EMBEDDINGS FOR SPACES OF LORENTZ-SOBOLEV TYPE

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Abstract. The purpose of this paper is to characterize all embeddings for versions of Besov and Triebel-Lizorkin spaces where the underlying Lebesgue space metric is replaced by a Lorentz space metric. We include two appendices, one on the relation between classes of endpoint Mikhlin-Hörmander type Fourier multipliers, and one on the constant in the triangle inequality for the spaces $L^{p,r}$ when $p < 1$.

1. Introduction

We consider Lorentz space variants of the classical function space scales of Sobolev, Besov and Triebel-Lizorkin spaces for distributions on $\mathbb{R}^d$. We use the traditional Fourier analytical definition (*cf.* [27]) and work with an inhomogeneous Littlewood-Paley decomposition $\{\Lambda_k\}_{k=0}^\infty$ which is defined as follows. Pick a C^∞ function β_0 such that $\beta_0(\xi) = 1$ for $|\xi| \le 3/2$ and $\beta_0(\xi) = 0$ for $|\xi| \ge 7/4$. For $k \ge 1$ let $\beta_k(\xi) = \beta_0(2^{-k}\xi) - \beta_0(2^{1-k}\xi)$. Define Λ_k via the Fourier transform by $\widehat{\Lambda_k f} = \beta_k \widehat{f}$, $k = 0, 1, 2, \dots$.

Let $\mathcal{Y}$ be a rearrangement invariant quasi-Banach space of functions on $\mathbb{R}^d$, and define

$$\|f\|_{B^s_q[\mathcal{Y}]} = \Big(\sum_k 2^{ksq}\big\|\Lambda_k f\|^q_{\mathcal{Y}}\Big)^{1/q}, \tag{1}$$

$$\|f\|_{F^s_q[\mathcal{Y}]} = \Big\|\Big(\sum_k |2^{ks}\Lambda_k f|^q\Big)^{1/q}\Big\|_{\mathcal{Y}}. \tag{2}$$

When the functors B^s_q and F^s_q are applied to the Lebesgue spaces $\mathcal{Y} = L^p$ one gets the usual classes of Besov spaces $B^s_q[L^p] \equiv B^s_{p,q}$ and Triebel-Lizorkin spaces $F^s_q[L^p] \equiv F^s_{p,q}$. Here we take for $\mathcal{Y}$ a Lorentz space $L^{p,r}$, see §2 for definitions and a review of basic facts. Of course $L^{p,p} = L^p$. It is also customary to write $B^s_q[L^{p,r}] = B^s_{(p,r),q}$, $F^s_q[L^{p,r}] = F^s_{(p,r),q}$, although for better readability we prefer the functorial notation. For $q = 2$ one obtains the Lorentz versions of the Hardy-Sobolev spaces, also denoted by $H^s_{(p,r)}$.

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For the range $1 < p < \infty$ the space $H^s_{(p,r)} \equiv F^s_2[L^{p,r}]$ can be identified with a variant of Bessel-potential spaces (*cf.* [21],ch.V), namely we have

$$\|f\|_{F^s_2[L^{p,r}]} \approx \|(I-\Delta)^{s/2} f\|_{p,r}, \quad 1 < p < \infty. \tag{3}$$

These spaces have been used repeatedly in the literature (see e.g. [22], [5], [14], [9]), although our original motivation came from a result about embeddings in [8]. Applications suggest natural questions about the relation between these spaces, in particular the relation between Besov and Lorentz-Sobolev spaces. We formulate our results for nonhomogeneous versions of the above spaces, but the proofs can be extended to cover homogeneous versions ($\dot{F}$, $\dot{B}$) versions as well (*cf.* [29]). Our two main theorems characterize all embeddings which involve a space in the $B^s_q[L^{p,r}]$ family and a space in the $F^s_q[L^{p,r}]$ family.

Theorem 1.1. *Let $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1, r_0, r_1 \leq \infty$. The embedding*

$$B^{s_0}_{q_0}[L^{p_0,r_0}] \hookrightarrow F^{s_1}_{q_1}[L^{p_1,r_1}] \tag{4}$$

holds if and only if one of the following six conditions is satisfied.

(i) $s_0 - s_1 > d/p_0 - d/p_1 > 0$.
(ii) $s_0 > s_1$, $p_0 = p_1$, $r_0 \leq r_1$.
(iii) $s_0 - s_1 = d/p_0 - d/p_1 > 0$, $q_0 \leq r_1$.
(iv) $s_0 = s_1$, $p_0 = p_1 \neq q_1$, $r_0 \leq r_1$, $q_0 \leq \min\{p_1, q_1, r_1\}$.
(v) $s_0 = s_1$, $p_0 = p_1 = q_1 \geq r_0$, $r_0 \leq r_1$, $q_0 \leq \min\{p_1, r_1\}$.
(vi) $s_0 = s_1$, $p_0 = p_1 = q_1 < r_0$, $r_0 \leq r_1$, $q_0 < p_1$.

Theorem 1.2. *Let $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1, r_0, r_1 \leq \infty$. The embedding*

$$F^{s_0}_{q_0}[L^{p_0,r_0}] \hookrightarrow B^{s_1}_{q_1}[L^{p_1,r_1}] \tag{5}$$

holds if and only if one of the following six conditions is satisfied.

(i) $s_0 - s_1 > d/p_0 - d/p_1 > 0$.
(ii) $s_0 > s_1$, $p_0 = p_1$, $r_0 \leq r_1$.
(iii) $s_0 - s_1 = d/p_0 - d/p_1 > 0$, $r_0 \leq q_1$.
(iv) $s_0 = s_1$, $p_0 = p_1 \neq q_0$, $r_0 \leq r_1$, $q_1 \geq \max\{p_0, q_0, r_0\}$.
(v) $s_0 = s_1$, $p_0 = p_1 = q_0 \leq r_1$, $r_0 \leq r_1$, $q_1 \geq \max\{p_0, r_0\}$
(vi) $s_0 = s_1$, $p_0 = p_1 = q_0 > r_1$, $r_0 \leq r_1$, $q_1 > p_0$.

Remark 1.3. The interesting cases deal with the critical relation

$$s_0 - s_1 = d/p_0 - d/p_1, \tag{6}$$

when $p_0 < p_1$, and when $p_0 = p_1$. The case $p_0 < p_1$ in (iii) of the two theorems sheds some light on the sharp embedding theorems by Jawerth [12] and Franke [7]. Part (iii) of Theorem 1.2 extends and improves Jawerth's theorem stating that $F^{s_0}_{q_0}[L^{p_0}] \hookrightarrow B^{s_1}_{p_0}[L^{p_1}]$ for any $q_0 \leq \infty$, under the assumption (6), $p_0 < p_1$. Part (iii) of Theorem 1.1 extends the dual version of Franke stating that $B^{s_0}_{p_1}[L^{p_0}] \hookrightarrow F^{s_1}_{q_1}[L^{p_1}]$, for any $q_1 > 0$, again under the assumption (6), $p_0 < p_1$. For the Hardy-Sobolev case, $q_1 = 2$, a partial result of Theorem 1.1, (iii) can be found in [8], under the additional assumption $r_0 \le r_1$.

Remark 1.4. We shall see in Appendix A that an application of parts (iii) of Theorems 1.1 and 1.2 in tandem is useful to compare sharp versions of the Hörmander multiplier theorem in [17] and [9].

Parts (iv), (v), (vi) of both theorems deal with the endpoint case $s_0 = s_1$, $p_0 = p_1$ in (6). The conditions on the q_i and r_i are now more restrictive. The sufficiency of the conditions in (iv), (v), (vi) for (4), (5), resp., follow from corresponding embedding results for the spaces $\ell^q(L^{p,r})$ and $L^{p,r}(\ell^q)$ for sequences of functions $f = \{f_k\}_{k=0}^\infty$. It turns out that these results can be reduced to two types of triangle inequalities for Lorentz spaces. We note that the two strict inequalities in parts (vi) of both theorems can be traced to the failure of a triangle inequality in $L^{1,\rho}$ for $\rho > 1$. While considering the results in parts (iv), (v) of the two theorems we came across the question on how the constants in a generalized triangle inequality for quasi-norms in $L^{p,\rho}$ depend on ρ when $p < 1$ and $p < \rho < \infty$. This dependence is not crucial to our results but may be interesting in its own right, and we include a result as Appendix B.

The above theorems are complemented by more straightforward results about embeddings within the $B^s_q[L^{p,r}]$ and $F^s_q[L^{p,r}]$ scales of spaces.

Theorem 1.5. *Let $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1, r_0, r_1 \le \infty$. The embedding*

$$B^{s_0}_{q_0}[L^{p_0,r_0}] \hookrightarrow B^{s_1}_{q_1}[L^{p_1,r_1}] \tag{7}$$

holds if and only if one of the following four conditions is satisfied.

(i) $s_0 - s_1 > d/p_0 - d/p_1 > 0$.
(ii) $s_0 > s_1$, $p_0 = p_1$, $r_0 \le r_1$.
(iii) $s_0 - s_1 = d/p_0 - d/p_1 > 0$, $q_0 \le q_1$.
(iv) $s_0 = s_1$, $p_0 = p_1$, $r_0 \le r_1$, $q_0 \le q_1$.

Theorem 1.6. *Let $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1, r_0, r_1 \le \infty$. The embedding*

$$F^{s_0}_{q_0}[L^{p_0,r_0}] \hookrightarrow F^{s_1}_{q_1}[L^{p_1,r_1}] \tag{8}$$

holds if and only if one of the following four conditions is satisfied.

(i) $s_0 - s_1 > d/p_0 - d/p_1 > 0$.
(ii) $s_0 > s_1$, $p_0 = p_1$, $r_0 \le r_1$.
(iii) $s_0 - s_1 = d/p_0 - d/p_1 > 0$, $r_0 \le r_1$.
(iv) $s_0 = s_1$, $p_0 = p_1$, $r_0 \le r_1$, $q_0 \le q_1$.

It is noteworthy that in statements (iii) of Theorem 1.5, for the critical relation (6) and $p_0 < p_1$ the parameters r_0, r_1 in the Besov-Lorentz embeddings can be chosen arbitrary. Likewise in Theorem 1.6, (iii), the parameters q_0, q_1 are arbitrary. This result can be quickly derived from Theorems 1.1 and 1.2 (see §6) and extends results by Jawerth [12] for the Lebesgue space cases $p_0 = r_0$, $p_1 = r_1$.

This paper. In §2 we shall review basic facts on Lorentz spaces and related spaces $\ell^q(L^{p,r})$ and $L^{p,r}(\ell^q)$ for sequences of functions $f = \{f_k\}_{k=0}^{\infty}$. In §3 we also discuss various examples demonstrating the sharpness of the results; see in particular the overview in §3.1 for a guide where to find the proof of each necessary condition. In §4 we prove embedding relations between $\ell^q(L^{p,r})$ and $L^{p,r}(\ell^q)$, for fixed p, r, which imply the sufficiency of the conditions in parts (iv)-(vi) of Theorems 1.1 and 1.2. In §5 we give the proofs of the Lorentz improvements of the embedding theorems by Franke and Jawerth (i.e. parts (iii) of Theorems 1.1 and 1.2). The proofs of sufficiency are concluded in §6. In Appendix A we discuss some classes of Fourier multipliers and state some open problems. In Appendix B we prove the above mentioned result on the constant for the triangle inequality for $L^{p,r}$ when $p < 1$, $r < \infty$.

2. Review of basic facts on Lorentz spaces

We review some basic facts about Lorentz spaces, and refer the reader to [3], [4], [11], [24] for more information.

2.1. *Lorentz spaces via the distribution function.* Let (X, μ) be a measure space. For a measurable function f we let

$$\mu_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}),$$

be the distribution function and $f^*(t) = \inf\{\alpha : \mu_f(\alpha) \le t\}$ be the nonincreasing rearrangement of $|f|$. We shall assume that μ is non-atomic (i.e. every set of positive measure has a subset of smaller positive measure).

For $0 < p, r < \infty$, the standard quasi-norm on the Lorentz space $L^{p,r}$ is given by

$$\|f\|_{p,r} = \Big(\frac{r}{p}\int [t^{1/p} f^*(t)]^r \frac{dt}{t}\Big)^{1/r}, \tag{9}$$

moreover $\|f\|_{p,\infty} = \sup_t t^{1/p} f^*(t)$, $0 < p < \infty$. There is also an alternative description via the distribution function, namely

$$\|f\|_{p,r} = \Big(r\int [\mu_f(\alpha)^{1/p}\alpha]^r \frac{d\alpha}{\alpha}\Big)^{1/r}, \tag{10}$$

and $\|f\|_{p,\infty} = \sup_\lambda \lambda \mu_f(\lambda)^{1/p}$. One checks this for simple functions first, and then applies the monotone convergence theorem. The analogue for the case $r = \infty$ is done in Stein-Weiss [24, p.191], and the case $r < \infty$, for simple functions relies on similar summation by parts arguments.

For later use we state the usual embedding for fixed p, namely $L^{p,r} \hookrightarrow L^{p,q}$ for $r \le q$. In fact there is the sharp inequality

$$\|f\|_{p,q} \le \|f\|_{p,r}, \quad 0 < r \le q \le \infty\,. \tag{11}$$

A proof using rearrangements is in Stein-Weiss [24, p.192], but the proof of (11) could also be based on (10), *cf.* Lemma B.2 in the appendix.

2.2. *Sequences of functions.* The study of function spaces crucially relies on the study of the sequence spaces $L^{p,r}(\ell^q)$ and $\ell^q(L^{p,r})$. We shall work with the quasi-norms

$$\|f\|_{L^{p,r}(\ell^q)} := \Big\| \Big(\sum_k |f_k|^q \Big)^{1/q} \Big\|_{p,r}, \tag{12a}$$

$$\|f\|_{\ell^q(L^{p,r})} := \Big(\sum_k \|f_k\|_{p,r}^q \Big)^{1/q}. \tag{12b}$$

Throughout the paper the domains of the sequences will usually be $\mathbb{N}$, but it could be any finite or countable set with counting measure.

2.3. *Powers.* It will be convenient to use formulas for the distribution and rearrangement functions of $|f|^\sigma$, for any $\sigma > 0$, namely

$$\mu_{|f|^\sigma}(\alpha) = \mu_f(\alpha^{1/\sigma}) \tag{13}$$

and

$$(|f|^\sigma)^*(t) = (f^*(t))^\sigma \tag{14}$$

These follow directly from the definition of distribution and rearrangement functions. An immediate consequence is

$$\||f|^\sigma\|_{L^{p/\sigma,r/\sigma}} = \|f\|_{L^{p,r}}^\sigma. \tag{15}$$

Moreover, for sequences of functions $f = \{f_k\}$,

$$\big\|\{|f_k|^\sigma\}\big\|_{L^{p/\sigma,r/\sigma}(\ell^{q/\sigma})} = \|f\|_{L^{p,r}(\ell^q)}^\sigma, \tag{16a}$$

$$\big\|\{|f_k|^\sigma\}\big\|_{\ell^{q/\sigma}(L^{p/\sigma,r/\sigma})} = \|f\|_{\ell^q(L^{p,r})}^\sigma\,. \tag{16b}$$

2.4. *Sums.* The expression (9) is not a norm unless $1 \le r \le p$. It is well known that the spaces $L^{p,r}$ are normable for $p > 1$ and $r \ge 1$; one replaces f^* by the maximal function f^{**} in the definition of the Lorentz spaces to get an equivalent expression which is a norm. We write

$$|\!|\!| f |\!|\!|_{p,r} = \Big(\int_0^\infty t^{r/p} f^{**}(t)^r \frac{dt}{t} \Big)^{1/r}.$$

We also use $|\!|\!| f |\!|\!|_{L^{p,r}(\ell^q)}$, $|\!|\!| f |\!|\!|_{\ell^q(L^{p,r})}$ for the expressions corresponding to (12), but with the **-functions. See [11] or [3]. The additivity property holds when the measure space is nonatomic, since in these cases we have a triangle inequality for

$$f \mapsto f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds = \sup_{E:\mu(E)\le t} \frac{1}{t}\int_E |f| d\mu \,. \tag{17}$$

See [11] or [3, ch.2]. The true norms can be used to prove duality theorems; one identifies the dual of $L^{p,q}$, $1 < p < \infty$, $1 \le q < \infty$ with $L^{p',q'}$. This also works on discrete spaces, with counting measure (see [3, ch.2.4]).

If we formulate the triangle inequality with the original quasi-norms in (9) we get for nonatomic μ,

$$\Big\| \sum_k f_k \Big\|_{p,r} \le C_{p,r} \sum_k \big\| f_k \big\|_{p,r}, \ 1 < p < \infty, \ r \ge 1, \tag{18}$$

with $C_{p,r} = C_p = (1 - p^{-1})^{-1}$. This is proved using the additivity of the functional in (17) in combination with Minkowski's and Hardy's inequalities ([3, p. 124]). Lorentz [15] showed that one can take $C_{p,r} = 1$ for $1 \le r \le p$. Barza, Kolyada and Soria [2] showed for $1 < p < r$ that the best constant $C_{p,r}$ in (18) is given by $(p/r)^{1/r}(p'/r')^{1/r'}$.

One can use (16) and (18) for the space $L^{p/u,r/u}$ to get

$$\Big\| \sum_k f_k \Big\|_{p,r} \lesssim_{p,r,u} \Big(\sum_k \big\| f_k \big\|_{p,r}^u \Big)^{1/u}, \ u < \min\{p, r, 1\}. \tag{19}$$

However this can be improved in some cases. The analogue of (18) *fails* for $L^{1,r}$, $r > 1$, (*cf.* [25] for a weaker substitute) but there is a different kind of triangle inequality for $p < 1$, for the pth power of $\|\cdot\|_{p,r}$, which gives

$$\Big\| \sum_k f_k \Big\|_{p,r} \le C(p,r) \Big(\sum_k \| f_k \|_{p,r}^p \Big)^{1/p}, \quad p < 1, \quad r \ge p; \tag{20}$$

here $C(p,r) \le (\frac{2-p}{1-p})^{1/p}$. This was proved for $r = \infty$ by Stein, Taibleson and Weiss [23], (see also Kalton [13], and unpublished work of Pisier and Zinn mentioned in [13]). It is easy to modify the proof in [23] to cover the cases $p < r < \infty$ with the same constant. However for $r = p$ one can put of course put $C(p,p) = 1$ in (20) which suggests that the behavior of $C(p,r)$ should improve for $r > p$ as r decreases. We shall prove such a result in Appendix B and show that for $0 < p < 1$, $p < r < \infty$

$$C(p,r) \le A^{1/p} \Big(\frac{1}{1-p} \Big)^{1/p-1/r} \Big(1 + \frac{p}{r} \log \frac{1}{1-p} \Big)^{1/p-1/r} \tag{21}$$

and A does not depend on p and r. The precise behavior of $C(p,r)$ is not relevant for the results in this paper, but (21) should be interesting in its own right. Note that the logarithmic term in (21) vanishes as $r \to p+$ and as $r \to \infty$. It would be interesting to get more precise information on $C(p,r)$, in particular one would like to know whether the logarithmic term is necessary for $p < r < \infty$.

2.4.1. *Computations of some lower bounds.* Suppose we are given $b > 0$ and sets A_j, indexed by $j \in \mathcal{Z} \subset \mathbb{Z}$ such that

$$\mu(A_j) \geq b\rho^j, \;\; j \in \mathcal{Z} \tag{22a}$$

for some $\rho > 1$. Assume that, for a nonnegative sequence $\beta \in \ell^r(\mathbb{Z})$,

$$|f(x)| \geq \sum_{j \in \mathcal{Z}} \beta(j)\rho^{-j/p} \mathbb{1}_{A_j}(x) \;\; \text{a.e.} \tag{22b}$$

Then

$$\|f\|_{p,r} \gtrsim \Big(\sum_{j \in \mathcal{Z}} |\beta(j)|^r \Big)^{1/r}. \tag{22c}$$

To see this observe that the distribution function satisfies $\mu_f(\beta(j)\rho^{-\frac{j+1}{p}}) \geq \mu(A_j) > b\rho^{j-1}$ and therefore $f^*(b\rho^{j-1}) \geq \beta(j)\rho^{-\frac{j+1}{p}}$ by definition of the rearrangement function. This easily implies (22c), under the assumption (22b).

2.4.2. *Computations of some upper bounds.* We now replace $\mathcal{Z}$ by $\mathbb{Z}$ and add the assumption that $n \mapsto \beta(n)2^{-n/p}$ *is nonincreasing.* Assume that $\{F_n\}_{n \in \mathbb{Z}}$ is a sequence of measurable sets such that

$$\mu(F_n) \leq B\rho^n, \;\; n \in \mathbb{Z}, \tag{23a}$$

for some $\rho > 1$, $B > 0$ and assume that,

$$|f(x)| \leq \sum_{n \in \mathbb{Z}} \beta(n)2^{-n/p} \mathbb{1}_{F_n}(x) \;\; \text{a.e.} \tag{23b}$$

Then

$$\|f\|_{p,r} \lesssim \Big(\sum_{n \in \mathbb{Z}} |\beta(n)|^r \Big)^{1/r}. \tag{23c}$$

To see this observe that

$$\mu_f(\beta(n)2^{-\frac{n-1}{p}}) \leq \mu(\cup_{j \leq n} F_j) \leq B \sum_{j \leq n} \rho^j \leq B(\rho-1)^{-1}\rho^{n+1}$$

and therefore $f^*(\frac{B}{\rho-1}\rho^{n+1}) \leq \beta(n)\rho^{-\frac{n-1}{p}}$. This easily implies (23c).

3. Necessary conditions

3.1. *Guide through this section.* Many examples for embedding relations of spaces of Hardy-Sobolev type have been discussed in the literature (e.g [26], [20]), and most of our examples are at least related to those earlier examples.

The necessity of the condition $p_0 \leq p_1$, and the necessity of the condition $r_0 \leq r_1$ in the case whenever $p_0 = p_1$, in all four Theorems 1.1, 1.2, 1.5, 1.6, is proved in §3.3. The necessity of the condition $s_0 - s_1 \geq d/p_0 - d/p_1 \geq 0$ in all four theorems is proved in §3.4.

Consider the case $s_0 - s_1 = d/p_0 - d/p_1 > 0$ which is case (iii) in all four theorems. The necessity of the condition $q_0 \le r_1$ in Theorem 1.1 (iii), the necessity of the condition $r_0 \le q_1$ in Theorem 1.2 (iii), the necessity of the condition $r_0 \le r_1$ in Theorem 1.5 (iii), and the necessity of the condition $q_0 \le q_1$ in Theorem 1.6 (iii), are all proved in §3.5.

Necessary conditions in the case $s_0 = s_1$ and $p_0 = p_1$. In Theorem 1.1 (iv), (v) the necessity of the condition $q_0 \le p_1$ is shown in §3.6.1, the necessity of the condition $q_0 \le q_1$ is shown in §3.6.2, and the necessity of the condition $q_0 \le r_1$ is shown in §3.5. In addition, for the case $p_1 = q_1 < r_0$ in part (vi), we must have the strict inequality in $q_0 < p_1$; this follows from (37) in §3.6.3.

In Theorem 1.2 (iv), (v) the necessity of the condition $q_1 \ge p_0$ is shown in §3.6.1, the necessity of the condition $q_0 \ge q_1$ is shown in §3.6.2, and the necessity of the condition $q_1 \ge r_0$ is shown in §3.5. Moreover, for the case $p_0 = q_0 > r_1$ in part (vi), we must have the strict inequality in $q_1 > p_0$; this follows from (38) in §3.6.3.

The necessity of the conditions $r_0 \le r_1$ in Theorems 1.5, (iv), and 1.6, (iv), is shown in §3.3 (as already pointed out). The necessity of the conditions $q_0 \le q_1$ in Theorems 1.5, (iv), and 1.6, (iv), is shown in §3.6.2.

3.2. *Preliminaries.* In what follows we let ψ_0 be a C^∞ function supported on $\{x : |x| \le 1/8\}$ such that $\widehat{\psi_0}(\xi) \ne 0$ for $|\xi| \le 2$ and such that

$$\int \psi_0(x)dx = 1, \quad \int \psi_0(x)x_1^{\nu_1} \dots x_d^{\nu_d} dx = 0, \qquad \text{for multiindices } \nu \text{ with } \nu_i \ge 0,\ 0 < \sum_{i=1}^d \nu_i \le M. \tag{24}$$

Here we assume that M is large, specifically given p, q, s, the condition

$$M > |s| + 100d \max\{1, 1/p, 1/q\}$$

will certainly be sufficient for our purposes. Let

$$\psi_k = 2^{kd}\psi_0(2^k\cdot) - 2^{(k-1)d}\psi_0(2^{k-1}\cdot), \quad k \ge 1, \tag{25a}$$

$$\mathcal{L}_k f = \psi_k * f \tag{25b}$$

We can arrange ψ_0 so that ψ_1 satisfies

$$\psi_1 * \psi_1(x) \ge c \text{ for } x \in I = (-\varepsilon, \varepsilon)^d \tag{26}$$

for some fixed $\varepsilon > 0$. We have (using Littlewood-Paley decompositions generated by dilates of compactly supported functions, aka local means in

[28])

$$\|f\|_{B^s_q[L^{p,r}]} \approx \Big(\sum_{k=0}^\infty 2^{ksq}\|\mathcal{L}_k f\|_{p,r}^q\Big)^{1/q},$$

$$\|f\|_{F^s_q[L^{p,r}]} \approx \Big\|\Big(\sum_{k=0}^\infty 2^{ksq}|\mathcal{L}_k f|^q\Big)^{1/q}\Big\|_{p,r}.$$

These equivalences follow from the standard corresponding statements for $p = r$ (*cf.* [28]) by real interpolation arguments.

We will repeatedly need the following straightforward lemma.

Lemma 3.1. *Let W be a finite collection of points in $\mathbb{R}^d$, with mutual distance at least 2^{-n}. Define $h(x) = \sum_{w\in W}\psi(2^n(x-w))$. Then*

$$\|\mathcal{L}_j h\|_\infty \lesssim \begin{cases} 2^{(n-j)(M+1)} & \text{if } j \ge n, \\ 2^{(j-n)(M+1-d)} & \text{if } n \ge j. \end{cases} \tag{27}$$

Moreover,

$$\|\mathcal{L}_j h\|_{p,r} \lesssim 2^{(n-j)M}(2^{-nd}\,\#W)^{1/p}, \quad \text{if } j \ge n. \tag{28a}$$

and for $j = n$ we have the equivalence

$$\|\mathcal{L}_n h\|_{p,r} \approx (2^{-nd}\#W)^{1/p}. \tag{28b}$$

For $j \le n$ let $W(j)$ be any maximal 2^{-j}-separated subset of W. Then

$$\|\mathcal{L}_j h\|_{p,r} \lesssim 2^{(j-n)(M-d)}(2^{-jd}\,\#W(j))^{1/p}, \quad \text{if } j \le n. \tag{28c}$$

Proof. Note that the upper bounds need to be proved only for $p = r$ as they follow then for all r by real interpolation. Let $h_w = \psi(2^n(\cdot - w))$. The derivation is straightforward; we use the moment condition on the convolution kernel ψ_j for the operator $\mathcal{L}_j$ to bound $|\mathcal{L}_j h_w(x)| \lesssim 2^{(n-j)(M+1)}$ for $j > n$. The corresponding L^p bound follows as the supports of $\mathcal{L}_j h_w$ are essentially disjoint. When $j \le n$ we use the moment condition on h_w to bound $|\mathcal{L}_j h_w(x)| \lesssim 2^{(j-n)(M+1)}$. The bound for $\mathcal{L}_j h$ follows since $\mathcal{L}_j h_w(x)$ is nonzero for at most $O(2^{(n-j)d})$ terms (and one gets improvements for sparse W). The corresponding L^p bound is then an immediate consequence.

In order to obtain the lower bound for $\|\mathcal{L}_n h\|_{p,r}$ we use the assumption (26) to see that $|\mathcal{L}_n h| \ge c$ on a set of measure $2^{-nd}\,\#W$. □

In what follows we shall denote by $B(x,\rho)$ the ball of radius ρ centered at x.

3.3. *Necessity of $p_0 \le p_1$, and of $r_0 \le r_1$ in the case $p_0 = p_1$.* Let ψ_1 be as in §3.2, $e_1 = (1, 0, \dots, 0\}$, and let

$$f(x) = \sum_{n=1}^\infty a_n \psi_1(x - ne_1)\,.$$

It is easy to see by (27) and Minkowski's inequality that for any M

$$|\mathcal{L}_k f(x)| \lesssim 2^{-kM} \sum_{n=1}^{\infty} |a_n| \mathbb{1}_{B(ne_1,1)}(x).$$

This implies $\|\mathcal{L}_k f\|_{p,r} \lesssim 2^{-kM} \|a\|_{\ell^p}$. We also have

$$|\mathcal{L}_1 f(x)| = \Big| \sum_{n=1}^{\infty} a_n \psi_1 * \psi_1(x - ne_1) \Big| \geq \sum_{n=1}^{\infty} |a_n| \mathbb{1}_{B(ne_1,\varepsilon)}(x)$$

which implies the lower bound $\|\mathcal{L}_1 f\|_{p,r} \gtrsim \|a\|_{\ell^{p,r}}$. It follows that $p_0 \leq p_1$ in all cases.

The same calculation proves the necessary condition $r_0 \leq r_1$ in the case $p_0 = p_1$.

3.4. *Necessity of* $s_0 - s_1 \geq d(1/p_0 - 1/p_1)$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be supported in the ball of radius 10^{-2} centered at 1. Let β_k be as in the definition of Λ_k in the introduction, so that for $k \geq 1$ we have $\beta_k(\xi) = 0$ when $2^{-k}|\xi| \notin [\frac{3}{4}, \frac{7}{4}]$ and $\beta_k(\xi) = 1$ when $2^{-k}|\xi| \in [\frac{7}{8}, \frac{3}{2}]$. Let $\omega_k = 2^{kd} \mathcal{F}^{-1}[\chi](2^k x)$ and notice that $\Lambda_k \omega_k = \omega_k$ and $\Lambda_j \omega_k = 0$ when $j \neq k$. We have by scaling $\|\Lambda \omega_k\|_{p,r} = 2^{k(d-d/p)} \|\mathcal{F}^{-1}[\chi]\|_{p,r}$ and thus any of the embeddings in the four theorems in the introduction requires $2^{k(s_1 - d/p_1)} \lesssim 2^{k(s_0 - d/p_0)}$ for $k \geq 0$. Hence $s_0 - s_1 \geq d(1/p_0 - 1/p_1)$.

3.5. *Necessary conditions for the case* $s_0 - s_1 = d(1/p_0 - 1/p_1) \geq 0$. Let $R \gg 8$ be large and let $\{n_l\}_{l=1}^\infty$ be an increasing sequence of integers which is sufficiently separated, i.e. such that $n_l \gg l \geq R$, $n_{l+1} - n_l \geq R$. Let $\mathfrak{N} := \{n_l : l \in \mathbb{N}\}$. Let $\{a_l\}_{l=1}^\infty$ be a decreasing sequence for which $l \mapsto 2^{n_l d/p}|a_l|$ is increasing. Define $\Psi_n(x) := \psi_1(2^n(x - 2^{-n}e_1))$, with ψ_1 as in §3.2 and

$$h_\gamma(x) = \sum_{l=1}^{\infty} a_l 2^{n_l \gamma} \Psi_{n_l}(x). \tag{29}$$

Lemma 3.2. *Let* $s \in \mathbb{R}$. *If the separation constant* R *in the definition of* $\mathfrak{N}$ *is sufficiently large then*

$$\big\| h_{-s+d/p} \big\|_{B_q^s[L^{p,r_0}]} \approx \|a\|_{\ell^q}, \tag{30}$$

$$\big\| h_{-s+d/p} \big\|_{F_\rho^s[L^{p,r}]} \approx \|a\|_{\ell^r}. \tag{31}$$

As an immediate consequence we get

Corollary 3.3. *Suppose* $s_0 - s_1 = d/p_0 - d/p_1 \geq 0$. *Then we have the following implications*

$$\begin{aligned}
B_{q_0}^{s_0}[L^{p_0,r_0}] \hookrightarrow B_{q_1}^{s_1}[L^{p_1,r_1}] &\implies q_0 \leq q_1, \\
F_{q_0}^{s_0}[L^{p_0,r_0}] \hookrightarrow F_{q_1}^{s_1}[L^{p_1,r_1}] &\implies r_0 \leq r_1, \\
B_{q_0}^{s_0}[L^{p_0,r_0}] \hookrightarrow F_{q_1}^{s_1}[L^{p_1,r_1}] &\implies q_0 \leq r_1, \\
F_{q_0}^{s_0}[L^{p_0,r_0}] \hookrightarrow B_{q_1}^{s_1}[L^{p_1,r_1}] &\implies r_0 \leq q_1.
\end{aligned}$$

Proof of Lemma 3.2. Let $u < \min\{p, q, r_0\}$. By (19) we have

$$\|h_\gamma\|_{B^s_q[L^{p,r_0}]} \lesssim \Big(\sum_{j=0}^\infty 2^{jsq}\Big(\sum_{l=1}^\infty \big\|a_l 2^{n_l\gamma}\mathcal{L}_j\Psi_{n_l}\big\|_{p,r_0}^u\Big)^{q/u}\Big)^{1/q}.$$

We use Lemma 3.1 for a singleton W to estimate for $\gamma = -s + d/p$, the right hand side in the last display by a constant times

$$\begin{aligned}
&\Big(\sum_{j=0}^\infty 2^{jsq}\Big(\sum_{l=1}^\infty |a_l|^u 2^{n_l(\gamma-d/p)u}2^{-|j-n_l|(M-2d)u}\Big)^{q/u}\Big)^{1/q}\\
&\lesssim \Big(\sum_{j=0}^\infty \Big(\sum_{l=1}^\infty |a_l|^u 2^{-|j-n_l|(M-2d-|s|)u}\Big)^{q/u}\Big)^{1/q}\\
&\lesssim \Big(\sum_{j=0}^\infty\sum_{l=1}^\infty |a_l|^q 2^{-|j-n_l|q}\Big)^{1/q} \lesssim \Big(\sum_{l=1}^\infty |a_l|^q\Big)^{1/q};
\end{aligned}$$

in this calculation we have used $M-2d-|s|>1$. We have proved the upper bound in (30).

For the lower bound we estimate

$$\|h_{-s+d/p}\|_{B^s_q[L^{p,r_0}]} \gtrsim \Big(\sum_k 2^{n_k sq}\big\|\mathcal{L}_{n_k}h_{-s+d/p}\|_{p,r_0}^q\Big)^{1/q} \ge cI - CII$$

where

$$\begin{aligned}
I &= \Big(\sum_k |a_k|^q 2^{n_k d/p}\|\mathcal{L}_{n_k}\Psi_{n_k}\|_{p,r_0}^q\Big)^{1/q},\\
II &= \Big(\sum_k 2^{n_k sq}\Big\|\sum_{\substack{l\ge 1:\\ l\ne k}} a_l 2^{-n_l s}2^{n_l d/p}\mathcal{L}_{n_k}\Psi_{n_l}\Big\|_{p,r_0}^q\Big)^{1/q}.
\end{aligned}$$

We have (using (26)) $2^{n_k d/p}\|\mathcal{L}_{n_k}\Psi_{n_k}\|_{p,r_0} \ge c > 0$ uniformly in k and therefore $I \gtrsim \|a\|_{\ell^q}$. The above computation for the upper bound also gives $II \lesssim 2^{-R}\|a\|_{\ell^q}$ and the lower bound in (30) follows if R is chosen sufficiently large.

We now turn to the proof of (31). For the upper bound we may assume without loss of generality that $\rho < \min\{1, r, p\}$. Then by Lemma 3.1

$$\Big(\sum_{j=0}^\infty\Big|2^{js}\sum_{l=1}^\infty a_l 2^{n_l(\frac{d}{p}-s)}\mathcal{L}_j\Psi_{n_l}(x)\Big|^\rho\Big)^{1/\rho} \le \mathcal{E}_1(x) + \mathcal{E}_2(x)$$

where

$$\mathcal{E}_1(x) = \Big(\sum_{j=0}^{\infty} 2^{js\rho} \sum_{\substack{l\in\mathbb{N} \\ n_l \le j}} |a_l|^\rho 2^{n_l(\frac{d}{p}-s)\rho} 2^{-(j-n_l)(M+1)\rho} \mathbb{1}_{B(0,2^{1-n_l})}(x) \Big)^{1/\rho},$$

$$\mathcal{E}_2(x) = \Big(\sum_{j=0}^{\infty} 2^{js\rho} \sum_{\substack{l\in\mathbb{N} \\ n_l > j}} |a_l|^\rho 2^{n_l(\frac{d}{p}-s)\rho} 2^{-(n_l-j)(M+1-d)\rho} \mathbb{1}_{B(0,2^{1-j})}(x) \Big)^{1/\rho}.$$

Interchanging the n_l, j summations and summing a geometric series (where $M+1 > |s|$) yields

$$\mathcal{E}_1(x) \le \Big(\sum_l |a_l|^\rho 2^{n_l \frac{d}{p}\rho} \mathbb{1}_{B(0,2^{1-n_l})}(x) \Big)^{1/p}$$

and, with the parameter $m = n_l - j$,

$$\mathcal{E}_2(x) \le \Big(\sum_{m=0}^{\infty} 2^{-m(M+1-d+s-\frac{d}{p})\rho} \mathcal{E}_{2,m}(x)^\rho \Big)^{1/\rho} \tag{33}$$

where

$$\mathcal{E}_{2,m}(x) = \Big(\sum_{\substack{l: \\ n_l \ge m}} |a_l|^\rho 2^{(n_l-m)\frac{d}{p}\rho} \mathbb{1}_{B(0,2^{1-n_l+m})}(x) \Big)^{1/\rho}.$$

We use (15), and §2.4.2 with the parameter $\rho = 2^{-d}$ and with exponents $(p/\rho, r/\rho)$ in place of (p,r), to get

$$\|\mathcal{E}_1\|_{p,r} = \|\mathcal{E}_1^\rho\|_{p/\rho,r/\rho}^{1/\rho} \lesssim \|a\|_{\ell^r}.$$

Similarly $\|\mathcal{E}_{2,m}\|_{p,r} \lesssim \|a\|_{\ell^r}$ uniformly in m, and then from (33) we also get $\|\mathcal{E}_2\|_{p,r} \lesssim \|a\|_{\ell^r}$.

For the lower bound

$$\Big(\sum_{j=0}^{\infty} \Big| 2^{js} \sum_{l=1}^{\infty} a_l 2^{n_l(\frac{d}{p}-s)} \mathcal{L}_j \Psi_{n_l}(x) \Big|^\rho \Big)^{1/\rho}$$
$$\ge \Big(\sum_{k=1}^{\infty} \Big| 2^{n_k s} \sum_{l=1}^{\infty} a_l 2^{n_l(\frac{d}{p}-s)} \mathcal{L}_{n_k} \Psi_{n_l}(x) \Big|^\rho \Big)^{1/\rho} \ge \mathcal{E}_3(x) - \mathcal{E}_4(x)$$

where

$$\mathcal{E}_3(x) = \Big(\sum_{k=1}^{\infty} \Big| a_k 2^{n_k \frac{d}{p}} \mathcal{L}_{n_k} \Psi_{n_k}(x) \Big|^\rho \Big)^{1/\rho},$$

$$\mathcal{E}_4(x) = \Big(\sum_{k=1}^{\infty} \Big| 2^{n_k s} \sum_{l \ne k} a_l 2^{n_l(\frac{d}{p}-s)} \mathcal{L}_{n_k} \Psi_{n_l}(x) \Big|^\rho \Big)^{1/\rho}.$$

Note that $\mathcal{L}_{n_k}\Psi_{n_k}(x) = \psi_1 * \psi_1(2^{n_k}x - e_1)$ and thus $|\mathcal{L}_{n_k}\Psi_{n_k}(x)| \geq c > 0$ on $B(2^{-n_k}e_1, 2^{-n_k}\varepsilon)$. Hence

$$\mathcal{E}_3(x) \geq c\Big(\sum_{k=1}^{\infty}\Big|a_k 2^{n_k\frac{d}{p}}\mathbb{1}_{B(2^{-n_k}e_1,2^{-n_k}\varepsilon)}(x)\Big|^{\rho}\Big)^{1/\rho},$$

which by §2.4.1 implies $\|\mathcal{E}_3\|_{p,r} \gtrsim \|a\|_{\ell^r}$. Analyzing the proof of the upper bound and taking into account the R-separation of the numbers n_l yields $\|\mathcal{E}_4\|_{p,r} \lesssim 2^{-R}\|a\|_{\ell^r}$ and for large R we get the lower bound in (31). □

3.6. *The case $s_0 = s_1 = s$, $p_0 = p_1 = p$.*

3.6.1. *Necessary conditions on (p, q_0) and (p, q_1) in Theorems 1.1, 1.2.* Let $R \gg 8$ be large and $\mathfrak{N} = \{n_l : l \in \mathbb{N}\}$ be as in §3.5, i.e. $l \mapsto n_l$ is increasing and R-separated.

Lemma 3.4. *Let $a = \{a\}_{l=1}^{\infty}$ be a sequence such that $l \mapsto |a_l|$ is nonincreasing. Let, for $\nu \in \mathbb{Z}^d$, $g_{l,\nu}(x) = \psi(2^{n_l}(x - (le_1 + 2^{-n_l}\nu)))$ and let*

$$g_l(x) = \sum_{\substack{2^3 \leq \nu_i \leq 2^{n_l - 3}\\ i=1,\dots,d}} g_{l,\nu}(x).$$

Define $f(x) = \sum_{l=1}^{\infty} 2^{-sn_l}a_l g_l(x)$. Then

$$\|f\|_{B^s_q[L^{p,r}]} \approx \Big(\sum_l |a_l|^q\Big)^{1/q}, \tag{34}$$

$$\|f\|_{F^s_q[L^{p,r}]} \approx \Big(\sum_{l=1}^{\infty} l^{\frac{r}{p}-1}|a_l|^r\Big)^{1/r}, \tag{35}$$

provided that the separation constant R is large enough.

Note that the expression of the right hand side of (35) is equivalent to the $\ell^{p,r}$ norm of a (if a is nonincreasing).

The lemma implies the following corollary relevant for Theorems 1.1 and 1.2.

Corollary 3.5.

$$B^s_{q_0}[L^{p,r_0}] \hookrightarrow F^s_{q_1}[L^{p,r_1}] \implies q_0 \leq p.$$
$$F^s_{q_0}[L^{p,r_0}] \hookrightarrow B^s_{q_1}[L^{p,r_1}] \implies p \leq q_1.$$

Proof of Lemma 3.4. By Lemma 3.1

$$\|\mathcal{L}_j g_l\|_{p,r} \lesssim 2^{-|n_l - j|(M-d)}, \tag{36a}$$

$$\|\mathcal{L}_{n_l} g_l\|_{p,r} \approx 1. \tag{36b}$$

We first establish the upper bound in (34) and let $u < \min\{p, q, r\}$. Estimate using (19), (36a) and then Hölder's inequality

$$\begin{aligned}\|f\|_{B^s_q[L^{p,r}]} &\lesssim \Big(\sum_j 2^{jsq}\Big(\sum_l |a_l|^u 2^{-sn_l u}\big\|\mathcal{L}_j g_l\big\|^u_{p,r}\Big)^{q/u}\Big)^{1/q} \\ &\lesssim \Big(\sum_j \Big(\sum_l |a_l|^u 2^{-(M-d-|s|)|j-n_l|u}\Big)^{q/u}\Big)^{1/q} \\ &\lesssim \Big(\sum_j \sum_l |a_l|^q 2^{-|j-n_l|q}\Big)^{1/q} \lesssim \Big(\sum_l |a_l|^q\Big)^{1/q}.\end{aligned}$$

where we used $M - d - |s| > 1$.

For the lower bound in (34) we have

$$\begin{aligned}\|f\|_{B^s_q[L^{p,r}]} &\gtrsim \Big(\sum_{k=1}^\infty 2^{n_k sq}\Big\|a_k 2^{-sn_k}\mathcal{L}_{n_k} g_{n_k} + \sum_{\substack{l\in\mathbb{N}\\ l\neq k}} a_l 2^{-sn_l}\mathcal{L}_{n_k} g_{n_l}\Big\|^q_{p,r}\Big)^{1/q} \\ &\geq c_I I - C_{II} II\end{aligned}$$

where

$$\begin{aligned}I &= \Big(\sum_{k=1}^\infty |a_k|^q \|\mathcal{L}_{n_k} g_{n_k}\|^q_{p,r}\Big)^{1/q} \\ II &\leq \Big(\sum_{k=1}^\infty 2^{n_k sq}\Big\|\sum_{\substack{l\in\mathbb{N}\\ l\neq k}} a_l 2^{-sn_l}\mathcal{L}_{n_k} g_{n_l}\Big\|^q_{p,r}\Big)^{1/q}\end{aligned}$$

By (36b) we get $I \gtrsim \|a\|_{\ell^q}$. Using the argument for the upper bound given above and taking account (36a) with $j = n_k \neq n_l$ yields $II \lesssim 2^{-R}\|a\|_{\ell^q}$. Thus if R is large enough we obtain the lower bound in (34).

We now prove the upper bound in (35). By the L^∞ bounds in Lemma 3.1 we have

$$\Big(\sum_{j=0}^\infty 2^{-jsq}|\mathcal{L}_j f(x)|^q\Big)^{1/q} \leq CG(x)$$

where $G(x) = \sum_{l=1}^\infty |a_l|\mathbb{1}_{Q_l}(x)$, with $Q_l = l + [-1/4, 1/4]^d$. The rearrangement of G satisfies

$$G^*(t) \leq \sum_{l=1}^\infty |a_l|\mathbb{1}_{((l-1)2^{-d}, l2^{-d}]}(t)$$

and we obtain

$$\|f\|_{F^s_q[L^{p,r}]} \lesssim \Big(\frac{r}{p}\int_0^\infty [t^{1/p}G^*(t)]^r \frac{dt}{t}\Big)^{1/r} \lesssim \Big(\sum_{l=1}^\infty l^{\frac{r}{p}-1}|a_l|^r\Big)^{1/r}.$$

For the lower bound we estimate

$$\|f\|_{F^s_q[L^{(p,r)}]} \ge \Big\| \Big(\sum_l 2^{n_l sq} |\mathcal{L}_{n_l} f|^q \Big)^{1/q} \Big\|_{L^{p,r}} \ge cI' - CII'$$

where

$$I' = \Big\| \Big(\sum_{l=1}^N \big| a_l \mathcal{L}_{n_l} g_{n_l} \big|^q \Big)^{1/q} \Big\|_{p,r},$$

$$II' = \Big\| \Big(\sum_{l=1}^N 2^{n_l sq} \Big| \sum_{\substack{k\ge 1:\\ k\ne l}} a_k 2^{-sn_k} \mathcal{L}_{n_l} g_{n_k} \Big|^q \Big)^{1/q} \Big\|_{p,r}.$$

Let $J_{l,\nu,\varepsilon} = \{x : |x - le_1 - \nu 2^{-n_l}| \le 2^{-n_l}\varepsilon\}$ and let $J_{l,\varepsilon}$ be the union of the $J_{l,\nu,\varepsilon}$ over all $\nu \in \mathbb{Z}^d$ with $8 \le \nu_i \le 2^{n_l-3}$ for $i = 1, \dots, d$. Notice that $J_{l,\varepsilon}$ is contained in a cube of sidelength $1/2$ centered at le_1. By the condition (26) we have $|\mathcal{L}_{n_l} g_{n_l,\nu}(x)| \ge c$ for $x \in J_{l,\nu,\varepsilon}$. We have

$$\Big(\sum_{l=1}^N \big| a_l \mathcal{L}_{n_l} g_{n_l}(x) \big|^q \Big)^{1/q} \ge G_{\text{low}}(x)$$

where

$$G_{\text{low}}(x) = \sum_{l=1}^N |a_l| \sum_{\nu=2^3}^{2^{n_l-3}} \mathbb{1}_{J_{l,\nu,\varepsilon}}(x).$$

Note that the measure of $J_{l,\varepsilon}$ is at least $c_0\varepsilon^d$ for some fixed positive c_0. Hence

$$G^*_{\text{low}}(t) \ge \sum_{l=1}^\infty |a_l| \mathbb{1}_{(c_0\varepsilon^d(l-1), c_0\varepsilon^d l]}(t)$$

and thus

$$I' \ge \|G_{\text{low}}\|_{L^{p,r}} \ge c' \Big(\sum_{l=1}^N l^{\frac{r}{p}-1} |a_l|^r \Big)^{1/r}.$$

For II' we get a better upper bound. By the argument for the upper bound above we obtain due to separateness condition of the n_l

$$II' \le C2^{-R} \Big(\sum_{l=1}^N l^{\frac{r}{p}-1} |a_l|^r \Big)^{1/r}.$$

Thus if R in our definition of the n_l is chosen large enough we get the lower bound

$$\|f\|_{F^s_q[L^{p,r}]} \ge c'' (\sum_{l=1}^N l^{\frac{r}{p}-1} |a_l|^r)^{1/r}$$

provided that the right hand side is finite. $\square$

3.6.2. *Conditions on q_0, q_1 when $p_0 = p_1$, $s_0 = s_1$.* Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be supported in $\{\xi : |\xi| \le 2^{-4}\}$ such that $\chi(0) = 1$, and $\widehat{\chi}(0) = 1$. Let, given $N \in \mathbb{N}$, $f(x) = \sum_{l=N+1}^{2N} 2^{-ls} a_l \eta_l(x)$ with $\eta_l = \mathcal{F}^{-1}[\chi(\cdot - 2^l e_1)]$. Using the support properties of β_k (*cf.* §3.4) we have, for $l > 1$, $\Lambda_l \eta_l = \eta_l$ and $\Lambda_k \eta_l = 0$ for $k \ne l$. Hence for large N

$$\|f\|_{B^s_{q_0}[L^{p,r}]} \approx \Big(\sum_{l=N+1}^{2N} |a_l|^{q_0} \Big)^{1/q_0},$$

$$\|f\|_{F^s_{q_1}[L^{p,r}]} \approx \Big(\sum_{l=N+1}^{2N} |a_l|^{q_1} \Big)^{1/q_1}.$$

This immediately yields that every of the embeddings $B^s_{q_0}[L^{p,r_0}] \hookrightarrow F^s_{q_1}[L^{p_1,r_1}]$, $F^s_{q_0}[L^{p,r_0}] \hookrightarrow B^s_{q_1}[L^{p_1,r_1}]$, $B^s_{q_0}[L^{p,r_0}] \hookrightarrow B^s_{q_1}[L^{p_1,r_1}]$, $B^s_{q_0}[L^{p,r_0}] \hookrightarrow B^s_{q_1}[L^{p_1,r_1}]$ implies that $q_0 \le q_1$.

3.6.3. *Necessary conditions on (p, r_0) and (p, r_1) in Theorems 1.1, 1.2, (vi).* We now show

$$B^s_p[L^{p,r_0}] \hookrightarrow F^s_p[L^{p,r_1}] \implies r_0 \le p. \tag{37}$$

$$F^s_p[L^{p,r_0}] \hookrightarrow B^s_p[L^{p,r_1}] \implies p \le r_1. \tag{38}$$

Proof of (37). Let $R, N \in \mathbb{N}$ be large. Consider

$$f(x) = \sum_{k=N+1}^{2N} 2^{-sRk} \sum_{l=0}^{4N} (1+|Rk-Rl|)^{-\frac{1}{p}} (\log(2+|Rk-Rl|))^{-\delta} f_{k,l}(x), \tag{39}$$

where

$$f_{k,l}(x) = \mathcal{F}^{-1}[\chi(\cdot - 2^{Rk} e_1)](x - Rle_1)$$

and χ is as in §3.6.2. Let $\varepsilon > 0$ such that $|\mathcal{F}^{-1}[\chi](x)| \ge c > 0$ on $[-\varepsilon, \varepsilon]^d$.

We have $\Lambda_{Rk} f_{k,l} = f_{k,l}$, and $\Lambda_j f_{k,l} = 0$ when $j \ne Rk$.

Let $\eta = \mathcal{F}^{-1}[\chi]$ then $f_{k,l}(x) = \eta(x - Rle_1) e^{2\pi i 2^{Rk} x_1}$. Hence

$$2^{sRk} |\Lambda_{Rk} f(x)| = \Big| \sum_{l \in \mathbb{Z}} (1 + |Rk - Rl|)^{-\frac{1}{p}} (\log(2 + R|k-l|))^{-\delta} \eta(x - Rle_1) \Big| \quad \text{if } N+1 \le l \le 2N$$

and $2^{sj} |\Lambda_j f(x)| = 0$ if $j \notin R\mathbb{N}$ or if $j \notin [N+1, 2N]$.

We argue by contradiction and assume $r_0 > p$. Choose δ so that $1/r_0 < \delta < 1/p$. We then have for fixed k that

$$\Big\| \sum_{l \in \mathbb{Z}} (1 + |Rk - Rl|)^{-\frac{1}{p}} (\log(2 + R|k-l|))^{-\delta} \eta(\cdot - Rle_1) \Big\|_{p,r_0} \lesssim 1$$

and hence

$$\|f\|_{B^s_p(L^{p,r_0})} \lesssim N^{1/p}. \tag{40}$$

To derive a lower bound for $\|f\|_{F^s_p[L^{p,\infty}]}$ we let

$$\mathcal{U}_\varepsilon = \{x : |x - k_0 Re_1| \le \varepsilon \text{ for some } k_0 \in [5N/4, 7N/4]\}.$$

Fix $x \in \mathcal{U}_\varepsilon$. Then $(\sum_j |2^{js}\Lambda_j f(x)|^p)^{1/p} \ge c_1 I(x) - c_2 II(x)$ where

$$I(x) = c\Big(\sum_{k=N+1}^{2N} (1+|Rke_1 - x|)^{-1}(\log(2+|Rke_1-x|))^{-\delta p}\Big)^{1/p}$$

and $II(x) =$

$$C\Big(\sum_{k=N+1}^{2N}\Big|\sum_{l\neq k_0}(1+R|k-l|)^{-1/p}(\log(2+R|k-l|))^{-\delta}\eta(x-Rle_1)\Big|^p\Big)^{1/p}.$$

Now $I(x) \gtrsim (R^{-1}\sum_{2\le l\le N/4} l^{-1}(\log l)^{-\delta p})^{1/p} \gtrsim R^{-1/p}(\log N)^{\frac{1-\delta p}{p}}$, as $\delta p < 1$. Using the decay of η we also get $II(x) \lesssim C_{N_1}R^{-N_1}(\log N)^{\frac{1-\delta p}{p}}$ for any $N_1 > 0$. Hence for R large we see that the measure of the subset of $\mathcal{U}_\varepsilon$ where $(\sum_j |2^{js}\Lambda_j f(x)|^p)^{1/p} \ge c(\log N)^{\frac{1-\delta p}{p}}$ is bounded below by times $c\varepsilon^d N$. Hence

$$\|f\|_{F^s_p(L^{p,r_1})} \gtrsim \|f\|_{F^s_p(L^{p,\infty})} \gtrsim_\varepsilon N^{1/p}(\log N)^{\frac{1-\delta p}{p}}. \tag{41}$$

Comparing (40) and (41), and choosing N large, we get a contradiction. Hence $r_0 \le p$. □

Proof of (38). Again we argue by contradiction and assume $r_1 < p$. Let δ be such that $1/p < \delta < 1/r_1$. Let f be as in (39). Since $\delta > 1/p$ we have, for any M_1,

$$\Big(\sum_k 2^{Rksp}|\Lambda_{Rk}f(x)|^p\Big)^{1/p} \le C_{M_1}\begin{cases}1 & \text{if } -N \le x \le 2N\\ (1+|x|)^{-M_1} & \text{otherwise.}\end{cases}$$

This gives

$$\Big\|\Big(\sum_k 2^{Rksp}|\Lambda_{Rk}f|^p\Big)^{1/p}\Big\|_{p,r_0} \lesssim N^{1/p}, \quad r_0 > 0. \tag{42}$$

On the other hand we claim that

$$2^{Rks}\|\Lambda_{Rk}f\|_{p,r_1} \gtrsim (\log N)^{-\delta+1/r_1}, \quad 5N/4 \le k \le 7N/4. \tag{43}$$

Let $\mathcal{V}_{\varepsilon,k,j} = \cup_{2^{j-1}\le k-l\le 2^j} B(Rle_1,\varepsilon)$, for j with $0 < 2^j \le N/4$. We have

$$\begin{aligned}2^{Rks}|\Lambda_{Rk}f(x)| &= \Big|\sum_{l=0}^{4N}(1+R|k-l|)^{-1/p}\log(2+R|k-l|)^{-\delta}\eta(x-Rle_1)\Big|\\ &\ge c_1 I_k(x) - C_1 II_k(x), \quad \text{with}\end{aligned}$$

$$I_k(x) = \sum_{0<2^j<N/4} (1+R2^j)^{-1/p} \log(2+R2^j)^{-\delta} \mathbb{1}_{\mathcal{V}_{\varepsilon,k,j}}(x),$$

$$II_k(x) = \sum_{\substack{0\le l\le 4N\\ |Rle_1-x|\ge R/2}} \frac{\log(2+R|k-l|)^{-\delta}}{(1+R|k-l|)^{1/p}} R^{-N_1}|x-Rle_1|^{-N_1}.$$

It is immediate that $\|II_k\|_p \lesssim R^{-N_1}$, for any N_1, and by interpolation also $\|II_k\|_{p,r_1} \lesssim R^{-N_1}$.

Notice that $\text{meas}(\mathcal{V}_{\varepsilon,k,j}) \ge c\varepsilon^d 2^j$. For the rearrangement of I_k we have

$$I_k^*(t) \ge c \sum_{0\le 2^j\le N/8} 2^{-j/p} R^{-1/p} [\log(R2^j)]^{-\delta}(t) \mathbb{1}_{[0,c\varepsilon^d 2^j]}(t)$$

and thus

$$\begin{aligned}\|I_k\|_{p,r_1} &\ge \Big(\sum_{0<2^j<N/8} \int_{c\varepsilon 2^{j-1}}^{c\varepsilon 2^j} \big[(R2^j)^{-1/p}(\log(R2^j))^{-\delta}\big]^{r_1} t^{r_1/p} \frac{dt}{t}\Big)^{1/r_1} \\ &\ge R^{-1/p} (\log N)^{\frac{1-\delta r_1}{r_1}}.\end{aligned}$$

The two estimates for $\|I_k\|_{p,r_1}$ and $\|II_k\|_{p,r_1}$ imply (43). Then also

$$\|f\|_{B_p^s(L^{p,r_1})} \gtrsim R^{-1/p} N^{1/p} (\log N)^{-\delta+1/r_1}. \tag{44}$$

Comparing (42) and (44), and choosing N large, we get a contradiction when $\delta < 1/r_1$. This means we must have $r_1 \ge p$ in (38). □

4. Sequences of vector-valued functions

In order to prove the positive results in parts (iv)-(vi) of Theorems 1.1 and 1.2 we derive corresponding embeddings for spaces of sequences $\ell^q(L^{p,r})$ and $L^{p,r}(\ell^q)$, for fixed p, r.

Proposition 4.1. *Let $0 < p < \infty$, $0 < q_0, q_1, r_0, r_1 \le \infty$ and assume $q_0 \le \min\{p, q_1, r_1\}$, $r_0 \le r_1$. The embedding*

$$\ell^{q_0}(L^{p,r_0}) \hookrightarrow L^{p,r_1}(\ell^{q_1})$$

holds in each of the following three cases:

(i) $p \ne q_1$, (ii) $p = q_1 \ge r_0$, (iii) $q_0 < p = q_1 < r_0$.

Proposition 4.2. *Let $0 < p < \infty$, $0 < q_0, q_1, r_0, r_1 \le \infty$, $r_0 \le r_1$ and $q_1 \ge \max\{p, q_0, r_0\}$. The embedding*

$$L^{p,r_0}(\ell^{q_0}) \hookrightarrow \ell^{q_1}(L^{p,r_1})$$

holds in each of the following three cases:

(i) $p \ne q_0$, (ii) $p = q_0 \le r_1$, (iii) $r_1 < q_0 = p < q_1$.

Remark 4.3. In both Proposition 4.1 and Proposition 4.2 the assumptions (i), (ii), (iii) cannot be improved (unless one imposes very restrictive conditions on the underlying measure spaces). This follows from the examples for the spaces $B^s_q[L^{p,r}]$, $F^s_q[L^{p,r}]$ discussed in §3, although one can give less technical examples for the propositions.

We split the proof into several lemmata.

Lemma 4.4. *Suppose $q \le r \le p$ or $q < p \le r$. Then*

$$\ell^q(L^{p,r}) \hookrightarrow L^{p,r}(\ell^q). \tag{45}$$

Proof. The asserted inequality is trivial when $p = q = r$. We may thus assume $p > q$. Then

$$\|f\|^q_{L^{p,r}(\ell^q)} = \Big\| \sum_k |f_k|^q \Big\|_{L^{p/q,r/q}} \lesssim \sum_k \big\| |f_k|^q \big\|_{L^{p/q,r/q}} = \sum_k \|f_k\|^q_{L^{p,r}}.$$

Here we have used the triangle inequality in (18), for the space $L^{p/q,r/q}$, and twice the formula (15). □

Lemma 4.5. *Suppose that either $p \le r \le q$ or $r \le p < q$. Then*

$$L^{p,r}(\ell^q) \hookrightarrow \ell^q(L^{p,r}).$$

Proof. We first consider the case $r < \infty$ and argue by duality. Recall that if A is a Banach space, A' its dual and $1 < u < \infty$ then the dual of $\ell^u(A)$ is $\ell^{u'}(A')$, with the natural pairing. Let $a < \min\{p, q, r\}$ and set $(P, Q, R) = (p/a, q/a, r/a)$ so that $1 < P, Q, R < \infty$. Since for $1 < P, R < \infty$ the dual of $L^{P,R}$ is $L^{P',R'}$ we see that

$$\begin{aligned}
\|f\|^a_{\ell^q(L^{p,r})} &= \Big(\sum_k \|f_k\|^q_{L^{p,r}} \Big)^{a/q} \\
&= \Big(\sum_k \||f_k|^a\|^Q_{L^{P,Q}} \Big)^{1/Q} \le \Big(\sum_k \vert\vert\vert |f_k|^a \vert\vert\vert^Q_{L^{P,Q}} \Big)^{1/Q} \\
&\lesssim \sup \Big\{ \sum_k \int |f_k(x)|^a g_k(x) d\mu(x) \,:\, \vert\vert\vert g \vert\vert\vert_{\ell^{Q'}(L^{P',R'})} \le 1 \Big\}
\end{aligned}$$

where the implicit constants depend on p, q, r. Now, let $\vert\vert\vert g \vert\vert\vert_{\ell^{Q'}(L^{P',R'})} \le 1$. Then

$$\begin{aligned}
\sum_k \int |f_k(x)|^a |g_k(x)| d\mu &\le \int \Big(\sum_k |f_k(x)|^{aQ} \Big)^{1/Q} \Big(\sum_k |g_k(x)|^{Q'} \Big)^{1/Q'} d\mu \\
&\lesssim \Big\| \Big(\sum_k |f_k|^{aQ} \Big)^{1/Q} \Big\|_{L^{P,R}} \Big\| \Big(\sum_k |g_k|^{Q'} \Big)^{1/Q'} \Big\|_{L^{P',R'}} \\
&\lesssim \Big\| \Big(\sum_k |f_k|^{aQ} \Big)^{1/aQ} \Big\|^a_{L^{aP,aR}} \vert\vert\vert g \vert\vert\vert_{\ell^{Q'}(L^{P',R'})} \lesssim \|f\|^a_{L^{p,r}(\ell^q)};
\end{aligned}$$

here we have used for the second to last inequality that

$$|||g|||_{L^{P',R'}(\ell^{Q'})} \lesssim \|g\|_{L^{P',R'}(\ell^{Q'})} \lesssim \|g\|_{\ell^{Q'}(L^{P',R'})},$$

by Lemma 4.4 since $Q' \le R' \le P'$ or $Q' < P' \le R'$. This completes the proof for $r < \infty$.

Next assume $r = \infty$, then also $q = \infty$. Clearly we have for any fixed k_0

$$\mu(\{x : |f_{k_0}(x)| > \alpha\}) \le \mu(\{x : \sup_k |f_k(x)| > \alpha\}).$$

Hence $\sup_k \sup_{\alpha>0} \alpha[\mu_{f_k}(\alpha)]^{1/p} \le \sup_{\alpha>0} \alpha[\mu_{\sup_k |f_k|}(\alpha)]^{1/p}$ which yields the case $r = q = \infty$. □

Next we state some weaker embedding properties for the case $p < q \le r$.

Lemma 4.6. *(i) Let $p < q \le r$ or $p = q = r$. Then*

$$\ell^p(L^{p,r}) \hookrightarrow L^{p,r}(\ell^q)\,.$$

(ii) Let $v < p \le r$. Then

$$\ell^v(L^{p,r}) \hookrightarrow L^{p,r}(\ell^p).$$

Proof. The statement is trivial for $q = r = p$. Let $p < q \le r$. We use the modified p/q-triangle inequality in $L^{p/q,s}$ for $s = r/q \ge 1$, as in (20), and estimate

$$\begin{aligned} \Big\| \Big(\sum_k |f_k|^q \Big)^{1/q} \Big\|_{L^{p,r}} &= \Big\| \sum_k |f_k|^q \Big\|_{L^{p/q,r/q}}^{1/q} \\ &\lesssim \Big(\Big(\sum_k \big\| |f_k|^q \big\|_{L^{p/q,r/q}}^{p/q} \Big)^{q/p} \Big)^{1/q} = \Big(\sum_k \|f_k\|_{L^{p,r}}^p \Big)^{1/p}. \end{aligned}$$

For (ii) we use the embedding $\ell^v \subset \ell^p$ and then the triangle inequality in $L^{p/v,r/v}$ (cf. (18)) to obtain

$$\begin{aligned} &\Big\| \Big(\sum_k |f_k|^p \Big)^{1/p} \Big\|_{L^{p,r}} \le \Big\| \Big(\sum_k |f_k|^v \Big)^{1/v} \Big\|_{L^{p,r}} \\ &\le \Big\| \sum_k |f_k|^v \Big\|_{L^{p/v,r/v}}^{1/v} \lesssim \Big(\sum_k \big\| |f_k|^v \big\|_{L^{p/v,r/v}} \Big)^{1/v} \lesssim \Big(\sum_k \|f_k\|_{L^{p,r}}^v \Big)^{1/v}. \end{aligned}$$ □

Lemma 4.7. *(i) Let $0 < r \le q < p$ or $r = p = q$. Then*

$$L^{p,r}(\ell^q) \hookrightarrow \ell^p(L^{p,r}).$$

(ii) Let $0 < r \le p < w$. Then

$$L^{p,r}(\ell^p) \hookrightarrow \ell^w(L^{p,r}).$$

Proof. The statement is trivial for $r = p = q$. If $0 < r \le q < p < \infty$, set $(P, Q, R) = (p/a, q/a, r/a)$ for some $a < \min\{p, q, r\}$ and argue by duality exactly as in the proof of Lemma 4.5, basing the argument on Lemma 4.6. □

Proof of Proposition 4.1. Let $q_0 \le \min\{p, q_1, r_1\}$. We distinguish the three cases according to whether p, q_1, or r_1 is the smallest exponent.

Case 1: $q_0 \le q_1 \le \min\{p, r_1\}$. If either $q_0 \le r_1 \le p$ or $q_0 < p \le r_1$ then $\ell^{q_0}(L^{p,r_1}) \hookrightarrow L^{p,r_1}(\ell^{q_0})$ by Lemma 4.4 and hence

$$\ell^{q_0}(L^{p,r_0}) \hookrightarrow \ell^{q_0}(L^{p,r_1}) \hookrightarrow L^{p,r_1}(\ell^{q_0}) \hookrightarrow L^{p,r_1}(\ell^{q_1}).$$

In the remaining subcase we have $q_0 = q_1 = p \le r_1$ and by assumption of the proposition we also have $r_0 \le p$. Thus $\ell^p(L^{p,r_0}) \hookrightarrow \ell^p(L^p) = L^p(\ell^p) \hookrightarrow L^{p,r_1}(\ell^p)$.

Case 2: $q_0 \le r_1 \le \min\{p, q_1\}$. Note that $\ell^{r_1}(L^{p,r_1}) \hookrightarrow L^{p,r_1}(\ell^{r_1})$ for $r_1 \le p$, again by Lemma 4.4. Thus we obtain

$$\ell^{q_0}(L^{p,r_0}) \hookrightarrow \ell^{r_1}(L^{p,r_1}) \hookrightarrow L^{p,r_1}(\ell^{r_1}) \hookrightarrow L^{p,r_1}(\ell^{q_1}).$$

In the third case $q_0 \le p \le \min\{q_1, r_1\}$. The embedding is trivial when $p = q_1 = r_1$. We distinguish three remaining subcases.

Case 3-1: $p < q_1 \le r_1$. We apply Lemma 4.6, (i), to get

$$\ell^{q_0}(L^{p,r_0}) \hookrightarrow \ell^{p}(L^{p,r_1}) \hookrightarrow L^{p,r_1}(\ell^{q_1}).$$

Case 3-2: $q_0 \le p = q_1 < r_1$. If $q_0 < p$ then by part (ii) of Lemma 4.6.

$$\ell^{q_0}(L^{p,r_0}) \hookrightarrow \ell^{q_0}(L^{p,r_1}) \hookrightarrow L^{p,r_1}(\ell^{p}).$$

If $q_0 = p$ then we have by assumption also $r_0 \le p$ and therefore $\ell^p(L^{p,r_0}) \hookrightarrow \ell^p(L^p) = L^p(\ell^p) \hookrightarrow L^{p,r_1}(\ell^p)$.

Case 3-3: $q_0 \le p \le r_1 < q_1$. Now observe that $\ell^p(L^{p,r_1}) \hookrightarrow L^{p,r_1}(\ell^{r_1})$ by Lemma 4.6, (i). Hence

$$\ell^{q_0}(L^{p,r_0}) \hookrightarrow \ell^{p}(L^{p,r_1}) \hookrightarrow L^{p,r_1}(\ell^{r_1}) \hookrightarrow L^{p,r_1}(\ell^{q_1})\,. \qquad \square$$

Proof of Proposition 4.2. The proof is 'dual' to the proof of Proposition 4.1. Formally the proof goes by reversing the arrows in the proof of Proposition 4.1 and replacing the subscripts $(0, 1)$ by $(1, 0)$. We now use Lemma 4.5 and Lemma 4.7 in place of Lemma 4.4 and Lemma 4.6. We run through the cases:

Case 1': $q_1 \ge q_0 \ge \max\{p, r_0\}$. If $p \le r_0 \le q_1$, or $r_0 \le p < q_1$ and the second of the following embeddings holds by Lemma 4.5:

$$L^{p,r_0}(\ell^{q_0}) \hookrightarrow L^{p,r_0}(\ell^{q_1}) \hookrightarrow \ell^{q_1}(L^{p,r_0}) \hookrightarrow \ell^{q_1}(L^{p,r_1}).$$

If $r_0 \le p = q_0 = q_1$ then also by assumption $r_1 \ge p$ and hence $L^{p,r_0}(\ell^{q_0}) \hookrightarrow L^p(\ell^p) = \ell^p(L^p) \hookrightarrow \ell^p(L^{p,r_1})$.

Case 2': $q_1 \ge r_0 \ge \max\{p, q_0\}$. First observe that Lemma 4.5 also implies $L^{p,r_0}(\ell^{r_0}) \hookrightarrow \ell^{r_0}(L^{p,r_0})$ for $p \ge r_0$. Hence

$$L^{p,r_0}(\ell^{q_0}) \hookrightarrow L^{p,r_0}(\ell^{r_0}) \hookrightarrow \ell^{r_0}(L^{p,r_0}) \hookrightarrow \ell^{q_1}(L^{p,r_1}).$$

The third case $q_1 \ge p \ge \max\{q_0, r_0\}$ is again split into three subcases (ignoring the trivial case $p = q_0 = r_0$).

Case 3-1': $p > q_0 \geq r_0$. We apply Lemma 4.7 to obtain

$$L^{p,r_0}(\ell^{q_0}) \hookrightarrow \ell^p(L^{p,r_0}) \hookrightarrow \ell^{q_1}(L^{p,r_1}).$$

Case 3-2': $q_1 \geq p = q_0 > r_1$. If $q_1 > p$ we get by part (ii) of Lemma 4.7,

$$L^{p,r_0}(\ell^{p}) \hookrightarrow \ell^{q_1}(L^{p,r_0}) \hookrightarrow \ell^{q_1}(L^{p,r_1}).$$

If $q_1 = p$ then by assumption $r_1 \geq p$ and therefore $L^p(\ell^{r_0}) \hookrightarrow L^p(\ell^p) = \ell^p(L^p) \hookrightarrow \ell^p(L^{p,r_1})$.

Case 3-3': $q_1 \geq p \geq r_0 > q_0$. We use that $L^{p,r_0}(\ell^{r_0}) \hookrightarrow \ell^p(L^{p,r_0})$, by Lemma 4.7. Hence

$$L^{p,r_0}(\ell^{q_0}) \hookrightarrow L^{p,r_0}(\ell^{r_0}) \hookrightarrow \ell^p(L^{p,r_0}) \hookrightarrow \ell^{q_1}(L^{p,r_1}). \qquad \square$$

We now get the statements (iv)-(vi) in Theorems 1.1 and 1.2.

Corollary 4.8. *Let $0 < p < \infty$, $0 < q_0, q_1, r_0, r_1 \leq \infty$.*

(i) Suppose that either $p \neq q_1$ or that $p = q_1 \geq r_0$. Then the embedding

$$B^s_{q_0}[L^{p,r_0}] \hookrightarrow F^s_{q_1}[L^{p,r_1}]$$

holds if and only if $q_0 \leq \min\{p, q_1, r_1\}$ and $r_0 \leq r_1$.

(ii) Let $r_0 > p$. Then the embedding

$$B^s_{q_0}[L^{p,r_0}] \hookrightarrow F^s_{p}[L^{p,r_1}]$$

holds if and only if $q_0 < p$ and $r_0 \leq r_1$.

Corollary 4.9. *Let $0 < p < \infty$, $0 < q_0, q_1, r_0, r_1 \leq \infty$.*

(i) Suppose either that $p \neq q_0$ or that $p = q_0 \leq r_1$. Then the embedding

$$F^s_{q_0}[L^{p,r_0}] \hookrightarrow B^s_{q_1}[L^{p,r_1}]$$

holds if and only if $q_1 \geq \max\{p, q_0, r_0\}$ and $r_0 \leq r_1$.

(ii) Let $r_1 < p$. Then the embedding

$$F^s_{p}[L^{p,r_0}] \hookrightarrow B^s_{q_1}[L^{p,r_1}]$$

holds if and only if $q_1 > p$ and $r_0 \leq r_1$.

Proof of Corollary 4.8 and Corollary 4.9. The positive results follow immediately from the corresponding results in Propositions 4.1 and 4.2 when applied to $\{f_k\}_{k=0}^{\infty}$ with $f_k = 2^{ks}\Lambda_k f$. The necessity of the conditions was proved in §3. $\square$

5. Embeddings of Jawerth-Franke type

Jawerth's and Franke's versions of the Sobolev embedding theorem were reproved by Vybíral [30] using sequence spaces which are discrete variants of Besov and Triebel-Lizorkin spaces. Here we do not introduce sequence spaces but nevertheless the proofs are inspired by [30].

Preliminary considerations. We first need a straightforward Lorentz space version of Peetre's maximal theorem.

Lemma 5.1. *Let $f_k \in \mathcal{S}'$ be such that $\widehat{f_k}$ is supported in $\{\xi : |\xi| \le 2^k\}$. Let*

$$\mathfrak{M}_k f_k(x) = \sup_{|h|\le d2^{-k}} |f_k(x+h)|.$$

Then for $0 < p < \infty$, $0 < q, r \le \infty$,

$$\big\|\{\mathfrak{M}_k f_k\}\big\|_{L^{p,r}(\ell^q)} \lesssim C_{p,q,r}\big\|\{f_k\}\big\|_{L^{p,r}(\ell^q)}.$$

Proof. Let M_{HL} be the Hardy-Littlewood maximal operator. We have

$$\|\{M_{HL} g_k\}\|_{L^{p_0,r_0}(\ell^{q_0})} \le C(p_0, r_0, q_0)\|\{g_k\}\|_{L^{p_0,r_0}(\ell^{q_0})} \tag{46}$$

for $1 < p_0, r_0, q_0 < \infty$. The version for $p_0 = r_0$ was proved by Fefferman and Stein [6] and the general version follows by real interpolation.

From [16] we have the inequality

$$\mathfrak{M}_k f_k(x) \le C_\rho \big(M_{HL}(|f_k|^\rho)(x)\big)^{1/\rho}$$

for all $\rho > 0$. We choose $\rho < \min\{p, q, r\}$, and apply (46) with $(p_0, r_0, q_0) = (p/\rho, r/\rho, q/\rho)$. Then

$$\begin{aligned}&\big\|\{\mathfrak{M}_k f_k\}\big\|_{L^{p,r}(\ell^q)} \lesssim \big\|\{(M_{HL}(|f_k|^\rho))^{1/\rho}\}\big\|_{L^{p,r}(\ell^q)}\\ &= \big\|\{M_{HL}(|f_k|^\rho)\}\big\|^{1/\rho}_{L^{\frac{p}{\rho},\frac{r}{\rho}}(\ell^{\frac{q}{\rho}})} \lesssim \big\|\{|f_k|^\rho\}\big\|^{1/\rho}_{L^{\frac{p}{\rho},\frac{r}{\rho}}(\ell^{\frac{q}{\rho}})} = \big\|\{f_k\}\big\|_{L^{p,r}(\ell^q)}. \qquad \square\end{aligned}$$

Theorem 5.2. *Suppose $0 < d(1/p_0 - 1/p_1) = s_0 - s_1$, $0 < q_0, q_1, r_0, r_1 \le \infty$. Then the embedding $F^{s_0}_{q_0}[L^{p_0,r_0}] \hookrightarrow B^{s_1}_{q_1}[L^{p_1,r_1}]$ holds if and only if $r_0 \le q_1$.*

Proof. The necessity of the condition $r_0 \le q_1$ has been established in §3.5.

Let $Q_k(x)$ be the unique dyadic cube of sidelength 2^{-k} which contains x, (the sides being half open intervals). Set

$$g_k(x) = 2^{ks_0} \sup_{y\in Q_k(x)} \Lambda_k f(y),$$

$$G(x) = \sup_k 2^{ks_0}\mathfrak{M}_k\Lambda_k f(x).$$

Clearly

$$g_k(x) \le 2^{ks_0}\,\mathfrak{M}_k\Lambda_k f(x) \le G(x)$$

and therefore $g_k^*(t) \le G^*(t)$.

Since g_k is constant on the dyadic cubes of sidelength 2^{-k} we see that g_k^* is constant on dyadic intervals of length 2^{-kd}. In particular

$$\begin{aligned}\|g_k\|_{p_1,r_1} &= \Big(\frac{r_1}{p_1}\int_0^\infty t^{r_1/p} g_k^*(t)^{r_1}\frac{dt}{t}\Big)^{1/r_1}\\ &\approx \Big(\sum_{n=1}^\infty \big[(2^{-kd}n)^{1/p_1} g_k^*(2^{-kd}(n-1))\big]^{r_1}\frac{2^{-kd}}{2^{-kd}n}\Big)^{1/r_1}.\end{aligned}$$

We now begin with the proof of the sufficiency of the condition $q_1 \geq r_0$. Since the $B^s_q(L^{p,r})$, $F^s_q(L^{p,r})$ norms increase when r decreases or when q decreases, it suffices to consider the case $q_1 = r_0 =: \rho$ and to prove for $0 < \rho \leq \infty$ and $r_1 < \rho$

$$F^{s_0}_\infty[L^{p_0,\rho}] \hookrightarrow B^{s_1}_\rho[L^{p_1,r_1}]. \tag{47}$$

We write the proof for $\rho < \infty$ but this is not essential as the case $\rho = \infty$ will only require notational changes.

Now

$$\begin{aligned}\|f\|_{B^{s_1}_\rho(L^{p_1,r_1})} &= \Big(\sum_k 2^{ks_1\rho}\|\Lambda_k f\|^\rho_{p_1,r_1}\Big)^{1/\rho} \\ &= \Big(\sum_k \|2^{ks_0}\Lambda_k f\|^\rho_{p_1,r_1} 2^{-kd(\frac{1}{p_0}-\frac{1}{p_1})\rho}\Big)^{1/\rho}\end{aligned}$$

Here we have used the relation $d/p_0 - d/p_1 = s_0 - s_1$. The last displayed expression is dominated by

$$\begin{aligned}&\Big(\sum_k 2^{-kd(\frac{1}{p_0}-\frac{1}{p_1})\rho}\|g_k\|^\rho_{p_1,r_1}\Big)^{1/\rho} \\ &\lesssim \Big(\sum_k 2^{-kd(\frac{1}{p_0}-\frac{1}{p_1})\rho}\Big[\sum_{n=1}^\infty (2^{-kd}n)^{r_1/p_1} g^*_k(2^{-kd}(n-1))^{r_1}\frac{1}{n}\Big]^{\rho/r_1}\Big)^{1/\rho} \\ &\leq \Big(\sum_k \Big(\sum_{n=1}^\infty n^{r_1(\frac{1}{p_1}-\frac{1}{p_0})-1} G^*(2^{-kd}(n-1))^{r_1}(2^{-kd}n)^{r_1/p_0}\Big)^{\rho/r_1}\Big)^{1/\rho}.\end{aligned}$$

The last expression is comparable to

$$\begin{aligned}&\Big(\sum_k \Big(\sum_{j=0}^\infty 2^{jr_1(1/p_1-1/p_0)} G^*(2^{-kd+j-1})^{r_1}(2^{-kd+j})^{r_1/p_0}\Big)^{\rho/r_1}\Big)^{1/\rho} \\ &\lesssim \Big(\sum_{j=0}^\infty 2^{jr_1(1/p_1-1/p_0)}\Big(\sum_k G^*(2^{-kd+j-1})^\rho(2^{-kd+j})^{\rho/p_0}\Big)^{r_1/\rho}\Big)^{1/r_1} \\ &\lesssim \Big(\sum_{j=0}^\infty 2^{jr_1(1/p_1-1/p_0)}\Big(\sum_k \int_{2^{-kd+j-1}}^{2^{-kd+j}} t^{\rho/p_0} G^*(t)^\rho \frac{dt}{t}\Big)^{r_1/\rho}\Big)^{1/r_1}.\end{aligned}$$

Here we have used the triangle inequality in L^{ρ/r_1} (as $\rho/r_1 \geq 1$). Now for fixed j the intervals $[2^{-kd+j-1}, 2^{-kd+j}]$ have disjoint interior and therefore the last expression is dominated by

$$\begin{aligned}&\Big(\sum_{j=0}^\infty 2^{jr_1(1/p_1-1/p_0)}\|G\|^{r_1}_{p_0,\rho}\Big)^{1/r_1} \\ &\lesssim \|G\|_{p_0,\rho} \lesssim \Big\|\sup_k 2^{ks_0}|\Lambda_k f|\Big\|_{p_0,\rho} = \|f\|_{F^{s_0}_\infty[L^{p_0,\rho}]}\end{aligned}$$

and (47) is proved. □

Theorem 5.3. *Suppose* $0 < d(1/p_0 - 1/p_1) = s_0 - s_1$, $0 < q_0, q_1, r_0, r_1 \le \infty$. *Then the embedding* $B^{s_0}_{q_0}[L^{p_0,r_0}] \hookrightarrow F^{s_1}_{q_1}[L^{p_1,r_1}]$ *holds if and only if* $q_0 \le r_1$.

Proof. For the necessity of the condition $q_0 \le r_1$ see §3.5. It now suffices to prove for any $q > 0$ and for $0 < \rho \le \infty$,

$$B^{s_0}_{\rho}[L^{p_0,\infty}] \hookrightarrow F^{s_1}_{q}[L^{p_1,\rho}]. \tag{48}$$

Assume that $f \in B^{s_0}_{\rho}[L^{p_0,\infty}]$. Define

$$h_k(x) = 2^{ks_1} \sum_{Q \in \mathfrak{Q}_k} \mathbb{1}_Q(x) \inf_Q \mathfrak{M}_k(\Lambda_k f)$$

where $\mathfrak{Q}_k$ denotes the grid of dyadic cubes with side length 2^{-k}. Then

$$\begin{aligned}\|f\|_{F^{s_1}_q(L^{p_1,\rho})} &= \Big\| \sum_k 2^{ks_1 q} |\Lambda_k f|^q \Big\|^{1/q}_{p_1/q,\rho/q} \le \Big\| \sum_k h_k^q \Big\|^{1/q}_{p_1/q,\rho/q} \\ &= \Big(\sup_{\|g\|_{(p_1/q)',(\rho/q)'}=1} \sum_k \int h_k(x)^q g(x)\, dx \Big)^{1/q}. \end{aligned} \tag{49}$$

Note that the rearrangement function of h_k is constant on the intervals $[2^{-kd}(n-1), 2^{-kd}n)$ for $n = 1, 2, \dots$. Thus for fixed k

$$\begin{aligned}&\Big| \int h_k(x)^q g(x) dx \Big| \le \int_0^\infty (h_k^q)^*(t) g^*(t) dt \\ &= \int_0^\infty (h_k^*(t))^q g^*(t) dt = \sum_{n=1}^\infty h_k^*(2^{-kd}(n-1)) \int_{2^{-kd}(n-1)}^{2^{-kd}n} g^*(t) dt \\ &\le \sum_{n=1}^\infty (h_k^*(2^{-kd}(n-1)))^q 2^{-kd} g^{**}(2^{-kd}n)\end{aligned}$$

We sum in k and get

$$\begin{aligned}&\sum_k \sum_{n=1}^\infty (h_k^*(2^{-kd}(n-1)))^q 2^{-kd} g^{**}(2^{-kd}n) \\ &\lesssim \sum_k \sum_{l=0}^\infty (h_k^*(2^{ld-kd-1}))^q 2^{ld-kd} g^{**}(2^{ld-kd}) \\ &= \sum_{l=0}^\infty \sum_k 2^{(ld-kd)q/p_0} 2^{k(s_0-s_1)q} h_k^*(2^{ld-kd-1})^q \times \\ &\qquad 2^{(ld-kd)(1-q/p_0)} g^{**}(2^{ld-kd}) 2^{-kd(1/p_0-1/p_1)q}\end{aligned}$$

where we have used $s_0 - s_1 = d/p_0 - d/p_1$. By Hölder's inequality the last displayed expression is dominated by

$$\sum_{l=0}^{\infty} \Big(\sum_k 2^{(ld-kd)\rho/p_0} 2^{k(s_0-s_1)\rho} h_k^*(2^{ld-kd-1})^\rho \Big)^{q/\rho} \times$$
$$\Big(\sum_k \Big[2^{(ld-kd)(1-q/p_0)} g^{**}(2^{ld-kd}) 2^{-kd(1/p_0-1/p_1)q} \Big]^{(\rho/q)'} \Big)^{1-q/\rho}$$
$$\lesssim \Big(\sum_k 2^{k(s_0-s_1)\rho} \sup_{m\ge 0} 2^{(md-kd-1)\rho/p_0} h_k^*(2^{md-kd-1})^\rho \Big)^{q/\rho} \times$$
$$\sum_{l=0}^{\infty} \Big(\sum_k \Big[2^{(ld-kd)(1-q/p_0)} g^{**}(2^{ld-kd}) 2^{-kd(1/p_0-1/p_1)q} \Big]^{(\rho/q)'} \Big)^{1-q/\rho}.$$

Now we have

$$\Big(\sum_k 2^{k(s_0-s_1)\rho} \sup_{m\ge 0} 2^{(md-kd-1)\rho/p_0} h_k^*(2^{md-kd-1})^\rho \Big)^{q/\rho}$$
$$\lesssim \Big(\sum_k 2^{ks_0\rho} \|2^{-ks_1} h_k\|_{p_0,\infty}^\rho \Big)^{q/\rho} \lesssim \Big(\sum_k 2^{ks_0\rho} \|\mathfrak{M}_k(\Lambda_k f)\|_{p_0,\infty}^\rho \Big)^{q/\rho}$$
$$\lesssim \Big(\sum_k 2^{ks_0\rho} \|\Lambda_k f\|_{p_0,\infty}^\rho \Big)^{q/\rho} = \|f\|^q_{B^{s_0}_\rho[L^{p_0,\infty}]}.$$

Finally we estimate, summing $\sum_{l=0}^\infty 2^{-ld(1/p_0-1/p_1)q} \lesssim 1$,

$$\sum_{l=0}^{\infty} \Big(\sum_k \Big[2^{(ld-kd)(1-q/p_0)} g^{**}(2^{ld-kd}) 2^{-kd(1/p_0-1/p_1)q} \Big]^{(\rho/q)'} \Big)^{1-q/\rho}$$
$$\lesssim \sup_{l\ge 0} \Big(\sum_k \Big[2^{(ld-kd)(1-q/p_0)} g^{**}(2^{ld-kd}) 2^{(ld-kd)(1/p_0-1/p_1)q} \Big]^{(\rho/q)'} \Big)^{1-q/\rho}$$
$$= \sup_{l\ge 0} \Big(\sum_k \big[2^{(kd-ld)(1-q/p_1)} g^{**}(2^{ld-kd}) \big]^{(\rho/q)'} \Big)^{1-\rho/q} \lesssim \|g\|_{(p_1/q)',(\rho/q)'} \lesssim 1.$$

Going back to (49) we see that $\|f\|_{F^{s_1}_q[L^{p_1,\rho}]} \lesssim \|f\|_{B^{s_0}_\rho[L^{p_0,\infty}]}$ and the proof of (48) is complete. □

6. Conclusion of the proofs of Theorems 1.1, 1.2, 1.5, 1.6

The necessity of all conditions was shown in §3. The proofs of the embeddings in parts (iv)-(vi) of Theorem 1.1 and Theorem 1.2 were shown in §4 (see Corollaries 4.8 and 4.9). The embeddings in part (iii) of Theorem 1.1 and Theorem 1.2 are covered by Theorem 5.3 and Theorem 5.2, respectively.

We consider part (ii) of Theorem 1.1. Let $s_0 > s_1$, $r_0 \le r_1$, $p_0 = p_1 = p$. Let $\varepsilon > 0$ such that $s_0 - \varepsilon > s_1$, and let $v < \min\{q_0, p_1, q_1, r_1\}$. By parts (iv) or (v) of Theorem 1.1 we have $B^{s_0-\varepsilon}_v[L^{p,r_0}] \hookrightarrow F^{s_0-\varepsilon}_{q_1}[L^{p,r_1}]$ and (ii) follows if we combine this with the trivial embeddings $B^{s_0}_{q_0}[L^{p,r_0}] \hookrightarrow B^{s_0-\varepsilon}_{v}[L^{p,r_0}]$ and $F^{s_0-\varepsilon}_{q_1}[L^{p,r_1}] \hookrightarrow F^{s_1}_{q_1}[L^{p,r_1}]$.

The proof of part (ii) of Theorem 1.2 is similar. Moreover if we use part (iii) in Theorems 1.1 and 1.2 the proofs of part (i) in those theorems follows the same pattern as above.

Finally we consider Theorems 1.5 and 1.6. Part (iv) of these theorems are proved by using embeddings of $L^{p,r}$ spaces and of ℓ^q spaces.

To see part (iii) of Theorem 1.5, assume $q_0 \le q_1$ and let $\widetilde{p}$ and $\widetilde{s}$ be such that $p_0 < \widetilde{p} < p_1$, $s_1 < \widetilde{s} < s_0$ and $\widetilde{s} - s_1 = d/\widetilde{p} - d/p_1$ and thus $s_0 - \widetilde{s} = d/p_0 - d/\widetilde{p}$. Pick $\widetilde{r}$ such that $q_0 \le \widetilde{r} \le q_1$ and then

$$B^{s_0}_{q_0}[L^{p_0,r_0}] \hookrightarrow F^{\widetilde{s}}_{\widetilde{q}}[L^{\widetilde{p},\widetilde{r}}] \hookrightarrow B^{s_1}_{q_1}[L^{p_1,r_1}]$$

for arbitrary $\widetilde{q}$, by Theorem 5.3 for the first embedding and Theorem 5.2 for the second.

To see part (iii) of Theorem 1.6 assume $r_0 \le r_1$. Pick $\widetilde{q}$ such that $r_0 \le \widetilde{q} \le r_1$ and then

$$F^{s_0}_{q_0}[L^{p_0,r_0}] \hookrightarrow B^{\widetilde{s}}_{\widetilde{q}}[L^{\widetilde{p},\widetilde{r}}] \hookrightarrow F^{s_1}_{q_1}[L^{p_1,r_1}]$$

for arbitrary $\widetilde{r}$, by Theorem 5.2 for the first embedding and Theorem 5.3 for the second.

Given parts (iv), (iii) of Theorems 1.5 and 1.6 the parts (i), (ii) in the noncritical ranges can be obtained by the argument above.

Appendix A. Remarks on Mikhlin-Hörmander multipliers

Part (iii) of Theorems 1.1 and 1.2 (i.e. Theorems 5.3 and 5.2) can be applied to clarify the connection between certain sharp versions of the Mikhlin-Hörmander multiplier theorem ([10]). Set $T_m f = \mathcal{F}^{-1}[m\widehat{f}]$. Let φ be a nontrivial radial smooth functions which is compactly supported in $\mathbb{R}^d \setminus \{0\}$. We first recall the endpoint bound

$$\|T_m\|_{L^p \to L^{p,2}} \lesssim \sup_{t>0} \|\varphi m(t\cdot)\|_{B^s_1[L^{d/s}]}, \quad s = d(1/p - 1/2),\ 1 < p \le 2, \tag{50}$$

which was proved by one of the authors in [17]. Moreover one gets $H^1 \to L^{1,2}$ boundedness under the condition $\sup_{t>0} \|\varphi m(t\cdot)\|_{B^{d/2}_1[L^2]} < \infty$, see [18]. Note that (50) immediately implies that

$$\|T_m\|_{L^p \to L^p} \lesssim \sup_{t>0} \|\varphi m(t\cdot)\|_{B^s_1[L^{d/s}]}, \quad d|1/p - 1/2| < s < d. \tag{51a}$$

Indeed, by the standard Sobolev imbedding theorem for Besov spaces we may assume that $s < d/2$. Define p_0 by $d(1/p_0 - 1/2) = s$, so that $1 < p_0 < p < 2$. Then (50) gives $L^{p_0} \to L^{p_0,2}$ boundedness, and by the Marcinkiewicz interpolation theorem, and a subsequent duality argument we get (51a).

A recent result by Grafakos and Slavíková [9] states that for $1 < p < \infty$

$$\|T_m\|_{L^p \to L^p} \lesssim \sup_{t>0} \|(I - \Delta)^{s/2}[\varphi m(t\cdot)]\|_{L^{d/s,1}}, \quad d|1/p - 1/2| < s < d. \tag{51b}$$

For $s<d$ the expression of the right hand side of the inequality is equivalent to $\sup_{t>0}\|\varphi m(t\cdot)\|_{F^s_2[L^{d/s,1}]}$, *cf.* (3). For fixed s the relation between the norms on the right hand side in (51a) and (51b) is not immediately clear. The spaces $F^s_2[L^{d/s,1}]$ and $B^s_1[L^{d/s}]$ are not comparable; we have $F^s_2[L^{d/s,1}]\not\subseteq B^s_1[L^{d/s}]$ by §3.6.2, and we get $B^s_1[L^{d/s}]\not\subseteq F^s_2[L^{d/s,1}]$ by the necessity of the condition $r_0\le r_1$ in §3.3. However we do have the embeddings

$$B^{s_3}_1[L^{d/s_3}]\hookrightarrow F^{s_2}_q[L^{d/s_2,1}]\hookrightarrow B^{s_1}_1[L^{d/s_1}],\quad s_1<s_2<s_3,\ q>0. \tag{52}$$

The first inclusion in (52) follows from Theorem 1.1 (iii) and the second from Theorem 1.2 (iii). Since both statements (51a), (51b) involve the same open s-interval we may apply (52) for $s_1>d|1/p-1/2|$ and $q=2$ to see that they cover L^p boundedness for exactly the same set of multiplier transformations.

Open problems.

A.1. Is there an endpoint inequality such as (50) in terms of localized $F^s_2[L^{d/s,1}]$ spaces, when $s=d|1/p-1/2|$?

A.2. It was proved in [19] that $\|T_m\|_{L^p\to L^p}\lesssim\sup_{t>0}\|\varphi m(t\cdot)\|_{B^{d|1/p-1/2|}_1[L^q]}$ for $1/q>|1/p-1/2|$, $1<p<\infty$ (see also [1], [19] for corresponding results on Hardy spaces). It would be interesting to know for which r (if any) the space $B^{d|1/p-1/2|}_1[L^q]$ in this result can be replaced with $H^{d|1/p-1/2|}_{(q,r)}\equiv F^{d|1/p-1/2|}_2[L^{q,r}]$.

Appendix B. On the Constant in the Triangle Inequality for $L^{p,r}$, $p<1$

In what follows we work with the quasinorm $\|\cdot\|_{p,r}$ on $L^{p,r}$ as defined in (9) or (10). The following result was referenced in §2.

Proposition B.1. *Let $0<p<1$, $p<r<\infty$. Then*

$$\Big\|\sum_k f_k\Big\|_{p,r}\le C(p,r)\Big(\sum_k\|f_k\|^p_{p,r}\Big)^{1/p},$$

where

$$C(p,r)\le A^{1/p}\Big(\frac{1}{1-p}\Big)^{1/p-1/r}\Big(1+\frac{p}{r}\log\frac{1}{1-p}\Big)^{1/p-1/r} \tag{53}$$

and A does not depend on p and r.

We shall need the following lemma. It can be used to proof the inequality (11) when applied in combination with (10).

Lemma B.2. *Let $g:\mathbb{R}^+\to\mathbb{R}$ be a Riemann integrable function and let $r<q$. Then for $0\le\alpha<\beta\le\infty$, $0<p<\infty$,*

$$\sup_{\alpha\le\sigma\le\beta}\sigma|g(\sigma)|^{1/p}\le\Big(q\int_\alpha^\beta\sigma^q|g(\sigma)|^{q/p}\frac{d\sigma}{\sigma}\Big)^{1/q}\le\Big(r\int_\alpha^\beta\sigma^r|g(\sigma)|^{r/p}\frac{d\sigma}{\sigma}\Big)^{1/r}.$$

Proof. We prove the second inequality, as the first one follows by letting $q \to \infty$. We may assume that g is a nonnegative step function on $[\alpha, \beta]$, i.e. there is a partition $\alpha = b_0 < b_1 < \cdots < b_N = \beta$ so that $g(s) = c_j$ if $b_{j-1} < s < b_j$; here $c_j \geq 0$. The inequality then reduces to

$$\Big(\sum_{j=1}^{N} c_j^{q/p} (b_j^q - b_{j-1}^q) \Big)^{1/q} \leq \Big(\sum_{j=1}^{N} c_j^{r/p} (b_j^r - b_{j-1}^r) \Big)^{1/r}$$

We set $v_j = c_j^{r/p}$, $a_j = b_j^r$, and $s = q/r$ so that $s > 1$, and see that the last inequality follows from

$$\Big(\sum_{j=1}^{N} v_j^s (a_j^s - a_{j-1}^s) \Big)^{1/s} \leq \sum_{j=1}^{N} v_j (a_j - a_{j-1}). \tag{54}$$

Since $s \geq 1$ we may (by the triangle inequality for the s-norms) replace $(a_j^s - a_{j-1}^s)$ on the left hand side of (54) with $(a_j - a_{j-1})^s$, and (54) follows from $\| \cdot \|_{\ell^s} \leq \| \cdot \|_{\ell^1}$. □

Proof of Proposition B.1. The proof is based on ideas in [25], [23]. For given $\alpha > 0$ we split $f_k = g_{k,\alpha} + b_{k,\alpha}$ where

$$g_{k,\alpha}(x) = \begin{cases} f_k(x) & \text{if } |f_k(x)| \leq \alpha \\ 0 & \text{if } |f_k(x)| > \alpha \end{cases}$$

and let $b_{k,\alpha} = f_k - g_{k,\alpha}$. Let

$$E_\alpha = \{x : b_{k,\alpha}(x) \neq 0 \text{ for some } k\}.$$

Then

$$\mu(\{x : |\sum_k f_k(x)| > \alpha\}) \leq \mu(E_\alpha) + \mu(\{x : \sum_k |g_{k,\alpha}(x)| > \alpha\})$$

and therefore

$$\begin{aligned} \Big\| \sum_k f_k \Big\|_{p,r}^p \leq & \Big(r \int_0^\infty \alpha^r \mu(E_\alpha)^{r/p} \frac{d\alpha}{\alpha} \Big)^{p/r} \\ & + \Big(r \int_0^\infty \alpha^r \Big[\mu\big(\{x : \sum_k |g_{k,\alpha}(x)| > \alpha\}\big) \Big]^{r/p} \frac{d\alpha}{\alpha} \Big)^{p/r}. \end{aligned} \tag{55}$$

Now

$$\mu(E_\alpha) \leq \sum_k \mu(\{x : b_{k,\alpha}(x) \neq 0\}) \leq \sum_k \mu_{f_k}(\alpha)$$

and hence

$$\begin{aligned} \Big(r \int_0^\infty \alpha^r \mu(E_\alpha)^{r/p} \frac{d\alpha}{\alpha} \Big)^{p/r} & \leq \Big(r \int_0^\infty \alpha^r \Big(\sum_k \mu_{f_k}(\alpha) \Big)^{r/p} \frac{d\alpha}{\alpha} \Big)^{p/r} \\ & \leq \sum_k \Big(r \int_0^\infty \alpha^r [\mu_{f_k}(\alpha)]^{r/p} \frac{d\alpha}{\alpha} \Big)^{p/r} \leq \sum_k \|f_k\|_{p,r}^p \end{aligned} \tag{56}$$

here we have used Minkowski's inequality in $L^{r/p}$ (and thus our assumption $r \geq p$).

We now further decompose $g_{k,\alpha} = l_{k,\alpha} + m_{k,\alpha}$ into a low and a middle part where for a suitable constant $B > 1$,

$$l_{k,\alpha}(x) = \begin{cases} f_k(x) & \text{if } |f_k(x)| \leq \alpha/B \\ 0 & \text{if } |f_k(x)| > \alpha/B \end{cases}$$

and

$$m_{k,\alpha}(x) = \begin{cases} 0 & \text{if } |f_k(x)| \leq \alpha/B \\ f_k(x) & \text{if } \alpha/B < |f_k(x)| \leq \alpha \\ 0 & \text{if } |f_k(x)| > \alpha \end{cases}$$

For a favorable choice for B see (60) below. Now

$$\mu(\{x : \sum_k |g_{k,\alpha}(x)| > \alpha\}) \leq \mu(\{x : \sum_k |l_{k,\alpha}(x)| > \alpha/2\}) + \mu(\{x : \sum_k |m_{k,\alpha}(x)| > \alpha/2\})$$

and therefore by Minkowski's inequality and a subsequent change of variable

$$\Big(r \int_0^\infty \alpha^r \Big[\mu(\{x : \sum_k |g_{k,\alpha}(x)| > \alpha\})\Big]^{r/p} \frac{d\alpha}{\alpha}\Big)^{p/r}$$

$$\leq 2^p \Big(r \int_0^\infty \alpha^r \Big[\mu(\{x : \sum_k |l_{k,2\alpha}(x)| > \alpha\})\Big]^{r/p} \frac{d\alpha}{\alpha}\Big)^{p/r} \tag{57a}$$

$$+ 2^p \Big(r \int_0^\infty \alpha^r \Big[\mu(\{x : \sum_k |m_{k,2\alpha}(x)| > \alpha\})\Big]^{r/p} \frac{d\alpha}{\alpha}\Big)^{p/r}. \tag{57b}$$

Next, by Tshebyshev's inequality

$$\begin{aligned} &\Big(r \int_0^\infty \alpha^r \Big[\mu(\{x : \sum_k |l_{k,2\alpha}(x)| > \alpha\})\Big]^{r/p} \frac{d\alpha}{\alpha}\Big)^{p/r} \\ &\leq \Big(r \int_0^\infty \alpha^r \Big[\frac{1}{\alpha} \int \sum_k |l_{k,2\alpha}| d\mu\Big]^{r/p} \frac{d\alpha}{\alpha}\Big)^{p/r} \\ &\leq \Big(r \int_0^\infty \Big[\sum_k \alpha^p \frac{1}{\alpha} \int_0^{2B^{-1}\alpha} \mu_{l_{k,2\alpha}}(\beta) d\beta\Big]^{r/p} \frac{d\alpha}{\alpha}\Big)^{p/r}; \end{aligned}$$

here we have used $\mu_{l_{k,2\alpha}}(\beta) = 0$ when $\beta > B^{-1}2\alpha$. We now use Minkowski's inequality in $L^{r/p}(d\alpha/\alpha)$. Since $|l_{k,2\alpha}| \leq |f_k|$ we see that the last displayed expression is dominated by

$$\begin{aligned}
&\sum_k \Big(r\int_0^\infty \Big[\alpha^p \frac{1}{\alpha}\int_0^{2B^{-1}\alpha} \mu_{f_k}(\beta)d\beta\Big]^{r/p}\frac{d\alpha}{\alpha}\Big)^{p/r} \\
&= \sum_k \Big(r\int_0^\infty \alpha^r \Big[\int_0^{2B^{-1}} \mu_{f_k}(s\alpha)ds\Big]^{r/p}\frac{d\alpha}{\alpha}\Big)^{p/r}.
\end{aligned}$$

We continue as in the proof of Hardy's inequalities and estimate using the integral Minkowski inequality

$$\begin{aligned}
&\Big(r\int_0^\infty \alpha^r \Big[\int_0^{2B^{-1}} \mu_{f_k}(s\alpha)ds\Big]^{r/p}\frac{d\alpha}{\alpha}\Big)^{p/r} \\
&\le \int_0^{2B^{-1}} s^{-p}\Big(\int_0^\infty r\beta^{r-1}\mu_{f_k}(\beta)^{r/p}d\beta\Big)^{p/r} = \frac{2^{1-p}B^{p-1}}{1-p}\|f_k\|^p_{L^{p,r}}
\end{aligned}$$

Thus, combining estimates we get

$$(57a) \le \frac{2}{1-p}B^{p-1}\sum_k \|f_k\|^p_{p,r}. \tag{58}$$

We now estimate the terms in (57b). We write $[0,\infty)$ as a union over the intervals $I_n = [B^n, B^{n+1}]$, $n\in\mathbb{Z}$ and apply Lemma B.2 to each interval. Then (57b) is estimated by

$$\begin{aligned}
&2^p\Big(\sum_{n\in\mathbb{Z}} r\int_{B^n}^{B^{n+1}} \alpha^r \Big[\mu(\{x : \sum_k |m_{k,2\alpha}(x)| > \alpha\})\Big]^{r/p}\frac{d\alpha}{\alpha}\Big)^{p/r} \\
&\le 2^p\Big(\sum_{n\in\mathbb{Z}} \Big[p\int_{B^n}^{B^{n+1}} \alpha^p \mu(\{x : \sum_k |m_{k,2\alpha}(x)| > \alpha\})\frac{d\alpha}{\alpha}\Big]^{r/p}\Big)^{p/r}.
\end{aligned}$$

We now define

$$f_{k,n}(x) = \begin{cases} f_k(x) & \text{if } B^n \le |f_k(x)| \le B^{n+2} \\ 0 & \text{otherwise} \end{cases}$$

and observe

$$|m_{k,2\alpha}(x)| \le |f_{k,n}(x)| \text{ if } B^n \le \alpha \le B^{n+1}.$$

Hence we get

$$\begin{aligned}
&\Big(r\int_0^\infty \alpha^r\Big[\mu(\{x:\sum_k |m_{k,2\alpha}(x)|>\alpha\})\Big]^{r/p}\frac{d\alpha}{\alpha}\Big)^{p/r}\\
&\le \Big(\sum_{n=-\infty}^{\infty}\Big[\int_0^\infty p\alpha^{p-1}\mu(\{x:\sum_k |f_{k,n}(x)|>\alpha\})d\alpha\Big]^{r/p}\Big)^{p/r}\\
&\le \Big(\sum_{n=-\infty}^{\infty}\Big\|\sum_k |f_{k,n}|\Big\|_p^r\Big)^{p/r} \le \Big(\sum_{n=-\infty}^{\infty}\Big[\sum_k \|f_{k,n}\|_p^p\Big]^{r/p}\Big)^{p/r}\\
&\le \sum_k\Big(\sum_{n=-\infty}^{\infty}\|f_{k,n}\|_p^r\Big)^{p/r}.
\end{aligned}$$

Here we have used the triangle inequality in L^p, $p<1$ and Minkowski's inequality for the sequence space $\ell^{r/p}$, $r\ge p$. Now

$$\begin{aligned}
\|f_{k,n}\|_p^p &\le \int_0^{B^n} p\alpha^{p-1}\mu_{f_k}(B)d\alpha + \int_{B^n}^{B^{n+2}} p\alpha^p\mu_{f_k}(\alpha)\frac{d\alpha}{\alpha}\\
&\le B^{np}\mu_{f_k}(B) + (\log B^2)^{1-p/r}\frac{p}{r^{p/r}}\Big(\int_{B^n}^{B^{n+2}} r\alpha^r\mu_{f_k}(\alpha)^{r/p}\frac{d\alpha}{\alpha}\Big)^{p/r},
\end{aligned}$$

by Hölder's inequality, and

$$\begin{aligned}
\Big(\sum_n \|f_{k,n}\|_p^r\Big)^{p/r} &\le \Big(\sum_n B^{nr}\mu_{f_k}(B)^{r/p}\Big)^{p/r}\\
&\qquad + (\log B^2)^{1-p/r}\frac{p}{r^{p/r}}\Big(\sum_n\int_{B^n}^{B^{n+2}} r\alpha^{r-1}\mu_{f_k}(\alpha)^{r/p}d\alpha\Big)^{r/p}\\
&\le \Big(1+\frac{p}{r^{p/r}}2^{p/r}(\log B^2)^{1-p/r}\Big)\Big(\int_0^\infty r\alpha^{r-1}\mu_{f_k}(\alpha)d\alpha\Big)^{p/r}.
\end{aligned}$$

Consequently

$$\text{(57b)} \le 2^p\Big(1+\frac{p}{r^{p/r}}2^{p/r}(2\log B)^{1-p/r}\Big)\sum_k \|f_k\|_{L^{p,r}}^p . \tag{59}$$

We now make the choice

$$B=(1-p)^{-\frac{p}{r(1-p)}} \tag{60}$$

so that

$$\frac{B^{p-1}}{1-p}=(1-p)^{-(1-p/r)}, \qquad \log B=\frac{p}{r(1-p)}\log\frac{1}{p-1}\,.$$

Combining the various estimates we obtain

$$\Big\| \sum_k f_k \Big\|_{L^{p,r}}^p \le \Big(1 + \frac{2}{(1-p)^{1-p/r}} + \frac{p2^{p+1}}{r^{p/r}} \frac{p}{r(1-p)} \log(\frac{1}{1-p})^{1-p/r}\Big) \sum_k \|f_k\|_{L^{p,r}}^p \tag{61}$$

which yields the lemma. □

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