AVERAGE DECAY ESTIMATES FOR FOURIER TRANSFORMS OF MEASURES SUPPORTED ON CURVES

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Abstract. We consider Fourier transforms $\widehat{\mu}$ of densities supported on curves in $\mathbb{R}^d$. We obtain sharp lower and close to sharp upper bounds for the decay rates of $\|\widehat{\mu}(R\cdot)\|_{L^q(S^{d-1})}$, as $R \to \infty$.

1. Introduction and Statement of Results

In this paper we investigate the relation between the geometry of a curve Γ in $\mathbb{R}^d$, $d > 2$, and the spherical L^q average decay of the Fourier transform of a smooth density μ compactly supported on Γ.

Let Γ be a smooth (C^∞) immersed curve in $\mathbb{R}^d$ with parametrization $t \to \gamma(t)$ defined on a compact interval I and let $\chi \in C^\infty$ be supported in the interior of I. Let $\mu \equiv \mu_{\gamma,\chi}$ be defined by

$$\langle \mu, f \rangle = \int f(\gamma(t))\chi(t)dt \tag{1.1}$$

and define by $\widehat{\mu}(\xi) = \int \exp(-i\langle \xi, \gamma(t)\rangle)\chi(t)dt$ its Fourier transform. For a large parameter R we are interested in the behavior of $\widehat{\mu}(R\omega)$ as a function on the unit sphere, in particular in the L^q norms

$$G_q(R) \equiv G_q(R; \gamma, \chi) := \Big(\int |\widehat{\mu}(R\omega)|^q d\omega \Big)^{1/q} \tag{1.2}$$

where $d\omega$ is the rotation invariant measure on S^{d-1} induced by Lebesgue measure in $\mathbb{R}^d$. The rate of decay depends on the number of linearly independent derivatives of the parametrization of Γ. Indeed if one assumes that for every t the derivatives $\gamma'(t)$, $\gamma''(t)$, ..., $\gamma^{(d)}(t)$ are linearly independent then from the standard van der Corput's lemma (see [20, page 334]) one gets $G_\infty(R) = \max_\omega |\widehat{\mu}(R\omega)| = O(R^{-1/d})$. If one merely assumes that at most $d-1$ derivatives are linearly independent then one cannot in general expect a decay of $G_\infty(R)$; one simply considers curves which lie in a hyperplane. However Marshall [15] showed that one gets an optimal estimate for the L^2 *average decay*, namely

$$G_2(R) = O(R^{-1/2}) \tag{1.3}$$

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as $R \to \infty$, for every compactly supported C^1 curve γ.

We are interested in estimates for the L^q average decay, for $2 < q < \infty$. If γ is a straight line such extensions fail, and additional conditions are necessary. Our first result addresses the case of nonvanishing curvature.

Theorem 1.1. *Suppose that for all $t \in I$ the vectors $\gamma'(t)$ and $\gamma''(t)$ are linearly independent. Then for $R \geq 2$*

(i)

$$G_q(R) \lesssim \begin{cases} R^{-1/2} (\log R)^{1/2-1/q} & \text{if } 2 \leq q \leq 4 \\ R^{-2/q} (\log R)^{1/q} & \text{if } 4 \leq q \leq \infty. \end{cases} \tag{1.4}$$

(ii) Suppose that there is $N \in \mathbb{N}$ so that for every $\omega \in S^{d-1}$ the function $s \mapsto \langle \omega, \gamma''(s) \rangle$ changes sign at most N times on I. Then

$$G_q(R) \lesssim R^{-1/2} \quad \text{if } 2 \leq q < 4 \tag{1.5}$$

and

$$G_q(R) \lesssim R^{-2/q} \quad \text{if } 4 < q \leq \infty. \tag{1.6}$$

Here and elsewhere the notation $a \lesssim b$ means $a \leq Cb$ for a suitable nonnegative constant C.

The L^4 estimate $G_4(R) = O(R^{-1/2}[\log R]^{1/4})$ is sharp even for nondegenerate curves, *cf.* Theorem 1.3 below. The estimate (1.5) is sharp and it is open whether for $q \neq 4$ there exists an example for which the logarithmic term in (1.4) is necessary.

The estimate (1.6) is sharp in the case where the curve lies in a two dimensional subspace. Under stronger nondegeneracy assumptions this estimate can be improved. In particular one is interested in the case of nondegenerate curves in $\mathbb{R}^d$, meaning that for all t the vectors $\gamma^{(j)}(t)$, $j = 1, \dots, d$, are linearly independent. In the case $d = 2$ we have of course the optimal bound $G_q(R) = O(R^{-1/2})$ for all $q \leq \infty$, by the well known stationary phase bound (for results for general curves in $\mathbb{R}^2$ and hypersurfaces in higher dimensions see [6] and references contained therein). The situation is more complicated for nondegenerate curves in higher dimensions, and Marshall [15] proved (essentially) optimal results for nondegenerate curves in $\mathbb{R}^d$ if $d = 3$ and $d = 4$.

We show that one gets close to optimal results for nondegenerate curves in all dimensions. Our method is different from the explicit computations in Marshall's paper and relies on a variable coefficient analogue of the Fourier restriction theorem due to Fefferman and Stein in two dimensions, see [11], and due to Drury [9] for curves in higher dimensions. The variable coefficient analogues are due to Carleson and Sjölin [7] (see also Hörmander [13]) in two dimensions and to Bak and Lee [3] in higher dimensions.

To formulate our result let, for $1 \le q \le \infty$,

(1.7)
$$\sigma_K(q) \equiv \sigma_K^d(q) = \begin{cases} \min_{\{k=2,\dots,d\}} \frac{1}{k} + \frac{k^2-k-2}{2kq}, & \text{for } K = d, \\ \min_{\{k=2,\dots,K\}} \left\{ \frac{1}{k} + \frac{k^2-k-2}{2kq},\ \frac{K}{q} \right\}, & \text{for } 2 \le K < d. \end{cases}$$

Theorem 1.2. *Suppose that for all $t \in I$ the vectors $\gamma'(t)$, ..., $\gamma^{(K)}(t)$ are linearly independent. Then for $R \ge 2$*

$$G_q(R) \le C_\sigma R^{-\sigma}, \quad \sigma < \sigma_K(q). \tag{1.8}$$

For integers $k \ge 1$ set

$$q_k := \frac{k^2+k+2}{2} \tag{1.9}$$

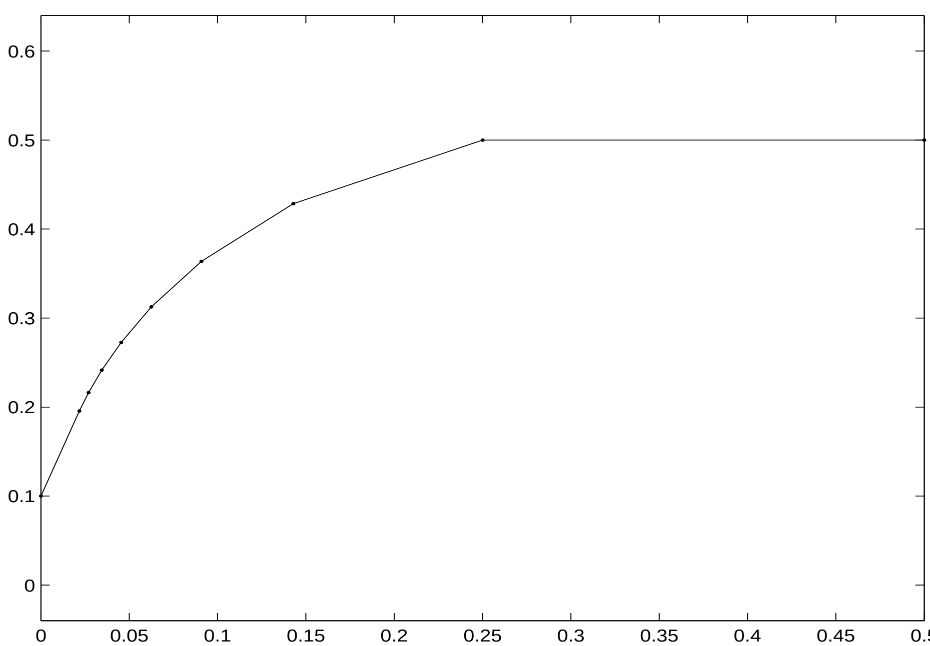
so that $q_1 = 2$, $q_2 = 4$, $q_3 = 7$, $q_4 = 11$. Observe that the set of points $(q^{-1}, \sigma_d^d(q))$, $q \ge 2$, is the broken line joining the points

$$(\frac{1}{q_1}, \frac{1}{q_1}), (\frac{1}{q_2}, \frac{2}{q_2}), \dots, (\frac{1}{q_k}, \frac{k}{q_k}), \dots, (\frac{1}{q_{d-1}}, \frac{d-1}{q_{d-1}}), (0, \frac{1}{d}),$$

while for $K < d$, the set of points $(q^{-1}, \sigma_K^d(q))$ is the concave broken line joining the points

$$(\frac{1}{q_1}, \frac{1}{q_1}), (\frac{1}{q_2}, \frac{2}{q_2}), \dots, (\frac{1}{q_k}, \frac{k}{q_k}), \dots, (\frac{1}{q_K}, \frac{K}{q_K}), (0, 0).$$

Furthermore observe that $\sigma_K^d(q) > 2/q$ if $3 \le K \le d$, and $q > 4$. The picture shows the graph $\{1/q, \sigma_K^K(q)\}$ as a function of $1/q$, for $K = 10$.



The graph $\{1/q, \sigma_{10}^{10}(q)\}$.

We emphasize that the graph of σ_K^d is slightly different for $d > K$, as then the left line segment connects (q_K^{-1}, Kq_K^{-1}) to $(0, 0)$.

Theorem 1.2 is sharp up only to endpoints, at least for nondegenerate curves (for which $\gamma'(t)$,..., $\gamma^{(d)}(t)$ are linearly independent), and also for some other cases where $K < d$, $\gamma'(t)$,..., $\gamma^{(K)}(t)$ are independent and γ lies in a K dimensional affine subspace. We note that for the case $K = d = 4$ Marshall [15] obtained the sharp bound $G_q(R) \lesssim R^{-\sigma_4^4(q)}$ when $4 < q < 7$ and $q > 7$; moreover $G_q(R) \lesssim R^{-\sigma_4^4(q)} \log^{1/q}(R)$ when $q = 4$ or $q = 7$ (the logarithmic term for the L^4 bound seems to have been overlooked in [15]).

We now state lower bounds for the average decay. The cutoff function χ is as in (1.1) (and $G_q(R)$ depends on χ).

Theorem 1.3. *Suppose that $2 \leq K \leq d$ and, for some $t_0 \in I$, the vectors $\gamma'(t_0)$, ..., $\gamma^{(K)}(t_0)$ are linearly independent. Then for suitable $\chi \in C_0^\infty$ there are $c > 0$, $R_0 \geq 1$ so that the following lower bounds hold for $R > R_0$.*

(i) If $2 \leq K \leq d - 1$ then

$$G_q(R) \geq cR^{-\sigma_K(q)}, \quad 2 < q < q_K; \tag{1.10}$$

moreover

$$G_q(R) \geq cR^{-\sigma_K(q)} \log^{1/q}(R), \quad q \in \{q_k : k = 2, \dots, K-1\}. \tag{1.11}$$

(ii) If $K = d$ then

$$G_q(R) \geq cR^{-\sigma_d(q)}, \quad 2 < q \leq \infty, \tag{1.12}$$

$$G_q(R) \geq cR^{-\sigma_d(q)} \log^{1/q}(R), \quad q \in \{q_k : k = 2, \dots, d-1\}. \tag{1.13}$$

(iii) If $2 \leq K \leq d - 1$ and, in addition, $\gamma^{(K+1)} \equiv 0$ then

$$G_q(R) \geq cR^{-\sigma_K(q)}, \quad 2 < q \leq \infty, \tag{1.14}$$

$$G_q(R) \geq cR^{-\sigma_K(q)} \log^{1/q}(R), \quad q \in \{q_k : k = 2, \dots, K\}. \tag{1.15}$$

Remark: A careful examination of the proof yields some uniformity in the lower bound. Assume that $\gamma^{(j)}(t_0) = e_j$, (the jth unit vector), $j = 1, \dots, K$, and $\|\gamma\|_{C^{K+3}(I)} \leq C_1$. Then there is $h = h(C_1) > 0$ so that for every smooth χ supported in $(-h, h)$ with Re $\chi(t) > c_1 > 0$ in $(-h/2, h/2)$ there exists an R_0 depending only on c_1, C_1, $\|\chi'\|_\infty$ and $\|\chi''\|_\infty$ so that the above lower bounds hold for $R \geq R_0$. We shall not pursue this point in detail.

Addendum: After the first version of this paper had been submitted we learned about the work of Arkhipov, Chubarikov and Karatsuba [1], [2] who proved sharp estimates for the $L^q(\mathbb{R}^d)$ norms of the Fourier transform of smooth densities on certain polynomial curves. We are grateful to Jong-Guk Bak who pointed out these references to us. The work of these authors shows that for, say $\gamma(t) = \sum_{k=1}^d t^k e_k$, $t \in [0, 1]$ the Fourier transform $\widehat{d\sigma}$ belongs to $L^q(\mathbb{R}^d)$ if and only if $q > q_d = (d^2 + d + 2)/2$. This result seems to have been overlooked until recently; it rules out an $L^{q_d'}$ endpoint bound for the Fourier restriction problem associated to curves, *cf.* a discussion in [16] and a remark in [3]). More can be said in two dimensions where the endpoint restricted weak type (4/3) inequality for the Fourier restriction operator is known to fail by a Kakeya set argument, see [4]. We note that the lower bound in chapter 2 of [2] is closely related to (1.12) and the method in [2] actually can be used to yield (1.12) for the curve $(t, ..., t^d)$ in the range $q \geq q_{d-1}$; vice versa one notices $\sigma_d(q_d) = d/q_d$ and integrates the lower bound for $R^{d-1}G_q^q(R)$ in R to obtain lower bounds for $\|\widehat{d\sigma}\|_{L^q(\mathbb{R}^d)}$.

A variant of an argument in [2] can be shown to close the ε gap between upper and lower bounds in some cases. We formulate one such result.

Theorem 1.4. *Suppose that γ is smooth and is either of finite type, or polynomial.*

Assume that $\gamma'(t), \dots, \gamma^{(K)}(t)$ are linearly independent, for every $t \in I$. Then the following holds:

(i) If $K = d$ then

$$G_q(R) \leq C_q R^{-\sigma_d(q)}, \quad q \geq 2, \quad q \notin \{q_k : k = 2, \dots, d-1\}, \tag{1.16}$$

and

$$G_q(R) \lesssim R^{-\sigma_d(q)} \log^{1/q}(R), \quad q \in \{q_k : k = 2, \dots, d-1\}. \tag{1.17}$$

(ii) If $2 \leq K \leq d-1$ then

$$G_q(R) \leq C_q R^{-\sigma_K(q)}, \quad q \geq 2, \quad q \notin \{q_k : k = 2, \dots, K\}. \tag{1.18}$$

and

$$G_q(R) \lesssim R^{-\sigma_K(q)} \log^{1/q}(R), \quad q \in \{q_k : k = 2, \dots, K\}. \tag{1.19}$$

It is understood that the implicit constants in (1.16) and (1.18) depend on q as $q \to q_k$. Note that in the finite type case (1.18) and (1.19) can be improved for $q > q_K$ since we have some nontrivial decay for $G_\infty(R)$. However, for the sharpness in the most degenerate case compare Theorem 1.3, part (iii).

Remark: The result of Theorem 1.4, for polynomial curves, could be used to obtain the upper bounds of Theorem 1.2, which involves a loss of R^ε, by a polynomial approximation argument. Note however, that such an argument requires upper bounds for derivatives of γ up to order $C + \varepsilon^{-1}$, as $\varepsilon \to 0$. An examination of the proof of Theorem 1.2 shows that one can get away with upper bounds for the derivatives up to order N where N depends on the dimension but not on ε.

Structure of the paper. In §2 we prove the estimates (1.5) and (1.6) which involve the assumption of $\langle \omega, \gamma''(s) \rangle$ not changing sign. Here we also discuss an application to some mixed norm inequalities for rotated measures. In §3 we prove Theorem 1.2 and (1.4). In §4 we give the proof of Theorem 1.4. In §5 we revisit some known asymptotic expansion with precise quantifications which are convenient for the proof of the lower bounds. The proof of the lower bounds of Theorem 1.3 is given in §6.

2. Upper bounds, I.

We shall now prove part (ii) of Theorem 1.1 (*i.e.* (1.5), (1.6)) under the less restrictive smoothness condition $\gamma \in C^2(I)$; we recall the assumptions that $\gamma'(t)$ and $\gamma''(t)$ are linearly independent and that we also require that the functions $s \mapsto \langle \omega, \gamma''(s)\rangle$ have at most a bounded number of sign changes on I. Note that this hypothesis is certainly satisfied if γ is a polynomial, or a trigonometric polynomial, or smooth and of finite type.

We need a result on oscillatory integrals which is a consequence of the standard van der Corput Lemma; it is also related to a more sophisticated statement on oscillatory integrals with polynomial phases in [18].

Let η be a C^∞ function with support in $(-1, 1)$ so that $\eta(s) = 1$ in $(-1/2, 1/2)$; we also assume that η' has only finitely many sign changes. Let $\eta_1(s) = \eta(s) - \eta(2s)$ (so that $1/4 \le |s| \le 1$ on $\operatorname{supp} \eta_1$) and let

$$\eta_l(s) = \eta_1(2^{l-1}s)$$

so that $2^{-l-1} \le |s| \le 2^{-l+1}$ on the support of η_l.

Lemma 2.1. *Let I be a compact interval and let $\chi \in C^1(I)$. Let $\phi \in C^2(I)$ and suppose that ϕ'' changes signs at most N times in I.*

Then, for $1 \le 2^l \le \lambda$,

$$\left| \int_I \eta_l(\phi'(s)) e^{i\lambda\phi(s)} \chi(s) ds \right| \le CN 2^l \lambda^{-1}.$$

Proof. We may decompose I into subintervals J_i, $1 \le i \le K$, $K \le 2N + 2$, so that both ϕ' and ϕ'' do not change sign in each J_i. Each interval J_i can be further decomposed into a bounded number of intervals $J_{i,k}$ so that $\eta'(\phi')$ is of constant sign in $J_{i,k}$. It suffices to estimate the integral $\mathcal{I}_{i,k}$ over $J_{i,k}$. By the standard van der Corput Lemma, the bound $\mathcal{I}_{i,k} = O(2^l/\lambda)$ follows if we can show that

$$\int_{J_{i,k}} \left| \partial_s \left(\eta_l(\phi'(s)) \chi(s) \right) \right| ds \le C$$

which immediately follows from

$$\int_{J_{i,k}} \left| 2^l \phi''(s) \eta_1'(2^l \phi'(s)) \right| ds \le C. \tag{2.1}$$

But by our assumption on the signs of ϕ', ϕ'', and η' the left hand side is equal to

$$\left| \int_{J_{i,k}} 2^l \phi''(s) \eta_1'(2^l \phi'(s)) ds \right| = \left| \int_{J_{i,k}} \partial_s \Big(\eta_l(\phi'(s)) \Big) ds \right| \le C.$$

□

Proof of (1.5) *and* (1.6). We may assume that Γ is parametrized by arc-length and that the support of χ is small (of diameter $\ll 1$). Determine the integer $M(R)$ by $2^M \le R < 2^{M+1}$. With η_l as above define for $l < M$

$$g_{R,l}(\omega) = \int e^{iR\langle\omega,\gamma(s)\rangle}\chi(s)\eta_l(\langle\omega,\gamma'(s)\rangle)ds$$

and for $l = M$ define $g_{R,M}$ similarly by replacing the cutoff $\eta_l(\langle\omega,\gamma'(s)\rangle)$ with $\eta(2^M\langle\omega,\gamma'(s)\rangle)$. We can decompose

$$\int e^{iR\langle\omega,\gamma(t)\rangle}\chi(t)dt = \sum_{l\le M} g_{R,l}(\omega)$$

and observe that $g_{R,l} = 0$ if $l \le -C$.

It follows from Lemma 2.1 that

$$\sup_{\omega\in S^{d-1}} |g_{R,l}(\omega)| \lesssim 2^l/R \tag{2.2}$$

We also claim that

$$\Big(\int |g_{R,l}(\omega)|^2 d\omega\Big)^{1/2} \lesssim \begin{cases} 2^l R^{-1} & \text{if } 2^l \le R^{1/2}, \\ 2^{-l}(1+\log(2^{2l}R^{-1}))^{1/2} & \text{if } 2^l \ge R^{1/2}. \end{cases} \tag{2.3}$$

Given (2.2) and (2.3) we deduce that

$$\begin{aligned} \|g_{R,l}\|_{L^q(S^{d-1})} &\le \|g_{R,l}\|^{2/q}_{L^2(S^{d-1})}\|g_{R,l}\|^{1-2/q}_{L^\infty(S^{d-1})} \\ &\lesssim \begin{cases} 2^l R^{-1} & \text{if } 2^l \le R^{1/2}, \\ 2^{l(1-4/q)}(1+\log(2^{2l}R^{-1}))^{1/q}R^{-1+2/q} & \text{if } 2^l \ge R^{1/2}. \end{cases} \end{aligned} \tag{2.4}$$

If $q \ne 4$ the asserted bound $O(R^{-2/q})$ bound follows by summing in l.

We now turn to the proof of (2.3). Note that (2.3) follows immediately from (2.2) if $2^l \le R^{1/2}$. Now let $2^l \ge R^{1/2}$. For the L^2 estimate in this range we shall just use the nonvanishing curvature assumption on Γ. We need to estimate the L^2 norm of $g_{R,l}$ over a small coordinate patch $\mathcal{V}$ on the sphere where we use a regular parametrization $y \to \omega(y)$, $y \in [-1,1]^{d-1}$; *i.e.*

$$\Big|\int u(y) g_{R,l}(\omega(y))\overline{g_{R,l}(\omega(y))}dy\Big| \lesssim 2^{-2l}(1+\log(2^{2l}R^{-1})), \tag{2.5}$$

where $u \in C_0^\infty$, so that $\omega(y) \in \mathcal{V}$ if $y \in \text{supp}(u)$. The left hand side of (2.5) can be written as

$$\begin{aligned} \mathcal{I}_l := \iint_{s_1,s_2}\int_y u(y)e^{iR\langle\omega(y),\gamma(s_1)-\gamma(s_2)\rangle}\chi(s_1)\overline{\chi(s_2)} \\ \times \eta(2^l\langle\omega(y),\gamma'(s_1)\rangle)\eta(2^l\langle\omega(y),\gamma'(s_2)\rangle)dy ds_1 ds_2 \end{aligned}$$

and we note that on the support of the amplitude we get that $\gamma'(s_i)$ is almost perpendicular to ω, *i.e.* we may assume by the assumption of small supports that there is a direction $w = (w_1, \dots, w_{d-1})$ so that

$$\Big|\sum_{\nu=1}^{d-1} w_\nu \partial_{y_\nu} \langle \omega(y), \gamma'(s)\rangle\Big| \ge 1/2$$

if $s \in \mathrm{supp}(\chi)$ and $y \in \mathrm{supp}(u)$. By a rotation in parameter space we may assume that

$$\big|\partial_{y_1} \langle \omega(y), \gamma'(s)\rangle\big| \ge 1/2 \quad \text{if } s \in \mathrm{supp}(\chi), \quad y \in \mathrm{supp}(u). \tag{2.6}$$

Now let for fixed unit vectors v_1, v_2 and $\delta > 0$

$$\mathcal{U}_\delta(v_1, v_2) = \{\omega \in S^{d-1} : |\langle \omega, v_1\rangle| \le \delta, |\langle \omega, v_2\rangle| \le \delta\}$$

and observe that the spherical measure of this region is at most $O(\delta)$; moreover this bound can be improved if $|v_1 - v_2|$ is $\ge \delta$. Namely if $\alpha(v_1, v_2)$ is the acute angle between v_1 and v_2 then

$$\mathrm{meas}(\mathcal{U}_\delta(v_1, v_2)) \lesssim \min\left\{\delta, \frac{\delta^2}{\sin \alpha(v_1, v_2)}\right\} \tag{2.7}$$

The condition (2.6) implies that

$$|\langle \partial_{y_1} \omega(y), \gamma(s_1) - \gamma(s_2)\rangle| \ge c|s_1 - s_2|$$

and given the regularity of the amplitude we can gain by a multiple integration by parts in y_1 provided that $|s_1 - s_2| \ge 2^l/R$; indeed we gain a factor of $O(R^{-1}|s_1 - s_2|^{-1}2^l)$ with each integration by parts. We obtain, for any N,

$$\mathcal{I}_l \lesssim \int_{s_1 \in \mathrm{supp}(\chi)} \bigg[\int_{|s_2 - s_1| \le c} \mathrm{meas}\,\big(\mathcal{U}_{2^{-l}}(\gamma'(s_1), \gamma'(s_2))\big) \times \min\{1, (R|s_1 - s_2|2^{-l})^{-N}\}\, ds_2\bigg]\, ds_1. \tag{2.8}$$

By the assumption that $|\gamma''(s)|$ is bounded below and γ' and γ'' are orthogonal we get as a consequence of (2.7)

$$\mathrm{meas}\,\big(\mathcal{U}_{2^{-l}}(\gamma'(s_1), \gamma'(s_2))\big) \le \min\{2^{-l}, 2^{-2l}|s_1 - s_2|^{-1}\}.$$

Now we use this bound and integrate out the s_2 integral in (2.8) and see that the main contribution comes from the region where $2^{-l} \le |s_1 - s_2| \le 2^l/R$ which yields the factor $\log(R2^{-2l})$ in (2.5). □

An application. We consider a C^2 curve $\gamma : [-a, a] \to \mathbb{R}^d$ with nonvanishing curvature and assume that, as in (1.5), the function $s \mapsto \langle \omega, \gamma''(s)\rangle$ has a bounded number of sign changes.

Let μ be the measure induced by the Lebesgue measure on Γ, multiplied by a smooth cutoff function. For every $\sigma \in SO\,(d)$ define μ_σ by $\mu_\sigma\,(E) = \mu\,(\sigma E)$ and for every test function f in $\mathbb{R}^d$

$$Tf\,(x, \sigma) = f * \mu_\sigma\,(x)\,.$$

We are interested in the $L^p(\mathbb{R}^d) \to L^s(SO(d), L^q(\mathbb{R}^d))$ mapping properties, in particular for $q = p' = p/(p-1)$. This question had been investigated in [19] for curves in the plane, with essentially sharp results in this case, see also [5]. The standard example, namely testing T on characteristic functions of balls of small radius yields the necessary condition $1 + (d-1)/q \geq d/p$. Setting $q = p'$ we see that the $L^p(\mathbb{R}^d) \to L^s(SO(d), L^{p'}(\mathbb{R}^d))$ fails for $p < (2d-1)/d$ (independent of s).

The approach in [19] together with the inequality (1.5) yields

$$\|Tf\|_{L^s(SO(d),L^{p'}(\mathbb{R}^d))} \leq C_p \|f\|_p, \quad p = \tfrac{2d-1}{d}, \quad s < \tfrac{4d-2}{d}. \tag{2.9}$$

Proof of (2.9). We imbed T in an analytic family of operators. After rotation and reparametrization (modfying the cutoff function) we may assume that $\gamma(t) = \sum_{j=1}^{d-1} \varphi_j(t) e_j + t e_d$, with $\varphi_j(0) = 0$. Let $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$ and define a distribution i_z by $\langle i_z, \chi \rangle = (\Gamma(z))^{-1} \int_0^{+\infty} \chi(t)\, t^{z-1} dt$. Then define μ_σ^z by $\widehat{\mu_\sigma^z}(\xi) = \widehat{\mu_\sigma}(\xi) \prod_{j=1}^d \widehat{i_z}(\langle \sigma\xi, e_j \rangle)$ and T^z by $T^z f(x,\sigma) = \mu_\sigma^z * f$. Following [19] one observes that $\mu_\sigma^{1+i\lambda}$ is a bounded function, namely we have

$$|\langle \mu^{1+i\lambda}, g \rangle| \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} |g(x_d e_d + \sum_{j=1}^{d-1} (y_j + \phi_j(x_d) e_j)| dy_1 \cdots dy_{d-1} dx_d \lesssim \|g\|_1$$

so that

$$T^{1+i\lambda} : L^1(\mathbb{R}^d) \to L^\infty(SO(d) \times \mathbb{R}^d). \tag{2.10}$$

We also have

$$T^{-\frac{1}{2d-2}+i\lambda} : L^2(\mathbb{R}^d) \to L^q(SO(d), L^2(\mathbb{R}^d)), \quad 2 \leq q < 4. \tag{2.11}$$

The implicit constants in both inequalities are at most exponential in λ. Thus we obtain the assertion (2.9) by analytic interpolation of operators.

To see (2.11) we observe that $\widehat{i_z}(\tau) = O(|\tau|^{-\operatorname{Re}(z)})$ and apply Plancherel's theorem and then Minkowski's integral inequality to bound for $\alpha > 0$

$$\begin{aligned} &\|T^{-\alpha+i\lambda} f\|_{L^q(L^2)}^2 \\ &= \Big(\int_{SO(d)} \Big(\int_{\mathbb{R}^d} |\widehat{f}(\xi) \widehat{\mu}_\sigma|^2 \prod_{j=1}^{d-1} |\widehat{i_{-\alpha+i\lambda}}(\langle \sigma\xi, e_j \rangle)|^2 d\xi \Big)^{q/2} d\sigma \Big)^{2/q} \\ &\lesssim \int_{\mathbb{R}^d} \big| \widehat{f}(\xi) \big|^2 \Big(\int_{SO(d)} \big| \widehat{\mu_\sigma}(\xi) \big|^q \prod_{j=1}^{d-1} |\langle \sigma\xi, e_j \rangle|^{\alpha q} d\sigma \Big)^{2/q} d\xi, \end{aligned}$$

and by (1.5) and the assumption $q < 4$ the last expression is dominated by a constant times

$$\int_{\mathbb{R}^d} \big| \widehat{f}(\xi) \big|^2 |\xi|^{2\alpha(d-1)} \Big(\int_{SO(d)} |\widehat{\mu_\sigma}(\xi)|^q d\sigma \Big)^{2/q} d\xi \lesssim \int_{\mathbb{R}^d} \big| \widehat{f}(\xi) \big|^2 |\xi|^{2\alpha(d-1)-1} d\xi.$$

For $\alpha = (2d-2)^{-1}$, this yields the bound (2.11). □

Remark. We do not know whether the index $s = \frac{4d-2}{d-1}$ in (2.9) is sharp. The following example only shows that we need $s \leqslant 10$ for $d = 3$. Let $\gamma(t) = \big(t, t^2, 0\big)$ and let χ_{B_δ} be a box centered at the origin with sides parallel to the axes and having sidelengths $1, 1$ and δ. A computation shows that $|T\chi_{B_\delta}(x,\sigma)| \geqslant c$ for σ in a set of measure ε^2 and x in a set of measure ε, for some small $\varepsilon > 0$. It follows that $p^{-1} \leq 2s^{-1} + q^{-1}$. For $p = 5/3$ and thus $p' = 5/2$ this yields $s \leq 10$.

3. Upper bounds, II

We are now concerned with the proof of Theorem 1.2 and the proof of part (i) of Theorem 1.1. For the latter we use a version of the Carleson-Sjölin theorem ([7], [13]), and for Theorem 1.2 we use a recent generalization due to Bak and Lee [3]. These we now recall.

Consider, for large positive R,

$$T_R f(x) = \int_{\mathbb{R}} e^{iR\phi(x,\,t)} a(x,\,t)\, f(t)\, dt$$

with real valued phase function $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R})$, and compactly supported amplitude $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. Assume the non-vanishing torsion condition

$$\det\big(\partial_t(\nabla_x\phi),\, \partial_t^2(\nabla_x\phi),\, \ldots,\, \partial_t^n(\nabla_x\phi)\big) \neq 0 \tag{3.1}$$

on the support of a. Then if $p^{-1} + n(n+1)(2q)^{-1} = 1$ and $q > (n^2+n+2)/2$ there is a constant C_q independent of f and of $R \geq 2$ such that

$$\|T_R f\|_{L^q(\mathbb{R}^n)} \leq C_q R^{-n/q} \|f\|_{L^p(\mathbb{R})}. \tag{3.2}$$

When $n = 2$ it is well known that a slight modification of Hörmander's proof ([13]) of the Carleson-Sjölin theorem gives the endpoint result

$$||T_R f||_{L^4(\mathbb{R}^n)} \lesssim R^{-1/2} \log^{1/4} R\, ||f||_{L^4(\mathbb{R})}\,; \tag{3.3}$$

see also [17], where a somewhat harder vector-valued analogue is proved.

In order to establish estimates (1.4) we need to show that under the assumption of linear independence of $\gamma'(t)$ and $\gamma''(t)$ (for each $t \in I$) that

$$G_4(R) \lesssim R^{-1/2}[\log R]^{1/4}. \tag{3.4}$$

To establish (1.8) under the assumption that the first K derivatives are linearly independent for every $t \in I$ we need to show that for any $2 \leq k \leq K$

$$G_q(R) \lesssim R^{-k/q}, \quad q > q_k, \tag{3.5}$$

where q_k is as in (1.9). All other estimates in (1.4), (1.8) follow by the usual convexity property of the L^p norm (*i.e.* $\|F\|_p \leq \|F\|_{p_0}^{1-\vartheta}\|F\|_{p_1}^{\vartheta}$ for $p^{-1} = (1-\vartheta)p_0^{-1} + \vartheta p_1^{-1}$).

Proof of (3.5) *and* (3.4). Let

$$F_R(\omega) = \int_{\mathbb{R}} e^{i\langle\omega,\gamma(t)\rangle} a(\omega)\chi(t) dt. \tag{3.6}$$

By compactness, we can suppose that χ is supported in $(-\varepsilon, \varepsilon)$, with ε as small as we need. Divide the sphere S^{d-1} into two subsets A and B; here in A, the unit normal to the sphere is essentially orthogonal to the span of the vectors $\gamma'(0), \dots, \gamma^{(k)}(0)$, and in B, the unit normal to the sphere is close to the span of $\gamma'(0), \dots, \gamma^{(k)}(0)$.

Now consider a coordinate patch V of diameter ε on A and parametrize it by $y \mapsto \omega(y)$ with $y \in \mathbb{R}^{d-1}$, y near Y_0. From the defining property of A, it follows that the vectors $\nabla_y \langle \omega(\cdot), \gamma^{(j)}(t)\rangle$, $j = 1, \dots, k$ are linearly independent when evaluated at y near Y_0, provided that $|t| < \varepsilon$. Therefore we can choose the parameterization $y = (x', y'') = (x_1, \dots, x_k, y_{k+1}, \dots, y_{d-1})$ in such a way that also the vectors $\nabla_{x'} \langle \omega(\cdot), \gamma^{(j)}(t)\rangle$, $j = 1, \dots, k$ are linearly independent. If we consider y'' as a parameter and we define

$$\phi^{y''}(x', t) = \langle \omega(x', y''), \gamma(t)\rangle,$$

then the phase functions $\phi^{y''}$ satisfy condition (3.1) uniformly in y''. We also have upper bounds for the higher derivatives in ω and γ which are uniform in y'' as well (here y'' is taken from a relevant compact set). Thus one can apply the Bak-Lee result (3.2) in k dimensions to obtain, for fixed y'',

$$\Big(\int \Big| \int_{\mathbb{R}} e^{iR\phi^{y''}(y', t)} a(\omega(y', y''))\chi(t)dt \Big|^q dy' \Big)^{1/q} \lesssim R^{-k/q}, \quad q > q_k. \tag{3.7}$$

An integration in y'' yields $\|F_R\|_{L^q(V)} \lesssim R^{-k/q}$ for $q > (k^2 + k + 2)/2$. Similarly if $k = 2$ and $q = 4$ we can apply (3.3) in two variables to obtain $\|F_R\|_{L^4(V)} \lesssim R^{-1/2} \log^{1/4} R$. This settles the main estimate for the $L^q(A)$ norm. As for contribution of the $L^q(B)$ norm we recall that the unit normal to the sphere is close to the span of $\gamma'(0), \dots, \gamma^{(k)}(0)$, and thus $\sum_{j=1}^k |\langle \omega, \gamma^{(j)}(t)\rangle| > 0$. Therefore we can apply van der Corput's lemma and obtain the L^∞ estimate

$$\|F_R\|_{L^\infty(B)} \lesssim R^{-1/k} \tag{3.8}$$

For $k = 2$, this completes the proof of the theorem. For $2 < k \leq K$, we argue by induction. We assume that the asserted estimate holds for $k - 1$, $(k \geq 3)$; that is

$$\|F_R\|_q \lesssim R^{-(k-1)/q} \qquad \text{for } q > q_{k-1} = \tfrac{k^2-k+2}{2} \tag{3.9}$$

where the implicit constants depend on q. If $\vartheta_k = 1 - q_{k-1}/q_k$ then we use the relation $q_k - q_{k-1} = k$ to verify that $(1 - \vartheta_k)(k-1)/q_{k-1} + \vartheta_k/k = k/q_k$. Thus by a convexity argument we see that a combination of (3.8) and (3.9) yields that

$$\|F_R\|_{L^q(B)} \lesssim R^{-k/q}, \quad \text{for } q > \tfrac{k^2+k+2}{2} = q_k.$$

Together with the corresponding bound for $\|F_R\|_{L^q(A)}$ proved above, this concludes the proof. $\square$

4. Upper bounds, III

We give the proof of Theorem 1.4 under the *finite type assumption.* By compactness, there is an integer $L \geq d$ and a constant $c > 0$ so that for every $s \in I$ and every $\theta \in S^{d-1}$ we have $\sum_{n=1}^{L} |\langle \gamma^{(n)}(s), \theta \rangle| \geq c$.

We shall argue by induction on k. By Theorem 1.2 the conclusion holds for $k = 2$. Assume $k > 2$, and that the desired inequalities are already proved for $2 \leq q \leq q_{k-1}$.

Let F_R be as in (3.6) and assume that the cutoff function χ is supported in $(-\varepsilon, \varepsilon)$. As in the proof of Theorem 1.2 we split the sphere into subsets A and B where in A, the unit normal to the sphere is almost perpendicular to the span of the vectors $\gamma'(t), \dots, \gamma^{(k)}(t)$, for all $|t| < \varepsilon$ and in B, the projections of the unit normals to the sphere to the span of $\gamma'(t), \dots, \gamma^{(k)}(t)$ have length $\geq c > 0$.

We shall estimate the $L^q(A \cap \Omega)$ norm of F_R on a small patch Ω on the sphere, and by further localization we may assume by the finite type assumption that there is an $n \leq L$ so that

$$|\langle \gamma^{(n)}(s), \theta \rangle| \geq c > 0, |s| \leq \varepsilon, \theta \in \Omega. \tag{4.1}$$

We distinguish between the case $n \geq k$ and $n < k$. First we assume $n \geq k$ (the main case). Then there is the pointwise bound

$$F_R(\theta) \lesssim \min \{1, H_R(\theta))^{-1}\} \tag{4.2}$$

where

$$H_R(\theta) = \min_{s \in I} \max_{1 \leq j \leq n} R^{1/j} |\langle \gamma^{(j)}(s), \theta \rangle|^{1/j}.$$

This is immediate from van der Corput's lemma; indeed the finite type assumption allows the decomposition of the interval $[-\varepsilon, \varepsilon]$ into a bounded number of subintervals so that on each subinterval all derivatives of $s \mapsto \langle \gamma^{(j)}(s), \theta \rangle$, $1 \leq j \leq n-1$ are monotone and one-signed. We now have to estimate the $L^q(A \cap \Omega)$ norm of the right hand side of (4.2).

For an $l > 0$ consider the set

$$\Omega_l(R) = \{\theta \in \Omega : H_R(\theta) \in [2^l, 2^{l+1})\}. \tag{4.3}$$

By (4.1) we have $|H_R(\theta)| \gtrsim R^{1/n}$. Thus only the values with

$$2^l \gtrsim R^{1/n} \tag{4.4}$$

are relevant (and likewise the set of $\theta \in \Omega$ for which $H_R(\theta) \lesssim 1$ is empty if R is large).

By the definition of H_R we can find a point $s_* = s_*(\theta)$ and an integer j_*, $1 \leq j^* \leq n$, so that $H_R(\theta) = |R\langle \gamma^{(j_*)}(s_*), \theta \rangle|^{1/j_*}$ and $|R\langle \gamma^{(j)}(s_*), \theta \rangle|^{1/j} \leq H_R(\theta)$ for all $\theta \in \Omega$ and all $j \leq n$. This implies

$$|\langle \gamma^{(j)}(s_*), \theta \rangle| \lesssim 2^{(l+1)j} R^{-1}, \text{ if } \theta \in \Omega_l(R), \quad j \leq n. \tag{4.5}$$

We shall now apply a nice idea of [1]: We divide our interval $(-\varepsilon, \varepsilon)$ into $O(2^l)$ intervals $I_{\nu,l}$ of length $\approx 2^{-l}$, with right endpoints t_ν, so that

$t_\nu - t_{\nu-1} \approx 2^{-l}$. The point s_* lies in one of these intervals, say in I_{ν_*}. We estimate $|\langle\gamma^{(j)}(t_{\nu_*}),\theta\rangle|^{1/j}$ in terms of $H_R(\theta)$. By a Taylor expansion we get (4.6)

$$\langle\gamma^{(j)}(t_{\nu_*}),\theta\rangle = \sum_{r=0}^{n-j-1}\langle\gamma^{(j+r)}(t_{\nu_*}),\theta\rangle\frac{(t_{\nu_*}-s_*)^r}{r!} + \langle\gamma^{(n)}(\widetilde{t}),\theta\rangle\frac{(t_{\nu_*}-s_*)^{n-j}}{(n-j)!}$$

where $\widetilde{t}$ is between s_* and t_{ν_*}. By (4.5) the terms in the sum are all $O(2^{lj}R^{-1})$. The remainder term is $O(2^{-l(n-j)})$ which is also $O(2^{lj}R^{-1})$, by the condition (4.4). Now define

$$\Omega_{\nu,l} = \{\theta\in\Omega : |\langle\gamma^{(j)}(t_\nu),\theta\rangle| \le C2^{lj}R^{-1}, j=1,\dots,n\}$$

and if C is sufficiently large then the set $\Omega_l(R)$ is contained in the union of the sets $\Omega_{\nu,l}$; the constant C can be chosen independently of l and R.

In view of the linear independence of the vectors $\gamma^{(j)}(t_\nu)$, $j=1,\dots,k$ and the condition $\theta\in A$, the measure of the set $\Omega_{\nu,l}$ is $O(\prod_{s=1}^k(2^{sl}R^{-1})) = O(2^{lk(k+1)/2}R^{-k})$, for every $1\le\nu\lesssim 2^l$, and thus the measure of the set $\Omega_l(R)$ is $O(2^{l(k^2+k+2)/2}R^{-k})$. On $\Omega_l(R)$ we have $|F_R(\theta)|\le H_R(\theta)^{-1}\lesssim 2^{-l}$. Therefore

$$\int_{\Omega\cap A}|F_R(\theta)|^q d\theta \lesssim \sum_{cR^{1/n}\le 2^l\le cR} 2^{-lq}2^{l(k^2+k+2)/2}R^{-k}, \tag{4.7}$$

which yields the endpoint bound

$$\Big(\int_{\Omega\cap A}|F_R(\theta)|^{q_k}d\theta\Big)^{1/q_k} \lesssim R^{-k/q_k}(\log R)^{1/q_k}.$$

Of course we also get (by using the same argument with just $k-1$ derivatives)

$$\int_{\Omega}|F_R(\theta)|^q d\theta \lesssim \sum_{cR^{1/n}\le 2^l\le cR} 2^{-lq}2^{l(k^2-k+2)/2}R^{1-k} \tag{4.8}$$

which yields the sharp $L^{q_{k-1}}(A\cap\Omega)$ bound. Now we consider q satisfying $q_{k-1}<q<q_k$, $k<d$ or $q_{d-1}<q<\infty$ and $K=d$. We distinguish the cases (i) $2^l<R^{1/k}$ and (ii) $2^l\ge R^{1/k}$. In the first case we use (4.7) while in the second case we use (4.8). Then in the case $k<d$

$$\Big(\int_\Omega|F_R(\theta)|^q d\theta\Big)^{1/q} \lesssim R^{-k}\Big(\sum_{2^l<R^{1/k}}2^{l(-2q+k^2+k+2)/2} + \sum_{2^l\ge R^{1/k}}2^{l(-2q+k^2-k+2)/2}R\Big)^{1/q}$$

which is bounded by $CR^{-1/k-(k^2-k-2)/2kq}$ if $q_{k-1}<q<q_k$. If $K=k=d$ then only values with $2^l\ge R^{1/d}$ are relevant and only the second sum in the last displayed line occurs. Thus if $K=d$ we obtain the estimate $CR^{-1/d-(d^2-d-2)/2dq}$ for $q>q_{d-1}$.

Now if $n < k$ one gets even better bounds; we use the induction hypothesis. First note that for $n = 1, 2$ integration by parts, or van der Corput's lemma, yields a better bound; therefore assume $n \geq 3$. We have the bounds $\|F_R\|_{L^\infty(\Omega)} \lesssim R^{-1/n}$ and $\|F_R\|_{L^{q_n}(\Omega)} \lesssim R^{-n/q_n}(\log R)^{1/q_n}$; the first one by van der Corput's lemma and the second one by the induction hypothesis. By convexity this yields $\|F_R\|_{L^{q_k}} \lesssim R^{-\alpha(k,n)} \log R^{1/q_k}$ where $\alpha(k, n) = n/q_k + (1-q_n/q_k)/n$ and one checks that $\alpha(k, n) = k/q_k + (k-n)(k+1-n)/(2nq_k)$ if $n < k$, so that one gets a better estimate. The case $q_{k-1} < q < q_k$, $n < k$ is handled in the same way. This yields the desired bounds for the $L^q(A)$ norm of F_R.

For the $L^q(B)$ bound we may use van der Corput's estimate with $\leq k$ derivatives to get an L^∞ bound $O(R^{-1/k})$; we interpolate this with the appropriate L^p bound for $q_{k-2} < p \leq q_{k-1}$ which holds by the induction hypothesis; the argument is similar to that in the proof of Theorem 1.2. This finishes the argument under the finite type assumption.

Modification for polynomial curves: If the coordinate functions γ_j are polynomials of degree $\leq L$ we need to take $n = L$ in the definition of $H_R(\theta)$. We use, for the case $l > 0$, the analogue of the Taylor expansion (4.6) up to order L with zero remainder term (again $n = L$). As above we obtain for $l > 0$ the bound

$$\int_{\Omega_l(R)} |F_R(\theta)|^q d\theta \lesssim 2^{-lq} R^{-k} \min\{2^{l(k^2+k+2)/2}, 2^{l(k^2-k+2)/2} R\}.$$

Summing in $l > 0$ works as before. However we also have a contribution from the set $\Omega_0(R) = \{\theta \in \Omega : H_R(\theta) \leq 1\}$. By the polynomial assumption a Taylor expansion (now about the point s_*, without remainder) is used to show that $\Omega_0(R)$ is contained in the subset of A where $|\langle \gamma^{(j)}(s_*), \theta\rangle| \leq CR^{-1}$, $j = 1, \dots, k$. This set has measure $O(R^{-k})$. Thus the desired bound for $l = 0$ follows as well. $\square$

5. Asymptotics for oscillatory integrals revisited

We examine the behavior of some known asymptotics for oscillatory integrals under small perturbations. This will be used in the subsequent section to prove the lower bounds of Theorem 1.3.

For $k = 2, 3, \dots$, there is the following formula for $\lambda > 0$:

$$\int_{-\infty}^{\infty} e^{i\lambda s^k} ds = \alpha_k \lambda^{-1/k}, \tag{5.1}$$

where

$$\alpha_k = \begin{cases} \frac{2}{k}\Gamma(\frac{1}{k}) \sin(\frac{(k-1)\pi}{2k}), & k \text{ odd}, \\ \frac{2}{k}\Gamma(\frac{1}{k}) \exp(i\frac{\pi}{2k}), & k \text{ even}. \end{cases} \tag{5.2}$$

(5.1) is proved by standard contour integration arguments and implies asymptotic expansions for integrals $\int e^{i\lambda s^k} \chi(s) ds$ with $\chi \in C_0^\infty$ (see *e.g.* §VIII.1.3 in [20], or §7.7 in [14]).

We need small perturbations of such results. In what follows we set $\|g\|_{C^m(I)} := \max_{0\le j\le m}\sup_{x\in I}|g^{(j)}(x)|$.

Lemma 5.1. *Let $0<h\le 1$, $I=[-h,h]$, $I^*=[-2h,2h]$ and let $g\in C^2(I^*)$. Suppose that*

$$h \le \frac{1}{10(1+\|g\|_{C^2(I^*)})} \tag{5.3}$$

and let $\eta\in C^1$ be supported in I and satisfy the bounds

$$\|\eta\|_\infty + \|\eta'\|_1 \le A_0, \text{ and } \|\eta'\|_\infty \le A_1. \tag{5.4}$$

Let $k\ge 2$ and define

$$I_\lambda(\eta,x) = \int \eta(s)\exp\big(i\lambda(\sum_{j=1}^{k-2} x_j s^j + s^k + g(s)s^{k+1})\big)ds \tag{5.5}$$

Let α_k be as in (5.2). Suppose $|x_j|\le \delta\lambda^{(j-k)/k}$, $j=1,\dots,k-2$. Then there is an absolute constant C so that, for $\lambda>2$,

$$|I_\lambda(\eta,x)-\eta(0)\alpha_k\lambda^{-1/k}| \le C[A_0\delta\lambda^{-1/k} + A_1\lambda^{-2/k}(1+\beta_k\log\lambda)];$$

here $\beta_2=1$, and $\beta_k=0$ for $k>2$.

Proof. We set $u(s):=s(1+sg(s))^{1/k}$; then

$$u'(s) = (1+sg(s))^{-1+1/k}(1+sg(s)+k^{-1}s)$$

and by our assumption on g we quickly verify that $(9/10)^{1/k}\le u'(s)\le (11/10)^{1/k}$ for $-h\le s\le h$. Thus u defines a valid change of variable, with $u(0)=0$ and $u'(0)=1$. Denoting the inverse by $s(u)$ we get

$$I_\lambda(\eta,x) = \int \eta_1(u)\exp\big(i\lambda(\sum_{j=1}^{k-2} x_j s(u)^j + u^k)\big)du$$

with $\eta_1(u)=\eta(s(u))s'(u)$. Clearly η_1 is supported in $(-2h,2h)$. We observe that

$$\|\eta_1\|_\infty + \|\eta_1'\|_1 \lesssim A_0, \qquad \text{and } \|\eta_1'\|_\infty \lesssim (A_0h^{-1}+A_1). \tag{5.6}$$

Indeed implicit differentiation and use of the assumption (5.3) reveals that $|s''(u)|\lesssim (1+\|g\|_\infty)\lesssim h^{-1}$. Taking into account the support properties of η_1 we obtain (5.6).

In order to estimate certain error terms we shall introduce dyadic decompositions. Let $\chi_0\in C_0^\infty(\mathbb{R})$ so that

$$\chi_0(s) = \begin{cases} 1, & \text{if } |s|\le 1/4,\\ 0, & \text{if } |s|\ge 1/2,\end{cases} \tag{5.7}$$

and $m\ge 1$, define

$$\chi_m(s) = \chi_0(2^{-m}s) - \chi_0(2^{-m+1}s), \qquad m\ge 1. \tag{5.8}$$

We now split

$$I_\lambda(\eta, x) = \eta_1(0)J_\lambda + \sum_{m\ge 0} E_{\lambda,m} + \sum_{m\ge 0} F_{\lambda,m}(x)$$

where J_λ is defined in (5.1) and

$$E_{\lambda,m} = \int (\eta_1(u) - \eta_1(0))\chi_m(\lambda^{1/k}u)e^{i\lambda u^k}\,du,$$

$$F_{\lambda,m}(x) = \int \eta_1(u)\Big(\exp(i\lambda(\sum_{j=1}^{k-2} x_j s(u)^j)) - 1\Big)\chi_m(\lambda^{1/k}u)e^{i\lambda u^k}\,du.$$

In view of (5.1) the main term in our asymptotics is contributed by $\eta_1(0)J_\lambda$ since $\eta_1(0) = \eta(0)$.

Now we estimate the terms $E_{\lambda,m}$. It is immediate that from an estimate using the support of the amplitude that

$$|E_{\lambda,0}| \le C\|\eta_1'\|_\infty \lambda^{-2/k}.$$

For $m \ge 1$ we integrate by parts once to get

$$E_{\lambda,m} = \frac{i}{k\lambda}\int \frac{d}{du}\big[(\eta_1(u) - \eta_1(0))u^{1-k}\chi_m(\lambda^{1/k}u)\big]e^{i\lambda u^k}\,du$$

and straightforward estimation gives

$$|E_{\lambda,m}| \le C\begin{cases} \|\eta_1'\|_\infty 2^{m(2-k)}\lambda^{-2/k}, & \text{if } 2^m \le \lambda^{1/k} \\ \|\eta_1\|_\infty 2^{m(1-k)}\lambda^{-1/k}, & \text{if } 2^m > \lambda^{1/k}. \end{cases}$$

Thus

$$\sum_m |E_{\lambda,m}| \le C(\|\eta_1\|_\infty \lambda^{-2/k}(1 + \beta_k \log\lambda).$$

We now show that

$$\sum_{m\ge 0} |F_{\lambda,m}(x)| \le C[\|\eta_1\|_\infty + \|\eta'\|_1]\delta\lambda^{-1/k} \tag{5.9}$$

and notice that only terms with $2^m\lambda^{-1/k} \le C$ occur in the sum.

Set $\zeta_{\lambda,x}(u) = \big(\exp(i\lambda(\sum_{j=1}^{k-2} x_j s(u)^j)) - 1\big)$. For the term $E_{\lambda,0}(x)$ we simply use the straightforward bound on the support of $\chi_0(\lambda^{1/k}\cdot)$ which is (in view of $|s(u)| \approx |u|$)

$$|\zeta_{\lambda,x}(u)| \le C\lambda \sum_{j=1}^{k-2} |x_j||\lambda^{-j/k}|$$

and since $|x_j| \le \delta\lambda^{-(k-j)/k}$ we get after integrating in u

$$|F_{\lambda,0}| \lesssim \|\eta_1\|_\infty \delta\lambda^{-1/k}.$$

For $m > 0$ we integrate by parts once and write

$$F_{\lambda,m} = ik^{-1}\lambda^{-1}\int \frac{d}{du}\big[u^{1-k}\chi_m(\lambda^{1/k}u)\eta_1(u)\zeta_{\lambda,x}(u))\big]e^{i\lambda u^k}\,du. \tag{5.10}$$

On the support of $\chi_m(\lambda^{1/k}\cdot)$,

$$|\zeta_{\lambda,x}(u)| \lesssim \lambda \sum_{j=1}^{k-2} \delta\lambda^{-(k-j)/k}(2^m\lambda^{-1/k})^j \lesssim \delta 2^{m(k-2)}$$
$$\big|\tfrac{d}{du}\big[u^{1-k}\chi_m(\lambda^{1/k}u)\big]\big| \lesssim \lambda 2^{-mk}$$

and also

$$|\zeta'_{\lambda,x}(u)| \lesssim \lambda \sum_{j=1}^{k-2} \delta\lambda^{-(k-j)/k}(2^m\lambda^{-1/k})^{j-1} \lesssim \delta 2^{m(k-3)}\lambda^{1/k}$$
$$|u^{1-k}\chi_m(\lambda^{1/k}u)| \lesssim 2^{-m(k-1)}\lambda^{(k-1)/k},$$

and thus we obtain the bound

$$\int \Big|\frac{d}{du}\big[u^{1-k}\chi_m(\lambda^{1/k}u)\zeta_{\lambda,x}(u)\eta_1(u)\big]\Big| du \lesssim [\|\eta_1\|_\infty + \|\eta_1'\|_1]\delta 2^{-m}\lambda^{(k-1)/k}.$$

Hence,

$$\sum_{m\geq 0} |F_{\lambda,m}| \lesssim \delta\lambda^{-1/k}$$

which completes the proof of (5.9). □

For the logarithmic lower bounds of $G_4(R)$ we shall need some asymptotics for modifications of Airy functions. Recall that for $t \in \mathbb{R}$ the Airy function is defined by the oscillatory integral

$$Ai(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i(\tfrac{x^3}{3} + \tau x))dx$$

and that for $t \to \infty$ we have

$$Ai(-t) = \pi^{-1/2}t^{-1/4}\cos(\tfrac{2}{3}t^{3/2} - \tfrac{\pi}{4})(1 + O(t^{-3/4})). \tag{5.11}$$

This statement can be derived using the method of stationary phase (combining expansions about the two critical points $\pm t^{1/2}$) or complex analysis arguments, *cf.* [10] or [21], p. 330, see also an argument in [12].

Let $g \in C^2([-1,1])$, and let $\varepsilon > 0$ be small, $\varepsilon \ll (1 + \|g\|_{C^2})^{-1}$. Let $\eta \in C_0^\infty$ with support in $(-\varepsilon,\varepsilon)$, so that $\eta(s) = 1$ for $|s| \leq \varepsilon/2$.

Lemma 5.2. *Define*

$$J(\lambda,\vartheta) = \int e^{i\lambda(\frac{s^3}{3} - \vartheta s)}e^{i\lambda g(s)s^4}\eta(s)ds \tag{5.12}$$

Then, for $0 < \vartheta < \varepsilon^2/2$ *and* $\lambda > \varepsilon^{-1}$

$$\begin{aligned} J(\lambda,\vartheta) &= \lambda^{-1/3}Ai(-\lambda^{2/3}\vartheta) + E_1(\lambda,\vartheta) \\ &= \pi^{-1/2}\lambda^{-1/2}\vartheta^{-1/4}\cos\big(\tfrac{2}{3}\lambda\vartheta^{3/2} - \tfrac{\pi}{4}\big) + E_2(\lambda,\vartheta) \end{aligned} \tag{5.13}$$

where, for $i = 1,2$

$$|E_i(\lambda,\vartheta)| \lesssim C_\varepsilon\big[\lambda^{-1}\vartheta^{-1} + \min\{\lambda\vartheta^{5/2}, \vartheta^{1/2}\}\big] \tag{5.14}$$

Proof. We split

$$J(\lambda,\vartheta)=\sum_{i=1}^{4}J_i(\lambda,\vartheta):=\sum_{i=1}^{4}\int e^{i\lambda(\frac{s^3}{3}-\vartheta s)}\zeta_i(s)\,ds$$

where

$$\zeta_1(s)=1,\qquad \zeta_2(s)=(\eta(s)-1),$$
$$\zeta_3(s)=\eta(s)(e^{i\lambda g(s)s^4}-1)\eta(C^{-1}\vartheta^{-1/2}s),$$
$$\zeta_4(s)=\eta(s)(e^{i\lambda g(s)s^4}-1)(1-\eta(C^{-1}\vartheta^{-1/2}s)).$$

where $C\ge\varepsilon^{-1}$. By a scaling we see that

$$J_1(\lambda,\vartheta)=\lambda^{-1/3}Ai(-\lambda^{2/3}\vartheta)$$

and we prove upper bounds for the error terms J_i, $i=2,3,4$. Let

$$\Phi(s)=-\vartheta s+s^3/3$$

then $\Phi'(s)=-\vartheta+s^2$ and in the support of ζ_2 we have $|\Phi'(s)|\ge c\varepsilon$. Thus by an integration by parts $J_2(\lambda,\vartheta)=O(\lambda^{-1})$. Note that ζ_3 is bounded and that also $|\zeta_3(s)|\lesssim\lambda|\vartheta|^2$. We integrate over the support of ζ_3 which is of length $O(\sqrt{b})$ and obtain $J_3(\lambda,\vartheta)=O(\min\{\vartheta^{1/2},\lambda\vartheta^{5/2}\})$. To estimate $J_4(\lambda,\vartheta)$ we argue by van der Corput's Lemma, for the phases Φ and its perturbation $\Psi(s):=\Phi(s)+s^4g(s)$. Thus we split

$$J_4(\lambda,\vartheta)=\sum_m\sum_\pm J_{4,m,\pm}(\lambda,\vartheta)$$

where we have set

$$J_{4,m,\pm}(\lambda,\vartheta)=\int e^{i\lambda\Psi(s)}\rho_{m,\pm}(s)\,ds-\int e^{i\lambda\Phi(s)}\rho_{m,\pm}(s)\,ds;$$

here $\rho_{m,+}(s)=\chi_{(0,\infty)}\eta(s)(1-\eta(C^{-1}\vartheta^{-1/2}s))\chi_m(C^{-1}\vartheta^{-1/2}s)$, χ_m is as in (5.8) and $2^m\vartheta^{1/2}\lesssim\varepsilon$ (in view of the condition on η). Let $\rho_{m,-}$ is analogously defined, with support on $(-\infty,0)$.

We argue as in the proof of Lemma 5.1. Note that now $|\Phi'(s)|\approx 2^{2m}\vartheta$, $\partial_s(g(s)s^4)=O(2^{3m}\vartheta^{3/2})$ and since $2^m\vartheta^{1/2}\lesssim\varepsilon$ we also have $|\Psi'(s)|\approx 2^{2m}\vartheta$. Moreover observe $\Phi''(s)=2s+O(s^2)$ so that van der Corput's lemma can be applied can be applied to the two integrals defining $J_{4,m,\pm}(\lambda,\vartheta)$. We obtain $J_{4,m,\pm}(\lambda,\vartheta)=O(\lambda^{-1}\vartheta^{-1}2^{-2m})$.

Finally, by (5.11) and (5.13), the difference of E_1 and E_2 is $O(\lambda^{-1}\vartheta^{-1})$. This concludes the proof. □

6. Lower bounds

For $w\in\mathbb{R}^d$ (usually restricted to the unit sphere), define

$$F_R(w)=\int\chi(t)e^{iR\langle\gamma(t),w\rangle}\,dt. \tag{6.1}$$

The following result establishes inequality (1.10) of Theorem 1.3.

Proposition 6.1. *Suppose that for some $t_0 \in I$ the vectors $\gamma'(t_0)$, ..., $\gamma^{(k)}(t_0)$ are linearly independent. Then $\chi \in C_0^\infty$ in (1.1) can be chosen so that, for sufficiently large R,*

$$\|F_R\|_{L^q(S^{d-1})} \geq CR^{-\frac{1}{k}-\frac{k^2-k-2}{2kq}}. \tag{6.2}$$

Proof. We may assume $t_0 = 0$. By a scaling and rotation we may assume that $\gamma^{(k)}(0) = e_k$. We shall then show the lower bound $|F_R(\omega)| \geq c_0 R^{-1/k}$ for a neighborhood of e_k which is of measure $\approx R^{-(k^2-k-2)/2k}$. Now let A_k be an invertible linear transformation which maps e_k to itself, and for $j = 1, \dots, k-1$ maps $\gamma^{(j)}(0)$ to e_j, $j = 1, \dots, k$. Then the map $\omega \to (A_k^*)^{-1}\omega/|(A_k^*)^{-1}\omega|$ defines a diffeomorphism from a spherical neighborhood of e_k to a spherical neighborhood of e_k. Thus we may assume for what follows that $\gamma : [-1, 1] \to \mathbb{R}^d$ satisfies

$$\gamma^{(j)}(0) = e_j, \qquad j = 1, \dots, k. \tag{6.3}$$

We may also assume that the cutoff function χ is supported in a small open interval $(-\varepsilon, \varepsilon)$ so that $\chi(0) = 1$.

As we have $\langle e_k, \gamma^{(k-1)}(0)\rangle = 0$ and $\langle e_k, \gamma^k(0)\rangle = 1$ we can use the implicit function theorem to find a neighborhood $\mathcal{W}_k$ of e_k and an interval $\mathcal{I}_k = (-\varepsilon_k, \varepsilon_k)$ containing 0 so that for all $w \in \mathcal{W}_k$ the equation $\langle w, \gamma^{(k-1)}(t)\rangle = 0$ has a unique solution $\widetilde{t}_k(w) \in \mathcal{I}_k$. This solution is also homogeneous of degree 0, i.e. $\widetilde{t}_k(sw) = \widetilde{t}_k(w)$ for s near 1), and we have $\widetilde{t}_k(e_k) = 0$.

Lemma 6.2. *There is $\varepsilon_0 > 0$, $R_0 > 1$, and $c > 0$ so that for all positive $\varepsilon < \varepsilon_0$ and all $R > R_0$ the following holds. Let*

$$U_{k,\varepsilon}(R) = \big\{\omega \in S^{d-1} : |\omega - e_k| \leq \varepsilon \\ \text{and } |\langle \omega, \gamma^{(j)}(\widetilde{t}_k(\omega))\rangle| \leq \varepsilon R^{(j-k)/k}, \text{ for } j = 1, \dots, k-2.\big\}$$

Then the spherical measure of $U_{k,\delta}(R)$ is at least $c\varepsilon^{d-1}R^{-\frac{k^2-k-2}{2k}}$.

Proof. In a neighborhood of e_k we parametrize the sphere by

$$\omega(y) = (y_1, \dots, y_{k-1}, \sqrt{1-|y|^2}, y_k, \dots, y_{d-1}).$$

We introduce new coordinates $z_1, \dots, z_{d-1}$ setting

$$z_j = \mathfrak{z}_j(y) = \begin{cases} \langle \omega(y), \gamma^{(j)}(\widetilde{t}_k(\omega(y)))\rangle, & j = 1, \dots, k-2, \\ y_j, & j = k-1, \dots, d-1. \end{cases} \tag{6.4}$$

Then it is easy to see that $\mathfrak{z}$ defines a diffeomorphism between small neighborhoods of the origin in $\mathbb{R}^{d-1}$; indeed the derivative at the origin is the identity map.

The spherical measure of $U_{k,\varepsilon}(R)$ is comparable to the measure of the set of $z \in \mathbb{R}^{d-1}$ satisfying $|z_j| \leq \varepsilon R^{(j-k)/k}$, for $j = 1, \dots, k-2$, and $|z_j| \leq \varepsilon$ for $k-1 \leq j \leq d-1$, and this set has measure $\approx \varepsilon^{d-1}R^{-\frac{k^2-k-2}{2k}}$. □

We now verify that for sufficiently small ε and sufficiently large R

$$|F_R(\omega)| \ge c_0 R^{-1/k}, \quad \omega \in U_{k,\varepsilon}(R), \tag{6.5}$$

with some positive constant c_0; by Lemma 6.2, this of course implies the bound (6.2). To see (6.5) we set

$$a_j(\omega) = \langle \omega, \gamma^{(j)}(\widetilde{t}_k(\omega))\rangle, \tag{6.6}$$

$s = t - \widetilde{t}_k(\omega)$ and expand

$$\langle \omega, \gamma(t)\rangle - \langle \omega, \gamma(\widetilde{t}_k(\omega))\rangle = \sum_{j=1}^{k-2} a_j(\omega)\frac{s^j}{j!} + a_k(\omega)\frac{s^k}{k!} + \mathcal{E}_k(\omega, s)s^{k+1}, \tag{6.7}$$

with $\mathcal{E}_k(\omega, s) = \int_{\sigma=0}^1 \frac{(1-\sigma)^k}{k!}\langle \omega, \gamma^{(k+1)}(\widetilde{t}_k(\omega) + \sigma s)\rangle\, d\sigma$. If ε is sufficiently small then we can apply Lemma 5.1 with $\omega \in U_{k,\varepsilon}(R)$, and the choice $\lambda = R\langle \omega, \gamma^{(k)}(\widetilde{t}_k(\omega))\rangle/k!$, and the lower bound (6.5) follows. □

We now formulate bounds for $q \ge (k^2+k+2)/2$ for the case that $\gamma^{(k+1)} \equiv 0$ for some $k < d$; this of course implies that the curve lies in a k-dimensional affine subspace.

Proposition 6.3. *Suppose that γ is a polynomial curve with $\gamma^{(k+1)} \equiv 0$ and suppose that for some $t_0 \in I$ the vectors $\gamma'(t_0)$, ..., $\gamma^{(k)}(t_0)$ are linearly independent. Then χ in* (1.1) *can be chosen so that for sufficiently large R*

$$\|F_R\|_{L^q(S^{d-1})} \ge C \begin{cases} R^{-k/q}[\log R]^{1/q}, & q = \frac{k^2+k+2}{2}, \\ R^{-k/q}, & q > \frac{k^2+k+2}{2}. \end{cases} \tag{6.8}$$

Proof. We first note that the assumption $\gamma^{(k+1)} \equiv 0$ implies that the curve is polynomial and for any fixed t_0 it stays in the affine subspace through $\gamma(t_0)$ which is generated by $\gamma^{(j)}(t_0)$, $j = 1, \dots, k$. We shall prove a lower bound for $\widehat{\mu}$ in a neighborhood of a vector $e \in S^{d-1}$ where e is orthogonal to the vectors $\gamma^{(j)}(t_0)$. After a rotation we may assume that

$$\gamma(t) = (\gamma_1(t), \dots, \gamma_k(t), 0, \dots, 0).$$

For $\omega \in S^{d-1}$, we split accordingly $\omega = (\omega', \omega'')$ with small $\omega' \in \mathbb{R}^k$, namely

$$|\omega'| \approx 2^{-l}$$

where $1 \ll R2^{-l} \ll R$. As before, we solve the first degree the equation $\langle \gamma^{(k-1)}(t), \omega\rangle = 0$ (observe that this is actually independent of ω'') with $t = \widetilde{t}_k(\omega')$; now $\widetilde{t}_k$ is homogeneous of degree 0 as a function on $\mathbb{R}^k$. Then

$$e^{-i\langle \omega, \gamma(\widetilde{t}_k(\omega))\rangle} F_R(\omega)$$
$$= \int \chi(\widetilde{t}_k(\omega') + s) \exp\Big(\sum_{j=1}^{k-2}\langle \omega, \gamma^{(j)}(\widetilde{t}_k(\omega'))\rangle \tfrac{s^j}{j!} + \langle \omega, \gamma^{(k)}(\widetilde{t}_k(\omega'))\rangle \tfrac{s^k}{k!}\Big) ds$$

If $k \ge 3$, let $V_{k,l}(R)$ be the subset of the unit sphere in $\mathbb{R}^k$ which consists of those $\theta \in S^{k-1}$ which satisfy the conditions

$$|\langle \gamma^{(\nu)}(\widetilde{t}_k(\theta)), \theta \rangle| \le \varepsilon (R2^{-l})^{\frac{k-\nu}{k}}, \quad \nu = 1, \dots, k-2.$$

Observe that the spherical measure of $V_{k,l}(R)$ (as a subset of S^{k-1}) is $(R2^{-l})^{-(k^2-k-2)/(2k)}$, by Lemma 6.2. Now, if $k = 2$, define

$$U_{2,l}(R) := \{\omega = (\omega', \omega'') \in S^{d-1} : |\omega - e_3| \le \delta, 2^{-l} \le |\omega'| < 2^{-l+1}\}.$$

If $3 \le k < d$ let

$$U_{k,l}(R) := \{\omega \in S^{d-1} : |\omega - e_{k+1}| \le \delta, 2^{-l} \le |\omega'| < 2^{-l+1}, \tfrac{\omega'}{|\omega'|} \in V_{k,l}(R)\}.$$

We need a lower bound for the spherical measure (on S^{d-1}) of $U_{k,l}(R)$ and using polar coordinates in $\mathbb{R}^k$ we see that it is at least

$$c\varepsilon^{d-1} 2^{-lk} (R2^{-l})^{-\frac{k^2-k-2}{2k}}.$$

If ε is small we obtain a lower bound $c(R2^{-l})^{-1/k}$ on this set; this follows from Lemma 5.1 with $\lambda \approx R2^{-l}$. Thus

$$\begin{aligned}\int_{U_{k,l}(R)} \big|F_R(\omega)|^q d\sigma(\omega) &\ge c_\varepsilon (R2^{-l})^{-q/k} 2^{-lk} (R2^{-l})^{-(k^2-k-2)/(2k)} \\ &= c_\varepsilon R^{-q/k-(k^2-k-2)/(2k)} 2^{l(q/k-(k^2+k+2)/(2k))}.\end{aligned}$$

As the sets $U_{k,l}(R)$ are disjoint in l we may now sum in l for $CR^{-1} \le 2^{-l} \le c$ for a large C and a small c. Then we obtain that $\sum_l \int_{U_{k,l}(R)} \big|F_R(\omega)|^q d\sigma(\omega)$ is bounded below by $cR^{-q/k-(k^2-k-2)/(2kq)}$, if $q < (k^2+k+2)/2$; this yields the bound that was already proved in Proposition 6.1. If $q > (k^2+k+2)/2$ then we get the lower bound cR^{-k} and for the exponent $q = (k^2+k+2)/2$ we obtain the lower bound $cR^{-k} \log R$. This yields (6.8). □

Proposition 6.4. *Suppose that $3 \le k \le d$ and that for some $t_0 \in I$ the vectors $\gamma'(t_0)$, ..., $\gamma^{(k)}(t_0)$ are linearly independent. Then $\chi \in C_0^\infty$ in* (1.1) *can be chosen so that for sufficiently large R*

$$\|F_R\|_{L^q(S^{d-1})} \ge CR^{-(k-1)/q}[\log R]^{1/q}, \ \textit{if } q = q_{k-1} = \tfrac{k^2-k+2}{2}. \tag{6.9}$$

Proof. We start with the same reductions as in the proof of Proposition 6.1, namely we may assume $t_0 = 0$ and $\gamma^{(j)}(0) = e_j$ for $j = 1, \dots, k$; we shall then derive lower bounds for $F_R(\omega)$ for ω near e_k. As before denote by $\widetilde{t}_k(\omega)$ the solution t of $\langle \gamma^{(k-1)}(t), \omega \rangle = 0$, for ω near e_k. We may use the expansion (6.7). Define the polynomial approximation

$$P_k(s, \omega) = P_k(s) = \sum_{j=1}^{k-2} a_j(\omega) \frac{s^j}{j!} + a_k(\omega) \frac{s^k}{k!}.$$

Note that $a_k(\omega)$ is near 1 if ω is near e_k. In what follows we shall only consider those ω with

$$a_{k-2}(\omega) < 0.$$

In our analysis we need to distinguish between the cases $k = 3$ and $k > 3$.

The case $k = 3$.

We let for small δ

$$\mathcal{U}_{R,j} = \{\omega \in S^{d-1} : |\omega - e_3| \le \delta,\ -2^{j+1}R^{-2/3} \le a_1(\omega) \le -2^j R^{-2/3}\}.$$

We wish to use the asymptotics of Lemma 5.2, with the parameters

$$\vartheta = \vartheta(\omega) = \frac{-2a_1(\omega)}{a_3(\omega)}$$

and $\lambda = Ra_3(\omega)/2$ ($\approx R$) to derive a lower bound on a portion of $\mathcal{U}_{R,j}$ whenever $\lambda^{-2/3} \ll \vartheta(\omega) \ll \lambda^{-6/11}$; *i.e.*

$$\delta^{-1} \le 2^j \le \delta\lambda^{4/33} \tag{6.10}$$

where δ is small (but independent of large λ).

The range (6.10) is chosen so that the error terms in (5.14) (with $\lambda \approx R$) are $\ll R^{-1/2}\vartheta^{-1/4}$ if δ is small; indeed the term $\lambda^{-1}\vartheta^{-1}$ is controlled by $C\delta^{3/4}\lambda^{-1/2}\vartheta^{-1/4}$ in view of the first inequality in (6.10) and the term $\lambda\vartheta^{5/2}$ is bounded by $C\delta^{11/4}\lambda^{-1/2}\vartheta^{-1/4}$ because of the second restriction. Since the main term in (5.13) can be written as

$$(2/\pi)^{1/2}R^{-1/2}a_3(\omega)^{-1/2}\vartheta(\omega)^{-1/4}\cos(\tfrac{1}{3}Ra_3(\omega)\vartheta(\omega)^{3/2} - \tfrac{\pi}{4})$$

it dominates the error terms in the range (6.10), provided that we stay away from the zeroes of the cosine term. To achieve the necessary further localization we let, for positive integers n,

$$\mathcal{U}_{R,j,n} = \{\omega \in \mathcal{U}_{R,j} : \left|\tfrac{1}{3}Ra_3(\omega)\vartheta(\omega)^{3/2} - \tfrac{\pi}{4} - \pi n\right| < \tfrac{\pi}{4}\}.$$

Let j be in the range (6.10). We use $|b^{3/2} - a^{3/2}| \approx (\sqrt{a} + \sqrt{b})|b - a|$ for $0 < b, a \ll 1$. Since $\vartheta(\omega)$ can be used as one of the coordinates on the unit sphere we see that the spherical measure of $\mathcal{U}_{R,j,n}$ is $\gtrsim \delta^2 R^{-2/3}2^{-j/2}$ for the about $2^{3j/2}$ values of n for which $n \approx 2^{3j/2}$, and on those disjoint sets $\mathcal{U}_{R,j,n}$ the value of $F_R(\omega)$ is $\ge cR^{-1/3}2^{-j/4}$.

This implies that, for j as in (6.10),

$$\operatorname{meas}\big(\{\omega \in \mathcal{U}_{R,j} : |F_R(\omega)| \ge c_\delta R^{-1/3}2^{-j/4}\}\big) \ge c'_\delta 2^j R^{-2/3},$$

and thus

$$\int_{\mathcal{U}_{R,j}} |F_R(\omega)|^4 d\sigma(\omega) \gtrsim R^{-2}.$$

Since the sets $\mathcal{U}_{R,j}$ are disjoint we may sum in j over the range (6.10) and obtain the lower bound $\|F_R\|_4 \gtrsim R^{-1/2}(\log R)^{1/4}$ (with an implicit constant depending on δ).

The case $k > 3$. We try to follow in spirit the proof of the case for $k = 3$. Notice that

$$P_k^{(k-2)}(s) = a_{k-2}(\omega) + a_k(\omega)s^2/2$$

has then two real roots, one of them being

$$s_1(\omega) = \Big(\frac{-2a_{k-2}(\omega)}{a_k(\omega)}\Big)^{1/2},$$

the other one $s_2 = -s_1$. The idea is now to use, for suitable ω, an asymptotic expansion for the part where s is close to s_1, and, unlike in the case $k = 3$, we shall now be able to neglect the contribution of the terms where s is near s_2. To achieve this we define, for $j = 1, \dots, k-3$,

$$\widetilde{a}_j(\omega) = P_k^{(j)}(s_1(\omega)) = a_j(\omega) + \sum_{\substack{1\le\nu\le k-2-j \\ \text{or } \nu=k-j}} \frac{a_{j+\nu}(\omega)}{\nu!}\Big(\frac{-2a_{k-2}(\omega)}{a_k(\omega)}\Big)^{\nu/2}. \tag{6.11}$$

We further restrict consideration to ω chosen in sets

(6.12)

$$\mathcal{V}_{k,j}(\delta) = \big\{\omega\in S^{d-1} : -2^{j+1}R^{-2/k} < a_{k-2}(\omega) < -2^j R^{-2/k}, |e_k-\omega|\le\delta,$$
$$|\widetilde{a}_\nu(\omega)| \le \delta |a_{k-2}(\omega)|^{\frac{\nu}{2k-2}} R^{-\frac{k-\nu-1}{k-1}}, 1\le\nu\le k-3\big\}.$$

We shall see that if we choose ω from one of the sets $\mathcal{V}_{k,j}(\delta)$ with small δ, and j not too large then the main contribution of the oscillatory integral comes from the part where $|s - s_1(\omega)| \le s_1(\omega)/2$. We shall reduce to an application of Lemma 5.1 to derive a lower bound for that part. For the remaining parts we shall derive smaller upper bounds using van der Corput's lemma.

For notational convenience we abbreviate

$$b := -a_{k-2}(\omega), \quad \widetilde{a}_\nu := \widetilde{a}_\nu(\omega), \quad s_1 := s_1(\omega), \quad \widetilde{t}_k := \widetilde{t}_k(\omega).$$

We now split

$$e^{-i\langle\omega,\gamma(\widetilde{t}_k(\omega))\rangle}F_R(\omega) = I_R(\omega) + E_R(\omega), \tag{6.13}$$

where

$$I_R(\omega) = \int \chi(\widetilde{t}_k+s)\chi_0(20\frac{s-s_1}{s_1})\exp(iR[P_k(s)+s^{k+1}\mathcal{E}_{k+1}(s,\omega)])ds.$$

Here χ_0 is as in (5.7) and thus the integrand is supported where $|s-s_1|\le s_1/40$.

Notice that $P_k^{(k-1)}(s_1) = s_1 a_k(\omega)$ and $P_k^{(k)}(s) \equiv a_k(\omega)$. Let

$$Q_{k-1}(s) = \sum_{\nu=1}^{k-3}\widetilde{a}_\nu\frac{(s-s_1)^\nu}{\nu!} + a_k s_1\frac{(s-s_1)^{(k-1)}}{(k-1)!};$$

then $P_k(s) - P_k(s_1) = Q_{k-1}(s) + a_k(s-s_1)^k/k!$.

Thus we can write

$$I_R(\omega) = \int \eta(s)e^{iR(Q_{k-1}(s)+a_k(s-s_1)^k/k!)}ds$$

with

$$\eta(s) = \chi(\widetilde{t}_k+s)\chi_0(10s_1^{-1}(s-s_1))\exp(iRs^{k+1}\mathcal{E}_{k+1}(s,\omega)).$$

Note that by (5.7) the function η is supported where $20s_1^{-1}|s-s_1|\le 1/2$, i.e. in $[s_1-h, s_1+h]$ with $h=s_1/40$. Clearly $\|\eta\|_\infty = O(1)$, and since $s_1\approx\sqrt{b}$ it is straightforward to check that

$$\|\eta'\|_\infty + b^{-1/2}\|\eta'\|_1 \lesssim (1+b^{-1/2}+Rb^{k/2}), \tag{6.14}$$

thus also

$$\|\eta\|_\infty + \|\eta'\|_1 \lesssim 1 \text{ if } b\le R^{-2/(k+1)}. \tag{6.15}$$

Moreover, if $g(s)=k^{-1}/s_1$ then we can write

$$RQ_{k-1}(s)+a_k\frac{(s-s_1)^k}{k!} \\ = \frac{Ra_k s_1}{(k-1)!}\Big(\sum_{\nu=1}^{k-3} x_\nu(s-s_1)^\nu + (s-s_1)^{k-1} + (s-s_1)^k g(s-s_1)\Big)$$

where $|x_\nu|\lesssim b^{-1/2}|\widetilde{a}_\nu|$. The conditions $|\widetilde{a}_\nu|\le \delta b^{\frac{\nu}{2k-2}}R^{-\frac{k-\nu-1}{k-1}}$ imply that

$$|x_\nu|\lesssim \delta(Rb^{1/2})^{-\frac{k-\nu-1}{k-1}}.$$

We of course have $\|g\|_{C^2([-h,h])}\le s_1^{-1}$ on $I^*=[-s_1/10, s_1/10]$; thus $h = s_1/40\le 10^{-1}(1+\|g\|_{C^2})^{-1}$.

Changing variables $\tilde{s}=s-s_1$ puts us in the position to apply Lemma 5.1 for perturbations of the phase $\tilde{s}\mapsto\lambda\tilde{s}^{k-1}$, with $\lambda := R|a_k|s_1 = R\sqrt{2a_kb}\approx Rb^{1/2}$, and we have the bounds $A_0\le C$ (if $b\le R^{-2/(k+1)}$ and $A_1\le(1+Rb^{k/2})$ for the parameters in Lemma 5.1. We thus obtain (*cf.* (5.2))

$$\big|I_R(\omega)-\alpha_{k-1}\chi(s_1(\omega))(R\sqrt{2a_kb})^{-1/(k-1)}\big| \\ \lesssim \delta(Rb^{1/2})^{-1/(k-1)} + b^{-1/2}(Rb^{1/2})^{-2/(k-1)}\log(Rb^{1/2}), \tag{6.16}$$

provided that $b\le R^{-2/(k+1)}\ll 1$. We wish to use this lower bound on the sets $\mathcal{V}_{k,j}(\delta)$. In order to efficiently apply (6.16) we shall choose j so that

$$R^{-\tau_1+2/k}\le 2^j\le R^{-\tau_2+2/k} \tag{6.17}$$

with τ_1,τ_2 satisfying

$$\frac{2}{k}>\tau_1>\tau_2>\frac{2}{k+1}.$$

so that the main term in (6.16) dominates the error terms.

We now need to bound from below the measure of the set $\mathcal{V}_{k,j}(\delta)$. We use the coordinates (6.4) on the sphere in a neighborhood of e_k. In view of the linear independence of $\gamma'',\dots,\gamma^{(k-1)}$ we can use the functions $a_j(\mathfrak{z}(y))$, $j\in\{1,\dots,k-2\}$, *cf.* (6.6), as a set of partial coordinates.

We may also change coordinates

$$(a_1,\dots,a_{k-3},a_{k-2})\mapsto(\widetilde{a}_1,\dots,\widetilde{a}_{k-3},a_{k-2}),$$

with $a_{k-2} \equiv -b$; here we use the shear structure of the (nonsmooth) change of variable (6.11). Thus, as in Lemma 6.2, we obtain a lower bound for the spherical measure of $\mathcal{V}_{k,j}(\delta)$, namely

$$\begin{aligned}\big|\mathcal{V}_{k,j}(\delta)\big| &\geq c\delta^{d-2}2^jR^{-\frac{2}{k}}\prod_{\nu=1}^{k-3}\Big(\big(2^jR^{-\frac{2}{k}}\big)^{\frac{\nu}{2(k-1)}}R^{-\frac{k-\nu-1}{k-1}}\Big)\\ &= c\delta^{d-2}2^jR^{-\frac{2}{k}}\big(2^jR^{-\frac{2}{k}}\big)^{\frac{(k-3)(k-2)}{4(k-1)}}R^{\frac{(k-3)(k-2)}{2(k-1)}-(k-3)}\\ &= c\delta^{d-2}2^{j\frac{k^2-k+2}{4(k-1)}}R^{-\frac{k^2-k-2}{2k}}\end{aligned}$$

after a little arithmetic. Thus

$$\big|\mathcal{V}_{k,j}(\delta)\big| \geq c\delta^{d-2}2^{jq_{k-1}/(2k-2)}R^{1/k-(k-1)/2}. \tag{6.18}$$

Now if δ is chosen small and then fixed, and R is chosen large then (6.16) implies the lower bound

$$|I_R(\omega)| \geq c_\delta\big(R\sqrt{2^jR^{-2/k}}\big)^{-1/(k-1)} = c_\delta 2^{-j/(2k-2)}R^{-1/k}, \quad \omega\in\mathcal{V}_{k,j}(\delta), \tag{6.19}$$

provided that $R^{-\tau_1+2/k} \leq 2^j \leq R^{-\tau_2+2/k}$. We shall verify that for $j \geq 0$

$$|E_R(\omega)| \lesssim R^{-1/k}\big(2^{-j/(k-2)} + 2^{-3j/(2k-6)}\big), \quad \omega\in\mathcal{V}_{k,j}(\delta), \tag{6.20}$$

and from (6.19) and (6.20) it follows that

$$|F_R(\omega)| \geq c_\delta 2^{-j/(2k-2)}R^{-1/k}, \quad \omega\in\mathcal{V}_{k,j}(\delta)$$

if $R^{-\tau_1+2/k} \leq 2^j \leq R^{-\tau_2+2/k}$. By (6.18) this implies for the same range a lower bound which is independent of j,

$$\int_{\mathcal{V}_{k,j}(\delta)}|F_R(\omega)|^{q_{k-1}}d\omega \geq c_\delta R^{-\frac{q_{k-1}}{k}-\frac{k^2-k-2}{2k}} = c_\delta R^{-(k-1)}.$$

We sum in j, $R^{-\tau_1+2/k} \leq 2^j \leq R^{-\tau_2+2/k}$; this yields, for large R,

$$\Big(\int_{\cup_j\mathcal{V}_{k,j}(\delta)}|F_R(\omega)|^{q_{k-1}}d\omega\Big)^{1/q_{k-1}} \geq c'_\delta R^{-(k-1)/q_{k-1}}\big(\log R\big)^{1/q_{k-1}}$$

which is the desired bound.

It remains to prove the upper bounds (6.20) for the error term E_R. It is given by

$$E_R(\omega) = e^{iRP_k(0)}\int\chi(\widetilde{t}_k+s)(1-\chi_0(20\frac{s-s_1}{s_1}))e^{iR\phi(s)}ds$$

where

$$\phi(s) = P_k(s) - P_k(0) + s^{k+1}\mathcal{E}_{k+1}(s).$$

We use a simple application of van der Corput's lemma. Write ϕ as

$$\phi(s) = Q_{k-1}(s) + a_k(s-s_1)^k/k! + s^{k+1}\mathcal{E}_{k+1}(s).$$

and observe

$$\phi^{(k-2)}(s) = a_k(s-s_1)(s+s_1)/2 + O(s^3),$$
$$\phi^{(k-3)}(s) = \widetilde{a}_{k-3} + a_k\frac{(s-s_1)^2}{2}\Big(\frac{2s_1}{3}+\frac{s}{3}\Big) + O(s^4).$$

The integrand of the integral defining E_R is supported where $|s-s_1|\ge s_1/80$, and $|s-s_1|\le c$ for small c. We see that

$$|\phi^{(k-2)}(s)| \ge c_0 b$$

if in addition $|s+s_1|\ge s_1/10$.

If $|s+s_1|\le s_1/10$, this lower bound breaks down; however, we have then

$$|\phi^{(k-3)}(s)| \ge cb^{3/2} - |\widetilde{a}_{k-3}(\omega)|.$$

Now on $\mathcal{V}_{k,j}(\delta)$ we have the restriction

$$|\widetilde{a}_{k-3}(\omega)| \le \delta b^{\frac{k-3}{2(k-1)}} R^{-\frac{2}{k-1}} \le \delta b^{3/2}$$

where the last inequality is equivalent to the imposed condition $b\ge R^{-2/k}$ (which holds when $j\ge 2$). Thus if δ is small we have $|\phi^{(k-3)}(s)|\approx b^{3/2}$ if $|s+s_1|\le s_1/10$.

We now split the integral into three parts (using appropriate adapted cut-off functions), namely where (i) $|s+s_1|\le s_1/10$, or (ii) $s+s_1\ge s_1/10$, or (iii) $s+s_1\le -s_1/10$. For parts (ii) and (iii) we can use van der Corput's lemma with $k-2$ derivatives and see that the corresponding integrals are bounded by $C(Rb)^{-1/(k-2)}$. Similarly for part (i), if $k>4$ we can use van der Corput's lemma with $(k-3)$ derivatives to see that the the corresponding integral is bounded by $C(Rb^{3/2})^{-1/(k-3)}$. The case $k=4$ requires a slightly different argument (as we do not necessarily have adequate monotonicity properties on ϕ'), however in the region (i) we now have $\phi''(s)=O(b)$, $|\phi'(s)|\gtrsim b^{3/2}$ and integrating by parts once gives the required bound $O((Rb^{3/2})^{-1})$ also in this case. Since $b\approx R^{-2/k}2^j$, the upper bound (6.20) follows. □

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