

A REMARK ON THE CONSTANT IN AN INEQUALITY BY FEFFERMAN AND STEIN

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ABSTRACT. This is an expository note on an inequality by Fefferman and Stein inequality which controls the L^p norm of the Hardy-Littlewood maximal function by the L^p norm of the $\#$ -function. We show that there is a linear growth of the constant in this inequality as $p \rightarrow \infty$. As a consequence one gets an interpolation result for operators which features a linear growth in p .

We consider the Fefferman–Stein $\#$ -function

$$f^\#(x) = \sup_Q \int_Q |f(x) - \int_Q f| dx$$

where the slashed integral denotes the average of f over the cube Q . The BMO -seminorm is defined by $\|f\|_{BMO} = \|f^\#\|_\infty$.

The following theorem is a variant of a result by Fefferman and Stein [4].

Theorem 1. *Let $0 < r_0 \leq p < \infty$ and assume that the Hardy-Littlewood maximal function Mf belongs to L^{r_0} and that f belongs to some L^q with $1 \leq q < \infty$. Then*

$$\|Mf\|_p \leq C(p, d) \|f^\#\|_p$$

where

$$C(p, d) \leq \begin{cases} 3^{\frac{d+1}{p}} 2^{3d+4} & \text{if } r_0 \leq p \leq 2 \\ 3^{\frac{d}{p}} 2^{3d+5} p & \text{if } 2 \leq p < \infty. \end{cases}$$

Here the Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f|$$

where the sup is taken over all cubes with sides parallel to the axes that contain x .

Remark. The original versions of the above theorems by Fefferman and Stein [4] were proved with a constant $C(d, p)$ which is exponential in p (i.e. $\lesssim C_d A^p$ with some $A > 1$). This exponential bound has been reproduced in the standard textbooks [5, 8, 9, 3, 6]. In this note we observe that a more careful choice of parameters yields the better $O(p)$ bound as $p \rightarrow \infty$. This is done by an adjustment of the usual method of good- λ inequalities. It may also be interesting to investigate the dependence of the constant of the dimension d , but we are not focusing on that here.

The inequality may be applied to interpolate between L^p and BMO .

Theorem 2. *Let $1 < p_0 < p < \infty$, let T be a linear operator defined on simple functions and suppose that for all simple f we have*

$$\begin{aligned} \|Tf\|_{p_0} &\leq A_0 \|f\|_{p_0} \\ \|Tf\|_{BMO} &\leq A_1 \|f\|_{\infty} \end{aligned}$$

Then for $p_0 \leq p < \infty$, T extends to a bounded operator on L^p with operator norm

$$\|T\|_{L^p \rightarrow L^p} \leq C_d p A_0^{p_0/p} A_1^{1-p_0/p}.$$

This is again variant of a theorem due to Fefferman–Stein [4] with an improved estimate for the dependence on p as $p \rightarrow \infty$. Theorem 2 is deduced from Theorem 1 by a linearization argument, just as in [4]. One considers for simple functions f

$$\mathfrak{T}f(x) = \frac{1}{|Q(x)|} \int_{Q(x)} e^{i\gamma(x,y)} \left[Tf(y) - \frac{1}{|Q(x)|} \int_{Q(x)} Tf(w) dw \right] dy$$

where γ is real-valued and measurable and $Q(x)$ is a cube depending measurably on x with $x \in Q(x)$. The assumptions on T together with the boundedness of the Hardy-Littlewood maximal operator imply that $\mathfrak{T}$ maps L^{p_0} into itself with operator norm $3^{d/p_0} A_0$ and L^∞ to L^∞ , with operator norm A_1 . An application of the Riesz-Thorin theorem to $\mathfrak{T}$ together with Theorem 1 shows the interpolation result.

Remark. A similar, slightly simpler interpolation inequality for $1 \leq p_0 < p < \infty$, namely

$$\|g\|_p \leq C_d p \|g\|_{p_0}^{p_0/p} \|g\|_{BMO}^{1-p_0/p}$$

has been shown by Chen and Zhu [2], using an inequality for the rearrangement g^* of a BMO -function. Assuming that $\lim_{R \rightarrow \infty} \int_{|f| > R} |f| dx < \infty$ they use the rearrangement $g^*(t) \leq C(g, n) \log(2 + \frac{1}{t})$ from the original paper by John and Nirenberg [7] to deduce their interpolation inequality.

1. CALDERÓN-ZYGMUND DECOMPOSITIONS

We start with the *Calderón-Zygmund decomposition at level $\lambda > 0$* . Let $f \in L^q$, $1 \leq q < \infty$. Considering $\frac{1}{|Q|} \int_Q |f| dx \leq \left(\frac{1}{|Q|} \int_Q |f|^q dx \right)^{1/q} \leq |Q|^{-1/q} \|f\|_p$ we see that $\lim_{|Q| \rightarrow \infty} \frac{1}{|Q|} \int_Q |f| dx = 0$. Let $\lambda > 0$ and consider the family of dyadic cubes such that $\frac{1}{|Q|} \int_Q |f| dx > \lambda$. The sidelengths of these cubes are bounded. Every cube that satisfies this inequality is contained in a maximal cube that also satisfies it. The diameters of these maximal cubes are bounded.

Now let $\mathfrak{Q}_\lambda$ be the collection of Calderón-Zygmund cubes at height λ , i.e. the collection of *maximal* dyadic cubes with the property that

$$\lambda < \int_Q |f|.$$

Since the dyadic parent cube Q^* of $Q \in \mathfrak{Q}_\lambda$ does not belong to $\mathfrak{Q}_\lambda$ one has

$$\lambda < \int_Q |f| \leq 2^d \lambda, \text{ and } \int_{Q^*} |f| \leq \lambda.$$

Note that the cubes in $\mathfrak{Q}_\lambda$ have disjoint interior. We will set

$$\Omega_\lambda = \bigcup_{Q \in \mathfrak{Q}_\lambda} Q.$$

Then there is a Lebesgue null set $\mathcal{N}$ so that

$$\{x : |f(x)| > \lambda\} \setminus \mathcal{N} \subset \Omega_\lambda \subset \{x : Mf(x) > \lambda\}$$

The inclusion on the right hand side follows from $\lambda < |Q|^{-1} \int_Q |f| dx$ for $Q \in \mathfrak{Q}_\lambda$ and the definition of the maximal function, and the left hand side follows by the Lebesgue differentiation theorem (note that if $x \notin \Omega_\lambda$ then $|Q|^{-1} \int_Q f \leq \lambda$ for all dyadic cubes containing x).

Lemma 3. For $\alpha > 0$,

$$\text{meas}(\{x : Mf(x) > \alpha\}) \leq 3^d |\Omega_{4^{-d}\alpha}|$$

Also

$$\|Mf\|_p^p \leq 3^d 4^{dp} \int_0^\infty p \mu^{p-1} |\Omega_\mu| d\mu.$$

Proof. Fix x such that $Mf(x) > \alpha$. Then there exists a cube R such that $|R|^{-1} \int_R |f| > \alpha$. Let n be such that the sidelength of R is between 2^{-n-1} and 2^{-n} . R is contained in the union of 2^d dyadic cubes of sidelength 2^{-n} . At least one of them, say Q must satisfy

$$\frac{1}{|R|} \int_{R \cap Q} |f(y)| dy > \frac{1}{2^d} \alpha.$$

We have $|Q| \leq 2^d |R|$ and therefore

$$\frac{1}{|Q|} \int_Q |f| dy \geq 2^{-d} \frac{1}{|R|} \int_{R \cap Q} |f(y)| dy \geq 4^{-d} \alpha$$

Hence Q is contained in a maximal cube in the Calderón-Zygmund decomposition of level $4^{-d}\alpha$. x may not belong to Q , but it belongs to the three-fold dilate of Q , call it Q^* . Hence $\{x : |Mf| > \alpha\}$ is contained in the union of 3-fold dilates of the cubes in $\mathfrak{Q}_{4^{-d}\alpha}$ which has measure at most $3^d |\Omega_{4^{-d}\alpha}|$.

For the L^p norm we estimate

$$\begin{aligned} \|Mf\|_p^p &= \int_0^\infty p \alpha^{p-1} |\{x : |Mf| > \alpha\}| d\alpha \\ &\leq \int_0^\infty p \alpha^{p-1} 3^d |\Omega_{4^{-d}\alpha}| d\alpha = 3^d 4^{dp} \int_0^\infty p \mu^{p-1} |\Omega_\mu| d\mu. \quad \square \end{aligned}$$

Notation. In what follows we use Calderón-Zygmund decompositions at two different levels $\lambda < \mu$. The cubes in $\mathfrak{Q}_\mu$ will be denoted by Q , and the cubes in $\mathfrak{Q}_\lambda$ will be denoted by W . We will put $\lambda = (1 - \varepsilon)\mu$. Note that in this scenario every $Q \in \mathfrak{Q}_\mu$ is contained in a unique cube in $\mathfrak{Q}_{(1-\varepsilon)\mu}$. For $\gamma > 0$ let

$$E_\gamma = \{x : f^\#(x) > \gamma\}$$

and let $E_\gamma^c = \mathbb{R}^d \setminus E_\gamma$. We will set $\gamma = \beta\mu$ where $\beta \ll 1$ (below we will determine $\beta = 2^{-d-4}p^{-1}$ when $p \geq 2$).

Lemma 4. *Let $0 < \beta < 1$, and $0 < \varepsilon < 1/2$. Then*

$$|\Omega_\mu \cap E_{\beta\mu}^c| \leq 2^d \varepsilon^{-1} \beta |\Omega_{(1-\varepsilon)\mu}|$$

Proof. Let $\tilde{\mathfrak{Q}}$ be the family of all $W \in \mathfrak{Q}_{(1-\varepsilon)\mu}$ which are not contained in $E_{\beta\mu}$. If $W \in \tilde{\mathfrak{Q}}$ then $f^\#(x_0) \leq \beta\mu$ for some $x_0 \in W$ and thus the mean oscillation for all cubes containing W is bounded by $\beta\mu$; in particular for the parent cube W^* of W we have

$$\int_{W^*} |f(x) - \int_{W^*} f| dx \leq \beta\mu.$$

We estimate for $W \in \tilde{\mathfrak{Q}}$

$$\begin{aligned} |W \cap \Omega_\mu| &= \sum_{Q \subset W} |Q| \leq \frac{1}{\mu} \sum_{Q \subset W} \int_Q |f(x)| dx \\ &\leq \frac{1}{\mu} \sum_{Q \subset W} \int_Q \left| f(x) - \int_{W^*} f + \int_{W^*} f \right| dx \\ &\leq \frac{1}{\mu} \sum_{Q \subset W} \int_Q \left| f(x) - \int_{W^*} f \right| dx + \frac{1}{\mu} \sum_{Q \subset W} |Q| \int_{W^*} |f| dx \\ &\leq \frac{1}{\mu} \int_{W^*} \left| f(x) - \int_{W^*} f \right| dx + \frac{1}{\mu} |W \cap \Omega_\mu| \int_{W^*} |f| dx \\ &\leq \frac{1}{\mu} |W^*| \beta\mu + \frac{1}{\mu} |W \cap \Omega_\mu| \mu(1 - \varepsilon) = |W^*| \beta + (1 - \varepsilon) |W \cap \Omega_\mu| \end{aligned}$$

This implies, for every $W \in \tilde{\mathfrak{Q}}$,

$$\varepsilon |W \cap \Omega_\mu| \leq 2^d \beta |W|$$

and summing over $W \in \tilde{\mathfrak{Q}}$ and dividing by ε yields the asserted inequality. $\square$

Now observe that the Calderón-Zygmund cubes W , which are not in $\tilde{\mathfrak{Q}}$, are contained in $E_{\beta\mu}$. This yields

Corollary 5. *For $\varepsilon \in (0, \frac{1}{2})$,*

$$|\Omega_\mu| \leq 2^d \varepsilon^{-1} \beta |\Omega_{(1-\varepsilon)\mu}| + |E_{\beta\mu}|.$$

We now estimate the quantity

$$I_R = \int_0^R p\mu^{p-1}|\Omega_\mu|d\mu$$

(and later let $R \rightarrow \infty$). Observe that I_N is finite, indeed by the assumption of $Mf \in L^{p_0}$ we have

$$\begin{aligned} I_R &\leq R^{p-r_0}p \int_0^\infty \mu^{r_0-1}|\Omega_\mu|d\mu \\ &\leq R^{p-r_0}p \int_0^\infty \mu^{r_0-1}|\{x : Mf(x) > \mu\}|d\mu \leq R^{p-r_0}(p/r_0)\|Mf\|_{r_0}^{r_0} < \infty \end{aligned}$$

By Lemma 4,

$$\begin{aligned} I_R &\leq 2^d\varepsilon^{-1}\beta \int_0^R p\mu^{p-1}|\Omega_{(1-\varepsilon)\mu}|d\mu + \int_0^R p\mu^{p-1}|E_{\beta\mu}|d\mu \\ &\leq 2^d\varepsilon^{-1}\beta(1-\varepsilon)^{-p} \int_0^{R(1-\varepsilon)} p\alpha^{p-1}|\Omega_\alpha|d\alpha + \beta^{-p} \int_0^R p\alpha^{p-1}|E_\alpha|d\alpha \\ &\leq 2^d\varepsilon^{-1}\beta(1-\varepsilon)^{-p}I_R + \beta^{-p}\|f^\#\|_p^p \end{aligned}$$

We choose

$$\varepsilon = \varepsilon(p) = \begin{cases} 1/2 & \text{if } p \leq 2 \\ 1/p & \text{if } p \geq 2. \end{cases}$$

Let $p \geq 2$. Then $\varepsilon(p)^{-1}(1-\varepsilon(p))^{-p} = p(1-\frac{1}{p})^{-p} < 8p$ and, (using that $|I_R| < \infty$, the inequality becomes

$$I_R(1-2^{d+3}p\beta) \leq \beta^{-p}\|f^\#\|_p^p.$$

Choosing $\beta = 2^{-d-4}p^{-1}$ we obtain

$$I_R \leq (2^{d+5}p)^p\|f^\#\|_p^p$$

We let $R \rightarrow \infty$ to deduce

$$\int_0^\infty p\mu^{p-1}|\Omega_\mu|d\mu \leq (2^{d+5}p)^p\|f^\#\|_p^p.$$

Finally

$$\|Mf\|_p^p \leq 3^d4^{dp} \int_0^\infty p\mu^{p-1}|\Omega_\mu|d\mu \leq 3^d4^{dp}(2^{d+5}p)^p\|f^\#\|_p^p = 3^d2^{(3d+5)p}p^p\|f^\#\|_p^p.$$

Take the p th root and the claimed estimate for $p \geq 2$ follows.

If $p \leq 2$ we put $\varepsilon = 1/2$ above and we get

$$I_R(1-2^{d+1+p}\beta) \leq \beta^{-p}\|f^\#\|_p^p.$$

and if we put $\beta = 2^{-d-2-p}$ we get

$$|I_R| \leq 2 \cdot 2^{(d+2+p)p}\|f^\#\|_p^p \leq 2 \cdot 2^{(d+4)p}\|f^\#\|_p^p$$

Continuing as above we get $\|Mf\|_p^p \leq 2 \cdot 3^d4^{dp}2^{(d+4)p}\|f^\#\|_p^p$ if $p \leq 2$. So

$$\|Mf\|_p \leq 2^{1/p}3^{d/p} \cdot 4^d2^{(d+4)}\|f^\#\|_p.$$

Comment (December 2025). Daniel Spector pointed out the early paper by Bennett and Sharpley [1] who show the inequality

$$(Mf)^*(t) \leq C \int_t^\infty (f^\sharp)^*(s) \frac{ds}{s} + (Mf)^*(\infty).$$

for the nondecreasing rearrangement functions of Mf and $f^\sharp$. Hence if $Mf \in L^{p_0}$ for some $p_0 < \infty$ then

$$\|Mf\|_p = \left(\int_0^\infty (Mf)^*(t)^p dt \right)^{1/p} \leq C \left(\int_0^\infty \left[\int_t^\infty (f^\sharp)^*(s) \frac{ds}{s} \right]^p dt \right)^{1/p}$$

and the Fefferman-Stein inequality with $O(p)$ dependence follows by an application of Hardy's inequality.

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