j}m \in [1/2, 2]$). The differentiability properties of the multiplier corresponding to $P_{\nu\mu}^{ml}$ satisfy the same bounds as a bump function adapted to R_μ^l.

(2.12) For each μ, μ' the set

$$\text{supp } (\widehat{P^l_{\nu\mu} f}) - \text{supp } (\widehat{P^l_{\nu'\mu'} f})$$

is contained in a rectangle $\widetilde{R}^l_{\mu\mu'}$ with sidelengths $C_2 2^{-l-j/2}$ and $C_2 2^{-j}$. The rectangle $\widetilde{R}^l_{\mu\mu'}$ is contained in an annulus $\{\xi : c_3 2^{l-j/2} \le |\xi| \le C_3 2^{l-j/2}\}$. The longer side is parallel to $u_\kappa = (-1, 2^{l-j/2}\kappa)$.

(2.13) Fix m, l. Then there is a constant C_4, independent of l, m such that each $\xi \in \mathbb{R}^2$ is contained in at most C_4 of the sets supp $(\widehat{P^{ml}_{\nu\mu} f})$ − supp $(\widehat{P^{ml}_{\nu'\mu'} f})$.

(2.14) Let $p^{ml}_\mu(\xi) = \phi(2^{l+j/2}\xi_1 - \mu)\phi(2^j|\xi| - m)$. Let

$$\mathfrak{S}^{ml\tau}_{\mu\mu'} = \{(\sigma, \rho) : \big(\text{supp } p^{ml}_\mu - \text{supp } p^{ml}_{\mu'}\big) \cap \text{supp } b^l_{\tau\sigma\rho} \ne \emptyset\}$$

and $\mathfrak{S}^{ml}_{\mu\mu'} = \cup_\tau \mathfrak{S}^{ml\tau}_{\mu\mu'}$. Then the cardinality of $\mathfrak{S}^{ml}_{\mu\mu'}$ is bounded, independently of m, l, μ, μ'. Likewise, if

$$\mathfrak{R}^{ml\kappa}_{\tau\sigma\rho} = \{(\mu, \mu') \in \mathfrak{A}_{l\kappa} \times \mathfrak{A}_{l\kappa} : \big(\text{supp } p^{ml}_\mu - \text{supp } p^{ml}_{\mu'}\big) \cap \text{supp } b^l_{\tau\sigma\rho} \ne \emptyset \text{ for some } \tau\}$$

and if $\mathfrak{R}^{ml}_{\tau\sigma\rho} = \cup_\kappa \mathfrak{R}^{ml\kappa}_{\tau\sigma\rho}$ then the cardinality of $\mathfrak{R}^{ml}_{\tau\sigma\rho}$ is bounded, independently of m, l, σ, ρ.

(2.15) The cardinality of the set

$$\mathfrak{T}^l_\kappa = \{\tau : \mathfrak{R}^{ml\kappa}_{\tau\sigma\rho} \ne \emptyset \text{ for some } (\sigma, \rho, m)\}$$

is bounded, independently of l.

For fixed m we now write

$$\begin{aligned} &\iint \Big| P^m T_t f(x) \Big|^2 w(x) dx \frac{dt}{t} \\ &= \iint \Big| \sum_\nu P^m_\nu T_t f(x) \Big|^2 w(x) dx \frac{dt}{t} \\ &= \iint \sum_{\nu, \nu'} P^m_\nu T_t f(x) \, \overline{P^m_{\nu'} T_t f(x)} \, w(x) dx \frac{dt}{t}. \end{aligned}$$

Then

$$\iint \sum_{\nu,\nu'} P^m_\nu T_t f(x) \, \overline{P^m_{\nu'} T_t f(x)} \, w(x) dx \frac{dt}{t} \le C \Big[J^m + \sum_{10 \le l \le j/2} |I^m_l| \Big] \tag{2.16}$$

where

$$J^m = \iint \sum_\nu \Big| P^m_\nu T_t f(x) \Big|^2 w(x) dx \frac{dt}{t}$$

and

$$I^m_l = \iint \sum_{(\nu,\nu') \in \mathcal{Z}_l} P^m_\nu T_t f(x) \, \overline{P^m_{\nu'} T_t f(x)} \, w(x) dx \frac{dt}{t}.$$

By (2.9)

$$I_l^m = \iint \sum_{\substack{(\kappa,\kappa')\\ |\kappa-\kappa'|\le 1}} \sum_{\substack{(\nu,\nu')\in\mathcal{Z}_l\\ \nu\in\mathfrak{Z}_{l\kappa}}} P_\nu^m T_t f(x)\, \overline{P_{\nu'}^m T_t f(x)}\, w(x) dx \frac{dt}{t}.$$

Therefore by (2.10)

$$I_l^m = \sum_\kappa I_{l\kappa}^m$$

where

$$I_{l\kappa}^m = \iint \sum_{\substack{(\nu,\nu')\in\mathcal{Z}_l\\ \nu\in\mathfrak{Z}_{l\kappa}}} \sum_{\substack{(\mu,\mu')\in\\ \mathfrak{A}_{l\kappa}\times\mathfrak{A}_{l\kappa}}} P_{\nu\mu}^{ml} T_t f(x) \overline{P_{\nu'\mu'}^{ml} T_t f(x)} w(x) dx \frac{dt}{t}.$$

Now we can write

$$I_{l\kappa}^m = \iint \sum_{\tau,\rho,\sigma} \sum_{\substack{(\nu,\nu')\in\mathcal{Z}_l\\ \nu\in\mathfrak{Z}_{l\kappa}}} \sum_{\substack{(\mu,\mu')\in\\ \mathfrak{A}_{l\kappa}\times\mathfrak{A}_{l\kappa}}} B_{\tau\sigma\rho}^l \big[P_{\nu\mu}^{kml} T_t f\, \overline{P_{\nu'\mu'}^{kml} T_t f}\big](x) w(x) dx \frac{dt}{t}$$

and we obtain using (2.7), (2.8), (2.14) and (2.15) together with various applications of the Cauchy-Schwarz inequality

$$\begin{aligned}
I_{l\kappa}^m &\le \sum_{\tau\in\mathfrak{T}_\kappa^l} \iint \sum_{\substack{(\nu,\nu')\in\mathcal{Z}_l\\ \nu\in\mathfrak{Z}_{l\kappa}}} \sum_{\substack{(\mu,\mu')\in\\ \mathfrak{A}_{l\kappa}\times\mathfrak{A}_{l\kappa}}} P_{\nu\mu}^{ml} T_t f(x)\, \overline{P_{\nu'\mu'}^{ml} T_t f(x)} \sum_{\substack{(\sigma,\rho)\\ \in\mathfrak{S}_{\mu\mu'}^{ml}}} (B_{\tau\sigma\rho}^l)^* w(x) dx \frac{dt}{t} \\
&\le \iint \Big(\sum_{\substack{(\mu,\mu')\in\\ \mathfrak{A}_{l\kappa}\times\mathfrak{A}_{l\kappa}}} \Big| \sum_{\substack{(\nu,\nu')\in\mathcal{Z}_l\\ \nu\in\mathfrak{Z}_{l\kappa}}} P_{\nu\mu}^{ml} T_t f(x)\, \overline{P_{\nu'\mu'}^{ml} T_t f(x)} \Big|^2 \Big)^{1/2} \\
&\qquad\qquad \times \Big(\sum_{\substack{(\mu,\mu')\in\\ \mathfrak{A}_{l\kappa}\times\mathfrak{A}_{l\kappa}}} \Big| \sum_{\tau\in\mathfrak{T}_\kappa^l} \sum_{\substack{(\sigma,\rho)\\ \in\mathfrak{S}_{\mu\mu'}^{ml}}} (B_{\tau\sigma\rho}^l)^* w(x) \Big|^2 \Big)^{1/2} dx \frac{dt}{t} \\
&\le C \iint \Big(\sum_{\substack{(\mu,\mu')\in\\ \mathfrak{A}_{l\kappa}\times\mathfrak{A}_{l\kappa}}} \Big| \sum_{\substack{(\nu,\nu')\in\mathcal{Z}_l\\ \nu\in\mathfrak{Z}_{l\kappa}}} |P_{\nu\mu}^{ml} T_t f(x)\, \overline{P_{\nu'\mu'}^{ml} T_t f(x)}|^2 \Big)^{1/2} \\
&\qquad\qquad \times \Big(\sum_{\tau\in\mathfrak{T}_\kappa^l} \sum_{\sigma,\rho} \sum_{\substack{(\mu,\mu')\\ \in\mathfrak{R}_{\tau\sigma\rho}^{ml}}} \big|(B_{\tau\sigma\rho}^l)^* w(x)\big|^2 \Big)^{1/2} dx \frac{dt}{t} \\
&\le C \iint \sum_{\nu\in\mathfrak{Z}_{l\kappa}} \sum_{\mu\in\mathfrak{A}_{l\kappa}} \big|P_{\nu\mu}^{ml} T_t f\big|^2 \sup_{\tau\in\mathfrak{T}_\kappa^l} \Big(\sum_{\sigma,\rho} \big|(B_{\tau\sigma\rho}^l)^* w(x)\big|^2 \Big)^{1/2} dx \frac{dt}{t}.
\end{aligned}$$

Now an application of Lemma 2.4 yields

$$\Big(\sum_{\sigma,\rho} \big|(B_{\tau\sigma\rho}^l)^* w(x)\big|^2 \Big)^{1/2} \le C \big(H_{\tau,4N}^l * |(Q_\tau^{jl})^* w|^2(x) \big)^{1/2}. \tag{2.17}$$

We summarize:

Lemma 2.5.

$$\int |\mathcal{G}^0 f(x)|^2 w(x)dx \;\le\; C\sum_m J^m + C\sum_m \sum_{10\le l\le j/2} |I_l^m|$$

where

$$J^m \;=\; \int \sum_\nu \int \big|P_\nu^m T_t f(x)\big|^2 \frac{dt}{t}\, w(x)dx$$

and

$$|I_l^m| \le C\int \sum_\kappa \sum_{\nu\in\mathfrak{Z}_{l\kappa}} \sum_{\mu\in\mathfrak{A}_{l\kappa}} \int \big|P_{\nu\mu}^{ml} T_t f(x)\big|^2 \frac{dt}{t} \sup_{\tau\in\mathfrak{T}_\kappa^l} \big(H_{\tau,4N}^l * |(Q_\tau^{jl})^* w|^2(x)\big)^{1/2} dx$$

We continue with the proof of Theorem 2.1. One checks by a straightforward integration by parts using (2.11) that if $\mu\in\mathfrak{A}_{l\kappa}$ and $\tau\in\mathfrak{T}_\kappa^l$ then

$$|\mathcal{F}^{-1}[\psi_I(t^{-1}\cdot)p_\mu^{ml}](x)| \;\le\; C H_{\tau,4N}^{kl}(x),$$

here $N\gg 2$ and p_μ^{ml} was defined in (2.14). Moreover the $t^{-1}dt$ measure of the set $\{t: P^m T_t \ne 0\}$ is bounded by $c2^{-j}$. Also observe that for $\tau\in\mathfrak{T}_\kappa^l$ the kernels $H_{\tau',4N}^l$ behave essentially the same; in fact

$$\sup_{\tau'\in\mathfrak{T}_\kappa^l} H_{\tau',4N}^l(x) \;\le\; C \inf_{\tau\in\mathfrak{T}_\kappa^l} H_{\tau,4N}^l(x). \tag{2.18}$$

An application of Lemma 2.4 yields

$$\int \sum_m \sum_{\nu\in\mathfrak{Z}_{l\kappa}} \sum_{\mu\in\mathfrak{A}_{l\kappa}} \int |P_{\nu\mu}^{ml} T_t f(x)|^2 \frac{dt}{t} w(x)dx \;\le\; C2^{-j}\int \sum_{\nu\in\mathfrak{Z}_{l\kappa}} |P_\nu f(x)|^2 H_{\tau,3N}^l * w(x)\,dx \tag{2.19}$$

for all $\tau\in\mathfrak{T}_\kappa^l$.

Next, the convolution kernel of P_ν is pointwise bounded by

$$\widetilde{H}_{j/2,3N}(x) \;=\; 2^{2(j/2)}(1+2^{j/2}|x|)^{-3N}$$

and by another application of Lemma 2.4 we obtain the inequality

$$\int \sum_\kappa \sum_{\nu\in\mathfrak{Z}_{l\kappa}} |P_\nu f(x)|^2 w(x)\,dx \;\le\; C\int |f(x)|^2 \widetilde{H}_{j/2,3N} * w(x)dx. \tag{2.20}$$

Since for large N

$$H_{\tau,4N}^l * \widetilde{H}_{j/2,4N}(x) \;\le\; C H_{\tau,3N}^l(x)$$

we obtain from (2.18), (2.19), (2.20) and Lemma 2.5 the estimate

$$\begin{aligned}
&\sum_{10\le l\le \frac{j}{2}} \sum_m I_l^m \\
&\le\; C2^{-j}\int |f(x)|^2 \sum_{10\le l\le\frac{j}{2}} \sup_{\substack{2^{l-j/2}\tau\\ \in[16^{-1},16]}} H_{\tau,3N}^l * \big(H_{\tau,3N}^l * |(Q_\tau^{jl})^* w(y)|^2\big)^{1/2}(x)\,dx \\
&\le\; C2^{-j}\int |f(x)|^2 \sqrt{j}\Big(\sum_{10\le l\le\frac{j}{2}} |\mathcal{W}_j w|^2\Big)^{1/2} dx
\end{aligned}$$

Similarly

$$\sum_m J^m \le C2^{-j} \int |f(x)|^2 \sup_\theta H^0_{\theta,N} * w(x)\, dx$$

and we obtain (2.2).

Finally let $\beta \in C_0^\infty(\mathbb{R}^2)$ such that $\beta(\xi) = 1$ if $1/4 \le |\xi| \le 4$ and $\beta(\xi) = 0$ if $|\xi| \notin (1/8, 8)$ and define the Littlewood-Paley operator L^k by

$$\widehat{L^k f}(\xi) = \beta(2^{-k}\xi)\widehat{f}(\xi).$$

Observe that

$$\mathcal{G}^k f = \mathcal{G}^k(L^{k-1}f + L^k f + L^{k+1}f).$$

Now $f \mapsto \{L^k f\}$ defines a vector-valued regular singular integral operator and there is the Córdoba-Fefferman weighted norm inequality

$$\int \sum_k |L^k f(x)|^2 w(x)\, dx \le C_s \int |f(x)|^2 M_s w(x) dx, \qquad s > 1,$$

see [10]. Consequently (2.2) yields also (2.3). □

Proof of Remark 2.3. This requires a modification of (2.17). Let

$$\widehat{\Gamma^l_{\tau\rho\sigma} f}(\xi) = \phi(2^{l+j/2}\langle \xi, \theta(\tau)\rangle - \rho)\phi(2^j\langle \xi, \theta^\perp(\tau)\rangle - \sigma)\widehat{f}(\xi);$$

then by definition $\Gamma^l_{\tau\rho\sigma} Q^{jl}_\tau = B^l_{\tau\sigma\rho}$. Now let $\mathfrak{U}^l_\tau$ be the set of all pairs (ρ, σ) such that $\Gamma^l_{\tau\rho\sigma} Q^{jl}_\tau \ne 0$. Then the cardinality of $\mathfrak{U}^l_\tau$ is bounded by $C2^{4l}$ where C does not depend on j or τ. Since the convolution kernel of $\Gamma^l_{\tau\rho\sigma} Q^{jl}_\tau$ is bounded by $CH^l_{\tau,N}$ we obtain

$$\Big(\sum_{(\rho,\sigma)\in\mathfrak{U}^l_\tau} \big|\Gamma^l_{\tau\rho\sigma} g(x)\big|^2\Big)^{1/2} \le C2^{2l} H^l_{\tau,N} * |g|(x).$$

An application of Lemma 2.4 and an interpolation argument then show that (2.17) can be replaced by

$$\Big(\sum_{\sigma,\rho} \big|(B^l_{\tau\sigma\rho})^* w(x)\big|^2\Big)^{1/2} \le C2^{4l(1/p-1/2)}\big(H^l_{\tau,N} * |Q^{jl}_\tau w|^p(x)\big)^{1/p} \tag{2.21}$$

if $1 \le p \le 2$. The rest of the proof requires only notational changes. □

Proof of Proposition 2.2. The convolution kernel of $Q^{jl,k}_\tau$ is bounded by a constant C_N times

$$\mathcal{H}^{l,k}_{\theta(\tau),N}(x) = 2^{2k+3l-3j/2}(1 + 2^{k+l-j/2}|\langle x, \theta(\tau)\rangle|)^{-N}(1 + 2^{k+2l-j}|\langle x, \theta(\tau)^\perp\rangle|)^{-N}$$

Moreover a straightforward calculation shows that

$$H^{l,k}_{\theta,N} * |\mathcal{H}^{l,k}_{\theta,N} * w|^2 \le C\mathfrak{M}_{j/2}[w^2]$$

pointwise which implies (2.5). For the same reason $\mathcal{W}_j^{kl}$ is uniformly bounded on L^∞ and the analogue of the inequality

$$\Big\|\Big(\sum_{0<l<j/2}|\mathcal{W}_j^{l,k}g|^2\Big)^{1/2}\Big\|_q \le Cj^{1/2-1/q}\|g\|_q \tag{2.22}$$

holds for $q=\infty$. On L^2 we use an orthogonality argument. Since for fixed k, j each $\xi\in R^2$ is contained in only a bounded number of the sets supp $\widehat{Q_\tau^{jl,k}g}$, $(0<l<j/2,\ |\tau|\le 2^{j/2-l})$ we obtain

$$\Big\|\Big(\sum_{0<l<j/2}|\mathcal{W}_j^{l,k}g|^2\Big)^{1/2}\Big\|_2 \le C\Big\|\Big(\sum_{0<l<j/2}\sum_\tau H^{l,k}_{\theta(\tau),N} * |(Q_\tau^{jl,k})^*g|^2\Big)^{1/2}\Big\|_2$$
$$\le C\Big(\sum_{0<l<j/2}\sum_\tau \|(Q_\tau^{jl,k})^*g\|_2^2\Big)^{1/2} \le C'\|g\|_2.$$

The desired bound (2.22) follows by interpolation.

In order to estimate the maximal function $\sup_k W_j^k g$ in L^q we dominate the sup by an ℓ^q norm and the asserted inequality follows from

$$\Big(\sum_k\Big\|\Big(\sum_{0<l<j/2}|\mathcal{W}_j^{l,k}g|^2\Big)^{1/2}\Big\|_q^q\Big)^{1/q} \le C(1+j)^{1/2}\|g\|_q. \tag{2.23}$$

Now the analogue of (2.23) for $q=\infty$ follows as before. For $q=2$ we use the observation that for fixed j, k the functions $q_\tau^{jl}(2^{-k}\cdot)$ are supported in an annulus $\{\xi : c_0 2^{k-j/2}\le|\xi|\le c_1 2^k\}$. Therefore the above L^2 argument yields now an additional factor of $\sqrt{1+j}$, proving (2.23) for $q=2$. □

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