

MOLLIFIED DISC MULTIPLIERS

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ABSTRACT. We prove a sharp boundedness result on $L^4(\mathbb{R}^2)$ for mollifications of the disc multiplier.

1. INTRODUCTION

Define the Fourier transform of a function $f \in L^1(\mathbb{R}^2)$ by $\mathcal{F}f(\xi) \equiv \widehat{f}(\xi) = \int f(y)e^{-i\langle y, \xi \rangle} dy$. For $1 \leq p \leq \infty$ let $M_p \equiv M_p(\mathbb{R}^2)$ be the usual Fourier multiplier space with norm

$$\|m\|_{M_p} = \sup_{\substack{f \in \mathcal{S} \\ \|f\|_p \leq 1}} \|\mathcal{F}^{-1}[m\widehat{f}]\|_p.$$

We often use the duality $\|m\|_{M_p} = \|m\|_{M_{p'}}$.

Consider the characteristic function $\mathbb{1}_D$ of the disc $D = \{\xi \in \mathbb{R}^2 : |\xi| \leq 1\}$. A famous result by C. Fefferman [14] says that $\mathbb{1}_D \in M_p$ if and only if $p = 2$. Therefore, by an application of the uniform boundedness principle and transference results ([27, Ch. VII]) one can show that L^p -convergence for the spherical summation of Fourier series in two dimensions fails for some $f \in L^p(\mathbb{T}^2)$ if $p \neq 2$. Naturally one is then led to consider regularizations of the disc multiplier $\mathbb{1}_D$.

We consider two such regularizations and ask for their precise behavior in M_p for $p \neq 2$. First, fix $\eta \in L^\infty(\mathbb{R})$ with support in $[-1/2, 1/2]$ and $\int \eta = 1$, and let $\delta \ll 1/2$ (see Section 3.1 for a further relaxation of the assumptions on η). We then set $\eta_\delta(u) = \delta^{-1}\eta(\delta^{-1}u)$ and $\omega_\delta = \mathbb{1}_{[-1,1]} * \eta_\delta$ and consider the radial extension of ω_δ as a multiplier in $\mathbb{R}^2$, thus producing the *mollified disc multipliers*

$$(1.1) \quad h_\delta(\xi) = \omega_\delta(|\xi|).$$

Note that $h_\delta(\xi) = 1$ for $|\xi| \leq 1 - \delta$ and $h_\delta(\xi) = 0$ for $|\xi| \geq 1 + \delta$, moreover h_δ is Lipschitz continuous with Lipschitz constant $\lesssim \delta^{-1}\|\eta\|_\infty$. One may want to choose for η a smooth function with compact support, but there are also interesting rougher choices such as $\eta = \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}$, in which case we get a

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trapezoidal (de la Vallée-Poussin type) profile with

$$h_\delta(\xi) = \begin{cases} 1 & \text{if } |\xi| < 1 - \frac{\delta}{2}, \\ -\frac{|\xi|}{\delta} + \frac{2+\delta}{2\delta} & \text{if } 1 - \frac{\delta}{2} \leq |\xi| \leq 1 + \frac{\delta}{2}, \\ 0 & \text{if } |\xi| \geq 1 + \frac{\delta}{2}. \end{cases}$$

Secondly, fix $\alpha \in (0, \frac{1}{2})$ and consider the well-studied *Bochner–Riesz multiplier* at index α ,

$$(1.2) \quad m_\alpha(\xi) = (1 - |\xi|^2)_+^\alpha.$$

Sharp results on both multipliers are known for $1 \leq p < 4/3$ and its dual range $p > 4$. We have $\|h_\delta\|_{M_p(\mathbb{R}^2)} \approx_p \delta^{-\alpha_p}$ with $\alpha_p = 2|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$ (at least for smooth η) and $m_\alpha \in M_p$ for $\alpha > \alpha_p$ (see [13], [8], [15], [12]). Regarding the $\alpha = \alpha_p$ endpoint for $1 \leq p < 4/3$ sharp weak type (p, p) estimates for m_{α_p} , and their strengthening via essentially optimal sparse domination results are known, and also optimal M_p results for multipliers such as $(1 - |\xi|^2)_+^{\alpha_p} |\log \frac{1}{|1 - |\xi|^2|}|^\beta$, with $\beta > \max\{\frac{1}{p}, 1 - \frac{1}{p}\}$. See [11, 10], [24, 25], [28], [3, 2]. For historical context see also [6, §9.3].

Much less is known about the behavior of the M_p -norms of h_δ and m_α in the range $4/3 \leq p \leq 4$ as $\delta \rightarrow 0+$, $\alpha \rightarrow 0+$. Some preliminary results can be obtained from a result of Córdoba [12] which we now describe. Let $\chi_0 \in C_c^\infty(\mathbb{R})$ be a nonnegative bump function supported in $(-1, 1)$. Then [12] provides the upper bound in

$$(1.3) \quad \|\chi_0(2^j(1 - |\cdot|^2))\|_{M_p(\mathbb{R}^2)} \approx j^{|\frac{1}{p} - \frac{1}{2}|}.$$

The lower bound in (1.3) also holds, by a variant of Fefferman’s argument for the disc multiplier [14] combined with sharp Minkowski dimension type lower bounds for the Besicovitch set by Keich [20]. A detailed proof of this lower bound, for nonnegative nonzero χ_0 can be found for example in [1, Lemma 2.3], see also [23, 16].

Assuming in addition $\text{supp}(\chi) \subset (\frac{1}{4}, 1)$ and summing estimates (1.3), with suitable choices of χ , yields for $4/3 \leq p \leq 4$ the upper bound in

$$(1.4) \quad (\log \frac{1}{\delta})^{|\frac{1}{p} - \frac{1}{2}|} \lesssim \|h_\delta\|_{M_p} \lesssim (\log \frac{1}{\delta})^{5|\frac{1}{p} - \frac{1}{2}|}.$$

Likewise we get $\|m_\alpha\|_{M_p} \lesssim \alpha^{-5|\frac{1}{p} - \frac{1}{2}|}$. For the lower bound in (1.4) one observes that $u_\delta(\xi) := h_\delta(\xi) - h_\delta((1 + \delta)\xi)$ is a multiplier of the form used in (1.3), and by the dilation invariance of the multiplier norms, $\|u_\delta\|_{M_p} \leq 2\|h_\delta\|_{M_p}$. For the cases $p = 4/3$ and $p = 4$, an asymptotic analysis yields that the convolution kernel $\mathcal{F}^{-1}[h_\delta]$ has $L^{4/3}$ norm $\approx (\log \frac{1}{\delta})^{3/4}$. Testing the operator on suitable Schwartz functions then leads to $\|m\|_{M_4} = \|m\|_{M_{4/3}} \gtrsim (\log \frac{1}{\delta})^{3/4}$. Our main result gives further improvements and completely closes the gap in (1.4) for the limiting cases $p = 4/3$ and $p = 4$.

Theorem 1.1. *There is a constant C such that for $0 < \delta \leq \frac{1}{2}$,*

$$C^{-1} \log \frac{1}{\delta} \leq \|h_\delta\|_{M_4(\mathbb{R}^2)} \leq C \log \frac{1}{\delta}.$$

The upper bounds in Theorem 1.1 follow from a more general result on sums of multipliers featured in (1.3), where we may also impose less restrictive regularity assumptions on the χ_j to obtain the rougher versions of Theorem 1.1. In what follows we denote by $L_a^4(\mathbb{R})$ the standard L^4 -based Sobolev (i.e. Bessel-potential) space on the real line.

Theorem 1.2. *Let $a > 1/2$ and let $\mathcal{U}$ be a bounded set of $L_a^4(\mathbb{R})$ functions supported in $(\frac{1}{8}, \frac{1}{2})$. Define the convolution operator T_j acting on $\mathcal{S}(\mathbb{R}^2)$ by*

$$\widehat{T_j f}(\xi) = \omega_j(2^j(1 - |\xi|))\widehat{f}(\xi)$$

with $\omega_j \in \mathcal{U}$. Then there is a constant C (depending only on $\mathcal{U}$) such that for $n \geq 1$, $N \geq 1$,

$$(1.5) \quad \left\| \sum_{n < j \leq n+N} T_j \right\|_{L^4 \rightarrow L^4} \leq C(n+N)^{1/4} N^{3/4}.$$

For the application of Theorem 1.2 to Theorem 1.1 we will need to use the continuous embedding $\text{Lip}(1) \hookrightarrow L_{\varepsilon+1/2}^4$ for $\varepsilon < 1/4$. When $N = 1$ in Theorem 1.2 we recover (1.3), but for larger N we get an improvement over just using the triangle inequality. Note that the Fourier transform $\widehat{T_j f}$ is supported in the intersection of a sector with the thin annulus $A_j = \{\xi : 1 - 2^{-j-3} \leq |\xi| \leq 1 - 2^{-j-1}\}$. A key idea is to somehow exploit the essential orthogonality of the operators T_j (or related versions of it) in L^2 , but it is not *a priori* obvious how to use it for the L^p inequalities.

Remark. Note an easy corresponding version for multipliers on the real line, namely $\sum_{j>2} \omega_j(2^j(1 - |\xi|))$ belongs to $M_p(\mathbb{R})$, for $1 < p < \infty$. We consider $m_\pm(\xi) = \sum_{j>2} \omega_j(2^j(1 \pm \xi))$ and observe that the multipliers $m_\pm(\xi \mp 1)$ satisfy the assumptions of the Marcinkiewicz multiplier theorem.

Theorem 1.2 implies the upper bounds in Theorem 1.1 and also better bounds for $\|m_\alpha\|_{M_4}$. After an interpolation with L^2 estimates we get the following improvement over (1.4). The argument for it is sketched in Section 3.

Corollary 1.3. *Let $0 < \delta, \alpha < \frac{1}{2}$ and $\frac{4}{3} \leq p \leq 4$. Then the following hold.*

(i)

$$\|h_\delta\|_{M_p(\mathbb{R}^2)} \lesssim (\log \frac{1}{\delta})^{4|\frac{1}{p} - \frac{1}{2}|}, \quad \|m_\alpha\|_{M_p(\mathbb{R}^2)} \lesssim \alpha^{-4|\frac{1}{p} - \frac{1}{2}|}.$$

(ii) *Define the logarithmic Bochner–Riesz multipliers by*

$$\gamma_\beta(\xi) = \begin{cases} |\log(1 - |\xi|^2)|^{-\beta} & \text{for } |\xi| < 1 \\ 0 & \text{for } |\xi| \geq 1. \end{cases}$$

Then $\gamma_\beta \in M_p$ for $\beta > 4|\frac{1}{p} - \frac{1}{2}|$.

Remark. The examples showing the lower bounds for the dual $L^{4/3}$ estimate in Theorem 1.1 are radial. Thus if $\mathcal{T}_\delta$ denotes the convolution operator with multiplier h_δ then

$$(1.6) \quad \|\mathcal{T}_\delta\|_{L^{4/3} \rightarrow L^{4/3}} \approx \|\mathcal{T}_\delta\|_{L_{\text{rad}}^{4/3} \rightarrow L^{4/3}}$$

and a similar statement for $p = 4$. This mirrors the equivalence of the $L^p \rightarrow L^p$ and the $L_{\text{rad}}^p \rightarrow L_{\text{rad}}^p$ operator norms in the previously mentioned results for the range $1 \leq p < 4/3$. However, in that range both operator norms are equivalent with $\|K_\delta\|_p$, the L^p -norm of the convolution kernel. The latter equivalence breaks down for the endpoint $p = 4/3$, indeed, as noted above, $\|K_\delta\|_{4/3} \approx (\log \frac{1}{\delta})^{3/4}$ but $\|\mathcal{T}_\delta\|_{L^p \rightarrow L^p} \approx \log \frac{1}{\delta}$. The analogue of the equivalence (1.6) for $4/3 < p < 4$ also breaks down, as does the analogous statement for the restricted weak-type category for $p = 4/3$ or $p = 4$. This follows by the positive results of Herz [19] and Chanillo [9] for radial functions and the negative results for general functions based on the Kakeya-type examples ([14], [1]).

Notation. Above and throughout this paper we write, for nonnegative quantities a, b , $a \lesssim b$ or $a \lesssim_L b$ to indicate $a \leq Cb$ for some constant C which may depend on some list L . We write $a \approx b$ to indicate that both $a \lesssim b$ and $b \lesssim a$ hold.

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2. LOWER BOUNDS FOR MOLLIFIED BALL MULTIPLIERS

Here we prove the lower bounds in Theorem 1.1. The arguments here work equally well in higher dimensions d for δ -regularizations of the ball multiplier and give the lower bound

$$(2.1) \quad \|h_\delta\|_{M_{p_d}} \geq c \log \frac{1}{\delta}, \quad p_d = \frac{2d}{d-1}$$

for some positive constant c and sufficiently small $\delta > 0$. We will derive such lower bounds by approximating the ball multiplier with its mollifications. Specifically we rely on a paper by Kenig and Tomas [21] who proved the failure of the weak type (p_d, p_d) property for the ball multiplier operator acting on radial functions.

We review a few formulas for radial multiplier transformations acting on radial functions and apply them to the ball multiplier and its mollification. Recall that for a radial function $f(x) = g(|x|)$ the Fourier transform is given by $\widehat{f}(\xi) = (2\pi)^{d/2} \mathcal{B}_g(|\xi|)$ where

$$\mathcal{B}_d g(\rho) = \int g(s) B_d(s\rho) \rho^{d-1} d\rho, \quad \text{with } B_d(\varrho) = \varrho^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(\varrho).$$

Here we adopt convenient notation in [17], noting that B_d corresponds up to a constant to the radial profile of the Fourier transform of surface measure on the unit sphere. The Fourier inverse is then given by $\mathcal{F}^{-1}f(\xi) = (2\pi)^{-d/2}\mathcal{B}_dg(|\xi|)$. Hence, for a radial multiplier $\omega(|\cdot|)$ we then have

$$(2.2a) \quad \mathcal{F}^{-1}[\omega(|\cdot|)\widehat{f}](x) = \int \mathcal{K}_\omega(r, s)s^{d-1}g(s)ds, \quad \text{with } r = |x|,$$

where

$$(2.2b) \quad \mathcal{K}_\omega(r, s) = \int_0^\infty \omega(\rho)B_d(r\rho)B_d(s\rho)\rho^{d-1}d\rho.$$

For the ball multiplier we use this for $\omega = \mathbb{1}_{[0,1]}$ or, equivalently for $\omega_0(\rho) = \mathbb{1}_{[-1,1]}$. Modifying a definition in [21] we set

$$(2.3) \quad g_\delta(s) = \begin{cases} s^{-\frac{d+1}{2}} \cos(s - \frac{d-1}{4}\pi), & \text{if } \delta^{-1/4} \leq s \leq \delta^{-1/2}, \\ 0 & \text{otherwise.} \end{cases}$$

and $f_\delta(y) = g_\delta(|y|)$. Note that

$$(2.4) \quad \|f_\delta\|_{p_d} \lesssim (\log \frac{1}{\delta})^{\frac{1}{p_d}}.$$

We will rely on a lower bound for the ball multiplier operator acting on the f_δ when evaluated on

$$(2.5) \quad \mathfrak{A}(\delta) = \{x : 2\delta^{-1/2} \leq |x| \leq \delta^{-1}\}.$$

Lemma 2.1. *Assume $d \geq 2$, $p_d = \frac{2d}{d+1}$. Then there exists $c > 0$ and $\delta_\circ > 0$ such that for all $0 < \delta \leq \delta_\circ$*

$$(2.6) \quad \left(\int_{\mathfrak{A}(\delta)} |\mathcal{F}^{-1}[\mathbb{1}_B \widehat{f}_\delta](x)|^{p_d} dx \right)^{1/p_d} \geq c(\log \frac{1}{\delta})^{1+\frac{1}{p_d}}.$$

Proof. We have to consider $\mathcal{K}_{\omega_0}(r, s)$ for $r \geq 2s$. This is (essentially) in [21]. From [31, §3.21 (5),(6)] we have

$$\frac{d}{dz}(z^d B_{d+2}(z)) = B_d(z)z^{d-1}, \quad \frac{d}{dz}[B_d(z)] = -zB_{d+2}(z)$$

and, following [21] an integration by parts yields

$$\begin{aligned} \mathcal{K}_{\omega_0}(r, s) &= s^{-d} \int_0^1 B_d(r\rho) \frac{d}{d\rho} [(s\rho)^d B_{d+2}(s\rho)] d\rho \\ &= s^{-d} B_d(r)s^d B_{d+2}(s) - s^{-d} \int_0^1 \frac{d}{d\rho} [B_d(r\rho)] (s\rho)^d B_{d+2}(s\rho) d\rho \\ &= B_d(r)B_{d+2}(s) + r^2 a(r, s) \end{aligned}$$

where $a(r, s) = \int_0^1 B_{d+2}(r\rho)B_{d+2}(s\rho)\rho^{d+1}d\rho$. This yields

$$\frac{\mathcal{K}_{\omega_0}(r, s) - B_d(r)B_{d+2}(s)}{r^2} = a(r, s) = a(s, r) = \frac{\mathcal{K}_{\omega_0}(s, r) - B_d(s)B_{d+2}(r)}{s^2}$$

and since $\mathcal{K}_\omega(r, s) = \mathcal{K}_\omega(s, r)$ for any ω we obtain

$$(2.7) \quad \mathcal{K}_{\omega_0}(r, s) = \frac{r^2}{r^2 - s^2} B_{d+2}(r) B_d(s) - \frac{s^2}{r^2 - s^2} B_d(r) B_{d+2}(s).$$

Hence, for $x \in \mathfrak{A}(\delta)$,

$$\mathcal{F}^{-1}[\mathbb{1}_B \widehat{f_\delta}](x) = \mathcal{T}_0 g_\delta(x) - \mathcal{T}_1 g_\delta(x)$$

where

$$\begin{aligned} \mathcal{T}_0 g_\delta(x) &= B_{d+2}(|x|) \int_0^\infty \frac{|x|^2}{|x|^2 - s^2} B_d(s) g_\delta(s) s^{d-1} ds, \\ \mathcal{T}_1 g_\delta(x) &= B_d(|x|) \int_0^\infty \frac{s^2}{|x|^2 - s^2} B_{d+2}(s) g_\delta(s) s^{d-1} ds. \end{aligned}$$

Observe that no singularities occur in these integrals since we assume $|x| \geq 2\delta^{-1/2}$ and g_δ is supported in $[\delta^{-1/4}, \delta^{-1/2}]$.

Using asymptotics for Bessel functions [27, ch. IV, Lemma 3.11] we have for $s \geq 1$

$$B_d(s) = \sqrt{\frac{2}{\pi}} \frac{\cos(s - \frac{d-1}{4}\pi)}{s^{(d-1)/2}} + O(s^{-(d+1)/2}).$$

From the definition of g_δ , $|\mathcal{T}_0 g_\delta(x)|$ is bounded below by

$$\left(\frac{2}{\pi}\right)^{1/2} |B_{d+2}(|x|)| \left(\int_{\delta^{-1/4}}^{\delta^{-1/2}} \frac{|x|^2}{|x|^2 - s^2} \cos^2(x - \frac{d-1}{4}\pi) \frac{ds}{s} - C \int_{\delta^{-1/4}}^{\delta^{-1/2}} s^{-2} ds \right)$$

and thus $|\mathcal{T}_0 g_\delta(x)| \gtrsim |B_{d+2}(x)| \log \frac{1}{\delta}$ for $x \in \mathfrak{A}(\delta)$. By the asymptotics for $B_{d+2}(r)$ we have

$$B_{d+2}(r) = \sqrt{\frac{2}{\pi}} \frac{\cos(r - \frac{d+1}{4}\pi)}{s^{(d+1)/2}} + O(s^{-(d+3)/2})$$

which leads to

$$(2.8) \quad \left(\int_{\mathfrak{A}(\delta)} |\mathcal{T}_0 g_\delta(x)|^{p_d} dx \right)^{\frac{1}{p_d}} \geq c(\log \frac{1}{\delta})^{1 + \frac{1}{p_d}}.$$

For $\mathcal{T}_1 g_\delta(x)$ we derive instead an upper bound. Using $B_d(r) = O(r^{-\frac{d-1}{2}})$, $B_{d+2}(s) = O(|s|^{-\frac{d+1}{2}})$ we get

$$|\mathcal{T}_1 g_\delta(x)| \lesssim \delta^{-1/2} |x|^{-\frac{d+3}{2}}$$

and a straightforward calculation yields

$$(2.9) \quad \left(\int_{\mathfrak{A}_\delta} |\mathcal{T}_1 g_\delta(x)|^{p_d} dx \right)^{\frac{1}{p_d}} \leq C,$$

uniformly in δ . Thus for small δ , (2.6) follows from (2.8) and (2.9). $\square$

Lower bounds for the mollified ball multipliers. For the lower bound $\approx \log \frac{1}{\delta}$ we may assume, by continuity considerations, that $\delta > 0$ is small. We do not need a strong assumption on η here and just assume that $\eta \in L^1$ supported on $[-\frac{1}{2}, \frac{1}{2}]$. By (2.4) the lower bound follows from $\|\mathcal{F}^{-1}[h_\delta f]\|_{p_d} \gtrsim (\log \frac{1}{\delta})^{1+1/p_d}$ and by Lemma 2.1 and the triangle inequality this is a consequence of

Lemma 2.2. *Assume $d \geq 2$, $p_d = \frac{2d}{d+1}$. Then there exists $c > 0$ and $\delta_o > 0$ such that for all $0 < \delta \leq 1/2$*

$$(2.10) \quad \left(\int_{\mathfrak{A}(\delta)} |\mathcal{F}^{-1}[(h_\delta - \mathbb{1}_B)\widehat{f}_\delta](x)|^{p_d} dx \right)^{1/p_d} \lesssim \log \frac{1}{\delta}$$

Proof. We use (2.2) for $h_\delta = \omega_\delta(|\cdot|)$ with

$$\omega_\delta(\rho) = \int \mathbb{1}_{[-1,1]}(\rho - u) \eta_\delta(u) du.$$

Note that

$$\mathbb{1}_{[-1,1]}(\rho - u) = \mathbb{1}_{[0,1+u]}(\rho) \text{ if } \rho > 0, |u| \leq 1$$

and we compute

$$\begin{aligned} \mathcal{K}_{\omega_\delta}(r, s) &= \int \eta_\delta(u) \int_0^{1+u} B_d(r\rho) B_d(s\rho) \rho^{d-1} d\rho du \\ &= \int \eta_\delta(u) \int_0^1 B_d(r(1+u)\varrho) B_d(s(1+u)\varrho) (1+u)^d \varrho^{d-1} d\varrho du, \end{aligned}$$

and thus

$$(2.11) \quad \mathcal{K}_{\omega_\delta}(r, s) = \int \eta_\delta(u) (1+u)^d \mathcal{K}_{\omega_0}(r(1+u), s(1+u)) du.$$

Since $\int \eta_\delta(u) du = 1$ we can write

$$\begin{aligned} &\mathcal{F}^{-1}[(h_\delta - \mathbb{1}_B)\widehat{f}_\delta](x) \\ &= \int \eta_\delta(u) \int \left(\mathcal{K}_{\omega_\delta}(|x|, s) - \mathcal{K}_{\omega_0}(|x|, s) \right) g_\delta(s) s^{d-1} ds du \\ &= \int u \eta_\delta(u) \int \frac{d}{dv} \left[(1+v)^d \mathcal{K}_{\omega_0}(|x|(1+v), s(1+v)) \right]_{v=1+\sigma u} d\sigma g_\delta(s) s^{d-1} ds du \end{aligned}$$

where we have used (2.11) and the fundamental theorem of calculus. Using ∂_1, ∂_2 as notation for derivatives with respect to the first and second entry of $\mathcal{K}_{h_0}(r, s)$ we have

$$\begin{aligned} E(v, r, s) &:= \frac{d}{dv} \left[(1+v)^d \mathcal{K}_{h_0}(r(1+v), s(1+v)) \right] \\ &= d(1+v)^{d-1} \mathcal{K}_{h_0} + r \partial_1 \mathcal{K}_{h_0} + s \partial_2 \mathcal{K}_{h_0} \Big|_{(r(1+v), s(1+v))}. \end{aligned}$$

We use (2.7) to compute these expressions and then use

$$|B_{d+2}(r)| + |B'_{d+2}(r)| \lesssim r^{-\frac{d+1}{2}}, \quad |B_d(r)| + |B'_d(r)| \lesssim r^{-\frac{d-1}{2}},$$

and similar estimates for the same expressions evaluated at s . The outcome is that

$$|E(v, r, s)| \lesssim r^{-\frac{d-1}{2}} s^{-\frac{d-1}{2}} \text{ provided that } 2s \leq r, |v| \leq \delta.$$

From $|g_\delta(s)s^{d-1}| \leq s^{\frac{d-3}{2}}$ we get

$$\begin{aligned} |\mathcal{F}^{-1}[(h_\delta - \mathbb{1}_B)\widehat{f}_\delta](x)| \\ \lesssim \int_{|u| \leq \delta} |u| |\eta_\delta(u)| |x|^{-\frac{d-1}{2}} \int_{\delta^{-1/4}}^{\delta^{-1/2}} \frac{ds}{s} du \lesssim |x|^{-\frac{d-1}{2}} \delta \log \frac{1}{\delta}. \end{aligned}$$

Since $(\int_{\mathfrak{A}(\delta)} |x|^{-\frac{d-1}{2}p_d} dx)^{1/p_d} = O(\delta^{-1})$ we obtain (2.10). $\square$

3. UPPER BOUNDS

3.1. The applications of Theorem 1.2. We show how Theorem 1.2 implies the asserted upper bounds in M_4 for the mollified disc multiplier and the two kinds of Bochner–Riesz multipliers.

We first consider $h_\delta(\xi) = \omega_\delta(|\xi|)$ with $\omega_\delta = \mathbb{1}_{[-1,1]} * \delta^{-1}\eta(\delta^{-1}\cdot)$, $\text{supp}(\eta) \subset [-\frac{1}{2}, \frac{1}{2}]$ and $\delta < 1/4$. We show the upper bound $\|h_\delta\|_{M_4} \lesssim \log \delta^{-1}$ under the less restrictive assumption $\eta \in L^r$ for $r > 4/3$. For the application of Theorem 1.2 it is convenient to scale and consider the multiplier $\tilde{h}_\delta(\xi) = \omega_\delta((1+2\delta)|\xi|)$ which is supported in $\{\xi : |\xi| \leq \frac{1+\delta}{1+2\delta}\}$ and has the same M_4 norm. We can write

$$(3.1) \quad \mathcal{F}^{-1}[\tilde{h}_\delta \widehat{f}] = P_0 f + \sum_{j \geq 1} T_j f$$

where P_0 is the operator of convolution with a Schwartz function and

$$\widehat{T_j f} = \chi_1(2^j(1 - |\xi|)) \tilde{h}_\delta(\xi) \widehat{f}(\xi)$$

with $\chi_1 \in C_c^\infty$, $\text{supp}(\chi_1) \in [\frac{1}{8}, \frac{1}{2}]$. Now

$$\begin{aligned} \chi_1(2^j(1 - |\xi|)) \tilde{h}_\delta(\xi) &= \chi_1(2^j(1 - |\xi|)) \int_{-1}^1 \delta^{-1} \eta(\delta^{-1}((1+2\delta)|\xi| - u)) du \\ &= \omega_j(2^j(1 - |\xi|)) \end{aligned}$$

where (with the change of variable $\sigma = 2^j(1 - \rho)$)

$$(3.2) \quad \omega_j(\sigma) = \chi_1(\sigma) \int_{-1}^1 \delta^{-1} \eta(\delta^{-1}(1+2\delta)(1 - 2^{-j}\sigma) - \delta^{-1}u) du.$$

Note that the integral in (3.2) is equal to 0 for $\frac{1+\delta}{1+2\delta} \leq 1 - 2^{-j-1}$, i.e. for $2^j > \delta^{-1}$. The integral is equal to 1 when $1 - 2^{-j}\sigma < \frac{1-\delta}{1+2\delta}$. Thus $\omega_j(2^j(1 - |\xi|)) = \chi_1(2^j(1 - |\xi|))$ if $1 - 2^{-j-3} \leq \frac{1-\delta}{1+2\delta}$ which holds when $2^j \leq$

$(24\delta)^{-1}$. It remains to determine the regularity of ω_j for $(24\delta)^{-1} \leq 2^j \leq \delta^{-1}$. Note that the integral in (3.2) is bounded and we get for the weak derivative

$$\frac{d}{d\sigma}\omega_j(\sigma) = \chi_j(\sigma)\eta((1+2\delta)^{\frac{1-2^{-j}\sigma}{\delta}} - \frac{1}{\delta})^{\frac{1+2\delta}{2j\delta}} + \chi_j'(\sigma)e_j(\sigma)$$

where e_j is uniformly bounded. Since now $2^j\delta \approx 1$ we have $\|\omega_j\|_{L_1^r} \lesssim \|\eta\|_r$. By the Sobolev embedding theorem, $L_1^r \hookrightarrow L_a^4$ for $a = 1 - \frac{1}{r} + \frac{1}{4}$ and since $r > 4/3$ we see that $a > 1/2$. Consequently we may apply Theorem 1.2 to the sum $\sum_{j=1}^N T_j$ with $n = 1$, $N = 1 + \lfloor \log \frac{2}{\delta} \rfloor$ and obtain the upper bounds in Theorem 1.1

For the Bochner–Riesz multipliers we make different choices of $\chi_j \in \mathcal{U}$ and write

$$\mathcal{F}^{-1}[m_\alpha \widehat{f}] = \widetilde{P}_0 f + \sum_{k=0}^{\infty} 2^{-k} \sum_{\frac{k}{\alpha} < j \leq \frac{(k+1)}{\alpha}} T_j f$$

where we now apply Theorem 1.2 with $N \sim \lfloor \alpha^{-1} \rfloor$, $n = \lfloor k\alpha^{-1} \rfloor$ and then sum in k , obtaining $\|m_\alpha\|_{M_4} \lesssim \sum_{k=0}^{\infty} 2^{-k} (\frac{k+1}{\alpha})^{1/4} (\frac{1}{\alpha})^{3/4} \sim \alpha^{-1}$. All other estimates in part (i) of Corollary 1.3 follow by interpolation with L^2 estimates and duality.

Finally, for the logarithmic Bochner–Riesz operators in part (ii) of Corollary 1.3 we decompose

$$\mathcal{F}^{-1}[\gamma_\beta \widehat{f}] = \widetilde{P}_0 f + \sum_{k=1}^{\infty} 2^{-k\beta} \sum_{2^{k+1} \leq j \leq 2^{k+1}} T_j f$$

for suitable choices of η_j in a bounded subset of C_c^∞ supported in $(\frac{1}{8}, 1)$. We then use Theorem 1.2 for the k -blocks $\sum_{2^{k+1} \leq j \leq 2^{k+1}} T_j$, with $n = 2^k = N$, and obtain the $L^4 \rightarrow L^4$ operator norm $O(2^k)$. By interpolation the $L^p \rightarrow L^p$ operator norm of these k -blocks is $O(2^{4k|1/p-1/2|})$ and since $\beta > 4|1/p-1/2|$ we can sum in k .

3.2. Proof of Theorem 1.2. We use a variant of arguments in [25] (and [7] on a related problem on weighted norm inequalities for Bochner–Riesz square functions). Observe that $\widehat{T_j f}$ is supported in the annulus

$$A_j = \{\xi : 1 - 2^{-j-3} \leq |\xi| \leq 1 - 2^{-j-1}\}.$$

The key is to use the essential disjointness of the annuli A_j so that some weak orthogonality property can be exploited. The orthogonality is used to prove an inequality for vector-valued operators:

Proposition 3.1. *Let $\mathcal{U}$, T_j be as in Theorem 1.2. Then for $f \in L^4(\mathbb{R}^2)$, $N \geq 2$,*

$$(3.3) \quad \left(\sum_{j=1}^N \|T_j f\|_4^4 \right)^{1/4} \lesssim N^{1/4} \|f\|_4.$$

Theorem 1.2 follows by crudely applying Hölder's inequality for the j -sum and interchanging sum. This gives

$$(3.4) \quad \begin{aligned} \left\| \sum_{n < j \leq n+N} T_j f \right\|_4 &\lesssim N^{3/4} \left(\sum_{n < j \leq n+N} \|T_j f\|_4^4 \right)^{1/4} \\ &\lesssim N^{3/4} (N+n)^{1/4} \|f\|_4 \end{aligned}$$

where in (3.4) we have applied Proposition 3.1 with N replaced with $N+n$.

3.2.1. Proof of Proposition 3.1 . Let $\chi_\circ$ be a C^∞ function supported in a ball of radius $\frac{1}{10}$ centered at $(0, 1)$. By a localization and rotation argument we may replace the operators T_j in the formulation of the theorem with $\mathcal{T}_j$ defined by

$$(3.5) \quad \mathcal{T}_j f(x) = \mathcal{B}_j[\omega_j, f](x) \text{ where } \widehat{\mathcal{B}_j[\omega, f]}(\xi) = \chi_\circ(\xi) \omega(2^j(1 - |\xi|)) \widehat{f}(\xi),$$

with $\|\omega_j\|_{L^4_{\varepsilon+1/2}} \lesssim 1$. Below it will be useful to view $\mathcal{T}_j$ as an bilinear operator acting on ω_j and f as in (3.5).

Observe that $\widehat{\mathcal{T}_j f}$ is supported in the intersection of a narrow sector with the annulus A_j .

Let $\vartheta \in C_c^\infty(\mathbb{R})$ satisfy $\sum_\nu \vartheta(s - \nu) = 1$ and $\text{supp}(\vartheta) \subset (-1, 1)$ and define

$$\widehat{T_{j,\nu} f} = \vartheta(2^{j/2} \xi_1 - \nu) \widehat{\mathcal{T}_j f}.$$

By the Fefferman argument in [15] (using the essential disjointness of the sets $\text{supp } \widehat{T_{j,\nu} f} + \text{supp } \widehat{T_{j,\nu'} f}$ due to the curvature of the circle),

$$(3.6) \quad \|\mathcal{T}_j f\|_4^4 = \left\| \sum_{\nu, \nu'} T_{j,\nu} f T_{j,\nu'} f \right\|_2^2 \lesssim \sum_{\nu, \nu'} \|T_{j,\nu} f T_{j,\nu'} f\|_2^2.$$

Let $\Gamma_j^0 = \{(\nu, \nu)\}$ and $\Gamma_j^m = \{(\nu, \nu') : 2^{m-1} \leq |\nu - \nu'| < 2^m\}$, where in this definition $1 \leq 2^{m-1} \leq 2^{\frac{j}{2}+C}$. If $R_{j,\nu}$ is the support of the multiplier for $T_{j,\nu}$ the distance of $R_{j,\nu}$ and $R_{j,\nu'}$ is $\approx 2^{m-\frac{j}{2}}$ provided that $m > C$.

For $(\nu, \nu') \in \Gamma_j^m$ we decompose

$$T_{j,\nu} f = \sum_\mu T_{j,\nu,\mu}^m f$$

where

$$\widehat{T_{j,\nu,\mu}^m f}(\xi) = \vartheta(2^{\frac{j}{2}+m} \xi_1 - \mu) \widehat{T_{j,\nu} f}(\xi).$$

We argue as in [25] and note that for fixed m the sets

$$\text{supp } \widehat{T_{j,\nu,\mu}^m f} + \text{supp } \widehat{T_{j,\nu',\mu'}^m f} \quad \text{with } (\nu, \nu') \in \Gamma_j^m$$

are essentially disjoint. Therefore, again by the Fefferman argument,

$$\sum_{\nu, \nu'} \|T_{j,\nu} f T_{j,\nu'} f\|_2^2 \lesssim \sum_{0 \leq m \leq \frac{j}{2}+C} \sum_{(\nu, \nu') \in \Gamma_j^m} \sum_{\mu, \mu'} \|T_{j,\nu,\mu}^m f T_{j,\nu',\mu'}^m f\|_2^2.$$

Now define, for $\kappa \in [-C2^{\frac{j}{2}-m}, C2^{\frac{j}{2}-m}] \cap \mathbb{Z}$,

$$\mathfrak{V}_{j,m}^\kappa = \{\mu \in \mathbb{Z} : |2^{-\frac{j}{2}-m}\mu - 2^{m-\frac{j}{2}}\kappa| \leq 2^{m-\frac{j}{2}+1}\}.$$

Note that for fixed j, m every μ is contained in $O(1)$ of the sets $\mathfrak{V}_{j,m}^\kappa$ (and contained in at least one of those). Moreover, for fixed j, m, μ , $T_{\nu,\mu}^j$ is nonzero only for $O(1)$ of the ν .

We can thus replace the previous bound by

$$\begin{aligned} & \sum_{\nu, \nu'} \|T_{j,\nu} f T_{j,\nu'} f\|_2^2 \\ & \lesssim \sum_{i \leq C} \sum_{0 \leq m \leq \frac{j}{2} + C} \sum_{-C'2^{m-\frac{j}{2}} \leq \kappa \leq C'2^{m-\frac{j}{2}}} \left\| \left(\sum_{\mu \in \mathfrak{V}_{j,m}^\kappa} \sum_{\nu} |T_{j,\nu,\mu}^m f|^2 \right)^{1/2} \left(\sum_{\mu' \in \mathfrak{V}_{j,m}^{\kappa+i}} \sum_{\nu'} |T_{j,\nu',\mu'}^m f|^2 \right)^{1/2} \right\|_2^2 \\ & \lesssim \sum_{0 \leq m \leq \frac{j}{2} + C} \sum_{-C'2^{m-\frac{j}{2}} \leq \kappa \leq C'2^{m-\frac{j}{2}}} \left\| \left(\sum_{\mu \in \mathfrak{V}_{j,m}^\kappa} \sum_{\nu} |T_{j,\nu,\mu}^m f|^2 \right)^{1/2} \right\|_4^4. \end{aligned}$$

Putting this into (3.6) we get

$$\begin{aligned} & \left(\sum_{j \leq N} \|\mathcal{T}_j f\|_4^4 \right)^{1/4} \lesssim \\ & \left(\sum_{0 \leq m \leq C + \frac{N}{2}} \sum_{2^{m-C} \leq j \leq N} \sum_{-C'2^{m-\frac{j}{2}} \leq \kappa \leq C'2^{m-\frac{j}{2}}} \left\| \left(\sum_{\mu \in \mathfrak{V}_{j,m}^\kappa} |T_{j,\nu,\mu}^m f|^2 \right)^{1/2} \right\|_4^4 \right)^{1/4}. \end{aligned}$$

We will pick up the term $N^{1/4}$ from the m -sum, and so, in order to get (1.5) it remains to prove, for fixed $m \leq C + N/2$,

$$(3.7) \quad \left(\sum_{\substack{j: \\ 2m-C \leq j \leq N}} \sum_{|\kappa| \leq C'2^{-m+\frac{j}{2}}} \left\| \left(\sum_{\mu \in \mathfrak{V}_{j,m}^\kappa} \sum_{\nu} |T_{j,\nu,\mu}^m f|^2 \right)^{1/2} \right\|_4^4 \right)^{1/4} \lesssim \|f\|_4,$$

with the implicit constant independent of m .

Inequality (3.7) follows by an interpolation argument. As indicated before it is now useful to consider the operators $T_{j,\nu,\mu}^m$ act as bilinear operators $\mathcal{B}_{j,\mu,\nu}^m$ acting on f and the ω_j .

Let $\Omega = \{\Omega_j\}_{j=1}^N$ in $\ell^\infty(L_a^r)$ where L_a^r is the usual Sobolev (or Bessel potential) space. Let $\chi_1 \in C^\infty$, $\chi_1 = 1$ on $(\frac{1}{8}, \frac{1}{2})$ and $\text{supp}(\chi_1) \subset (\frac{1}{16}, \frac{3}{4})$, and define $\mathcal{B}_{j,\mu,\nu}^m(f, \Omega)$ by

$$\begin{aligned} \widehat{\mathcal{B}_{j,\mu,\nu}^m(f, \Omega)} &= h_{m,j,\mu,\nu}[\Omega] \widehat{f} \text{ where } h_{m,j,\mu,\nu}[\Omega](\xi) = \\ & \chi_0(\xi) \vartheta(2^{\frac{j}{2}}\xi_1 - \nu) \vartheta(2^{\frac{j}{2}+m}\xi_1 - \mu) \chi_1(2^j(1 - |\xi|)) \Omega_j(2^j(1 - |\xi|)). \end{aligned}$$

Taking Ω_j such that $\Omega_j = \omega_j$ we see that $T_{j,\nu,\mu}^m f = \mathcal{B}_{j,\mu,\nu}^m(f, \Omega)$. Inequality (3.7) is then the special case with $q = 4$ of

$$(3.8) \quad \left(\sum_{2m-C \leq j \leq N} \sum_{|\kappa| \leq C'2^{-m+\frac{j}{2}}} \left\| \left(\sum_{\mu \in \mathfrak{V}_{j,m}^\kappa} \sum_{\nu} |\mathcal{B}_{j,\nu,\mu}^m(f, \Omega)|^2 \right)^{1/2} \right\|_q^q \right)^{1/q} \\ \lesssim \sup_j \|\Omega_j\|_{L_a^r} \|f\|_q, \quad 2 < q \leq \infty, \quad a > 1 - \frac{2}{q}, \quad r = \frac{2q}{q-2}.$$

This inequality expresses the $\ell^\infty(L_a^r) \times L^q \rightarrow \ell^q(L^q(\ell^2))$ boundedness of a bilinear operator which can be interpolated using the complex interpolation method. Here one combines standard results for interpolation of bilinear operators [4, §4.4] and interpolation of Sobolev spaces ([4, §6]). Here for the limiting result $q = 2$ we use L^∞ in place of L_a^r . We use the interpolation in conjunction with a Sobolev embedding theorem. Observe that for $2 < q < \infty$, $\theta = 1 - 2/q$ and $a > 1 - 2/q$ the complex interpolation space $[L^\infty, L_{1+\varepsilon}^2]_\theta$ contains L_a^r with $r = \frac{2q}{q-2}$ provided that $\varepsilon > 0$ is such that $(1 + \varepsilon)(1 - 2/q) < a$.

For $q = 2$ we aim for the $\ell^\infty(L^\infty) \times L^2 \rightarrow \ell^2(L^2(\ell^2))$ boundedness. We thus prove the inequality

$$(3.9) \quad \left(\sum_{2m-C \leq j \leq N} \sum_{|\kappa| \leq C'2^{-m+\frac{j}{2}}} \left\| \left(\sum_{\mu \in \mathfrak{V}_{j,m}^\kappa} \sum_{\nu} |\mathcal{B}_{j,\nu,\mu}^m(f, \Omega)|^2 \right)^{1/2} \right\|_2^2 \right)^{1/2} \\ \lesssim \sup_j \|\Omega_j\|_{L^\infty} \|f\|_2.$$

This holds by almost-orthogonality. Indeed, for fixed m , the non-empty supports $\mathcal{R}_{j,\nu,\mu}^m$ are contained in A_j and have all finite overlap, i.e.

$$\sup_{\xi} \sum_{j,\kappa,\mu,\nu} \mathbb{1}_{\mathcal{R}_{j,\nu,\mu}^m}(\xi) < \infty$$

which implies (3.9).

For $q = \infty$ we prove $\ell^\infty(L_{1+\varepsilon}^2) \times L^\infty \rightarrow \ell^\infty(L^\infty(\ell^2))$ boundedness; i.e. we show the inequality

$$(3.10) \quad \sup_{2m-C \leq j \leq N} \sup_{|\kappa| \leq C'2^{m-\frac{j}{2}}} \left\| \left(\sum_{\mu \in \mathfrak{V}_{j,m}^\kappa} \sum_{\nu} |\mathcal{B}_{j,\nu,\mu}^m(f, \Omega)|^2 \right)^{1/2} \right\|_\infty \\ \lesssim \sup_j \|\Omega_j\|_{L_{1+\varepsilon}^2(\mathbb{R})} \|f\|_\infty.$$

For fixed j, m, κ this becomes essentially an L^∞ estimate for equally-spaced square functions. This type of L^∞ bound was first noticed by Carleson (in unpublished work), see [12] and [22]. For a version close to what we need here see [25].

For fixed j, m, κ the multipliers $h_{m,j,\mu,\nu}$ with $\mu \in \mathfrak{V}_{j,m}^\kappa$ are supported in essentially disjoint rectangles of sidelength $2^{-j/2-m}$ and 2^{-j} , with the

longer side perpendicular to $\mathbf{n}_{j,\nu,\mu}$. The angle between $\mathbf{n}_{j,\nu,\mu}$ and $\mathbf{n}_{j,\mu',\nu'}$ for $\mu, \mu' \in \mathfrak{V}_{j,m}^\kappa$ is $O(2^{-\frac{j}{2}+m})$ which is comparable to the ratio of the length of the shorter side to the length of the longer side. This means that for fixed j, m, κ these rectangles have essentially the same orientation. Each is contained in a rectangle with short side of length $C2^{-j}$ parallel to a unit vector $\mathbf{n}_\kappa$ and a long side of length $C2^{-m-j/2}$ parallel to the orthogonal unit vector $\mathbf{n}_\kappa^\perp$. Moreover $\mathbf{n}_\kappa$ is normal to the circle in the angular region associated with $\mathfrak{V}_{j,m}^\kappa$.

Now let $\Lambda_\kappa \equiv \Lambda_{\kappa,m,j} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation with $\Lambda_\kappa \mathbf{n}_\kappa = 2^j \mathbf{n}_\kappa$ and $\Lambda_\kappa \mathbf{n}_\kappa^\perp = 2^{m+\frac{j}{2}} \mathbf{n}_\kappa^\perp$. Let, for $(\mu, \nu) \in \mathfrak{V}_{j,m}^\kappa$,

$$a_{m,j,\mu,\nu}[\omega](\xi) = h_{m,j,\mu,\nu}[\omega](L_\kappa^{-1}\xi).$$

The multipliers $a_{m,j,\mu,\nu}[\omega]$ live in balls of bounded diameter and, given fixed m, κ each of those balls nontrivially intersects the sets $\text{supp}(a_{m,j,\mu,\nu})$ only for a bounded number of pairs (μ, ν) . Also,

$$(3.11) \quad \sup_{(\mu,\nu) \in \mathfrak{V}_{j,m}^\kappa} \|a_{m,j,\mu,\nu}[\omega]\|_{L_s^2(\mathbb{R}^2)} \lesssim \left(\sum_{\mu,\nu} |c_{\mu,\nu}|^2 \right)^{1/2} \sup_{\mu,\nu} \|\omega\|_{L_s^2(\mathbb{R})},$$

for $s = 0, 1, 2, \dots$, a straightforward consequence of the chain rule.

Now to prove (3.10), we use duality in ℓ^2 to see that for each j, m, x it suffices to show that for $\varepsilon > 0$

$$(3.12) \quad \left| \sum_{(\mu,\nu) \in \mathfrak{V}_{j,m}^\kappa} c_{\mu,\nu} \mathcal{F}^{-1}[a_{m,j,\mu,\nu}] * f(x) \right| \lesssim \|\{c_{\mu,\nu}\}\|_{\ell^2(\mathbb{Z})} \|\omega\|_{L_{1+\varepsilon}^2} \|f\|_\infty.$$

As in the standard proof for Bernstein's inequality the left hand side of (3.12) is dominated by

$$\begin{aligned} & \left(\int \left| \sum_{(\mu,\nu) \in \mathfrak{V}_{j,m}^\kappa} c_{\mu,\nu} \mathcal{F}^{-1}[a_{m,j,\mu,\nu}[\omega]](y) \right|^2 (1+|y|)^{2+2\varepsilon} dy \right)^{\frac{1}{2}} \left(\int \frac{|f(x-y)|^2}{(1+|y|)^{2+2\varepsilon}} dy \right)^{\frac{1}{2}} \\ & \lesssim_\varepsilon \left\| \sum_{(\mu,\nu) \in \mathfrak{V}_{j,m}^\kappa} c_{\mu,\nu} a_{m,j,\mu,\nu}[\omega] \right\|_{L_{1+\varepsilon}^2(\mathbb{R}^2)} \|f\|_\infty \end{aligned}$$

and the proof of (3.12) is concluded after noting that

$$\left\| \sum_{(\mu,\nu) \in \mathfrak{V}_{j,m}^\kappa} c_{\mu,\nu} a_{m,j,\mu,\nu}[\omega] \right\|_{L_s^2(\mathbb{R}^2)} \lesssim_s \left(\sum_{\mu,\nu} |c_{\mu,\nu}|^2 \right)^{1/2} \sup_{\mu,\nu} \|\omega\|_{L_s^2}.$$

This inequality in turn is verified for $s = 0, 1, 2, \dots$ by using (3.11) and the described disjointness of supports. It then follows for all $s \geq 0$ by interpolation. This finishes the proof of (3.12) and hence (3.10). $\square$

4. CONCLUDING REMARKS

4.1. *Other regularizations.* In the definition of h_δ it was convenient to regularize the characteristic function of an interval and then to use that in the radial variable. Alternatively, we can also choose $\tilde{h}_\delta(\xi) = \mathbb{1}_D * v_\delta$ where $v_\delta(\xi) = \delta^{-2}v(\delta^{-1}\xi)$ and v is a radial C_c^∞ function with $\int v(\xi)d\xi = 1$. Theorem 1.1 remains valid with $\tilde{h}_\delta$ in place of h_δ since $\|\tilde{h}_\delta - h_\delta\|_{M_4} \lesssim (\log \frac{1}{\delta})^{1/4}$ for small δ ; this can be seen by applying either Córdoba's argument in [12] or Theorem 1.2 with $N = 1$. We remark that while Córdoba relies explicitly on an L^2 estimate for a Nikodym maximal function, this is avoided here (indeed already avoided in [25]). However it was noted in [5] that the Fefferman argument from [15], as applied in [25], [7] and here, already implies the L^2 estimate for the Nikodym maximal function.

4.2. *Remarks on higher dimensions.* In this paper we have focused on the upper bounds entirely in the two-dimensional situation. In the complement of the range $p \in [\frac{2d}{d+1}, \frac{2d}{d-1}]$ only partial results for Bochner–Riesz and mollified disc multipliers are known for $d \geq 3$, and much research is currently being done in this area. We refer to [26], [29, 28] for a discussion of the Bochner–Riesz and related conjectures. We will not review the long list of important partial results here and refer for the most recent progress on the Bochner–Riesz problem to [18], in conjunction with recent work on the Fourier restriction conjecture [30], and the references in those papers. Before the resolution of the (non-endpoint) Bochner–Riesz conjecture sharp results in the range $\frac{2d}{d+1} \leq p \leq \frac{2d}{d-1}$, analogous to Theorem 1.1, seem currently completely out of reach in dimension $d \geq 3$.

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