

TWO WEIGHT ESTIMATES FOR DIFFERENCE QUOTIENTS

CARLOS PÉREZ, ANDREAS SEEGER, BRIAN STREET, PO-LAM YUNG

Dedicated to Neil Trudinger on the occasion of his 85th birthday

ABSTRACT. We prove local and global two weight estimates in which we bound difference quotients of a function in terms of certain weighted L^p norms of its gradient.

1. INTRODUCTION

In the study of partial differential equations, the weak differentiability of solutions may often be deduced through a consideration of their difference quotients. The classic monograph [26] by Gilbarg and Trudinger already highlights the following basic facts:

- The first order difference quotients of a $W^{1,p}$ function are uniformly in L^p (see Lemma 7.23 in [26]);
- Conversely, if the first order difference quotients of an L^p function are uniformly in L^p , then $u \in W^{1,p}$ (see Lemma 7.24 in [26]).

More recently, other interesting and useful characterizations of membership of ∇f in L^p have been found which involve first order differences, or difference quotients. Let us define for $x \neq y$

$$\mathfrak{D}_b f(x, y) = \frac{|f(y) - f(x)|}{|y - x|^{1+b}} \quad (1)$$

and set $\mathfrak{D}_b f(x, x) = 0$. Let $b = \gamma/p$. One might try to measure the L^p norm of $(x, y) \mapsto \mathfrak{D}_{\gamma/p} f(x, y)$ with respect to the measure $|y - x|^{\gamma-d} dx dy$ which reduces to the γ -independent expression

$$\left(\iint_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^{d+p}} dx dy \right)^{1/p}.$$

However, it is well known that if this integral is finite for a locally integrable function f , then f must be constant almost everywhere [7]. The failure of any such L^p estimate can be remedied by replacing the $L^p(|x - y|^{\gamma-d} dx dy)$ norm with a corresponding Marcinkiewicz $(L^{p,\infty})$ quasinorm. In particular, for $1 < p < \infty$, $\gamma \neq 0$ and $f \in C_c^\infty(\mathbb{R}^d)$ the $L^{p,\infty}$ quasi-norm of $\mathfrak{D}_{\gamma/p} f$ in $\mathbb{R}^d \times \mathbb{R}^d$ with respect to the measure $|y - x|^{\gamma-d} dx dy$ is bounded by $C \|\nabla f\|_{L^p(\mathbb{R}^d)}$, i.e.

$$\sup_{\lambda > 0} \lambda^p \iint_{\substack{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d: \\ \mathfrak{D}_{\gamma/p} f(x,y) > \lambda}} |y - x|^{\gamma-d} dx dy \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)}^p. \quad (2)$$

For the case $\gamma = d$, in which one considers Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$, (2) was shown by Brezis, Van Schaftingen and Yung [11] who also covered the case $p = 1$. For the case $\gamma = -p$ (in $(-\infty, -1)$), in which one considers differences instead of difference quotients, (2) had already been shown by Nguyen [42] (and independently by A. Ponce and J. Van Schaftingen). For other choices of γ it was shown by Brezis, Van Schaftingen and two of the authors [9] that inequality (2)

holds for $1 < p < \infty$, $\gamma \neq 0$ and for $p = 1$ and $\gamma \in \mathbb{R} \setminus [-1, 0]$, and these γ -ranges are sharp. The results can be extended to functions in the homogeneous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^d)$, and to functions of bounded variation when $p = 1$; moreover for $f \in \dot{W}^{1,p}$ one gets equivalences of quasi-norms (see [9]) and thus the desired characterizations of $\dot{W}^{1,p}$. The results for $\gamma = d$ were used in [12] to extend the scope of Gagliardo–Nirenberg inequalities in [8]. Further developments can be found in [5, 16–19, 29, 39, 43, 50, 51], and the surveys [10, 44].

The purpose of this paper is to extend the weak type inequality (2) to versions with possibly different weight functions on either side of the inequality, so called *two-weight inequalities*. We also cover local versions where $\mathbb{R}^d$ may be replaced by a suitable open subset $\Omega \subset \mathbb{R}^d$.

Given a nonnegative measurable function u on Ω we define a measure μ_γ^u on $\Omega \times \Omega$ by

$$d\mu_\gamma^u(x, y) = u(x)|y - x|^{\gamma-d} dx dy, \quad (3)$$

on $\Omega \times \Omega$, with u a non-negative measurable function on Ω . Even though $u(x)|y - x|^{\gamma-d}$ may not be locally integrable, (3) defines an inner regular measure [23, Proposition 412Q] (i.e. $\mu_\gamma^u(E) = \sup_{K \subset E} \mu_\gamma^u(K)$ where the supremum is taken over all compact subsets of E).

For $f \in C^1(\Omega)$ we are interested in a priori μ_γ^u bounds for the super-level sets $\{(x, y) \in \Omega \times \Omega : \mathfrak{D}_{\gamma/p} f(x, y) > \lambda\}$ where ∇f is measured in a weighted $L^p(w)$ norm with respect to a nonnegative weight w .

We shall assume that the weight pair (u, w) belongs to the Muckenhoupt class $A_p(\Omega)$. Recall that for $1 < p < \infty$, a pair of non-negative measurable functions (u, w) on $\mathbb{R}^d$ is said to be in $A_p(\Omega)$, written $(u, w) \in A_p(\Omega)$, if

$$[u, w]_{A_p(\Omega)} := \sup_B \frac{1}{|B|} \int_{B \cap \Omega} u(x) dx \left(\frac{1}{|B|} \int_{B \cap \Omega} w(y)^{-p'/p} dy \right)^{p/p'} < \infty, \quad (4)$$

with the supremum taken over all balls B in $\mathbb{R}^d$. For $p = 1$ a pair of weights (u, w) belongs to the class $A_1(\Omega)$ if there is a constant K such that for any ball $B \subset \mathbb{R}^d$,

$$\frac{1}{|B|} \int_{B \cap \Omega} u(x) dx \leq K \operatorname{essinf}_{y \in B \cap \Omega} w(y). \quad (5)$$

We denote the infimum of all possible constants K by $[u, w]_{A_1}$. For $\Omega = \mathbb{R}^d$ we recover the standard Muckenhoupt conditions for pairs of weights in $\mathbb{R}^d$. Membership of (u, w) in $A_p(\Omega)$ implies that $u \in L^1_{\text{loc}}(\Omega)$ is locally integrable and that $w(y) > 0$ for almost every $y \in \Omega$; for this and other basic facts we refer to §IV.1 and §IV.5 in [24] (cf. also §2.1 below).

We will study our problem in appropriate *convex* domains Ω , and with weight pairs $(u, w) \in A_p(\Omega)$. In the class of admissible domains we wish to include $\mathbb{R}^d$ itself, as well as balls and cubes in $\mathbb{R}^d$; however domains which are long and thin may pose problems. To formulate this quantitatively we denote for a domain $\Omega \subset \mathbb{R}^d$ the diameter by $\operatorname{diam}(\Omega) = \sup_{x, y \in \Omega} |x - y|$, and define the *ball deviation* of Ω as

$$\operatorname{dev}(\Omega) = \sup_{x \in \Omega} \sup_{0 < r \leq \operatorname{diam}(\Omega)} \frac{|B(x, r)|}{|B(x, r) \cap \Omega|}.$$

In two dimensions it may be interpreted as the square of an eccentricity of Ω . Note that $\operatorname{dev}(\Omega) \geq 1$ for all domains and $\operatorname{dev}(\mathbb{R}^d) = 1$. Balls and cubes have ball deviation uniformly bounded by a dimensional constant. All bounded convex domains in $\mathbb{R}^d$ have finite ball deviation. Among

unbounded domains, open convex cones have finite ball deviation while infinite strips of finite width do not have finite ball deviation.

We will prove a priori estimates for the difference quotient operator $\mathfrak{D}_b$ when acting on functions in $C^1(\Omega)$. By completion the a priori estimates extend to larger classes of functions but we will not explicitly address the question of what happens for a general function in a weighted Sobolev space with weight w . For $b = \gamma/p$ we want to estimate the measure μ_γ^u of the sets

$$E_{\lambda,b}(f) = \{(x, y) \in \Omega \times \Omega : \mathfrak{D}_b f(x, y) > \lambda\}.$$

Theorem 1.1. *Let $d \geq 1$, $\Omega \subset \mathbb{R}^d$ be a convex domain with finite ball deviation and $f \in C^1(\Omega)$. Assume $1 \leq q \leq p < \infty$, $p > d(q-1)$ and $(u, w) \in A_q(\Omega)$. If $\gamma \in \mathbb{R} \setminus [-qd, 0]$ then for all $\lambda > 0$,*

$$\mu_\gamma^u(E_{\lambda, \gamma/p}(f)) \lesssim_{d, \gamma, p, q} [u, w]_{A_q(\Omega)} \left(\frac{\text{dev}(\Omega)}{\lambda} \right)^p \int_\Omega |\nabla f(y)|^p w(y) dy.$$

The implicit constant depends only on d, γ, p, q and is independent of Ω , the weights u and w , and the function f .

By specializing to the case $q = p$, we obtain:

Corollary 1.2. *Let $d \geq 1$, $\Omega \subset \mathbb{R}^d$ be a convex domain with finite ball deviation and $f \in C^1(\Omega)$. If $\gamma \in \mathbb{R} \setminus [-pd, 0]$ and $1 \leq p < \frac{d}{d-1}$ (in particular $1 \leq p < \infty$ when $d = 1$) then for all $(u, w) \in A_p(\Omega)$ and all $\lambda > 0$,*

$$\mu_\gamma^u(E_{\lambda, \gamma/p}(f)) \lesssim_{d, p, \gamma} [u, w]_{A_p(\Omega)} \left(\frac{\text{dev}(\Omega)}{\lambda} \right)^p \int_\Omega |\nabla f(y)|^p w(y) dy. \quad (6)$$

For the case $q = 1$ in Theorem 1.1 we obtain:

Corollary 1.3. *Let $d \geq 1$, $\Omega \subset \mathbb{R}^d$ be a convex domain with finite ball deviation and $f \in C^1(\Omega)$. Let $1 \leq p < \infty$ and $(u, w) \in A_1(\Omega)$. If $\gamma \in \mathbb{R} \setminus [-d, 0]$, then for all $\lambda > 0$,*

$$\mu_\gamma^u(E_{\lambda, \gamma/p}(f)) \lesssim_{d, p, \gamma} [u, w]_{A_1(\Omega)} \left(\frac{\text{dev}(\Omega)}{\lambda} \right)^p \int_\Omega |\nabla f(y)|^p w(y) dy.$$

In particular, for any non-negative locally integrable function u on Ω , we have

$$\mu_\gamma^u(E_{\lambda, \gamma/p}(f)) \lesssim_{d, p, \gamma} \left(\frac{\text{dev}(\Omega)}{\lambda} \right)^p \int_\Omega |\nabla f(y)|^p M_\Omega u(y) dy.$$

The second statement holds since the pair $(u, M_\Omega u)$ belongs to the class $A_1(\Omega)$. It would be interesting to find new necessary conditions for the ranges of p and $\gamma \in [-qd, 0]$. Some further perspectives on this are offered in Section 4 below.

Weighted analogs of inequality (2) with one weight $u = w$ were considered by Li–Yang–Yuan–Zhang–Zhao [39]. Their results rely on the wavelet analysis ideas by Cohen–Dahmen–Daubechies–DeVore [14] and raise questions about possible improvements in the range of negative γ in Corollary 1.2. We note that the example in §4.3.1 indicates a fundamental difference between the one-weight and two-weight theories for our difference quotients.

Notation. Denote by $|E|$ the Lebesgue measure of a set E in $\mathbb{R}^d$ and by χ_E its characteristic function. For a set E of positive finite Lebesgue measure let

$$\text{av}_E f = \frac{1}{|E|} \int_E f dx$$

be the average of f over E with respect to the Lebesgue measure. Sometimes we write $u(E)$ for the integral $\int_E u$. A ball centered at x and of radius r is denoted by $B(x, r)$. For $t > 0$, the dilation of a ball B by t times around its center will be denoted tB . Let $L^{p,\infty}(X, \mu)$ denote the Marcinkiewicz (or weak type p) space on the measure space (X, μ) which consists of all measurable F on X such that

$$[F]_{L^{p,\infty}(X,\mu)}^p := \sup_{\lambda>0} \lambda^p \mu(\{x \in X : |F(x)| > \lambda\})$$

is finite. $F \mapsto [F]_{L^{p,\infty}}$ is a quasi-norm which is homogeneous of degree 1. Typically we will consider $X = \Omega \times \Omega$ with μ given by μ_γ^u as in (3). We also use the notation $L^{p,\infty}(v)$ if v is a weight function and $d\mu = v(x)dx$ with dx Lebesgue measure. Similarly, for the usual L^p spaces we concurrently use the notations $L^p(v)$ and $L^p(\mu)$ if $d\mu = vdx$. The side length of a cube Q will be denoted $\ell(Q)$.

2. MAIN ESTIMATES

2.1. Preliminaries on two weight A_p conditions. The significance of the $A_p(\Omega)$ class arises from a well-known result by Muckenhoupt for the Ω -variant M_Ω of the Hardy-Littlewood maximal operator, given by

$$M_\Omega f(x) = \sup_{x \in B} \frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy. \quad (7)$$

Namely we have $[u, w]_{A_p(\Omega)} < \infty$ if and only if the operator M_Ω maps $L^p(\Omega, w)$ to $L^{p,\infty}(\Omega, u)$, and then

$$\|M_\Omega\|_{L^p(\Omega, w) \rightarrow L^{p,\infty}(\Omega, u)} \approx [u, w]_{A_p}^{1/p}. \quad (8)$$

In other words

$$\sup_{\lambda>0} \lambda^p \int_{\{x: M_\Omega f(x) > \lambda\}} u(x) dx \lesssim C^p \int |f(y)|^p w(y) dy$$

and the infimum over admissible C in this inequality is comparable to $[u, w]_{A_p}^{1/p}$. The implicit constants in (8) depend on the dimension. There is also an analogous result for the *centered version*

$$M_\Omega^{\text{cent}} f(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r) \cap \Omega} |f(y)| dy;$$

moreover we have analogues where in both versions balls are replaced by cubes.

From [24], the $A_p(\Omega)$ class of weights is characterized by the condition

$$\left(\frac{1}{|B|} \int_{B \cap \Omega} f(y) dy \right)^p u(B \cap \Omega) \leq C \int_{B \cap \Omega} f(y)^p w(y) dy \quad (9)$$

for any ball B and nonnegative measurable f ; and the best constant C is equal to $[u, w]_{A_p(\Omega)}$. For $p = 1$ this characterization is immediate. For $p > 1$ one direction follows from Hölder's inequality.

$$\begin{aligned} \left(\frac{1}{|B|} \int_{B \cap \Omega} f(y) dy \right)^p u(B \cap \Omega) \\ \leq \left(\int_{B \cap \Omega} f(y)^p w(y) dy \right) \left(\frac{1}{|B|} \int_{B \cap \Omega} w(y)^{-p'/p} dy \right)^{p/p'} \left(\frac{1}{|B|} \int_{B \cap \Omega} u(x) dx \right), \end{aligned}$$

and the other one follows by testing (9) with $f = w^{-p'/p}$.

A useful form of this inequality says that for any $t \geq 1$ and any ball B , we have

$$\begin{aligned} \left(\frac{1}{|B|} \int_{B \cap \Omega} f dy \right)^p u((tB) \cap \Omega) &\leq t^{pd} \left(\frac{1}{|tB|} \int_{(tB) \cap \Omega} f \chi_B dy \right)^p u(tB \cap \Omega) \\ &\leq t^{pd} [u, w]_{A_p} \int_{B \cap \Omega} f^p w dy. \end{aligned} \quad (10)$$

To see this, we just apply (9) to tB and $f \chi_B$ in place of B and f .

2.2. The main technical ingredients. We provide the main technical ingredients in the proofs of Theorem 1.1. In this section we assume that Ω is a *general measurable subset of $\mathbb{R}^d$ with positive Lebesgue measure*. Define, for $b \in \mathbb{R}$, an operator T_b acting on measurable nonnegative functions g by

$$T_b g(x, r) = \frac{1}{r^{d+b}} \int_{B(x, r) \cap \Omega} g(z) dz, \quad x \in \Omega, r > 0. \quad (11)$$

We estimate this for the parameter $b = \gamma/p$ and choose as the measure on $\Omega \times (0, \infty)$

$$d\nu_\gamma^u = u(x) r^{\gamma-1} dx dr.$$

As μ_γ^u above, ν_γ^u is an inner regular measure, for all $\gamma \in \mathbb{R}$ [23, Proposition 412Q].

Theorem 2.1. *Let $d \geq 1$, and let $\Omega \subset \mathbb{R}^d$ have positive Lebesgue measure. Let $1 \leq p < \infty$, and let g be a non-negative measurable function on Ω .*

(a) *If $(u, w) \in A_p(\Omega)$, then for any $\gamma > 0$, we have*

$$[T_{\gamma/p} g]_{L^{p,\infty}(\Omega \times (0,\infty), \nu_\gamma^u)} \lesssim_{d,\gamma,p} [u, w]_{A_p(\Omega)}^{1/p} \|g\|_{L^p(\Omega, w)}.$$

(b) *If $1 \leq q \leq p$ and $(u, w) \in A_q(\Omega)$, then for any $\gamma < -qd$ we have*

$$[T_{\gamma/p} g]_{L^{p,\infty}(\Omega \times (0,\infty), \nu_\gamma^u)} \lesssim_{d,\gamma,p,q} [u, w]_{A_q(\Omega)}^{1/p} \|g\|_{L^p(\Omega, w)}.$$

Proof. For $\lambda > 0$, let

$$E(\lambda) = \{(x, r) \in \Omega \times (0, \infty) : T_{\gamma/p} g(x, r) > \lambda\}.$$

The case $\gamma > 0$. By the inner regularity of ν_γ^u it suffices to prove that for any compact subset $\mathcal{K}$ of $E(\lambda)$, we have

$$\nu_\gamma^u(\mathcal{K}) \lesssim_{d,\gamma,p} [u, w]_{A_p(\Omega)} \lambda^{-p} \int_{\Omega} |g(y)|^p w(y) dy. \quad (12)$$

The set $\{B(x, r) \times (0, 2r) : (x, r) \in \mathcal{K}\}$ covers $\mathcal{K}$, and hence admits a finite subcover, say

$$B(x_i, r_i) \times (0, 2r_i), \text{ for } i = 1, \dots, N.$$

Let $\mathcal{C}_{\text{fin}}$ be the collection of open balls $B(x_i, r_i)$, $i = 1, \dots, N$. Since there are only finitely many balls in $\mathcal{C}_{\text{fin}}$, each of which has a finite radius, the Vitali covering selection algorithm (see e.g. [47] Chapter I, Section 3) gives a disjoint subcollection $\mathcal{I}'$ of $\mathcal{C}_{\text{fin}}$ such that every $B \in \mathcal{C}_{\text{fin}}$ is contained in $5B'$ for some $B' \in \mathcal{I}'$, with $r_B \leq 2r_{B'}$ (we write r_B for the radius of B). Now if $(x, r) \in \mathcal{K}$, then $(x, r) \in B(x_i, r_i) \times (0, 2r_i)$ for some i , with $B(x_i, r_i) \in \mathcal{C}_{\text{fin}}$, so together with $x \in \Omega$ we obtain $(x, r) \in (B \cap \Omega) \times [0, 2r_B]$ for some $B \in \mathcal{C}_{\text{fin}}$. Using this together with the covering property of $\mathcal{I}'$, we obtain

$$\mathcal{K} \subset \bigcup_{B \in \mathcal{C}_{\text{fin}}} (B \cap \Omega) \times [0, 2r_B] \subset \bigcup_{B' \in \mathcal{I}'} (5B' \cap \Omega) \times [0, 4r_{B'}].$$

(In short, if a ball B is contained in a dilation of another ball B' , then the Carleson tent $B \times [0, 2r_B]$ over B is contained in the dilation of the Carleson tent $B' \times [0, 2r_{B'}]$ over B' .) It follows that

$$\begin{aligned} \int_{\mathcal{K}} u(x) r^{\gamma-1} dx dr &\leq \sum_{B' \in \mathcal{I}'} \int_{5B' \cap \Omega} u(x) dx \int_0^{4r_{B'}} r^{\gamma-1} dr \\ &\lesssim_{\gamma} \sum_{B' \in \mathcal{I}'} r_{B'}^{\gamma} u(5B' \cap \Omega). \end{aligned}$$

We used $\gamma > 0$ in the last inequality. Now observe $r_{B'}^{\gamma} \approx_{d,p} \left(\frac{1}{|B'|} r_{B'}^{d+\frac{\gamma}{p}} \right)^p$ and using that $B' \in \mathcal{I}'$, we obtain

$$\begin{aligned} \int_{\mathcal{K}} u(x) r^{\gamma-1} dx dr &\lesssim_{d,\gamma,p} \frac{1}{\lambda^p} \sum_{B' \in \mathcal{I}'} \left(\frac{1}{|B'|} \int_{B' \cap \Omega} g(z) dz \right)^p u(5B' \cap \Omega) \\ &\lesssim_{d,p} \frac{[u, w]_{A_p(\Omega)}}{\lambda^p} \sum_{B' \in \mathcal{I}'} \int_{B' \cap \Omega} g(z)^p w(z) dz \lesssim \frac{[u, w]_{A_p(\Omega)}}{\lambda^p} \int_{\Omega} |g(z)|^p w(z) dz. \end{aligned}$$

Here the second inequality is from (10), and the third inequality uses the disjointness of the balls in $\mathcal{I}'$. We established (12) in the case $\gamma > 0$.

The case $\gamma < -qd$ with $q \in [1, p]$.

Let $\mathcal{K} \subset E(\lambda)$ be a compact set. Again, by inner regularity of ν_{γ}^u it suffices to show that

$$\nu_{\gamma}^u(\mathcal{K}) \lesssim_{d,\gamma,p,q} [u, w]_{A_q(\Omega)} \lambda^{-p} \int_{\Omega} |g(y)|^p w(y) dy. \quad (13)$$

For every (x, r) in $\mathcal{K}$ there exists an open ball $B(x, r)$ centered at x with radius r such that

$$\frac{1}{r^{d+\frac{\gamma}{p}}} \int_{B(x,r) \cap \Omega} g(z) dz > \lambda$$

The open sets $B(x, r) \times (\frac{r}{2}, \infty)$ cover $\mathcal{K}$ and therefore there is a finite subcover, i.e. there exists a finite family $\mathcal{C}_0$ of balls B of radius r_B such that $r_B^{-d-\gamma/p} \int_{B \cap \Omega} g dz > \lambda$, and $\mathcal{K} \subset \bigcup_{B \in \mathcal{C}_0} Z_B$, with $Z_B = (B \cap \Omega) \times (r_B/2, \infty)$.

We use a reverse Vitali selection procedure. We select disjoint balls $B_1, \dots, B_N$, such that the center of B_j is x_j , its radius is r_j , $B_j \in \mathcal{C}_{j-1}$, the radius of B_j is the *minimum* of all radii of balls in $\mathcal{C}_{j-1}$, and $\mathcal{C}_j$ consists of all balls in $\mathcal{C}_{j-1}$ which are disjoint from B_j (by induction these balls have radius at least r_j and these balls are disjoint from all already selected balls $B_1, \dots, B_j$).

Let

$$S(B_j) = \left\{ (x, \rho) \in \Omega \times [r_j/2, \infty) : |x - x_j| \leq 6\rho \right\}.$$

We claim that if B is any ball in $\mathcal{C}_{j-1}$ such that $B \cap B_j \neq \emptyset$, then

$$Z_B = (B \cap \Omega) \times (r_B/2, \infty) \subset S(B_j).$$

Indeed, let $x \in B$, $z \in B \cap B_j$ then $|x - x_j| \leq |x - z| + |z - x_j| \leq 2r_B + r_{B_j} \leq 3r_B$; hence if $\rho \geq r_B/2$ then $|x - x_j| \leq 3r_B \leq 6\rho$ and thus $Z_B \subset S(B_j)$.

In particular we get $\mathcal{K} \subset \bigcup_{j=1}^N S(B_j)$ and therefore $\nu_\gamma^u(\mathcal{K}) \leq \sum_{j=1}^N \nu_\gamma^u(S(B_j))$.

We fix j and let for $n = 0, 1, 2, \dots$

$$S_n(B_j) = \{(x, \rho) \in \Omega \times \mathbb{R} : |x - x_j| \leq 6\rho, 2^{n-1}r_j \leq \rho < 2^n r_j\}.$$

Then $S(B_j) = \bigcup_{n=0}^\infty S_n(B_j)$.

Now since $B_j \in \mathcal{C}_0$ we may use the condition $r_j^{dp+\gamma} \leq \lambda^{-p}(\int_{B_j \cap \Omega} g)^p$ and estimate

$$\begin{aligned} \nu_\gamma^u(S_n(B_j)) &\leq \int_{2^{n-1}r_j}^{2^n r_j} \rho^{\gamma-1} \int_{B(x_j, 6\rho) \cap \Omega} u(x) dx d\rho \\ &\lesssim_\gamma (2^n r_j)^\gamma \int_{B(x_j, 6 \cdot 2^n r_j) \cap \Omega} u(x) dx \\ &\lesssim_{d, \gamma, p} 2^{n\gamma} \left(\lambda^{-1} \frac{1}{|B_j|} \int_{B_j \cap \Omega} g(z) dz \right)^p u((6 \cdot 2^n B_j) \cap \Omega). \end{aligned}$$

If $\gamma < -qd$, with $q \in [1, p]$, we first apply Hölder's inequality in g :

$$\left(\frac{1}{|B_j|} \int_{B_j \cap \Omega} g(x) dx \right)^p \leq \left(\frac{1}{|B_j|} \int_{B_j \cap \Omega} g(x)^{p/q} dx \right)^q \quad (14)$$

and then use (10) with exponent q ,

$$u((6 \cdot 2^n B_j) \cap \Omega) \left(\frac{1}{|B_j|} \int_{B_j \cap \Omega} g(y)^{p/q} dy \right)^q \lesssim 2^{ndq} [u, w]_{A_q(\Omega)} \int_{B_j \cap \Omega} g(y)^p w(y) dy. \quad (15)$$

We obtain

$$\nu_\gamma^u(S_n(B_j)) \lesssim 6^q 2^{n(\gamma+qd)} \lambda^{-p} [u, w]_{A_q(\Omega)} \int_{B_j \cap \Omega} g(y)^p w(y) dy$$

and, since $\gamma < -qd$, we may sum in n and get

$$\nu_\gamma^u(S(B_j)) \lesssim_{d, \gamma, p, q} \lambda^{-p} [u, w]_{A_q(\Omega)} \int_{B_j \cap \Omega} g(y)^p w(y) dy.$$

By the disjointness of the B_j ,

$$\nu_\gamma^u(\mathcal{K}) \leq \sum_{j=1}^N \nu_\gamma^u(S(B_j)) \lesssim \frac{[u, w]_{A_q(\Omega)}}{\lambda^p} \sum_{j=1}^N \int_{B_j \cap \Omega} g(y)^p w(y) dy \leq \frac{[u, w]_{A_q(\Omega)}}{\lambda^p} \int_{\Omega} g(y)^p w(y) dy.$$

and (13) is proved. $\square$

For any $b \in \mathbb{R}$, any nonnegative measurable function g on Ω and any $k \geq 2$ define

$$T_b^k g(x, y) = \frac{2^{kd}}{|y - x|^{d+b}} \int_{B(y, 2^{-k}|y-x|) \cap \Omega} g(z) dz. \quad (16)$$

Note that here we are breaking the symmetry in x, y by integrating over the ball of radius $2^{-k}|x-y|$ centered at y while the weight function u depends on x . We use the inner regular measure μ_γ^u as defined in (3). As before the relevant parameter will be $b = \gamma/p$.

Theorem 2.2. *Let $d \geq 1$, $\Omega \subset \mathbb{R}^d$ be a measurable subset of positive Lebesgue measure, $k \geq 2$, and g be a non-negative measurable function on Ω . Let $1 \leq q \leq p < \infty$ and $(u, w) \in A_q(\Omega)$. Then, for any $\gamma \in \mathbb{R} \setminus [-qd, 0]$,*

$$[T_{\gamma/p}^k g]_{L^{p,\infty}(\Omega \times \Omega, \mu_\gamma^u)} \lesssim_{d,\gamma,p,q} 2^{\frac{k d(q-1)}{p}} [u, w]_{A_q(\Omega)}^{1/p} \|g\|_{L^p(\Omega, w)}.$$

The implicit constant is independent of k .

Proof. Fix $k \geq 2$. For any $x, y \in \Omega$, let $B_{x,y}^k := B(y, 2^{-k}|y - x|)$. For $\lambda > 0$, let

$$E_k(\lambda) = \{(x, y) \in \Omega \times \Omega : x \neq y, T_{\gamma/p}^k g(x, y) > \lambda\}.$$

We can rewrite this as

$$E_k(\lambda) = \left\{ (x, y) \in \Omega \times \Omega : x \neq y, 2^{k\gamma} \text{rad}(B_{x,y}^k)^\gamma < \left(\frac{c_d}{\lambda} \frac{1}{|B_{x,y}^k|} \int_{B_{x,y}^k \cap \Omega} g(z) dz \right)^p \right\}$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$.

The case $\gamma > 0$. Let $\mathcal{I}$ be the collection of balls $B = B_{x,y}^k$ for which $(x, y) \in E_k(\lambda)$.

Arguing as in the proof of Theorem 2.1 (this time using a Whitney decomposition of $\mathcal{U} = \{(x, y) \in \Omega \times \Omega : x \neq y\}$ and the local integrability of $(x, y) \mapsto |x - y|^{\gamma-d}$ away from the diagonal) it suffices to show for any compact set $\mathcal{K} \subset E_k(\lambda)$ that

$$\mu_\gamma^u(\mathcal{K}) \lesssim_{d,\gamma,p,q} \lambda^{-p} [u, w]_{A_q(\Omega)} \int_{\Omega} |g(x)|^p w(x) dx. \quad (17)$$

Now $(x, y) \in 2^{k+1}B_{x,y}^k \times B_{x,y}^k$, and so if $(x, y) \in E_k(\lambda)$, then $(x, y) \in ((2^{k+1}B) \cap \Omega) \times (B \cap \Omega)$ for some $B \in \mathcal{I}$. By the compactness of $\mathcal{K}$ there exists a finite collection $\mathcal{C}_{\text{fin}} \subset \mathcal{I}$ such that the open sets $(2^{k+1}B \times B)$ cover $\mathcal{K}$ and since $\mathcal{K} \subset \Omega \times \Omega$ we have

$$\mathcal{K} \subset \bigcup_{B \in \mathcal{C}_{\text{fin}}} (2^{k+1}B \cap \Omega) \times (B \cap \Omega).$$

By Vitali covering, there exists a disjoint subcollection $\mathcal{I}'$ of $\mathcal{C}_{\text{fin}}$ such that every $B \in \mathcal{C}_{\text{fin}}$ is contained in $3B'$ for some $B' \in \mathcal{I}'$. Hence

$$\mathcal{K} \subset \bigcup_{B' \in \mathcal{I}'} (6 \cdot 2^k B' \cap \Omega) \times 3B'.$$

Now for $B' \in \mathcal{I}'$, we estimate

$$\mu_\gamma^u((6 \cdot 2^k B' \cap \Omega) \times 3B') = \int_{(6 \cdot 2^k B') \cap \Omega} u(x) \int_{3B'} |x - y|^{\gamma-d} dy dx.$$

Note that since $\gamma > 0$, for any $x \in \mathbb{R}^d$,

$$\int_{3B'} |x - y|^{\gamma-d} dy \lesssim_{d,\gamma} r_{B'}^\gamma \left(1 + \frac{|x - c_{B'}|}{r_{B'}} \right)^{\gamma-d},$$

where $c_{B'}$ is the center of B' and $r_{B'}$ is the radius of B' ; this can be verified by splitting into the case $x \in 6B'$ and $x \notin 6B'$ respectively, and the case $x \in 6B'$ requires using the condition $\gamma > 0$. As a result,

$$\mu_\gamma^u((6 \cdot 2^k B' \cap \Omega) \times (3B' \cap \Omega)) \lesssim_{d,\gamma} r_{B'}^\gamma \int_{(6 \cdot 2^k B') \cap \Omega} u(x) \left(1 + \frac{|x - c_{B'}|}{r_{B'}} \right)^{\gamma-d} dx.$$

We decompose

$$6 \cdot 2^k B' \subseteq B' \cup \bigcup_{n=1}^{k+3} (2^n B' \setminus 2^{n-1} B')$$

and use

$$1 + \frac{|x - c_{B'}|}{r_{B'}} \approx \begin{cases} 1 & \text{if } x \in B' \\ 2^n & \text{if } x \in (2^n B') \setminus (2^{n-1} B') \end{cases}.$$

Since $B' \in \mathcal{I}$ we have

$$r_{B'}^\gamma \lesssim_{d,p} 2^{-k\gamma} \lambda^{-p} \left(\frac{1}{|B'|} \int_{B' \cap \Omega} g(x) dx \right)^p,$$

and therefore

$$\begin{aligned} & r_{B'}^\gamma \int_{(6 \cdot 2^k B') \cap \Omega} u(x) \left(1 + \frac{|x - c_{B'}|}{r_{B'}} \right)^{\gamma-d} dx \\ & \lesssim_{d,p} 2^{-k\gamma} \lambda^{-p} \left(\frac{1}{|B'|} \int_{B' \cap \Omega} g(x) dx \right)^p \sum_{n=0}^{k+3} 2^{n(\gamma-d)} u((2^n B') \cap \Omega) \\ & \lesssim_{d,p} 2^{-k\gamma} \lambda^{-p} \left(\frac{1}{|B'|} \int_{B' \cap \Omega} g(x)^{p/q} dx \right)^q \sum_{n=0}^{k+3} 2^{n(\gamma-d)} u((2^n B') \cap \Omega), \end{aligned}$$

with Hölder's inequality (14). Using (10) as in (15), this shows that

$$\begin{aligned} \mu_\gamma^u((6 \cdot 2^k B' \cap \Omega) \times (3B' \cap \Omega)) & \lesssim_{d,\gamma,p} 2^{-k\gamma} \lambda^{-p} [u, w]_{A_q(\Omega)} \sum_{n=0}^{k+3} 2^{n(\gamma+d(q-1))} \int_{B' \cap \Omega} g(x)^p w(x) dx \\ & \lesssim_{d,\gamma,p,q} \lambda^{-p} 2^{kd(q-1)} [u, w]_{A_q(\Omega)} \int_{B' \cap \Omega} g(x)^p w(x) dx. \end{aligned}$$

Summing over the disjoint $B' \in \mathcal{I}'$ we obtain

$$\mu_\gamma^u(\mathcal{K}) \lesssim_{d,\gamma,p,q} \lambda^{-p} 2^{kd(q-1)} [u, w]_{A_q(\Omega)} \int_{\Omega} g(x)^p w(x) dx$$

Taking the supremum over compact $\mathcal{K} \subset E_k(\lambda)$ gives the desired estimate for $\gamma > 0$.

The case $\gamma < -qd$. Fix $k \geq 2$. Let

$$E_k(\lambda) = \left\{ (x, y) \in \Omega \times \Omega : x \neq y, \frac{2^{kd}}{|y - x|^{d+\frac{\gamma}{p}}} \int_{B(y, 2^{-k}|y-x|) \cap \Omega} g(x) dx > \lambda \right\}.$$

Let $\mathcal{I}_k$ be the collection of open balls $B(y, 2^{-k}|y-x|)$ for which $(x, y) \in E_k(\lambda)$. We denote by c_B the center of the ball B and by r_B its radius. Then

$$r_B^\gamma < \lambda^{-p} 2^{-k\gamma} \left(\frac{1}{r_B^d} \int_{B \cap \Omega} g(x) dx \right)^p \quad (18)$$

for every $B \in \mathcal{I}_k$, and

$$E_k(\lambda) \subset \bigcup_{B \in \mathcal{I}_k} (\Omega \setminus 2^{k-1} \overline{B}) \times B.$$

For any compact subset $\mathcal{K}$ of $E_k(\lambda)$, there exists a finite subcollection $\mathcal{C}_{\text{fin}} \subset \mathcal{I}_k$ such that

$$\mathcal{K} \subset \bigcup_{B \in \mathcal{C}_{\text{fin}}} Z_{k,B} \text{ where } Z_{k,B} := (\Omega \setminus 2^{k-1} \overline{B}) \times B.$$

As in the proof of Theorem 2.1 we use a reverse Vitali selection procedure, selecting a disjoint collection $B_1, \dots, B_N$ from $\mathcal{C}_{\text{fin}}$ such that for every $B \in \mathcal{C}_{\text{fin}}$, there exists $j \in \{1, \dots, N\}$ for which $B \cap B_j \neq \emptyset$, and $r_B \geq r_{B_j}$. Write

$$\mathfrak{S}_k(B_j) = \{(x, y) \in \Omega \times \mathbb{R}^d : |y - c_{B_j}| \leq 3 \cdot 2^{-(k-2)}|x - y| \text{ and } 2^{-(k-2)}|x - y| \geq r_{B_j}\}.$$

If $B \in \mathcal{C}_{\text{fin}}$ and $B \cap B_j \neq \emptyset$ with $r_B \geq r_{B_j}$, we claim that $Z_B \subset \mathfrak{S}_k(B_j)$. Indeed, for any $(x, y) \in Z_B$, one has

$$2^{k-1}r_B < |x - c_B| \leq |x - y| + |y - c_B| \leq |x - y| + r_B$$

which implies, since $k \geq 2$, that

$$|x - y| > (2^{k-1} - 1)r_B \geq 2^{k-2}r_B.$$

If $r_B \geq r_{B_j}$, then this implies $|x - y| \geq 2^{k-2}r_{B_j}$, i.e. $2^{-(k-2)}|x - y| \geq r_{B_j}$; also, if we pick a point $z \in B \cap B_j \neq \emptyset$, then

$$|y - c_{B_j}| \leq |y - z| + |z - c_{B_j}| \leq 2r_B + r_{B_j} \leq 3r_B \leq 3 \cdot 2^{-(k-2)}|x - y|,$$

proving that $Z_{k,B} \subset \mathfrak{S}_k(B_j)$. It follows that $\mathcal{K} \subset \bigcup_{j=1}^N \mathfrak{S}_k(B_j)$.

We note that for $(x, y) \in \mathfrak{S}_k(B_j)$ we have $|x - y| \geq 2^{k-2}r_{B_j}$. Let

$$\mathfrak{S}_{0,k}(B_j) = \{(x, y) : 2^{k-2}r_{B_j} \leq |x - y| \leq 2^{k-1}r_{B_j}, |y - c_{B_j}| \leq 3 \cdot 2^{-(k-2)}|x - y|\}$$

and for $n = 1, 2, \dots$

$$\mathfrak{S}_{n,k}(B_j) = \{(x, y) : 2^{n+k-2}r_{B_j} \leq |x - y| \leq 2^{n+k-1}r_{B_j}, |y - c_{B_j}| \leq 3 \cdot 2^{-(k-2)}|x - y|\}$$

Then

$$\mathfrak{S}_k(B_j) \subset \bigcup_{n=0}^{\infty} \mathfrak{S}_{n,k}(B_j).$$

We estimate $\mu_\gamma^u(\mathfrak{S}_{0,k})$. Note that $(x, y) \in \mathfrak{S}_{0,k}$ implies $2^{k-2}r_{B_j} \leq |x - y| \leq 2^{k-1}r_{B_j}$ and

$$|y - c_{B_j}| \leq 3 \cdot 2^{-(k-2)}|x - y|,$$

which implies $|x - c_{B_j}| \leq (2^{k-1} + 6)r_{B_j} \leq 2^{k+1}r_{B_j}$ (since $2^{k+1} - 2^{k-1} = 3 \cdot 2^{k-1} \geq 6$ for $k \geq 2$). In other words, $x \in 2^{k+1}B_j$. Also, $|y - c_{B_j}| \leq 3 \cdot 2^{-(k-2)}2^{k-1}r_{B_j} \leq 6r_{B_j}$. As a result,

$$\begin{aligned} \mu_\gamma^u(\mathfrak{S}_{0,k}(B_j)) &\leq \int_{(2^{k+1}B_j) \cap \Omega} u(x) \int_{\substack{y \in \mathbb{R}^d : |y - c_{B_j}| \leq 6r_{B_j} \\ 2^{k-2}r_{B_j} \leq |x - y| \leq 2^{k-1}r_{B_j}}} |x - y|^{\gamma-d} dy dx \\ &\lesssim_{d,\gamma} u((2^{k+1}B_j) \cap \Omega) (2^k r_{B_j})^{\gamma-d} r_{B_j}^d \\ &\lesssim_{d,\gamma,p} 2^{-kd} \lambda^{-p} \left(\frac{1}{|B_j|} \int_{B_j \cap \Omega} g(x) dx \right)^p u((2^{k+1}B_j) \cap \Omega) \end{aligned}$$

where we have used (18) in the last inequality. From Hölder's inequality and again (10) for q instead of p

$$\left(\frac{1}{|B_j|} \int_{B_j \cap \Omega} g(x) dx \right)^p u((2^{k+1}B_j) \cap \Omega) \lesssim_{d,q} 2^{kdq} \int_{B_j \cap \Omega} g(y)^p w(y) dy$$

and hence summing over the disjoint B_j 's we obtain

$$\sum_{j=1}^N \mu_\gamma^u(\mathfrak{S}_{0,k}(B_j)) \lesssim_{d,\gamma,p} 2^{kd(q-1)} \lambda^{-p} [u, w]_{A_q(\Omega)} \int_{\Omega} g(y)^p w(y) dy.$$

Similarly for $n \geq 1$ we observe that if $2^{n+k-2}r_{B_j} \leq |x - y| \leq 2^{n+k-1}r_{B_j}$ and $|y - c_{B_j}| \leq 3 \cdot 2^{-(k-2)}|x - y|$ then $|x - c_{B_j}| \leq (2^{n+k-1} + 6)r_{B_j} \leq 2^{n+k}r_{B_j}$ and also $|y - c_{B_j}| \leq 6 \cdot 2^n r_{B_j}$. Hence,

$$\begin{aligned} \mu_\gamma^u(\mathfrak{S}_{n,k}(B_j)) &\leq \int_{(2^{n+k}B_j) \cap \Omega} u(x) \int_{\substack{y \in \mathbb{R}^d: |y - c_{B_j}| \leq 6 \cdot 2^n r_{B_j} \\ 2^{n+k-2}r_{B_j} \leq |x - y| \leq 2^{n+k-1}r_{B_j}}} |x - y|^{\gamma-d} dy dx \\ &\lesssim_{d,\gamma} u((2^{n+k}B_j) \cap \Omega) (2^{n+k}r_{B_j})^{\gamma-d} (2^n r_{B_j})^d \\ &\lesssim_{d,\gamma,p} 2^{n\gamma} 2^{-kd} u((2^{n+k}B_j) \cap \Omega) (2^k r_{B_j})^\gamma \\ &\lesssim_{d,\gamma,p} 2^{n\gamma} 2^{-kd} u((2^{n+k}B_j) \cap \Omega) \lambda^{-p} \left(\frac{1}{|B_j|} \int_{B_j \cap \Omega} g(x) dx \right)^p \end{aligned}$$

by (18). By Hölder's inequality and (10),

$$u((2^{n+k}B_j) \cap \Omega) \left(\frac{1}{|B_j|} \int_{B_j \cap \Omega} g(x) dx \right)^p \leq 2^{(n+k)dq} [u, w]_{A_q(\Omega)} \int_{B_j \cap \Omega} g(y)^p w(y) dy$$

and therefore

$$\mu_\gamma^u(\mathfrak{S}_{n,k}(B_j)) \lesssim_{d,\gamma,p,q} 2^{n(dq+\gamma)} 2^{kd(q-1)} \lambda^{-p} [u, w]_{A_q(\Omega)} \int_{B_j \cap \Omega} g(y)^p w(y) dy. \quad \square$$

Since $\gamma < -dq$ we may sum in n and then sum over $j = 1, \dots, N$ using disjointness of the B_j 's. We then obtain the desired bound for $\mu_\gamma^u(\mathcal{K})$.

3. PROOF OF THEOREM 1.1

In this proof we shall now have to assume that Ω is a convex domain of bounded ball deviation. As a replacement of the fundamental theorem of calculus in one dimension we shall use a “sub-representation formula” for these domains (see e.g. Lemma 7.16 of [26] or more generally Section 8.1 from [1], or the proof of equation (4.5) in [32]) which is well known to lead to Poincaré-type inequalities. We refer the reader to [20] for extensions and applications of the classical Martio–Reshetnyak pointwise estimate for John domains, which is proven in detail in [28]. We also note that there are improved versions where the gradient is replaced by appropriate fractional difference quotients, with the Bourgain–Brezis–Mironescu factor $(1 - s)$ in front. These can be found in [34, 35], with applications to pointwise bounds for maximal rough singular integrals. For more about Poincaré inequalities on metric measure spaces see [33].

For the convenience of the reader we include a proof which tracks the dependency on the ball deviation of Ω .

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a convex domain with bounded ball deviation. Let $x, y \in \Omega$ with $x \neq y$ and $\mathcal{B}_{x,y}$ be the ball in $\mathbb{R}^d$ centered at $\frac{x+y}{2}$ with radius $r_{x,y} = \frac{|x-y|}{2}$ and $\mathcal{R}_{x,y} = \mathcal{B}_{x,y} \cap \Omega$. Let ϖ_d be*

the surface measure of the unit sphere in $\mathbb{R}^d$. Then for $f \in C^1(\Omega)$

$$|f(x) - \text{av}_{\mathcal{R}_{x,y}} f| \leq \frac{2^d \text{dev}(\Omega)}{\varpi_d} \int_{\mathcal{R}_{x,y}} \frac{|\nabla f(z)|}{|x - z|^{d-1}} dz. \quad (19)$$

and

$$|f(x) - f(y)| \leq \frac{2^d \text{dev}(\Omega)}{\varpi_d} \left(\int_{\mathcal{R}_{x,y}} \frac{|\nabla f(z)|}{|x - z|^{d-1}} dz + \int_{\mathcal{R}_{x,y}} \frac{|\nabla f(z)|}{|y - z|^{d-1}} dz \right). \quad (20)$$

Proof. To prove (19) we fix x and y and let, for $\omega \in \mathbb{S}^{d-1}$, $r_\omega = \sup\{r \geq 0 : x + r\omega \in \mathcal{R}_{x,y}\}$. Then we have $r_\omega \leq |x - y| = \text{diam}(\mathcal{R}_{x,y})$, and for every $s \in [0, r_\omega]$, we have

$$\begin{aligned} |f(x) - f(x + s\omega)| &= \left| \int_0^s \omega \cdot \nabla f(x + t\omega) dt \right| \\ &\leq \int_0^{r_\omega} |\nabla f(x + t\omega)| dt. \end{aligned}$$

Hence

$$\begin{aligned} |f(x) - \text{av}_{\mathcal{R}_{x,y}} f| &= \left| f(x) - \frac{1}{|\mathcal{R}_{x,y}|} \int_{\mathbb{S}^{d-1}} \int_0^{r_\omega} f(x + s\omega) s^{d-1} ds d\omega \right| \\ &\leq \frac{1}{|\mathcal{R}_{x,y}|} \int_{\mathbb{S}^{d-1}} \int_0^{r_\omega} |f(x) - f(x + s\omega)| s^{d-1} ds d\omega \\ &\leq \frac{1}{|\mathcal{R}_{x,y}|} \int_{\mathbb{S}^{d-1}} \int_0^{r_\omega} \int_0^{r_\omega} |\nabla f(x + t\omega)| dt s^{d-1} ds d\omega \\ &= \frac{1}{|\mathcal{R}_{x,y}|} \int_{\mathbb{S}^{d-1}} \frac{r_\omega^d}{d} \int_0^{r_\omega} |\nabla f(x + t\omega)| dt d\omega \\ &\leq \frac{\text{diam}(\mathcal{R}_{x,y})^d}{d|\mathcal{R}_{x,y}|} \int_{\mathbb{S}^{d-1}} \int_0^{r_\omega} |\nabla f(x + t\omega)| dt d\omega. \end{aligned}$$

Since

$$\int_{\mathcal{R}_{x,y}} \frac{|\nabla f(z)|}{|z - x|^{d-1}} dz = \int_{\mathbb{S}^{d-1}} \int_0^{r_\omega} |\nabla f(x + t\omega)| dt d\omega$$

and since ϖ_d/d is the d -dimensional volume of the unit ball,

$$\frac{\text{diam}(\mathcal{R}_{x,y})^d}{d|\mathcal{R}_{x,y}|} = \frac{2^d \text{rad}(\mathcal{B}_{x,y})^d}{d|\mathcal{B}_{x,y} \cap \Omega|} = \frac{2^d |\mathcal{B}_{x,y}|}{\varpi_d |\mathcal{B}_{x,y} \cap \Omega|} \leq \frac{2^d \text{dev}(\Omega)}{\varpi_d}$$

we obtain (19).

Since $\mathcal{R}_{x,y} = \mathcal{R}_{y,x}$ we obtain from (19)

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - \text{av}_{\mathcal{R}_{x,y}} f| + |f(y) - \text{av}_{\mathcal{R}_{y,x}} f| \\ &\leq \frac{2^d \text{dev}(\Omega)}{\varpi_d} \left(\int_{\mathcal{R}_{x,y}} \frac{|\nabla f(z)|}{|x - z|^{d-1}} dz + \int_{\mathcal{R}_{x,y}} \frac{|\nabla f(z)|}{|y - z|^{d-1}} dz \right) \end{aligned}$$

which is (20). $\square$

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a convex domain with bounded ball deviation. Let $f \in C^1(\Omega)$ and $x, y \in \Omega$. With T_b as in (11) and T_b^k as in (16) we have*

$$\mathfrak{D}_{\gamma/p} f(x, y) \lesssim_{d, \gamma, p} \text{dev}(\Omega) \left(\sum_{k \geq 2} 2^{-k} T_{\gamma/p}^k |\nabla f|(x, y) + \sum_{k \geq 0} 2^{-k} 2^{-k \frac{\gamma}{p}} T_{\gamma/p} |\nabla f|(x, 2^{-k}|y - x|) \right). \quad (21)$$

Proof. We let $\mathcal{B}_{x,y}$ and $\mathcal{R}_{x,y} = \mathcal{B}_{x,y} \cap \Omega$ be as in Lemma 3.1. We dyadically decompose $\mathcal{B}_{x,y}$ into the annuli

$$\mathcal{A}_{x,y}^k = B(x, 2^{-k}|y-x|) \setminus B(x, 2^{-k-1}|y-x|)$$

and estimate

$$\frac{|f(y) - f(x)|}{|y-x|} \lesssim_d \frac{\text{dev}(\Omega)}{|y-x|} \sum_{k=0}^{\infty} \left(\int_{\mathcal{A}_{y,x}^k \cap \Omega} \frac{|\nabla f(z)|}{|y-z|^{d-1}} dz + \int_{\mathcal{A}_{x,y}^k \cap \mathcal{R}_{x,y}} \frac{|\nabla f(z)|}{|x-z|^{d-1}} dz \right)$$

and thus

$$\begin{aligned} \frac{|f(y) - f(x)|}{|y-x|^{1+\frac{\gamma}{p}}} &\lesssim_d \text{dev}(\Omega) \left(\sum_{k \geq 0} 2^{-k} \frac{2^{kd}}{|y-x|^{d+\frac{\gamma}{p}}} \int_{B(y, 2^{-k}|y-x|) \cap \Omega} |\nabla f(z)| dz \right. \\ &\quad \left. + \sum_{k \geq 0} 2^{-k} \frac{2^{kd}}{|y-x|^{d+\frac{\gamma}{p}}} \int_{B(x, 2^{-k}|y-x|) \cap \mathcal{R}_{x,y}} |\nabla f(z)| dz \right). \end{aligned} \quad (22)$$

We obtain with $r_k(x, y) = 2^{-k}|x-y|$,

$$\frac{2^{kd}}{|y-x|^{d+\frac{\gamma}{p}}} \int_{B(x, 2^{-k}|y-x|) \cap \Omega} |\nabla f(z)| dz \lesssim_d 2^{-k\frac{\gamma}{p}} T_{\gamma/p}[|\nabla f|](x, r_k(x, y)).$$

For $k = 0, 1, 2$, we bound

$$\frac{2^{kd}}{|y-x|^{d+\frac{\gamma}{p}}} \int_{B(y, 2^{-k}|y-x|) \cap \mathcal{R}_{x,y}} |\nabla f(z)| dz \lesssim_d T_{\gamma/p}[|\nabla f|](x, |x-y|).$$

However for larger k we need to invoke the operators $T_{\gamma/p}^k$ and obtain

$$\frac{2^{kd}}{|y-x|^{d+\frac{\gamma}{p}}} \int_{B(y, 2^{-k}|y-x|) \cap \mathcal{R}_{x,y}} |\nabla f(z)| dz \leq T_{\gamma/p}^k[|\nabla f|](x, y), \quad k \geq 2.$$

Combining the three previous displays with (22) leads to (21). $\square$

To deal with the k -sums in (21), we use the following lemma, which is a substitute for the failure of triangle inequality for the $L^{p,\infty}$ quasi-norm. Instead we could also use, for $p = 1$ a summing lemma of Stein and N. Weiss [48] and for $p > 1$ the equivalence of the seminorm to a norm [36].

Lemma 3.3. *Let $1 \leq p < \infty$ and $\varepsilon > 0$. Then there is $C = C_\varepsilon > 0$ independent of p such that for any sequence of non-negative functions G_k on the same measure space (X, μ) , we have*

$$\left[\sum_{k=1}^{\infty} 2^{-k\varepsilon} G_k \right]_{L^{p,\infty}(X,\mu)} \leq C \sup_{k \geq 1} [G_k]_{L^{p,\infty}(X,\mu)}.$$

Proof. For $\lambda > 0$, since $1 = \sum_{k=1}^{\infty} \frac{6}{\pi^2} \frac{1}{k^2}$, we see that the set $\{x \in X : \sum_{k=1}^{\infty} 2^{-k\varepsilon} G_k(x) > \lambda\}$ is contained in the union of the sets $\{x \in X : G_k(x) > \frac{6}{\pi^2} \frac{\lambda 2^{k\varepsilon}}{k^2}\}$. As $G_k \in L^{p,\infty}$ we get for all $\lambda > 0$,

$$\mu \left\{ x \in X : \sum_{k=1}^{\infty} 2^{-k\varepsilon} G_k(x) > \lambda \right\} \leq \sum_{k=1}^{\infty} \left(\frac{\pi^2 k^2}{6 \cdot 2^{k\varepsilon}} \right)^p \lambda^{-p} [G_k]_{L^{p,\infty}(X,\mu)}^p$$

and the assertion follows from $\sum_{k=1}^{\infty} \left(\frac{k^2}{2^{k\varepsilon}} \right)^p \leq \left(\sum_{k=1}^{\infty} \frac{k^2}{2^{k\varepsilon}} \right)^p$, with $C = \frac{\pi^2}{6} \sum_{k=1}^{\infty} k^2 2^{-k\varepsilon}$. $\square$

Proof of Theorem 1.1. Assume $1 \leq q \leq p < \infty$ and $(u, w) \in A_q(\Omega)$. We use Lemma 3.2 and let $\varepsilon = 1 - \frac{(q-1)d}{p} > 0$, so that $2^{-k} = 2^{-k\varepsilon} 2^{-k\frac{(q-1)d}{p}}$. We estimate the first expression on the right hand side of (21) using Lemma 3.3 and Theorem 2.2

$$\begin{aligned} \left[\sum_{k \geq 2} 2^{-k} T_{\gamma/p}^k |\nabla f|(x, y) \right]_{L^{p,\infty}(\Omega \times \Omega, \mu_\gamma^u)} &\lesssim \sup_{k \geq 2} \left[2^{-k\frac{(q-1)d}{p}} T_{\gamma/p}^k |\nabla f|(x, y) \right]_{L^{p,\infty}(\Omega \times \Omega, \mu_\gamma^u)} \\ &\lesssim_{d,\gamma,p,q} [u, w]_{A_q(\Omega)}^{1/p} \|\nabla f\|_{L^p(\Omega, w)}, \end{aligned}$$

For the second expression on the right hand side of (21) we use Lemma 3.3 again to estimate

$$\begin{aligned} \left[\sum_{k \geq 0} 2^{-k} 2^{-k\gamma/p} T_{\gamma/p} |\nabla f|(x, 2^{-k}|x - y|) \right]_{L^{p,\infty}(\Omega \times \Omega, \mu_\gamma^u)} \\ \lesssim \sup_{k \geq 0} \left[2^{-k\gamma/p} T_{\gamma/p} |\nabla f|(x, 2^{-k}|x - y|) \right]_{L^{p,\infty}(\Omega \times \Omega, \mu_\gamma^u)} \end{aligned}$$

which is dominated by a constant times

$$\sup_{k \geq 0} \left[2^{-k\gamma/p} T_{\gamma/p} |\nabla f|(x, 2^{-k}r) \right]_{L^{p,\infty}(\Omega \times (0,\infty), \nu_\gamma^u)} = \left[T_{\gamma/p} |\nabla f|(x, r) \right]_{L^{p,\infty}(\Omega \times (0,\infty), \nu_\gamma^u)}.$$

(The last equality is by a change of variable formula that accounts for the scaling in r .) By Theorem 2.1 the last term is $\lesssim_{d,\gamma,p,q} [u, w]_{A_q(\Omega)}^{1/p} \|\nabla f\|_{L^p(\Omega, w)}$ (we also used $[u, w]_{A_p(\Omega)} \leq [u, w]_{A_q(\Omega)}$ in the case $\gamma > 0$). This finishes the proof of Theorem 1.1. $\square$

4. FURTHER DISCUSSION AND PERSPECTIVES

4.1. Weighted Bourgain–Brezis–Mironescu formula. As $s \rightarrow 1^-$ the fractional $\dot{W}^{s,p}$ seminorms of any non-constant C^∞ function tend to ∞ . Bourgain–Brezis–Mironescu [4] who showed that

$$\lim_{s \rightarrow 1^-} (1-s) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(y) - f(x)|^p}{|y - x|^{sp+d}} dx dy = k(d, p) \int_{\mathbb{R}^d} |\nabla f(x)|^p dx \quad (23)$$

for some constant $k(d, p)$ depending only on d and p . Thus the blowup of the double integral as $s \rightarrow 1^-$ can be fixed by multiplying it with a factor $1 - s$. This formula has been extended in many directions (see e.g. [3, 6, 21, 37, 49]), and more recently even with pointwise analogues using Spector’s nonlinear “fractional gradient” operator

$$\mathcal{D}^s f(x) = \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^{d+s}} dy,$$

from [45]. Namely, letting I_s denote the Riesz potential operator, defined for $0 < s < d$ by

$$I_s f(x) = \int_{\mathbb{R}^d} |x - y|^{s-d} f(y) dy,$$

there is the pointwise limit

$$\lim_{s \rightarrow 1^-} (1-s) I_s (\mathcal{D}^s f)(x) = k(d, 1) I_1(|\nabla f|)(x),$$

which was recently established in [13].

Theorem 1.1 is partially motivated by a local weighted Bourgain–Brezis–Mironescu inequality for the case $p = 1$, with A_1 pairs of weights of the form $(\mu, M\mu)$, which can be found in [37, Theorem

2.1]. Here we let Q be a cube and let ν be a Radon measure on $\mathbb{R}^d$, $d \geq 1$. Then, for every $0 < s < 1$ and $1 \leq p < \infty$, there exists a dimensional constant $c_d > 0$ such that

$$\int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} dy d\nu(x) \leq \frac{c_d}{1-s} (\sqrt{d}, \ell(Q))^{(1-s)p} \int_Q |\nabla f(x)|^p M(\chi_Q \nu)(x) dx \quad (24)$$

for every $f \in C^1(Q)$. If we omit the factor $(1-s)^{-1}$ the corresponding inequality for $s = 1$ tends to be false for non-constant f (recall the already mentioned case whenever ν is Lebesgue measure, see [7]). (See also the previous work [41, Theorem 6.2] with the constant $s^{-1}(1-s)^{-1}$ in front).

It should be interesting to further pursue the study of such weighted estimates with the correct $(1-s)^{-1}$ blowup, and for general pairs of weights (u, w) in the classes $A_q(Q)$.

4.2. Difference quotients and maximal functions. Historically, unweighted estimates for difference quotients have been studied, in the case $1 < p < \infty$, via the classical estimate

$$\frac{|f(x) - f(y)|}{|x - y|} \lesssim_d (M|\nabla f|(x) + M|\nabla f|(y)) \quad (25)$$

which seems to be written down first in Bojarski [2]. It is valid at least for all $f \in C^1(\mathbb{R}^d)$, $x \neq y$, and can be found in [38]. P. Hajlasz used this estimate in [27] as the key tool to define Sobolev spaces $W^{1,p}$ on a metric space with a Borel measure for $p \in (1, \infty)$. Its various uses have been discussed for example in [42], [15], [11] and [9]. In one dimension, only one of the terms on the right hand side of (25) is present (which is immediate from the fundamental theorem of calculus). In two and higher dimensions arguments in [9, 11, 42] via (25) (relying on the method of rotations in the unweighted case) are insufficient in the weighted setting and do not recover Theorem 1.1. For illustration we consider the case of $\Omega = \mathbb{R}^d$.

Proposition 4.1. *For $d \geq 1$, $\gamma \neq 0$, $1 < p < \infty$,*

$$\mu_\gamma^u(\{(x, y) : \mathfrak{D}_{\gamma/p} f(x, y) > \lambda\}) \lesssim_{d, \gamma, p} \lambda^{-p} \int_{\mathbb{R}^d} |\nabla f(x)|^p M^2 u(x) dx \quad (26)$$

for all $f \in C^1(\mathbb{R}^d)$. In dimension $d = 1$, $M^2 u$ on the right hand side of (26) can be replaced with Mu .

Note that $Mu(x) \leq M^2 u(x)$ for almost every $x \in \mathbb{R}^d$. In light of the case $d = 1$ and Corollary 1.3 it would be desirable to try to prove a stronger inequality with $M^2 u$ replaced by Mu also in dimension $d \geq 2$.

Proof. Using (25), we have

$$[\mathfrak{D}_{\gamma/p} f]_{L^{p, \infty}(d\mu_\gamma^u)} \lesssim_d \left[\frac{M|\nabla f|(x)}{|y - x|^{\frac{\gamma}{p}}} \right]_{L^{p, \infty}(d\mu_\gamma^u)} + \left[\frac{M|\nabla f|(y)}{|y - x|^{\frac{\gamma}{p}}} \right]_{L^{p, \infty}(d\mu_\gamma^u)}. \quad (27)$$

For the first term on the right hand side of (27) we have a good bound, involving only one maximal function on the right hand side, namely we can prove

$$\left[\frac{M|\nabla f|(x)}{|y - x|^{\frac{\gamma}{p}}} \right]_{L^{p, \infty}(\mathbb{R}^d \times \mathbb{R}^d, d\mu_\gamma^u)} \lesssim_{\gamma, p} \|\nabla f\|_{L^p(\mathbb{R}^d, Mu(x) dx)}. \quad (28)$$

To see this let $\mathcal{E}_1 = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y|^{-\gamma/p} M|\nabla f|(x) > \lambda\}$. Note that for each fixed $x \in \mathbb{R}^d$

$$\int_{\{y: |x-y|^{-\gamma/p} Mf(x) > \lambda\}} |x - y|^{\gamma-d} dy \lesssim_{\gamma,d} \left(\frac{Mf(x)}{\lambda} \right)^p.$$

This works by direct calculation for both positive and negative γ (but not for $\gamma = 0$). Integrating in x we obtain

$$\mu_\gamma^u(\mathcal{E}_1) \lesssim_{d,\gamma} \lambda^{-p} \int M(|\nabla f|)(x)^p u(x) dx$$

and we can invoke the well-known Fefferman–Stein inequality [22]: if $1 < p < \infty$ there exists a constant C_d such that for all non-negative measurable functions g on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} M g(x)^p u(x) dx \leq C_d p' \int_{\mathbb{R}^d} |g(y)|^p M u(y) dy. \quad (29)$$

Applying this to $g = |\nabla f|$, we obtain

$$\int_{\mathcal{E}_1} u(x) dx dy \lesssim_d p' \lambda^{-p} \int_{y \in \mathbb{R}^d} |\nabla f|(y)^p M u(y) dy,$$

proving (28).

However, for the second term on the right hand side of (27) which is only present in dimensions $d \geq 2$, we can only prove

$$\left[\frac{M|\nabla f|(y)}{|y - x|^{\frac{\gamma}{p}}} \right]_{L^{p,\infty}(\mathbb{R}^d \times \mathbb{R}^d, \mu_\gamma^u)} \lesssim_{d,p} \|\nabla f\|_{L^p(\mathbb{R}^d, M^2 u)} \quad (30)$$

with two maximal functions on the right hand side, which gives rise to the two maximal functions on the right hand side of (26). Let $\mathcal{E}_2 = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y|^{-\gamma/p} M|\nabla f|(y) > \lambda\}$. (30) will follow if we can prove

$$\mu_\gamma^u(\mathcal{E}_2) = \int_{y \in \mathbb{R}^d} \int_{x: |x-y|^{-\gamma/p} M|\nabla f|(y) > \lambda} |x - y|^{\gamma-d} u(x) dx dy \lesssim_{d,p} \lambda^{-p} \int |\nabla f|(y)^p M^2 u(y) dy. \quad (31)$$

For fixed y we estimate the inner integral. Assuming $M|\nabla f|(y) < \infty$ we let, for $n \in \mathbb{N}$,

$$\mathcal{A}_{n,\gamma}(y) = \{x \in \mathbb{R}^d : \lambda 2^{n-1} < |x - y|^{-\gamma/p} M(|\nabla f|)(y) \leq \lambda 2^n\}.$$

Then, for both positive and negative γ ,

$$\begin{aligned} \int_{x \in \mathcal{A}_{n,\gamma}(y)} |x - y|^{\gamma-d} u(x) dx &\lesssim \left(\frac{M(|\nabla f|)(y)}{\lambda 2^n} \right)^{\frac{p}{\gamma}(\gamma-d)} \int_{|x-y| \lesssim_{d,\gamma} \left(\frac{M(|\nabla f|)(y)}{\lambda 2^n} \right)^{\frac{p}{\gamma}}} u(x) dx \\ &\lesssim 2^{-np} \left(\frac{M(|\nabla f|)(y)}{\lambda} \right)^p M u(y). \end{aligned}$$

We integrate in y and sum in n to obtain

$$\mu_\gamma^u(\mathcal{E}_2) \lesssim_d \int M(|\nabla f|)(y)^p M u(y) dy$$

and another application of the Fefferman–Stein inequality (29) yields (31) and then (30). $\square$

Remark. Let ν be a nonnegative Radon measure on $\mathbb{R}^d$. In Proposition 4.1 the pairs (u, Mu) and $(u, M^2 u)$ may also be replaced by $(\nu, M\nu)$ and $(\nu, M^2 \nu)$, respectively, if we replace the measure μ_γ^u with $|x - y|^{\gamma-d} d\nu(x)$. One uses here that the classical Fefferman–Stein inequality [22] also holds for general measures ν .

There are various refinements of (25), which are or may be useful in applications. One involves more regular maximal functions (leading for example to a version of Bressan's problem with Hardy spaces, see [30, §5.1]). Another potentially useful replacement of (25) (which had been included in a preliminary unpublished preprint version of [37], arXiv 2302.14029) is contained in the following proposition.

Proposition 4.2. *For all $f \in C^1(\mathbb{R}^d)$, all cubes $Q \subset \mathbb{R}^d$, and for a.e. $(x, y) \in Q \times Q$,*

$$\frac{|f(x) - f(y)|}{|x - y|} \lesssim_d \min_{z \in \{x, y\}} M(|\nabla f| \chi_Q)(z)^{\frac{1}{d}} \max_{z \in \{x, y\}} M(|\nabla f| \chi_Q)(z)^{1 - \frac{1}{d}}. \quad (32)$$

(25) follows from (32) by Young's inequality, i.e. $ab \leq a^p/p + b^{p'}/p'$ for $a, b \geq 0$, and $p > 1$. Other extensions of (25) in different directions will be included in the upcoming work [25].

For the proof of Proposition 4.2 we consider again, for $0 < s < d$, the standard Riesz potential of a non-negative measure μ , defined by $I_s \mu(x) = \int_{\mathbb{R}^d} |x - y|^{s-d} d\mu(y)$. We have the following lemma:

Lemma 4.3. *Let Q be a cube in $\mathbb{R}^d$, μ be a non-negative measure and $0 < s < d$. Then for almost every $x \in Q$,*

$$I_s(\mu \chi_Q)(x) \leq \frac{c_d}{s} \mu(Q)^{\frac{s}{d}} M(\mu \chi_Q)(x)^{\frac{d-s}{d}}. \quad (33)$$

In particular,

$$I_s(\mu \chi_Q)(x) \leq \frac{c_d}{s} \ell(Q)^s M(\mu \chi_Q)(x). \quad (34)$$

The second estimate (34) is a version of Hedberg's inequality [31]. However, (33) is sharper and it is more relevant for proving (32).

Proof of Lemma 4.3. We may assume that $\mu(Q) \neq 0$, otherwise there is nothing to prove. Let $Q \subset \mathbb{R}^d$ be a fixed cube and $x \in Q$. To prove (33) we let $t > 0$ and define $Q_{x,t}$ as the cube with center at x and side length $2t^{-\frac{1}{d-s}}$. Then using the layer cake formula, we obtain

$$\begin{aligned} \int_Q \frac{d\mu(y)}{|x - y|^{d-s}} &= \int_0^\infty \mu\left(\left\{y \in Q : \frac{1}{|x - y|^{d-s}} > t\right\}\right) dt = \int_0^\infty \mu\left(\left\{y \in Q : |x - y| < t^{-\frac{1}{d-s}}\right\}\right) dt \\ &\leq \int_0^\infty \min\left\{\mu(Q), \frac{\mu(Q_{x,t})}{|Q_{x,t}|} |Q_{x,t}|\right\} dt \leq \int_0^\infty \min\{\mu(Q), M(\mu \chi_Q)(x) 2^d t^{-\frac{d}{d-s}}\} dt \\ &= \int_0^{2^{d-s}(M(\mu \chi_Q)(x)/\mu(Q))^{\frac{d-s}{d}}} \mu(Q) dt + 2^d \int_{2^{d-s}(M(\mu \chi_Q)(x)/\mu(Q))^{\frac{d-s}{d}}}^\infty M(\mu \chi_Q)(x) t^{-\frac{d}{d-s}} dt \\ &= \frac{2^{d-s}d}{s} \mu(Q)^{\frac{s}{d}} (M(\mu \chi_Q)(x))^{\frac{d-s}{d}}. \end{aligned} \quad \square$$

Proof of Proposition 4.2. For $d = 1$ the proof is immediate from $|f(x) - f(y)| \leq \int_x^y |f'(t)| dt \leq |x - y| M(|f'| \chi_Q)(x)$. In what follows we assume $d \geq 2$.

Let Q be a cube in $\mathbb{R}^d$ and $f \in W^{1,1}(Q)$. For any cube $Q \subset \mathbb{R}^d$ and any Lebesgue points x, y of f in Q , there is a closed cube $R \subset Q$ containing both x and y , for which $\ell(R) \leq |x - y|$. We will use the well-known subrepresentation formula

$$|f(x) - \text{av}_R f| \leq c_d I_1(|\nabla f| \chi_R)(x)$$

(see e.g. Lemma 7.16 of [26]). Hence by (33)

$$|f(x) - \text{av}_R f| \leq c_d \left(\int_R |\nabla f(z)| dz \right)^{\frac{1}{d}} M(|\nabla f| \chi_R)(x)^{1-\frac{1}{d}}.$$

If we apply this formula with y in place of x and observe that $|f(x) - f(y)|$ is dominated by $|f(x) - \text{av}_R f| + |f(y) - \text{av}_R f|$ we obtain

$$|f(x) - f(y)| \leq 2c_d \left(\int_R |\nabla f|(z) dz \right)^{\frac{1}{d}} \max_{z \in \{x, y\}} M(|\nabla f| \chi_Q)(z)^{1-\frac{1}{d}}.$$

Furthermore, since $x, y \in R$,

$$\int_R |\nabla f(z)| dz \leq \int_{B(x, \ell(R))} |\nabla f(z)| \chi_Q(z) dz \lesssim_d \ell(R)^d M(|\nabla f| \chi_Q)(x)$$

and similarly the same estimate with x replaced by y . Thus

$$\left(\int_R |\nabla f(z)| dz \right)^{\frac{1}{d}} \lesssim_d \ell(R) \min_{z \in \{x, y\}} M(|\nabla f| \chi_Q)(z)^{\frac{1}{d}}$$

and (32) follows since $\ell(R) \leq |x - y|$. $\square$

4.3. More results for $d = 1$. The following elementary results on $\mathfrak{D}_{\gamma/p}$ for $d = 1$, $\gamma < 0$ have been assisted by ChatGPT5.5-Pro. Again, they raise corresponding questions in higher dimensions.

4.3.1. Failure of a two-weight A_p result for $\gamma = -p$. We report a necessary condition in the two weight case which does not seem to have an analogue in the one weight theory. Let $d = 1$, $p = q \geq 1$, $\gamma = -p$, then inequality (6) fails for some $(u, w) \in A_p(\mathbb{R})$. Namely, let $u(x) = (1 + |x|)^{p-1}$, $w(y) = (1 + |y|)^\beta$ for $\beta > p - 1$; then $(u, w) \in A_p(\mathbb{R})$ (note that this fails for $\beta = p - 1$ when $p > 1$). Let $f \in C^\infty(\mathbb{R})$ such that $f(y) = 0$ for $y < -1$ and $f(y) = 1$ for $y > 1$ so that $\int |f'|^p w < \infty$. For $\lambda < 1$ the set $E_{\lambda, -1}$ contains $A_\lambda = \{(x, y) : x < -1, y > 1\}$ which satisfies $\mu_{-p}^u(A_\lambda) = \infty$ for $\lambda < 1$.

4.3.2. A two-weight A_q inequality in one dimension. In Theorem 1.1 the range of γ can be improved for $d = 1$ and $q < p$. Namely for $\gamma < 0$,

$$\mu_\gamma^u(E_{\lambda, \gamma/p}(f)) \lesssim [u, w]_{A_q(\mathbb{R})} \lambda^{-p} \int_{\mathbb{R}} |f'(y)|^p w(y) dy, \quad 1 \leq q < p < \infty. \quad (35)$$

We argue as in §4.2 above (see also [9, 42]). Since $|f(x) - f(y)| \leq |x - y| M[f'](x)$ for $d = 1$ we have $\mathfrak{D}_{\gamma/p} f(x, y) \leq |x - y|^{-\gamma/p} M[f'](x)$. If $\mathfrak{D}_{\gamma/p} f(x, y) > \lambda$ then $|x - y| \geq \lambda^{-p/\gamma} M[f'](x)^{p/\gamma}$; hence

$$\mu_\gamma^u(E_{\lambda, \gamma/p}(f)) \leq \int_{\mathbb{R}} u(x) \int_{\{y: |x-y| \geq \lambda^{-p/\gamma} M[f'](x)^{p/\gamma}\}} |x - y|^{\gamma-1} dy dx \lesssim \lambda^{-p} \int_{\mathbb{R}} |M[f'](x)|^p u(x) dx.$$

By the weak type inequality (8), with q in place of p , and subsequent Marcinkiewicz interpolation with the L^∞ bound we get $\|Mg\|_{L^p(u)} \lesssim_{p,q} [u, w]_{A_q}^{1/p} \|g\|_{L^p(w)}$ for $q < p < \infty$, and (35) follows.

ACKNOWLEDGEMENTS

This work was initiated during a visit of Pérez to the Mathematical Sciences Institute at the Australian National University, whose support we gratefully acknowledge. Pérez is partially supported by the Spanish government through the grant PID2023-146646NB-I00 and by Severo Ochoa accreditation CEX2021-001142-S, both at BCAM, and also by the Basque Government through grant IT1615-22 at the University of the Basque Country and by the BERC programme 2022-2025 at BCAM. Seeger was supported in part by NSF Grant 2348797, and Yung supported by a Discovery Project DP250103744 from the Australian Research Council.

REFERENCES

- [1] David R. Adams and Lars Inge Hedberg, *Function spaces and potential theory*, Grundlehren der Mathematischen Wissenschaften, vol. 314, Springer-Verlag, Berlin, 1996.
- [2] Bogdan Bojarski, *Remarks on some geometric properties of Sobolev mappings*, Functional analysis & related topics (Sapporo, 1990), 1991, pp. 65–76.
- [3] ———, *Remarks on the Bourgain-Brezis-Mironescu approach to Sobolev spaces*, Bull. Pol. Acad. Sci. Math. **59** (2011), no. 1, 65–75.
- [4] Jean Bourgain, Haïm Brezis, and Petru Mironescu, *Another look at Sobolev spaces*, Optimal control and partial differential equations, IOS, Amsterdam, 2001, pp. 439–455.
- [5] Jean Bourgain and Hoai-Minh Nguyen, *A new characterization of Sobolev spaces*, C. R. Math. Acad. Sci. Paris **343** (2006), no. 2, 75–80.
- [6] Denis Brazke, Armin Schikorra, and Po-Lam Yung, *Bourgain-Brezis-Mironescu convergence via Triebel-Lizorkin spaces*, Calc. Var. Partial Differential Equations **62** (2023), no. 2, Paper No. 41, 33.
- [7] Haim Brezis, *How to recognize constant functions. A connection with Sobolev spaces*, Uspekhi Mat. Nauk **57** (2002), no. 4(346), 59–74 (Russian); English transl., Russian Math. Surveys **57** (2002), no. 4, 693–708.
- [8] Haïm Brezis and Petru Mironescu, *Gagliardo-Nirenberg inequalities and non-inequalities: the full story*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **35** (2018), no. 5, 1355–1376.
- [9] Haïm Brezis, Andreas Seeger, Jean Van Schaftingen, and Po-Lam Yung, *Families of functionals representing Sobolev norms*, Anal. PDE **17** (2024), no. 3, 943–979.
- [10] ———, *Sobolev spaces revisited*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **33** (2022), no. 2, 413–437.
- [11] Haïm Brezis, Jean Van Schaftingen, and Po-Lam Yung, *A surprising formula for Sobolev norms*, Proc. Natl. Acad. Sci. USA **118** (2021), no. 8, e2025254118.
- [12] ———, *Going to Lorentz when fractional Sobolev, Gagliardo and Nirenberg estimates fail*, Calc. Var. Partial Differential Equations **60** (2021), no. 4, Paper No. 129, 12.
- [13] Alejandro Claros and Carlos Pérez, *Bourgain-Brezis-Mironescu formula for Riesz Potentials* (2026). arXiv:2604.06827.
- [14] Albert Cohen, Wolfgang Dahmen, Ingrid Daubechies, and Ronald DeVore, *Harmonic analysis of the space BV*, Rev. Mat. Iberoamericana **19** (2003), no. 1, 235–263.
- [15] Gianluca Crippa and Camillo De Lellis, *Estimates and regularity results for the DiPerna-Lions flow*, J. Reine Angew. Math. **616** (2008), 15–46.
- [16] Feng Dai, Xiaosheng Lin, Dachun Yang, Wen Yuan, and Yangyang Zhang, *Poincaré inequality meets Brezis-Van Schaftingen-Yung formula on metric measure spaces*, J. Funct. Anal. **283** (2022), no. 9, Paper No. 109645, 52.
- [17] ———, *Brezis-Van Schaftingen-Yung formulae in ball Banach function spaces with applications to fractional Sobolev and Gagliardo-Nirenberg inequalities*, Calc. Var. Partial Differential Equations **62** (2023), no. 2, Paper No. 56, 73.
- [18] Oscar Domínguez and Mario Milman, *New Brezis-Van Schaftingen-Yung-Sobolev type inequalities connected with maximal inequalities and one parameter families of operators*, Adv. Math. **411** (2022), Paper No. 108774, 76.

- [19] Oscar Domínguez, Andreas Seeger, Brian Street, Jean Van Schaftingen, and Po-Lam Yung, *Spaces of Besov-Sobolev type and a problem on nonlinear approximation*, J. Funct. Anal. **284** (2023), no. 4, Paper No. 109775, 50.
- [20] Irene Drelichman and Ricardo G. Durán, *Improved Poincaré inequalities with weights*, J. Math. Anal. Appl. **347** (2008), no. 1, 286–293.
- [21] ———, *The Bourgain-Brézis-Mironescu formula in arbitrary bounded domains*, Proc. Amer. Math. Soc. **150** (2022), no. 2, 701–708.
- [22] Charles Fefferman and Elias M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.
- [23] D. H. Fremlin, *Measure theory. Vol. 4*, Torres Fremlin, Colchester, 2006. Topological measure spaces. Part I, II. Corrected second printing of the 2003 original.
- [24] José García-Cuerva and José L. Rubio de Francia, *Weighted norm inequalities and related topics*, Vol. 116, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática, 104.
- [25] Iker Gardeazabal, Emiel Lorist, and Carlos Pérez, *Self-improving properties of generalized Poincaré inequalities*, in preparation.
- [26] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer, Cham, 1977.
- [27] Piotr Hajlasz, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), no. 4, 403–415.
- [28] ———, *Sobolev inequalities, truncation method, and John domains*, Papers on analysis, Rep. Univ. Jyväskylä Dep. Math. Stat., vol. 83, Univ. Jyväskylä, Jyväskylä, 2001, pp. 109–126.
- [29] Qingsong Gu and Po-Lam Yung, *A new formula for the L^p norm*, J. Funct. Anal. **281** (2021), no. 4, Paper No. 109075, 19.
- [30] Mahir Hadžić, Andreas Seeger, Charles K. Smart, and Brian Street, *Singular integrals and a problem on mixing flows*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **35** (2018), no. 4, 921–943.
- [31] Lars Inge Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.
- [32] Juha Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.
- [33] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson, *Sobolev spaces on metric measure spaces*, New Mathematical Monographs, vol. 27, Cambridge University Press, Cambridge, 2015. An approach based on upper gradients.
- [34] Cong Hoang, Kabe Moen, and Carlos Pérez, *Pointwise estimates for rough operators with applications to Sobolev inequalities*, Journal d'Analyse Mathématique **155** (2025), no. 1, 43–74.
- [35] ———, *A New Look at the Subrepresentation Formulas*, La Matematica **4** (2025), no. 2, 293–309. Special Volume: Analysis & Applications of PDEs in Honor of David R. Adams.
- [36] Richard A. Hunt, *On $L(p, q)$ spaces*, Enseign. Math. (2) **12** (1966), 249–276.
- [37] Ritva Hurri-Syrjänen, Javier C. Martínez-Perales, Carlos Pérez, and Antti V. Vähäkangas, *On the weighted inequality between the Gagliardo and Sobolev seminorms*, Israel J. Math. **263** (2024), 1–10, DOI 10.1007/s11856-025-2834-y.
- [38] Juha Kinnunen, Juha Lehrbäck, and Antti Vähäkangas, *Maximal function methods for Sobolev spaces*, Mathematical Surveys and Monographs, vol. 257, American Mathematical Society, Providence, RI, 2021.
- [39] Yinqin Li, Dachun Yang, Wen Yuan, Yangyang Zhang, and Yirui Zhao, *Sharp Weighted Cohen–Dahmen–Daubechies–DeVore Inequality with Applications to (Weighted) Critical Sobolev Spaces, Gagliardo–Nirenberg Inequalities, and Muckenhoupt Weights*, Sci. China Math. **69** (2026).
- [40] Emiel Lorist and Carol Wagenaar, *The two weight fractional Poincaré–Sobolev sandwich*. arXiv:2604.0841.
- [41] Kim Myrskyläinen, Carlos Pérez, and Julian Weigt, *Weighted fractional Poincaré inequalities via isoperimetric inequalities*, Calc. Var. Partial Differential Equations **63** (2024), no. 8, Paper No. 205, 32.
- [42] Hoai-Minh Nguyen, *Some new characterizations of Sobolev spaces*, J. Funct. Anal. **237** (2006), no. 2, 689–720.
- [43] ———, *Further characterizations of Sobolev spaces*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 1, 191–229.
- [44] ———, *Characterizations of the Sobolev norms and the total variation via nonlocal functionals, and related problems*, C. R. Math. Acad. Sci. Paris **363** (2025), 1429–1455.
- [45] Daniel Spector, *An optimal Sobolev embedding for L^1* , J. Funct. Anal. **279** (2020), no. 3, 108559, 26.
- [46] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, vol. No. 30, Princeton University Press, Princeton, NJ, 1970.

- [47] ———, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993.
- [48] E. M. Stein and N. J. Weiss, *On the convergence of Poisson integrals*, Trans. Amer. Math. Soc. **140** (1969), 35–54.
- [49] Hans Triebel, *Limits of Besov norms*, Arch. Math. (Basel) **96** (2011), no. 2, 169–175.
- [50] Chenfeng Zhu, Dachun Yang, and Wen Yuan, *Generalized Brezis–Seeger–Van Schaftingen–Yung formulae and their applications in ball Banach Sobolev spaces*, Calc. Var. Partial Differential Equations **62** (2023), no. 8, Paper No. 234, 76.
- [51] ———, *Brezis–Seeger–Van Schaftingen–Yung-type characterization of homogeneous ball Banach Sobolev spaces and its applications*, Commun. Contemp. Math. **26** (2024), no. 8, Paper No. 2350041, 48.

C. PÉREZ, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE BASQUE COUNTRY, IKERBASQUE (BASQUE FOUNDATION FOR SCIENCE) AND BCAM – BASQUE CENTER FOR APPLIED MATHEMATICS, BILBAO, SPAIN

Email address: `cperez@bcamath.org`

A. SEEGER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, 480 LINCOLN DRIVE, MADISON, WI, 53706, USA

Email address: `seeger@math.wisc.edu`

B. STREET, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, 480 LINCOLN DRIVE, MADISON, WI, 53706, USA

Email address: `street@math.wisc.edu`

P.-L. YUNG, MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA ACT 2601, AUSTRALIA, CNRS-ANU INTERNATIONAL RESEARCH LABORATORY FAMSI, (FRANCE AUSTRALIA MATHEMATICAL SCIENCES AND INTERACTIONS), AND DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG

Email address: `PoLam.Yung@anu.edu.au`

Email address: `plyung@math.cuhk.edu.hk`