POWER WEIGHT INEQUALITIES FOR SPHERICAL MAXIMAL FUNCTIONS

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ABSTRACT. This paper is about spherical maximal functions with general dilation sets acting on functions in weighted $L^p(|x|^\alpha)$ spaces. Aside from endpoint cases, a complete description of the allowable ranges of p, α is given in terms of the Legendre–Assouad function of the dilation set. This settles, up to endpoints, an open problem of Duoandikoetxea and Seijo.

1. INTRODUCTION

Let $d \geq 2$. For a locally integrable function $f : \mathbb{R}^d \to \mathbb{C}$ the average over the sphere S^{d-1} of radius t centered at $x \in \mathbb{R}^d$ is given by

$$A_t f(x) = f * \sigma_t(x) = \int f(x + t\omega) d\sigma(\omega),$$

where σ is the normalized surface measure and the dilate σ_t is defined by $\langle \sigma_t, f \rangle = \langle \sigma, f(t\cdot) \rangle$. The average $A_t f(x)$ is well-defined for almost every $x \in \mathbb{R}^d$. Given a set $\mathcal{E}$ of radii in $(0, \infty)$, the maximal function $M_{\mathcal{E}}$ is given by

$$M_{\mathcal{E}} f(x) = \sup_{t \in \mathcal{E}} |A_t f(x)|$$

which yields a measurable function at least for continuous f. If $\mathcal{E}$ has an accumulation point at 0, then size estimates for the maximal operator can be used to prove pointwise convergence results for $A_t f(x)$ as $t \to 0$ within $\mathcal{E}$. For the full spherical maximal operator, corresponding to $\mathcal{E} = (0, \infty)$, $L^p(\mathbb{R}^d)$ boundedness holds if and only if $p > 1 + \frac{1}{d-1}$; this is due to Stein [19] for $d \geq 3$ and to Bourgain [4] for $d = 2$.

The case of general $\mathcal{E} \subset (0, \infty)$ was considered in [18], where it was shown that the boundedness range depends on a variant of Minkowski dimension. Equip $(0, \infty)$ with the metric

$$d_\times(s, t) = |\log_2(s/t)|. \tag{1.1}$$

We denote the diameter of a set $J \subset (0, \infty)$ by $|J|_\times$. Note that dilations $t \mapsto \lambda t$ with $\lambda > 0$ are isometries on $(0, \infty)$, so that e.g. $|[R, 2R]|_\times = 1$

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for all $R > 0$. For a set $E \subset (0, \infty)$ with bounded diameter we denote by $N(E, \delta)$ the *entropy numbers*, i.e. for each $\delta > 0$ the minimum number of intervals of diameter δ required to cover E (where diameter is always taken with respect to $d_\times$).

With $\mathcal{E} \subset (0, \infty)$ fixed, define

$$\beta = \beta_{\mathcal{E}} = \limsup_{\delta \to 0} \frac{\sup_{|J|_\times = 1} \log N(\mathcal{E} \cap J, \delta)}{\log(\delta^{-1})},$$

where the supremum is taken over intervals $J \subset (0, \infty)$ of diameter one. This is a natural extension of the standard (upper) Minkowski dimension to the case of unbounded sets; indeed, β is equal to the Minkowski dimension if $\mathcal{E}$ has bounded diameter (with respect to $d_\times$). The result in [18] states that $M_{\mathcal{E}}$ is bounded on L^p if

$$p > p_\beta = 1 + \tfrac{\beta}{d-1}$$

and unbounded if $p < p_\beta$ (see also [17], [15] for the discussion of various endpoint problems).

In this paper we are interested in power weight inequalities of the form

$$\int |M_{\mathcal{E}} f(x)|^p |x|^\alpha \, dx \lesssim \int |f(y)|^p |y|^\alpha \, dy. \tag{1.2}$$

Thus for the weight $w_\alpha(x) = |x|^\alpha$ we ask for which $p \in [1, \infty]$ and which $\alpha \in \mathbb{R}$ the sublinear operator $M_{\mathcal{E}}$ maps $L^p(w_\alpha)$ to itself. Our main result answers this question for every given $\mathcal{E} \subset (0, \infty)$ up to endpoint cases in the parameters p and α.

More precisely, we will determine the closure of the "type set"

$$\mathfrak{T}_{\mathcal{E}} = \{(\tfrac{1}{p}, \tfrac{\alpha}{p}) \in [0,1] \times \mathbb{R} \,:\, M_{\mathcal{E}} \text{ is bounded on } L^p(w_\alpha)\}, \tag{1.3}$$

which is in one-to-one correspondence with the closure of the set of all $(\frac{1}{p}, \alpha)$ for which $M_{\mathcal{E}}$ is bounded on $L^p(w_\alpha)$. Observe that $\mathfrak{T}_{\mathcal{E}}$ is convex (for the short proof see §2.1).

To formulate our results we need another quantification of dimension besides β, called the *Legendre–Assouad function* which was first introduced in [3, 2] for subsets of $[1, 2]$. For $\rho \in \mathbb{R}$ we let

$$\nu^\sharp(\rho) = \nu^\sharp_{\mathcal{E}}(\rho) = \limsup_{\delta \to 0} \frac{\log\big(\sup_{\delta \le |J|_\times \le 1} |J|_\times^{-\rho} N(\mathcal{E} \cap J, \delta)\big)}{\log(\delta^{-1})}, \tag{1.4}$$

where the supremum is taken over intervals $J \subset (0, \infty)$ of $d_\times$-diameter in $[\delta, 1]$. This is a natural extension of the original definition from [3, 2] to unbounded sets. One can show ([2, §2]) that $\nu^\sharp$ equals the Legendre transform of the function

$$\nu(\theta) = -(1-\theta) \dim_{\mathrm{A},\theta} \mathcal{E},$$

where $\theta \mapsto \dim_{\mathrm{A},\theta} \mathcal{E}$ is the Assouad spectrum of $\mathcal{E}$ with respect to the metric $d_\times$ (see Fraser–Yu [10], Fraser [8]).

Recall from [2] that $\nu^\sharp$ is convex and increasing, and $\nu^\sharp(\rho) = \beta$ for $\rho \le 0$. Now consider the generalized inverse of the increasing function $\nu^\sharp$ which is defined for $s \ge \beta$ as

$$(\nu^\sharp)^\dagger(s) = \sup\{\rho \ge 0 \,:\, \nu^\sharp(\rho) \le s\}.$$

For $p \ge p_\beta = 1 + \frac{\beta}{d-1}$ let

$$U(p) = (d-1)(p-1) - \beta, \tag{1.5}$$

$$L(p) = (d-1)(p-2) - (\nu^\sharp)^\dagger\big((d-1)(p-1)\big). \tag{1.6}$$

Theorem 1.1. *Let $d \ge 2$, $\mathcal{E} \subset (0,\infty)$ nonempty. Then*

$$\overline{\mathfrak{T}_\mathcal{E}} = \{\big(\tfrac{1}{p}, \tfrac{\alpha}{p}\big) \in [0,1] \times \mathbb{R} \,:\, p \ge p_\beta,\ L(p) \le \alpha \le U(p)\}. \tag{1.7}$$

Figure 1 visualizes a typical set $\overline{\mathfrak{T}_\mathcal{E}}$. Note that $\overline{\mathfrak{T}_\mathcal{E}}$ is not necessarily a polygon, since $\nu^\sharp|_{[0,\infty)}$ can be any nonnegative increasing convex function such that $\nu^\sharp(\rho) = \rho$ for $\rho \ge 1$ (see [2, Thm. 1.2 (ii)]).

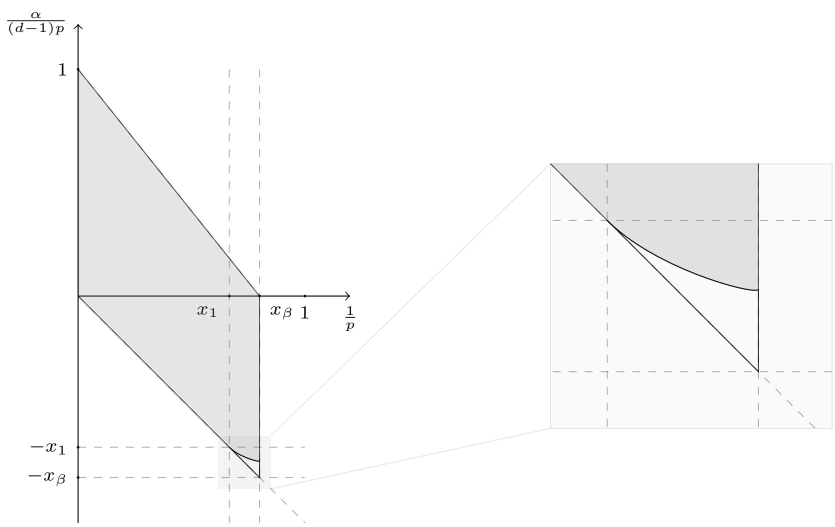

FIGURE 1. Typical shape of the type set $\mathfrak{T}_\mathcal{E}$ (shaded). The containing trapezoid is defined by the necessary conditions $p \ge p_\beta$ and (1.10). Here $x_1 = \frac{1}{p_1}$, $x_\beta = \frac{1}{p_\beta}$.

Note that if $\rho_* = \sup\{\rho : \nu^\sharp(\rho) = \beta\}$, then $0 \le \rho_* \le \beta$, and by convexity of $\nu^\sharp$ it is continuous and strictly increasing for $\rho \ge \rho_*$. Thus $(\nu^\sharp)^\dagger$ is equal to the inverse of $\nu^\sharp|_{[\rho_*,\infty)}$. Note also that $\nu^\sharp(\rho) = \rho$ for $\rho \ge \gamma$, where $\gamma \ge \beta$ is the quasi-Assouad dimension of $\mathcal{E}$ (see [12, 9, 8]). Finally, note $L(p_\beta) = 1 - d + \beta - \rho_*$ and $L(p) = 1 - d$ for $p \ge p_\gamma$.

We may also state Theorem 1.1 with a more concise, albeit implicit condition. Namely, if we set

$$\Theta(p,\alpha) = \max\big\{\alpha + \beta,\ \nu^\sharp\big((d-1)(p-2) - \alpha\big)\big\}, \tag{1.8}$$

then

$$\overline{\mathfrak{T}_{\mathcal{E}}} = \big\{(\tfrac{1}{p}, \tfrac{\alpha}{p}) \in [0,1] \times \mathbb{R} : p \geq 1 + (d-1)^{-1}\Theta(p,\alpha)\big\}. \tag{1.9}$$

See §2.2 for details on this equivalence. Our proof of Theorem 1.1 will rely on the implicit formulation.

We briefly discuss Theorem 1.1 in the context of the previous literature. In the unweighted case $\alpha = 0$ we recover the result from [18]. Examples from the unweighted theory show that the condition $p \geq p_\beta$ is necessary. Since $\nu^\sharp(\rho) \geq \max(\rho, \beta)$ for all $\rho \in \mathbb{R}$ (see [3, Lemma 2.1]), we get the necessary condition

$$-(d-1) \leq \alpha \leq (d-1)(p-1) - \beta, \tag{1.10}$$

which was also known previously.

The case of the full maximal operator with $\mathcal{E} = (0, \infty)$ was almost fully resolved by Duoandikoetxea and Vega [7], where it was shown that boundedness holds for the full maximal operator with $\mathcal{E} = (0,\infty)$ when $1-d < \alpha < (d-1)p-d$ and $p > 1+\frac{1}{d-1}$. This matches our condition since in this case $\nu^\sharp(\rho) = \max(1,\rho)$. See also the recent endpoint results for $\alpha = 1-d$ by Juyoung Lee [11], for $d = 2$, $p > 2$, and $d \geq 3$, $p \geq 2$. Sharp results for the full spherical maximal operator acting on radial functions can be found in [5], see also [13]. In the lacunary case $\mathcal{E} = \{2^k : k \in \mathbb{Z}\}$ it was proved in [7] that $M_{\mathcal{E}}$ is bounded on $L^p(w_\alpha)$ if and only if $1-d \leq \alpha < (d-1)(p-1)$, $p > 1$.

For more general sets $\mathcal{E} \subset (0,\infty)$ the maximal operators $M_{\mathcal{E}}$ were investigated by Duoandikoetxea and Seijo [6]. They showed that $M_{\mathcal{E}}$ fails to be $L^p(w_\alpha)$-bounded if $\alpha < 1-d$ or if $\alpha > (d-1)(p-1)-\beta$, and proved a positive result if $1-d < \alpha < (d-1)(p-1)-\beta$, and $p > p_1 = 1+\frac{1}{d-1}$. Theorem 1.1 shows that the latter p-range can be improved to $p > p_\gamma = 1 + \frac{\gamma}{d-1}$.

Less was known in the remaining cases $p_\beta \leq p \leq p_\gamma$. The zoomed-in region in Figure 1 shows a typical behavior in this range. For a class of "regular sets" $\mathcal{E}$ an essentially sharp result for this range was obtained in [6]. In our terminology, these are precisely the sets where β is equal to the Assouad dimension and in this case $\nu^\sharp(\rho) = \max(\rho,\beta)$; specific examples include self-similar Cantor sets. Duoandikoetxea and Seijo [6] also showed that there are sets E_1, E_2 with $\beta_{E_1} = \beta_{E_2}$ but with a different α-range for $L^p(w_\alpha)$ boundedness for certain values of p. In particular, they investigated the model examples $\mathcal{E} = \{1 + n^{-a} : n \in \mathbb{N}\}$, proved an essentially sharp result for $a = 1$, but obtained only partial results for $a \neq 1$. These sets are all special cases of *Assouad regular sets*; for the definition we refer to §3. There we further explore our boundedness condition for Assouad regular sets, and then also for finite unions of those. However, Theorem 1.1 goes far beyond these classes and settles the problem on $L^p(w_\alpha)$-boundedness, up to endpoints, for *all* dilations sets.

A key point in the proof is the discussion of two families of examples, one of them having appeared previously in papers on L^p-improving bounds for spherical maximal operators in [1, 14]. These yield the required lower bounds on the maximal function proved in §4. The upper bounds for the proof of Theorem 1.1 are given in §5.

Notation. We write, for nonnegative quantities u, v, $u \lesssim v$ or $u \lesssim_L v$ to indicate $u \le Cv$ for some constant C which may depend on some list L. We write $u \approx v$ to indicate that both $u \lesssim v$ and $v \lesssim u$ hold.

2. Preliminaries

2.1. *Type set convexity.* We show that the type set $\mathfrak{T}_{\mathcal{E}}$ as defined in (1.3) is convex. Consider the operator $\widetilde{M}_{u,\mathcal{E}}$ defined by

$$\widetilde{M}_{u,\mathcal{E}} f(x) = w_u(x) M_{\mathcal{E}}[f w_{-u}](x)$$

and note that the $L^p(w_\alpha)$ boundedness of $M_{\mathcal{E}}$ is equivalent to the boundedness of $\widetilde{M}_{u,\mathcal{E}}$ on unweighted L^p if $u = \alpha/p$. Hence the convexity of $\mathfrak{T}_{\mathcal{E}}$ can be seen by applying Stein's analytic interpolation theorem [21] to the family of linear operators $T_{u,\mathcal{E}} f(x) = |x|^u A_{t(x)}[f|\cdot|^{-u}](x)$ where $u = (1-z)u_0 + z u_1$ with $u_0, u_1 \in \mathfrak{T}_{\mathcal{E}}$ and $0 \le \operatorname{Re}(z) \le 1$, and $t(x)$ is a measurable function of radii with $t(x) \in \mathcal{E}$. Alternatively (and equivalently) one can also argue more directly by applying the Stein–Weiss theorem for interpolation with changes of measures [20] to the operator $A_{t(\cdot)}$.

2.2. *An equivalent condition.* We will prove the equivalence of the two type set descriptions (1.7) and (1.9). The condition $p \ge 1 + (d-1)^{-1}\Theta(p, \alpha)$ in (1.9) can be written as

$$\max\big\{\alpha + \beta,\, \nu^\sharp\big((d-1)(p-2) - \alpha\big)\big\} \le (d-1)(p-1). \tag{2.1}$$

Thus we need to check that (2.1) is equivalent with the conditions $p \ge p_\beta = 1 + \frac{\beta}{d-1}$, $L(p) \le \alpha \le U(p)$. The condition $\alpha \le U(p)$ coincides with the condition $\alpha + \beta \le (d-1)(p-1)$. Thus it only remains to show that

$$\nu^\sharp\big((d-1)(p-2) - \alpha\big) \le (d-1)(p-1) \iff p \ge p_\beta,\ \alpha \ge L(p). \tag{2.2}$$

Recall that by definition of $(\nu^\sharp)^\dagger(s) = \sup\{\rho : \nu^\sharp(\rho) \le s\} \ge 0$ for $s \in [\beta, \infty)$ and since $\nu^\sharp$ is continuous and increasing, we have for all $s \ge \beta$ and $\rho \in \mathbb{R}$:

$$\nu^\sharp(\rho) \le s \quad \iff \quad \rho \le (\nu^\sharp)^\dagger(s).$$

Setting $\rho = (d-1)(p-2) - \alpha$ and $s = (d-1)(p-1)$ and noting that $\nu^\sharp(\rho) \le s$ implies $\beta \le s$ (i.e. $p \ge p_\beta$), we obtain (2.2).

2.3. *Scaling and a global to local reduction.* It will be useful to observe that the maximal operator enjoys dilation invariance. Precisely, for every $\lambda > 0$ and every $\mathcal{E} \subset (0, \infty)$ we have

$$\|M_{\lambda\mathcal{E}}\|_{L^p(w_\alpha) \to L^p(w_\alpha)} = \|M_{\mathcal{E}}\|_{L^p(w_\alpha) \to L^p(w_\alpha)}. \tag{2.3}$$

Moreover, Duoandikoetxea and Seijo [6, Lemma 4] observed that for $\alpha < 0$ the global bounds for $M_{\mathcal{E}}$ follow from the uniform bounds for the spherical maximal operators associated to the sets $\mathcal{E}_R = R^{-1}(\mathcal{E} \cap [R, 2R]) \subset [1, 2]$, provided we assume L^p-boundedness for the unweighted operator.

Lemma 2.1 ([6]). *Let $p \in [1, \infty]$. Then for $\alpha < 0$ there exists a constant $C_{\alpha,p} < \infty$ such that*

$$\|M_{\mathcal{E}}\|_{L^p(w_\alpha)\to L^p(w_\alpha)} \le C_{\alpha,p}\big(\|M_{\mathcal{E}}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} + \sup_{R>0} \|M_{\mathcal{E}_R}\|_{L^p(w_\alpha)\to L^p(w_\alpha)}\big).$$

3. Examples of type sets

In this section we provide further illustrations of special cases of Theorem 1.1.

3.1. *Assouad regular sets.* We examine the boundedness condition in Theorem 1.1 for some additional examples of sets $\mathcal{E}$. The Assouad spectrum of $\mathcal{E} \subset (0, \infty)$ (taken with respect to the appropriate metric (1.1)) will be denoted by $\dim_{A,\theta} \mathcal{E}$. Note that it equals the Assouad spectrum taken with respect to the Euclidean metric whenever $\mathcal{E}$ is compactly supported in $(0, \infty)$. We say that a set $\mathcal{E}$ is (β, γ)-*Assouad regular* (also just *(quasi)-Assouad regular*) if

$$\dim_{\mathrm{A},\theta} \mathcal{E} = \min\left(\tfrac{\beta}{1-\theta}, \gamma\right),$$

Here γ denotes the quasi-Assouad dimension. In this case the Legendre–Assouad function is given by

$$\nu_{\mathcal{E}}^{\sharp}(\rho) = \begin{cases} (1 - \frac{\beta}{\gamma})\rho + \beta & \text{if } \rho \le \gamma, \\ \rho & \text{if } \rho > \gamma. \end{cases}$$

This class of sets played a special role in several previous papers, see e.g. [1, 14, 3, 2, 16]. If $\mathcal{E}$ is (β, γ)-Assouad regular, then the boundedness condition of Theorem 1.1 can be summarized as follows:

(i) If $p > p_\gamma = 1 + \frac{\gamma}{d-1}$, then $M_{\mathcal{E}}$ is bounded on $L^p(w_\alpha)$ if

$$1 - d < \alpha < (d-1)(p-1) - \beta.$$

This part holds for all sets $\mathcal{E}$ with quasi-Assouad dimension γ and also fully describes up to endpoints the boundedness in the case of the "regular sets" from [6] where $\beta = \gamma$.

(ii) If $p_\beta < p \le p_\gamma$, then $M_{\mathcal{E}}$ is bounded on $L^p(w_\alpha)$ if

$$1 - d + \beta\,\tfrac{\gamma-(d-1)(p-1)}{\gamma-\beta} < \alpha < (d-1)(p-1) - \beta.$$

These conditions are sharp up to endpoints. The type set $\mathfrak{T}_{\mathcal{E}}$ in this case takes the shape of a polygon. If $\beta < \gamma$, then the lower boundary in the zoomed-in portion of Figure 1 consists of a piecewise linear function with two pieces (the 'kink' occurs at $1/p = 1/p_\gamma$ and corresponds to the transition from case (i) to case (ii)).

Remark 3.1. The sequence sets $E_a = \{1 + n^{-a} : n \in \mathbb{N}\} \subset [1, 2]$ for $a > 0$ are examples of Assouad regular sets with $\beta = \frac{1}{1+a}$ and $\gamma = 1$, and so are the corresponding 'global' sets $\mathcal{E}_a = \cup_{k\in\mathbb{Z}} 2^k E_a \subset (0, \infty)$. The middle-third Cantor set in $[1, 2]$ is an example of an Assouad regular set with $\beta = \gamma = \log_3 2$ (and in this case our result is already covered by [6]).

3.2. *Finite unions.* The boundedness condition becomes more complicated if we consider finite unions of Assouad-regular sets. Let $\mathcal{E} = \cup_{j=1}^N \mathcal{E}_j$ where $\mathcal{E}_j$ is a (β_j, γ_j)-Assouad regular set. Then $\beta = \beta_{\mathcal{E}}$ equals $\max_{j=1,\dots,N} \beta_j$ and $M_{\mathcal{E}}$ is bounded on $L^p(w_\alpha)$ if $p > p_\beta$ and $L(p) < \alpha < (d-1)(p-1) - \beta$, where

$$L(p) = 1 - d + \max\Big\{0, \max_{\substack{j=1,\dots,N\\ \gamma_j > \beta_j}} \beta_j \,\frac{\gamma_j - (d-1)(p-1)}{\gamma_j - \beta_j}\Big\}$$

and this condition is sharp up to endpoints. Since

$$\max_{j=1,\dots,N} \|M_{\mathcal{E}_j}\|_{p\to p} \le \|M_{\mathcal{E}}\|_{p\to p} \le \sum_{j=1}^N \|M_{\mathcal{E}_j}\|_{p\to p},$$

this is an immediate consequence of the Assouad regular case. The type set $\mathfrak{T}_{\mathcal{E}}$ in this case again takes the form of a polygon with an increased number of edges on the lower boundary in the zoomed-in region of Figure 1.

4. Lower bounds

In this section we prove the inclusion '$\subset$' in Theorem 1.1. The necessary condition $\alpha \le (d-1)(p-1) - \beta$ can be found in the paper by Duoandikoetxea and Seijo [6] and is obtained by testing $M_{\mathcal{E}}$ on χ_δ, the characteristic function of the ball of radius δ, centered at the origin.

It remains to prove the necessary condition $\alpha \ge L(p)$ in Theorem 1.1 which by (2.2) is equivalent to

$$\nu^\sharp\big((d-1)(p-2) - \alpha\big) \le (d-1)(p-1) \tag{4.1}$$

(see (1.9)). To this end it will suffice to prove that

$$\|M_E\|^p_{L^p(w_\alpha)\to L^p(w_\alpha)} \gtrsim 2^{-j(d-1)(p-1)} N(E \cap I, 2^{-j}) |I|^{(d-1)(2-p)+\alpha} \tag{4.2}$$

for all $E \subset [1, 2]$ and all $j \ge 1$ and all intervals $I \subset [1, 2]$ of length $2^{-j} \le |I| \le 1$. Indeed, using (2.3) that would imply that for every $\mathcal{E} \subset (0, \infty)$ and every $R > 0$ and $j \ge 1$, $2^{-j} \le |I| \le 1$,

$$\begin{aligned}\|M_{\mathcal{E}}\|^p_{L^p(w_\alpha)\to L^p(w_\alpha)} &\ge \|M_{\mathcal{E}_R}\|^p_{L^p(w_\alpha)\to L^p(w_\alpha)}\\ &\ge 2^{-j(d-1)(p-1)} N(\mathcal{E}_R \cap I, 2^{-j}) |I|^{(d-1)(2-p)+\alpha},\end{aligned}$$

where $\mathcal{E}_R = R^{-1}(\mathcal{E} \cap [R, 2R])$. By dilation invariance of the metric (1.1), $N(\mathcal{E}_R \cap I, 2^{-j}) = N(\mathcal{E} \cap J, 2^{-j})$ for $J = RI \cap [R, 2R]$ and $|J|_\times = |I|_\times \approx |I|$. Taking a supremum over j, I and $R > 0$ and comparing to the definition (1.4) therefore gives (4.1).

It remains to show (4.2). Notice that for all f supported in

$$\mathfrak{A} = \{y : 1 \le |y| \le 2\}$$

we have $\|f\|_{L^p(w_\alpha)} \approx \|f\|_p$. Observe that (4.2) follows from the following:

Lemma 4.1. *Let $E \subset [1,2]$ and $0 < k \le j$, and $I \subset [1,2]$ an interval of length 2^{-k}. Then*

$$\sup_{\substack{\|f\|_p \le 1 \\ \text{supp }(f) \subset \mathfrak{A}}} 2^{k\frac{\alpha}{p}} \Big(\int_{|x| \le 2^{-k}} |M_E f(x)|^p dx \Big)^{\frac{1}{p}} \tag{4.3}$$
$$\gtrsim 2^{-j(d-1)(1-\frac{1}{p})} N(E \cap I, 2^{-j})^{\frac{1}{p}} 2^{-k(\frac{\alpha}{p} + (d-1)(\frac{2}{p}-1))}.$$

Remark 4.2. The case $k = j/2$ in (4.3) corresponds to a standard Knapp-type example which is essentially in [6].

In the proof of Lemma 4.1 we may assume that $E \cap I$ is non-empty since otherwise there is nothing to show. In this case choose an element $a \in E \cap I$. We further distinguish the cases $0 < k \le j/2$ and $j/2 < k \le j$. We let $\tau_j(I)$ be a maximal 2^{-j}-separated set in $E \cap I$, so that

$$\#\tau_j(I) \approx N(E \cap I, 2^{-j}).$$

4.1. *Proof of Lemma 4.1 for the case* $0 < k \le j/2$. Let $\varepsilon > 0$ be small and

$$U(a,t) = \{(x', x_d) : (t - \varepsilon 2^{-j})^2 \le |x'|^2 + |x_d - a|^2 \le (t + \varepsilon 2^{-j})^2,\ |x'| \le 2^{-k}\}$$

which is a 2^{-j}-neighborhood of a subset with diameter $O(2^{-k})$ of the t-sphere centered at $(0, a)$.

We note that

$$U(a,t) \cap U(a,t') = \emptyset \quad \text{for } t, t' \in I,\ |t - t'| \ge 2^{-j}. \tag{4.4}$$

Let

$$Q(a) = \{(y', y_d) : |y_d - a| \le \varepsilon^{-1} 2^{-j},\ |y'| \le \varepsilon^{-1} 2^{k-j}\}.$$

Note that $Q(a)$ is comparable to a box with $(d-1)$ long horizontal sides of length 2^{k-j} and one short vertical side of length 2^{-j} around the point $(0, a)$. Let $f_Q = \mathbb{1}_{Q(a)}$. We observe

$$\|f_Q\|_p \lesssim_\varepsilon 2^{(k-j)(d-1)/p} 2^{-j/p}.$$

Given ε be small enough, we will show that for $x \in U(a,t)$ there is a subset $S(x)$ of the sphere of radius t centered at x which is of diameter $\approx \varepsilon 2^{k-j}$ and such that $S(x) \subset Q(a)$.

Let $x \in U(a,t)$ and consider the unit vectors

$$\theta_{x,w'} = \frac{(-x' + w', a - x_d)}{\sqrt{|x' - w'|^2 + (a - x_d)^2}}.$$

Consider the subset $S(x)$ of the unit sphere, defined by

$$S(x) = \{\theta_{x,w'} : |w'| \le 2^{k-j}\}$$

Then the spherical surface measure of $S(x)$ is $\gtrsim 2^{(k-j)(d-1)}$ (uniformly in $x \in U(a,t)$, and $1 \le t \le 2$).

We claim that for $x \in U(a,t)$

$$\{x + t\theta_{x,w'} : |w'| \le 2^{k-j}\} \subset Q(a)\,. \tag{4.5}$$

We let $e_d = (0,\dots,0,1)$ and let Π be the map $(x', x_d) \mapsto x'$. We write

$$x + t\theta_{x,w'} - ae_d = x + t\theta_{x,0} - ae_d + t(\theta_{x,w'} - \theta_{x,0})$$

and compute

$$x + t\theta_{x,0} - ae_d = \frac{(x', x_d - a)}{\sqrt{|x'|^2 + (a-x_d)^2}}\big(\sqrt{|x'|^2 + (a-x_d)^2} - t\big)$$

which by definition of $U(a,t)$ implies $|x + t\theta_{x,0} - ae_d| \le 2^{-j}$.

Next,

$$\big|\Pi(\theta_{x,w'} - \theta_{x,0})\big| = \Big|\frac{(-x'+w')\sqrt{|x'|^2 + (a-x_d)^2} + x'\sqrt{|x'-w'|^2 + (a-x_d)^2}}{\sqrt{|x'|^2 + (a-x_d)^2}\sqrt{|x'-w'|^2 + (a-x_d)^2}}\Big|$$

$$\le \frac{|w'|\sqrt{|x'|^2 + (a-x_d)^2} + |x'|\big|\sqrt{|x'|^2 + (a-x_d)^2} - \sqrt{|x'-w'|^2 + (a-x_d)^2}\big|}{(a-x_d)^2}$$

and clearly $|\Pi(\theta_{x,w'} - \theta_{x,0})| \lesssim 2^{k-j}$. Moreover, using for $b, \tilde b > 0$ the identity

$$|b^{-1/2} - \tilde b^{-1/2}| = |b - \tilde b| b^{-1/2} \tilde b^{-1/2} (b^{1/2} + \tilde b^{1/2})^{-1}$$

we obtain

$$\begin{aligned}
&\big|\langle \theta_{x,w'} - \theta_{x,0}, e_d\rangle\big| \\
&\quad = (a - x_d)\big|(|x'-w'|^2 + (a-x_d)^2)^{-\frac12} - (|x'|^2 + (a-x_d)^2)^{-\frac12}\big| \\
&\quad \lesssim \big||x'-w'|^2 - |x'|^2\big| \lesssim \big(|w'|^2 + 2|x'||w'|\big)\,.
\end{aligned}$$

Now $|x'||w'| \lesssim 2^{-k} \cdot 2^{k-j} \lesssim 2^{-j}$ and $|w'|^2 \lesssim 2^{2k-2j}$, and since $k \le j/2$ we also have $|w'|^2 \lesssim 2^{-j}$. Hence $|\langle \theta_{x,w'} - \theta_{x,0}, e_d\rangle| \lesssim 2^{-j}$. This means (with the appropriate choice of $\varepsilon > 0$ in the definition of $Q(a)$) that (4.5) is proved. This implies

$$|A_t f(x)| \gtrsim 2^{(k-j)(d-1)} \text{ for } x \in U(a,t).$$

Now let

$$U(a) = \bigcup_{t \in \tau_j(I)} U(a,t).$$

Then by (4.4),

$$\Big(\int_{U(a)} \Big[\sup_{s \in E \cap I} |A_s f_Q(x)|\Big]^p |x|^\alpha dx\Big)^{1/p} \ge \Big(\sum_{t \in \tau_j(I)} \int_{U(a,t)} |A_t f_Q(x)|^p |x|^\alpha dx\Big)^{1/p}$$

$$\gtrsim 2^{-k\alpha/p} 2^{(k-j)(d-1)} \Big(\sum_{t \in \tau_j(I)} |U(a,t)|\Big)^{1/p}$$

$$\gtrsim 2^{-k\alpha/p} 2^{(k-j)(d-1)} N(E \cap I, 2^{-j})^{1/p} 2^{-j/p} 2^{-k(d-1)/p}$$

and thus, with $|I| = 2^{-k}$

$$\frac{\|M_E f_Q\|_{L^p(w_\alpha)}}{\|f_Q\|_{L^p(w_\alpha)}} \gtrsim \frac{2^{-k\alpha/p} 2^{(k-j)(d-1)} N(E\cap I, 2^{-j})^{1/p} 2^{-j/p} 2^{-k(d-1)/p}}{2^{(k-j)(d-1)/p} 2^{-j/p}}$$
$$\gtrsim 2^{-j(d-1)(1-\frac{1}{p})} N(E\cap I, 2^{-j})^{\frac{1}{p}} |I|^{\frac{\alpha}{p}+(d-1)(\frac{2}{p}-1)}.$$

4.2. *Proof of Lemma 4.1 for the case $j/2 < k \le j$.* Let

$$\mathcal{U}(a) = \{(y', y_d) : |y'| \le \varepsilon^{-1} 2^{k-j},\ (a-2^{-j})^2 \le |y'|^2 + |y_d|^2 \le (a+2^{-j})^2\}$$

which is a 2^{-j}-neighborhood of a subset with diameter $O(2^{k-j})$ of the a-sphere centered at $(0,0)$. Let $g_{\mathcal{U}} = \mathbb{1}_{\mathcal{U}(a)}$. Then

$$\|g_{\mathcal{U}}\|_p \lesssim_\varepsilon 2^{(k-j)(d-1)/p} 2^{-j/p}.$$

For $t \in I$, let

$$\mathcal{Q}(a,t) = \{(x', x_d) : \varepsilon 2^{-k-1} \le |x'| \le \varepsilon 2^{-k},\ |x_d - a + t| \le \varepsilon 2^{-j}\}$$

which is comparable to a rectangle in $\{x : |x| \approx \varepsilon 2^{-k}\}$, with $(d-1)$ long horizontal sides of length $\approx \varepsilon 2^{-k}$ and one short vertical side of length $\varepsilon 2^{-j}$. Observe

$$\mathcal{Q}(a,t) \cap \mathcal{Q}(a,t') = \emptyset \text{ for } t, t' \in I,\ |t-t'| \ge 2^{-j}. \tag{4.6}$$

We show that

$$A_t g_{\mathcal{U}}(x) \gtrsim 2^{(k-j)(d-1)} \quad \text{for } x \in \mathcal{Q}(a,t). \tag{4.7}$$

This is a calculation in [14, §5] which for convenience of the reader we reproduce with our current parameters.

Let $x \in \mathcal{Q}(a,t)$, $\omega = (\omega', \omega_d) \in S^{d-1}$, with $|\omega'| \le \varepsilon 2^{k-j}$ and $\omega_d = \sqrt{1-|\omega'|^2}$. We wish to show that $x + t\omega \in \mathcal{U}(a)$. To see this we compute

$$|x+t\omega|^2 = |x'|^2 + x_d^2 + t^2 + 2t\langle x', \omega'\rangle + 2t x_d \omega_d$$
$$= |x'|^2 + (x_d+t)^2 + 2t\langle x', \omega'\rangle + 2t x_d(\sqrt{1-|\omega'|^2} - 1).$$

Since $|x_d + t - a| \le \varepsilon 2^{-j}$ and $|x'|^2 \le \varepsilon^2 2^{-2k} \le \varepsilon^2 2^{-j}$, we get

$$||x'|^2 + (x_d+t)^2 - a^2| \le 6\varepsilon 2^{-j},$$
$$|2t\langle x', \omega'\rangle| \le 4|x'||\omega'| \le 4\varepsilon 2^{-j},$$

and

$$|2t x_d(\sqrt{1-|\omega'|^2} - 1)| \le 2(|t-a| + \varepsilon 2^{-j})|\omega'|^2 \le 2(2^{-k} + \varepsilon 2^{-j})\varepsilon 2^{k-j} \le 4\varepsilon 2^{-j}.$$

Hence

$$||x+t\omega|^2 - a^2| \le 14\varepsilon 2^{-j},$$

and thus if $\varepsilon \le 10^{-2}$ we have $||x+t\omega| - a| \le 2^{-j}$. Also, since $k \ge j/2$,

$$|x' + t\omega'| \le |x'| + 2|\omega'| \le 2^{-k} + 2^{k-j} \le 2^{k-j+1}$$

so that altogether $x + t\omega \in \mathcal{U}(a)$. This establishes (4.7).

Now let $\mathcal{Q}(a) = \cup_{t\in\tau_j(I)}\mathcal{Q}(a,t)$. Then by (4.7)

$$\begin{aligned}
&\Big(\int_{\mathcal{Q}(a)}\Big[\sup_{s\in\mathcal{E}_R\cap I}|A_s g_{\mathcal{U}}(x)|\Big]^p|x|^\alpha dx\Big)^{1/p} \\
&\quad\geq \Big(\sum_{t\in\tau_j(I)}\int_{\mathcal{Q}(a,t)}|A_t g_{\mathcal{U}}(x)|^p|x|^\alpha dx\Big)^{1/p} \\
&\quad\gtrsim 2^{-k\alpha/p}2^{(k-j)(d-1)}\Big(\sum_{t\in\tau_j(I)}|\mathcal{Q}(a,t)|\Big)^{1/p} \\
&\quad\gtrsim 2^{-k\alpha/p}2^{(k-j)(d-1)}N(E\cap I,2^{-j})^{1/p}2^{-j/p}2^{-k(d-1)/p}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\|M_E g_{\mathcal{U}}\|_{L^p(w_\alpha)}}{\|g_{\mathcal{U}}\|_{L^p(w_\alpha)}} &\gtrsim \frac{2^{-k\alpha/p}2^{(k-j)(d-1)}N(E\cap I,2^{-j})^{1/p}2^{-j/p}2^{-k(d-1)/p}}{2^{(k-j)(d-1)/p}2^{-j/p}} \\
&\gtrsim 2^{-j(d-1)(1-\frac{1}{p})}N(E\cap I,2^{-j})^{\frac{1}{p}}|I|^{\frac{\alpha}{p}+(d-1)(\frac{2}{p}-1)}.
\end{aligned}$$

5. Upper bounds

In this section we prove the upper bounds in Theorem 1.1, i.e. that $M_{\mathcal{E}}$ is bounded if $\alpha < (d-1)(p-1)-\beta$ and

$$\nu_{\mathcal{E}}^\sharp((d-1)(p-2)-\alpha) < (d-1)(p-1) \tag{5.1}$$

(note here we used the equivalent condition from (1.8)). We may also assume that $\alpha < 0$ since the case $\alpha \geq 0$ was already handled in [6]. For the same reason we may also assume $p \leq 1 + \frac{1}{d-1} \leq 2$.

As a consequence of the above and §2.3, it will now suffice to show boundedness of $M_{\mathcal{E}}$ on $L^p(w_\alpha)$ under the additional assumption $\mathcal{E} = E \subset [1,2]$. To be precise, we need to prove the following

Proposition 5.1. *Let $p \in [1,2]$ and $\alpha \leq 0$. Assume that $E \subset [1,2]$ is non-empty and there exists $\varepsilon > 0$ and a constant $A \geq 1$ such that*

$$\sup_{0<\delta<1}\sup_{\delta\leq|I|\leq 1} N(E\cap I,\delta)\delta^{(d-1)(p-1)-\varepsilon}|I|^{\alpha+(d-1)(2-p)} \leq A^p. \tag{5.2}$$

Then $\|M_E\|_{L^p(w_\alpha)\to L^p(w_\alpha)} \lesssim A$ with the implicit constant only depending on p, α, d.

Remark 5.2. Note that since $E \subset [1,2]$ here, it makes no difference whether we use the Euclidean metric or the metric (1.1) in (5.2).

Proposition 5.1 completes the proof of Theorem 1.1. Since we may assume $\alpha < 0$ and $p \leq 2$, (5.1) implies that there exists $\varepsilon > 0$ and $A \geq 1$ so that for every $R > 0$,

$$\sup_{0<\delta<1}\sup_{\delta\leq|I|\leq 1} N(\mathcal{E}_R\cap I,\delta)\delta^{(d-1)(p-1)-\varepsilon}|I|^{\alpha+(d-1)(2-p)} \leq A^p.$$

By Proposition 5.1 applied to each $E = \mathcal{E}_R$ we then obtain

$$\|M_{\mathcal{E}_R}\|_{L^p(w_\alpha)\to L^p(w_\alpha)} \lesssim A,$$

for every $R > 0$, which by Lemma 2.1 implies that $M_{\mathcal{E}}$ is bounded on $L^p(w_\alpha)$ as required.

5.1. *Proof of Proposition 5.1.* The arguments here are close to [7, 6]. We first observe that the assumption (5.2) implies

$$p > p_\beta = 1 + \tfrac{\beta}{d-1}$$

by considering the case $|I| = 1$ in the supremum and also

$$\alpha > 1 - d$$

by considering the case $|I| = \delta$ in the supremum. We split

$$f = f_{\mathrm{low}} + f_0 + f_{\mathrm{high}}, \tag{5.3}$$

where f_{low} lives on $\{y : |y| < 1/4\}$, f_0 lives on $\{y : 1/2 \le |y| \le 4\}$, and f_{high} lives on $\{y : |y| > 4\}$. Then $M_E f_{\mathrm{low}}(x) = 0$ for $|x| > 9/4$ and for $|x| < 1/2$. Thus for f_{low} the $L^p(w_\alpha)$ bound follows from the unweighted bound. Indeed,

$$\begin{aligned}\|M_E f_{\mathrm{low}}\|_{L^p(w_\alpha)} &\lesssim_\alpha \Big(\int_{\frac12\le|x|\le\frac94} |M_E f_{\mathrm{low}}(x)|^p dx\Big)^{\frac1p} \\ &\lesssim \|f_{\mathrm{low}}\|_p \lesssim \|f_{\mathrm{low}}\|_{L^p(w_\alpha)} \lesssim \|f\|_{L^p(w_\alpha)},\end{aligned} \tag{5.4}$$

where we have used $\alpha \le 0$ and $p > p_\beta$.

We split $f_{\mathrm{high}} = \sum_{k=2}^\infty f^k$ where $f^k = f \mathbb{1}_{2^k<|y|\le 2^{k+1}}$. Let

$$\mathfrak{A}_k = \{x : 2^k - 2 \le |x| \le 2^{k+1} + 2\}.$$

Then $M_E f^k(x) = 0$ for $x \in \mathfrak{A}_k^\complement$. Again, for all $\alpha \in \mathbb{R}$ we can use the unweighted inequality to estimate

$$\begin{aligned}\|M_E f_{\mathrm{high}}\|_{L^p(w_\alpha)} &\le \Big(\int_{|x|\ge 2} \Big|\sum_{k=2}^\infty \mathbb{1}_{\mathfrak{A}_k}(x) M_E f^k(x)\Big|^p |x|^\alpha dx\Big)^{\frac1p} \\ &\lesssim \Big(\sum_{k=2}^\infty 2^{k\alpha} \int |M_E f^k(x)|^p dx\Big)^{\frac1p} \\ &\lesssim \Big(\sum_{k=2}^\infty 2^{k\alpha} \int |f^k(x)|^p dx\Big)^{\frac1p} \lesssim \|f\|_{L^p(w_\alpha)}.\end{aligned} \tag{5.5}$$

The estimate for $M_E f_0$ is more interesting; only here the weight plays an essential role. Let $\phi_0 \in C_c^\infty$ be supported in $\{x : |x| < 1/4\}$ such that $\int \phi_0(x) x^\iota \, dx = 0$ for all multiindices $\iota \in \mathbb{N}_0^d$ with $1 \le \sum_{j=1}^d \iota_j \le N_0$ where $N_0 \gg \frac{d-1}{2}$. For $j \ge 1$ let $\phi_j(x) = 2^{jd}\phi_0(2^j x) - 2^{(j-1)d}\phi_0(2^{j-1}x)$ and set

$$K^j(x) = \sigma * \phi_j(x)$$

and $K_t^j(x) = t^{-d}K^j(t^{-1}x)$. We then have the decomposition $\sigma_t = \sum_{j=0}^{\infty} K_t^j$ which has the behavior of a dyadic frequency decomposition but with strong spatial localization. Set $A_t^j f = f * K_t^j$.

Lemma 5.3. *Let $\alpha \geq 1-d$. Then for $j \geq 0$,*

$$\Big(\int_{|x|\le 2^{-j}} \sup_{1\le t\le 2} \big|A_t^j f_0(x)\big|^p |x|^\alpha dx\Big)^{\frac1p} \lesssim 2^{-j\frac{d-1+\alpha}{p}} \|f_0\|_p. \tag{5.6}$$

Proof. For $2^{j-3} \le \nu \le 2^{j+5}$ let $\mathbb{1}_\nu$ be the characteristic function of the thin annulus $\mathcal{R}_\nu = \{y : \big||y| - \nu 2^{-j}\big| \le 2^{-j}\}$ and $J_\nu = [(\nu-2)2^{-j}, (\nu+3)2^{-j}]$. Let $B_j = \{x : |x| \le 2^{-j}\}$. Then

$$\begin{aligned}
&\Big(\int_{B_j} \sup_{1\le t\le 2} |A_t^j f_0(x)|^p |x|^\alpha dx\Big)^{\frac1p} \\
&= \Big(\int_{B_j} \sup_{1\le t\le 2} \Big|A_t^j[\sum_\nu (f_0\mathbb{1}_\nu)](x)\Big|^p |x|^\alpha dx\Big)^{\frac1p} \\
&\lesssim \Big(\sum_\nu \int_{B_j} \sup_{t\in J_\nu} \big|A_t^j (f_0\mathbb{1}_\nu)(x)\big|^p |x|^\alpha dx\Big)^{\frac1p}.
\end{aligned} \tag{5.7}$$

Now use the pointwise bound

$$\sup_{t\in J_\nu} \big|A_t^j (f_0\mathbb{1}_\nu)(x)\big| \le |A_{\nu 2^{-j}}^j (f_0\mathbb{1}_\nu)(x)| + \int_{J_\nu} \Big|\tfrac{d}{ds} A_s^j (f_0\mathbb{1}_\nu)(x)\Big| ds\,. \tag{5.8}$$

We interpolate inequalities for $\alpha = 1-d$ and for $\alpha = 0$.

For $\alpha = 1-d$ we have, for $1 \le p < \infty$,

$$\Big(\int_{B_j} \Big|A_{2^{-j}\nu}^j (f_0\mathbb{1}_\nu)(x)\Big|^p |x|^{1-d} dx\Big)^{\frac1p} \lesssim \|f_0\mathbb{1}_\nu\|_p, \tag{5.9a}$$

$$\Big(\int_{B_j} \Big[\int_{J_\nu} \big|\tfrac{d}{ds} A_s^j (f_0\mathbb{1}_\nu)(x)\big| ds\Big]^p |x|^{1-d} dx\Big)^{\frac1p} \lesssim \|f_0\mathbb{1}_\nu\|_p, \tag{5.9b}$$

which follow by interpolation between $p=1$ and the corresponding inequality for $p=\infty$. For $p=1$ we use that $\|K_t^j\|_\infty + 2^{-j}\|\frac{d}{dt}K_t^j\|_\infty = O(2^j)$ and $\int_{B_j} |x|^{1-d} dx = O(2^{-j})$.

For $\alpha = 0$ and $1 \le p \le 2$ we have

$$\Big(\int_{B_j} \big|A_{2^{-j}\nu}^j [f_0\mathbb{1}_\nu](x)\big|^p dx\Big)^{\frac1p} \lesssim 2^{-j(d-1)(1-\frac1p)} \|f_0\mathbb{1}_\nu\|_p, \tag{5.10a}$$

$$\Big(\int_{B_j} \Big(\int_{J_\nu} \big|\tfrac{d}{ds} A_s^j [f_0\mathbb{1}_\nu](x)\big| ds\Big)^p dx\Big)^{\frac1p} \lesssim 2^{-j(d-1)(1-\frac1p)} \|f_0\mathbb{1}_\nu\|_p. \tag{5.10b}$$

Again this follows by interpolation between $p=1$ and $p=2$. For $p=1$ we just use $\|K_t^j\|_1 + 2^{-j}\|\frac{d}{dt}K_t^j\|_1 = O(1)$. For $p=2$ we use $|\widehat{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{d-1}{2}}$ and thus, with $\frac{d-1}{2} \ll N_0 \ll N$,

$$|\widehat{K_1^j}(\xi)| \lesssim (2^{-j}|\xi|)^{N_0}(1+(2^{-j}|\xi|))^{-N}(1+|\xi|)^{-\frac{d-1}{2}} \lesssim 2^{-j\frac{d-1}{2}}. \tag{5.11a}$$

We have $\frac{d}{dt}\widehat{K_t^j}(\xi) = \langle 2^{-j}\xi, \nabla\phi(2^{-j}t\xi)\rangle\widehat{\sigma}(t\xi) + \phi(2^{-j}t\xi)\langle\xi, \nabla\widehat{\sigma}\rangle(t\xi)$ and obtain

$$\big|\tfrac{d}{dt}\widehat{K_t^j}(\xi)\big| \lesssim 2^{-j\frac{d-3}{2}}, \quad t \approx 1. \tag{5.11b}$$

Since we also integrate over J_ν the interpolation with change of weight ([20]) together with (5.8) leads to

$$\Big(\int_{B_j} \sup_{t\in J_\nu} \big|A_t^j[f_0\mathbb{1}_\nu](x)\big|^p |x|^\alpha dx\Big)^{\frac1p} \lesssim 2^{-j\frac{d-1+\alpha}{p}} \|f_0\mathbb{1}_\nu\|_p$$

and then also

$$\Big(\int_{B_j} \big|\sup_{1/4\le t\le 4} \big|A_t^j f_0(x)\big|^p |x|^\alpha dx\Big)^{\frac1p} \lesssim 2^{-j\frac{d-1+\alpha}{p}} \|f\|_p. \qquad \square$$

Lemma 5.4. *Let $E \subset [1,2]$. For $0 < k \le j$,*

$$\begin{aligned}\Big(\int_{2^{-k-1}\le|x|\le 2^{-k}} \sup_{t\in E} |A_t^j f_0(x)|^p|x|^\alpha dx\Big)^{\frac1p} \\ \lesssim 2^{-j\frac{d-1}{p'}} \sup_{\substack{I\subset[1,2]\\ |I|\approx 2^{-k}}} \big[N(E\cap I, 2^{-j})2^{-k((d-1)(2-p)+\alpha)}\big]^{\frac1p} \|f_0\|_p\,.\end{aligned} \tag{5.12}$$

Proof. We let $I_{k,\mu} = [2^{-k}(\mu-1), 2^{-k}(\mu+2)]$. For each μ with $E\cap I_{k,\mu} \ne \emptyset$, denote by $\Gamma_{k,\mu,j}$ a collection of intervals of length 2^{-j} covering $E\cap I_{k,\mu}$ such that $N(E\cap I_{k,\mu}, 2^{-j}) \approx \#\Gamma_{k,\mu,j}$. If $J\in\Gamma_{k,\mu,j}$ we let t_J be the center of J.

We first prove that for $1\le p\le 2$, $k<j$, $B_k = \{x : |x| \le 2^{-k}\}$,

$$\begin{aligned}\Big(\int_{B_k\setminus B_{k+1}} \sup_{t\in E\cap I_{k,\mu}} |A_t^j f_0|^p dx\Big)^{\frac1p} \\ \lesssim 2^{-j(d-1)(1-\frac1p)} 2^{-k(d-1)(\frac2p-1)} N(E\cap I_{k,\mu}, 2^{-j})^{\frac1p} \|f_0\|_p.\end{aligned} \tag{5.13}$$

We use the pointwise estimate

$$\begin{aligned}&\sup_{t\in E\cap I_{k,\mu}} |A_t^j f_0(x)| \\ &\le \Big(\sum_{J\in\Gamma_{k,\mu,j}} |A_{t_J}^j f_0(x)|^p\Big)^{\frac1p} + \Big(\sum_{J\in\Gamma_{k,\mu,j}} \Big(\int_{|s|\le 2^{-j}} |\tfrac{d}{ds}A_{t_J+s}^j f_0(x)| ds\Big)^p\Big)^{\frac1p}.\end{aligned} \tag{5.14}$$

By the integral Minkowski inequality we need to show

$$\begin{aligned}\Big(\sum_{J\in\Gamma_{k,\mu,j}} \int_{B_k\setminus B_{k+1}} |A_{t_J}^j f_0|^p dx\Big)^{\frac1p} \\ \lesssim 2^{-j(d-1)(1-\frac1p)} 2^{-k(d-1)(\frac2p-1)} N(E\cap I_{k,\mu}, 2^{-j})^{\frac1p} \|f_0\|_p\end{aligned} \tag{5.15a}$$

and

$$\text{(5.15b)}\quad 2^{-j} \sup_{|s|\le 2^{-j}} \Big(\sum_{J\in\Gamma_{k,\mu,j}} \int_{B_k\setminus B_{k+1}} |\tfrac{d}{ds} A^j_{t_J+s} f_0|^p dx \Big)^{\frac1p} \lesssim 2^{-j(d-1)(1-\frac1p)} 2^{-k(d-1)(\frac2p-1)} N(E\cap I_{k,\mu}, 2^{-j})^{\frac1p} \|f_0\|_p\,.$$

We give the proof of (5.15a) and omit the completely analogous proof of (5.15b). These inequalities follow from the cases $p=1$ and $p=2$ by interpolation. For $p=1$ we estimate

$$\int_{B_k} |A^j_{t_J} f_0(x)|^p dx \le \int |f_0(y)| \int_{B_k} |K^j_{t_J}(x-y)| dx\, dy \lesssim 2^{-k(d-1)} \|f_0\|_1$$

where we have used that $\{x\in B_k : ||x-y|-t_J| \lesssim 2^{-j}\}$ has measure $O(2^{-k(d-1)}2^{-j})$ and $\|K^j_t\|_\infty = O(2^j)$ for $1\le t\le 2$. (5.15a) follows for $p=1$ by summing over the intervals in $\Gamma_{k,\mu,j}$. For $p=2$ we just use $\|A^j_{t_J}\|_{L^2\to L^2} = O(2^{-j(d-1)/2})$ (see (5.11a)) and (5.15a) for $p=2$ follows. Analogous arguments apply to (5.15b) and thus one gets (5.13).

Let

$$f_{k,\mu}(y) = f_0(y) \mathbb{1}_{2^{-k}\mu \le |y| \le 2^{-k}(\mu+1)}(y)$$

and observe that

$$A^j_t f(x) = A^j_t \Big[\sum_{m=\mu-2}^{\mu+2} f_{k,m} \Big](x) \quad \text{ for } |x|\le 2^{-k} \text{ and } t\in I_{k,\mu}.$$

Moreover $w_\alpha(x) \approx 2^{-k\alpha}$ for $x\in B_k\setminus B_{k+1}$. Hence,

$$\begin{aligned}
&\Big(\int_{B_k\setminus B_{k+1}} \sup_{t\in E} |A^j_t f_0(x)|^p |x|^\alpha dx \Big)^{\frac1p} \\
&\lesssim 2^{-k\frac{\alpha}{p}} \Big(\int_{B_k} \sum_\mu \sup_{t\in E\cap I_{k,\mu}} \Big| A^j_t [\sum_{m=\mu-2}^{\mu+2} f_{k,m}](x) \Big|^p dx \Big)^{\frac1p} \\
&\lesssim 2^{-k\frac{\alpha}{p}} 2^{-j(d-1)(1-\frac1p)} 2^{-k(d-1)(\frac2p-1)} \sup_\mu N(E\cap I_{k,\mu}, 2^{-j})^{\frac1p} \Big(\sum_\mu \Big\| \sum_{m=\mu-2}^{\mu+2} f_{k,m} \Big\|_p^p \Big)^{\frac1p} \\
&\lesssim 2^{-j(d-1)(1-\frac1p)} 2^{-k(\frac{\alpha}{p}+(d-1)(\frac2p-1))} \sup_\mu N(E\cap I_{k,\mu}, 2^{-j})^{\frac1p} \|f\|_p
\end{aligned}$$

and the lemma is proved. □

To conclude the proof of Proposition 5.1 we set

$$M^j_E f(x) = \sup_{t\in E} |A^j_t f(x)|.$$

By Lemma 5.3 and Lemma 5.4,

$$\begin{aligned}
&\|M_E^j f_0\|_{L^p(w_\alpha)} \\
&\qquad \lesssim 2^{-j(d-1)(1-\frac{1}{p})} \sum_{k=0}^{j} \sup_{|I|=2^{-k}} N(E\cap I, 2^{-j})^{\frac{1}{p}} 2^{-k((d-1)(\frac{2}{p}-1)+\frac{\alpha}{p})} \|f_0\|_p \\
&\qquad \lesssim (1+j) 2^{-j\frac{\varepsilon}{p}} A \|f\|_{L^p(w_\alpha)}
\end{aligned}$$

where in the last estimate we have used the main hypothesis (5.2). Finally, we may sum in j and then combine the resulting estimate for $M_E f_0$ with the estimates (5.4) for $M_E f_{\text{low}}$ and (5.5) for $M_E f_{\text{high}}$ to complete the proof of Proposition 5.1. □

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