FAILURE OF STABILITY OF A MAXIMAL OPERATOR BOUND FOR PERTURBED NEVO–THANGAVELU MEANS

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ABSTRACT. Let G be a two-step nilpotent Lie group, identified via the exponential map with the Lie-algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_2$. We consider maximal functions associated to spheres in a d-dimensional linear subspace H, dilated by the automorphic dilations. L^p boundedness results for the case where $H = \mathfrak{g}_1$ are well understood. Here we consider the case of a tilted hyperplane $H \neq \mathfrak{g}_1$ which is not invariant under the automorphic dilations. In the case of Métivier groups it is known that the L^p-boundedness results are stable under a small linear tilt. We show that this is generally not the case for other two-step groups, and provide new necessary conditions for L^p boundedness. We prove these results in a more general setting with tilted versions of submanifolds of $\mathfrak{g}_1$.

1. INTRODUCTION

Let G be a finite dimensional two-step nilpotent Lie group, which via the exponential map we identify with its Lie algebra $\mathfrak{g}$. We fix the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ where $\dim \mathfrak{g}_1 = d$, $\dim \mathfrak{g}_2 = m$, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_2$ and $\mathfrak{g}_2$ is contained in the center. By the Baker-Campbell-Hausdorff formula and the two-step assumption the group law is given by

$$(X, U) \cdot (Y, V) = (X + Y, U + V + \tfrac{1}{2}[X, Y])$$

where the commutator $[X, Y]$ belongs to $\mathfrak{g}_2$. On G a natural group of automorphic dilations is given for $t > 0$ by $\delta_t : (X, U) \mapsto (tX, t^2U)$. For every linear functional $\vartheta \in \mathfrak{g}_2^*$, and for $X \in \mathfrak{g}_1$ the functional $\mathcal{J}_X^\vartheta : Y \to \frac{1}{2}\vartheta([X, Y])$ belongs to $\mathfrak{g}_1^*$ and depends linearly on X and ϑ. The linear map $\mathcal{J}^\vartheta : \mathfrak{g}_1 \to \mathfrak{g}_1^*$ given by $\mathcal{J}^\vartheta[X] = \mathcal{J}_X^\vartheta$ depends linearly on ϑ and can be identified with the bilinear form $(X, Y) \mapsto \frac{1}{2}\vartheta([X, Y])$.

Let H be a d-dimensional plane which is transversal to $\mathfrak{g}_2$, i.e. H is given as

$$H = \{(X, \Lambda(X)),\ X \in \mathfrak{g}_1\}$$

and $\Lambda : \mathfrak{g}_1 \to \mathfrak{g}_2$ is linear. We assume that a standard scalar product is defined on $\mathfrak{g}_1$ and define a measure ς^Λ by

$$\langle f, \varsigma^\Lambda \rangle = \frac{1}{\varsigma(S^{d-1})} \int_{S^{d-1}} f(X, \Lambda(X)) d\varsigma$$

J.R. was supported in part by the Ewha Womans University Research Grant of 2025. A.S. was supported in part by NSF Grant DMS-2348797.

where ς denotes surface measure on the unit sphere $S^{d-1} = \{X : |X| = 1\}$. Define the dilate ς_t^Λ of ς^Λ by $\langle f, \varsigma_t^\Lambda\rangle = \int f(tX, t^2U) d\varsigma^\Lambda$. For Schwartz functions $f \in \mathcal{S}(\mathfrak{g})$ we are then interested in the maximal function generated by the non-commutative convolutions $f * \varsigma_t$. That is, we ask for L^p-boundedness properties of the maximal operator $\mathcal{M}^\Lambda$, defined by

$$\mathcal{M}^\Lambda f = \sup_{t>0} |f * \varsigma_t^\Lambda|.$$

This problem was first proposed by Nevo and Thangavelu in [9], for the Heisenberg groups, with $\Lambda = 0$ and Σ the sphere $\{|X| = 1\}$ in $\mathfrak{g}_1$. For $d \geq 3$, and $\Lambda = 0$ an optimal boundedness result was proved by Müller and the second author in [7] for Métivier groups (i.e. in the case that for every nonzero $\vartheta \in \mathfrak{g}_2^*$ the linear map $\mathcal{J}^\vartheta : \mathfrak{g}_1 \to \mathfrak{g}_1^*$ is an isomorphism); in this case the maximal operator is bounded on $L^p(G)$ if and only if $p > \frac{d}{d-1}$. Independently, this result was also obtained for the Heisenberg groups $\mathbb{H}_n$ ($n \geq 2$) by Narayanan and Thangavelu [8], using a different approach. More recently we showed in [13] that for $\Lambda = 0$ the maximal operator $\mathcal{M}^0$ is L^p bounded for $p > \frac{d}{d-1}$ on all two-step groups with $d > 2$. The L^p-boundedness in the case $d = 2$, in particular the case of the Heisenberg group $\mathbb{H}_1$, remains open (see however positive results for the case of Heisenberg-radial functions in [2], [5]).

The question about L^p-boundedness of the perturbed maximal operators $\mathcal{M}^\Lambda$ was first raised by Müller and the second author in [7]. Satisfactory results for the case of Métivier groups were obtained using an L^2 local smoothing estimate in [12] (see also related results in [4]); for another approach on the Heisenberg groups based on fixed time L^p-regularity results via decoupling see [11, 1]. We remark that the condition $p > \frac{d}{d-1}$ is always necessary for L^p-boundedness. This follows from a variant of Stein's example [14] (as was already noted in [6, 13]). It turns out that on the Métivier groups the above results for $\Lambda = 0$ remain true under small perturbations [12]. That is, for small $\|\Lambda\|$ the operator $\mathcal{M}^\Lambda$, is still L^p bounded for $p > \frac{d}{d-1}$ when $d > 2$. The purpose of this paper is to investigate this stability phenomenon for other two-step groups, outside the Métivier class. One might have expected that L^p boundedness of $\mathcal{M}^\Lambda$ with small but nonzero Λ still holds for all two-step groups when $p > \frac{d}{d-1}$, but we show in this paper that this is not the case.

We investigate this phenomenon in a more general setting. In what follows we fix a nontrivial, nonnegative $\chi \in C_c^\infty(\mathfrak{g}_1)$ and surface measure $d\sigma$ on a k-dimensional C^1-submanifold Σ of $\mathfrak{g}_1$. Let $d\mu = \chi d\sigma$. For $t > 0$ we define the measure μ_t by

$$\langle f, \mu_t\rangle = \int f(tX, t^2\Lambda(X))\chi(X) d\sigma(X).$$

We are then interested in lower bounds for the maximal function $\mathfrak{M}f = \sup_{t>0} |f * \mu_t|$; these follow from lower bounds for a localized operator. Let

$I \subset (0,\infty)$ a compact interval of positive length and define, for continuous functions f with compact support, the maximal operator $\mathfrak{M}_I \equiv \mathfrak{M}_I^\Lambda$ by

$$\mathfrak{M}_I f(X,U) = \sup_{t\in I} |f * \mu_t(X,U)|. \tag{1.1}$$

For $\omega^\circ \in \Sigma$ let $(T_{\omega^\circ}\Sigma)^0 = \{\phi \in \mathfrak{g}_1^* : \phi(v) = 0 \text{ for all } v \in T_{\omega^\circ}\Sigma\}$, the annihilator of the tangent space $T_{\omega^\circ}\Sigma$. For $\vartheta \in \mathfrak{g}_2^*$ consider

$$\mathbb{V}_{\Lambda,\vartheta} = \mathrm{range}(\mathcal{J}^\vartheta) + \mathbb{R}(\vartheta \circ \Lambda)\,. \tag{1.2}$$

which is a linear subspace of $\mathfrak{g}_1^*$. Our unboundedness results rely on the following hypothesis, for $1 \le r \le k$.

Hypothesis $\mathcal{H}(r)$. There exist $\vartheta \in \mathfrak{g}_2^*$ and $\omega^\circ \in \Sigma$ satisfying $\chi(\omega^\circ) \ne 0$ such that

- $\vartheta \circ \Lambda \neq 0$.
- $\mathbb{V}_{\Lambda,\theta} \cap (T_{\omega^\circ}\Sigma)^0 = \{0\}$
- $\dim(\mathbb{V}_{\Lambda,\vartheta}) = r$.

Note that the second condition implies that there exists a scalar product on $\mathfrak{g}_1^*$ with respect to which $\mathbb{V}_{\lambda,\theta}$ is orthogonal to $(T_{\omega^\circ}\Sigma)^0$, and the condition means that $\mathbb{V}_{\lambda,\theta}$ is a subspace of $((T_{\omega^\circ}\Sigma)^0)^\perp$ which is identified with $T^*_{\omega^\circ}\Sigma$.

Theorem 1.1. *Assume $d \ge 2$, $\Lambda \ne 0$, $1 \le r \le k \le d$ and that Hypothesis $\mathcal{H}(r)$ holds.*

(i) If $2 \le r \le k$ and $\mathfrak{M}_I$ extends to a bounded operator on $L^p(G)$ then $p > \frac{r+1}{r}$.

(ii) If $r = 1$ and $\mathfrak{M}_I$ extends to a bounded operator on $L^p(G)$ then $p = \infty$.

Remarks.

(a) Because of the skew-symmetry of the bilinear form $(X,Y) \mapsto \vartheta([X,Y])$ the dimension of range $\mathcal{J}^\vartheta$ is always even. That is, in part (ii) of Theorem 1.1 where $r = 1 \le k \le d$ we get unboundedness for all $p < \infty$ if there exists $\vartheta \in \mathfrak{g}_2^*$ such that $\vartheta \circ \Lambda \ne 0$ and $\mathcal{J}^\vartheta = 0$. For $m = 1$ this corresponds to the commutative Euclidean case. The main tool used in the proof of this unboundedness result is the (complement of the) Nikodym set in the plane [10, 3], and we exploit that in some situations the nonisotropic dilations in some compact t-interval have an effect which is similar to rotations.

(b) Theorem 1.1 applies with $k = d$ if $\Lambda \ne 0$ and $d\mu = \chi dX$ where dX is Lebesgue measure on $\mathfrak{g}_1$.

(c) Theorem 1.1 applies with $k = d - 1$ if $d\mu = \chi d\sigma$ where σ is surface measure on the smooth boundary of a convex domain Ω in $\mathfrak{g}_1$. In this case every linear hyperplane is the cotangent space to $\partial\Omega$ at some point ω°, i.e., $\mathbb{V}_{\Lambda,\theta}$ is a linear subspace of one of those cotangent spaces. We note that $\mathfrak{M}_I$ fails to be bounded for $p \le \frac{d}{d-1}$ as can be seen by a variant of the familiar example by Stein [14]. The new examples proving the necessity of the condition $p > \frac{r+1}{r}$ in Theorem 1.1 are thus relevant for the cases $2 \le r \le d-2$.

(d) In case (c) above, under the additional assumption that 0 is in the interior of Ω and that $\partial\Omega$ has nonvanishing curvature everywhere, we have a tilted version of the setup in our previous paper [13]. Theorem 1.1 shows that the hypothesis $\Lambda = 0$ in the general result in [13] cannot be dropped. In a subsequent paper we intend to prove satisfactory upper bounds for the cases $r \le d-2$, under the nonvanishing Gaussian curvature assumption on $\partial\Omega$.

(e) Again for the case (c), we conjecture that stability holds in the cases $r = d-1$ and $r = d$; this appears to be a difficult problem. Moreover, even in the Métivier case (where $r = d$ since the $\mathcal{J}^\vartheta$ are invertible for $\vartheta \neq 0$) the L^p boundedness for $p > \frac{d}{d-1}$ was established in [12] only for sufficiently small $\|\Lambda\|$, and it would be interesting to settle the general case.

Outline. In §2 we discuss coordinates on the group and show that it suffices to prove the lower bounds in the case $m = 1$. In §3 we show unboundedness for $p \le \frac{r+1}{r}$ and in §4 we treat the special case $r = 1$.

Acknowledgements. We thank the referee for a careful reading of the paper and useful suggestions. J.R. was supported in part by the Ewha Womans University Research Grant of 2025. A.S. was supported in part by NSF Grant 2348797.

2. Preliminary reductions

2.1. *Coordinates on G.* Choosing coordinates on $\mathfrak{g}$ we may identify $\mathfrak{g}_1$ with $\mathbb{R}^d$ and $\mathfrak{g}_2$ with $\mathbb{R}^m$. We denote coordinates x on G by $x = (\underline{x}, \overline{x}) \in \mathbb{R}^d \times \mathbb{R}^m$; then the group law becomes

$$(\underline{x}, \overline{x}) \cdot (\underline{y}, \overline{y}) = (\underline{x} + \underline{y}, \overline{x} + \overline{y} + \underline{x}^\intercal \vec{J} \underline{y}),$$

where $\underline{x}^\intercal \vec{J} \underline{y} = (\underline{x}^\intercal J_1 \underline{y}, \dots \underline{x}^\intercal J_m \underline{y})$ and $J_1, \dots, J_m$ are $d \times d$ skew-symmetric matrices. Our convolution operator is then written as

$$Af(x,t) := f * \mu_t(x) = \int f(\underline{x} - t\omega, \overline{x} - t\underline{x}^\intercal \vec{J}\omega - t^2 \Lambda\omega) d\mu(\omega). \tag{2.1}$$

The linear map $\Lambda : \mathbb{R}^d \to \mathbb{R}^m$ is given by $\Lambda \underline{y} = \sum_{i=1}^m (\lambda_i^\intercal \underline{y}) e_{d+i}$ where $e_{d+1}, \dots, e_{d+m}$ is the standard basis of $\mathbb{R}^m$, and $\lambda_i \in \mathbb{R}^d$, $i = 1, \dots, m$. For $1 \le i \le m$, let S_i be the $(d+1) \times d$ matrix given by

$$S_i = \begin{pmatrix} J_i \\ \lambda_i^\intercal \end{pmatrix}. \tag{2.2}$$

If $\vartheta \in \mathfrak{g}_2^*$ is given by $\vartheta(e_{d+i}) = \theta_i$, $i = 1, \dots, m$, then the dimension of the space $\mathbb{V}_{\Lambda,\vartheta}$ in Theorem 1.1 is equal to

$$r(\vartheta) = \operatorname{rank}\big(\sum_{i=1}^{m} \theta_i S_i\big). \tag{2.3}$$

2.2. *Scaling.* A calculation shows that $f * \mu_{st}(x) = [f(\delta_s \cdot)] * \mu_t(\delta_{1/s} x)$ and thus $\mathfrak{M}_{sI} f(x) = \mathfrak{M}_I[f(\delta_s \cdot)](\delta_{1/s} x)$ which shows $\|\mathfrak{M}_{sI}\|_{L^p \to L^p} = \|\mathfrak{M}_I\|_{L^p \to L^p}$. Since for $I_1 \subset I_2$ we have $\|\mathfrak{M}_{I_2}\|_{L^p \to L^p} \geq \|\mathfrak{M}_{I_1}\|_{L^p \to L^p}$ we may assume, after a finite decomposition of a t-interval and scaling, that $I \subseteq [1-\varepsilon, 1+\varepsilon]$ for some small $\varepsilon > 0$.

2.3. *Reduction to the case $m = 1$.* We show that in order to prove the unboundedness results in Theorem 1.1 it suffices to do this for the case $m = 1$.

Let G be a general two-step nilpotent group of dimensions $d + m$, $m > 1$, which we have identified with $\mathbb{R}^d \times \mathbb{R}^m$ as above. By the definition of r in Hypothesis $\mathcal{H}(r)$, there exists $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{S}^{m-1} \setminus \{0\}$ such that $\operatorname{rank}(\sum_{i=1}^m \theta_i S_i) = r$ and $\sum_{i=1}^m \theta_i \lambda_i^\intercal \neq 0$. Choose unit vectors $(b^i)_{2 \leq i \leq m}$ in $\mathbb{R}^m$ such that $\theta, b^2, \dots, b^m$ are mutually orthogonal. Let R_θ be the $m \times m$ rotation matrix with columns $\theta, b^2, \dots, b^m$. Then the averaging operator A_t is expressed as

$$A_t f(\underline{x}, \overline{x}) = \int f\big(\underline{x} - t\omega, R_\theta\big(R_\theta^\intercal \overline{x} - t R_\theta^\intercal\big((\underline{x}, t)^\intercal \vec{S}\omega\big)\big)\big) d\mu(\omega) \tag{2.4}$$

with $(\underline{x}, t)^\intercal \vec{S}\omega = \sum_{i=1}^m (\underline{x}^\intercal J_i \omega + t\lambda_i^\intercal \omega) e_{d+i}$. Note that

$$\langle R_\theta^\intercal\big((\underline{x}, t)^\intercal \vec{S}\omega\big), e_{d+1}\rangle = (\underline{x}, t)^\intercal \big(\sum_{i=1}^m \theta_i S_i\big)\omega.$$

Thus, after a conjugation with a rotation in $\mathbb{R}^m$, we may assume that

$$R_\theta = I_m, \ \operatorname{rank}(S_1) = r \text{ and } \lambda_1^\intercal \neq 0.$$

Let $I \subset (0, \infty)$ be a compact subinterval. We now consider a class of compactly supported smooth functions $f = \widetilde{f} \otimes h$ given by

$$f(\underline{y}, \overline{y}_1, \overline{y}') = \widetilde{f}(\underline{y}, \overline{y}_1) h(\overline{y}'),$$

with $\widetilde{f} \in C_c^\infty(\mathbb{R}^{d+1})$, $h \in C_c^\infty(\mathbb{R}^{m-1})$. For f in this class we have

$$A_t f(x) = \int \widetilde{f}(\underline{x} - t\omega, x_{d+1} - t(x, t)^\intercal S_1 \omega) h\big(\sum_{i=2}^m (\overline{x}_i - t(\underline{x}, t)^\intercal S_i \omega) e_{d+i}\big) d\mu(\omega)$$

with $t(x, t)^\intercal S_1 \omega) = t\underline{x}^\intercal J_1 \omega + t^2 \lambda_1^\intercal \omega$. Define

$$\widetilde{A}_t \widetilde{f}(\underline{x}, x_{d+1}) = \int \widetilde{f}(\underline{x} - t\omega, x_{d+1} - t(x, t)^\intercal S_1 \omega) d\mu(\omega). \tag{2.5}$$

Let $\widetilde{B}$ be a ball in $\mathbb{R}^{d+1}$ and B' be a ball in $\mathbb{R}^{m-1}$. Choose $h \in C_c^\infty(\mathbb{R}^{m-1})$ so that $h \equiv 1$ on a large compact set, specifically

$$h(\sum_{i=2}^m (\overline{x}_i - t(\underline{x}, t)^\intercal S_i \omega) e_{d+i}) = 1$$

for every $\omega \in \mathrm{supp}(\mu)$, $t \in I$, $x \in \widetilde{B} \times B'$. Then

$$\big\| \mathbb{1}_{\widetilde{B}\times B'} \sup_{t\in I} |A_t f| \big\|_{L^p(\mathbb{R}^{d+m})} = |B'|^{1/p} \big\| \mathbb{1}_{\widetilde{B}} \sup_{t\in I} |\widetilde{A}_t \widetilde{f}| \big\|_{L^p(\mathbb{R}^{d+1})}, \tag{2.6}$$

Denote by $\widetilde{G}$ denote the $d+1$-dimensional two-step group with group law $(\underline{x}, x_{d+1}) \cdot (\underline{y}, y_{d+1}) = (\underline{x} + \underline{y}, x_{d+1} + y_{d+1} + \underline{x}^\intercal J_1 \underline{y})$. From (2.6) we conclude that the desired unboundedness on G would follow once on $\widetilde{G}$, the local maximal operator $g \to \mathbb{1}_{\widetilde{B}} \sup_{t\in I} |\widetilde{A}_t g|$ is shown to be unbounded on $L^p(\widetilde{G})$; here $\widetilde{B} \subset \widetilde{G}$ is a suitable ball, and $I \subset (0, \infty)$ is a compact interval. Below we will consider $\sup_{t\in I} |A_t g|$ for compactly supported functions g so that the maximal function is then supported on compact sets; thus the characteristic function of the ball $\widetilde{B}$ will can then be dropped in the definition of the local maximal function.

3. Unboundedness for $p \le \frac{r+1}{r}$

In what follows we prove that our (local) maximal operator is not bounded on L^p if $p \le \frac{r+1}{r}$, showing part (i) of Theorem 1.1, for $r \ge 2$. By the reduction in §2.3 we may assume $m = 1$, $\Lambda(\underline{x}) = \lambda^\intercal \underline{x}$ where λ is a nonzero vector in $\mathbb{R}^d$. We thus write $J = J_1$ and $\lambda = \lambda_1$ in (2.5) and consider the maximal function $\mathfrak{M}_I g = \sup_{t\in I} |A_t g|$ where $I = [1-\varepsilon_0, 1+\varepsilon_0]$ with $\varepsilon_0 < \frac{1}{2}$,

$$A_t g(\underline{x}, x_{d+1}) = \int g(\underline{x} - t\omega, x_{d+1} - t\underline{x}^\intercal J\omega - t^2 \lambda^\intercal \omega) \chi(\omega) d\sigma(\omega). \tag{3.1}$$

Moreover $S^\intercal = \begin{pmatrix} J^\intercal & \lambda \end{pmatrix}$ and r is the rank of $S^\intercal$. We have $0 \ne \lambda \in \mathrm{range}(S^\intercal)$. Let $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ be the orthogonal projection to the range of $S^\intercal$.

By assumption, there exists $\omega^\circ \in \Sigma$ such that $\chi(\omega^\circ) \ne 0$, and range$(S^\intercal)$ is contained in the tangent space $T_{\omega^\circ}\Sigma$. Let $\widetilde{\Sigma}$ be a neighborhood (in Σ) of ω° such that $|\chi(\omega)| \ge c > 0$ for $\omega \in \widetilde{\Sigma}$. By choosing $\widetilde{\Sigma}$ sufficiently small we may assume that there is a parametrization of $\widetilde{\Sigma}$ with $u \in \mathbb{R}^k$ close to the origin and $\Gamma(0) = \omega^\circ$ such that $\{\frac{\partial \Gamma}{\partial u_i}(0)\}_{i=1}^k$ is an orthogonal basis of $T_{\omega^\circ}\Sigma$, and such that $\frac{\partial \Gamma}{\partial u_1}(0) = \frac{\lambda}{|\lambda|}$ and $\{\frac{\partial \Gamma}{\partial u_i}(0)\}_{i=1}^r$ is a basis of range$(S^\intercal)$. We split the parameters as $u = (u', u'') \in \mathbb{R}^r \times \mathbb{R}^{k-r}$; in the case $k = r$ the variable u'' is not present (which requires a slight notational modification in what follows).

We use the implicit function theorem to solve for fixed x, t the equation $\Pi(\underline{x} - t\omega) = 0$. More precisely we solve for u' the equation

$$\Pi(\underline{x} - t\Gamma(u', u'')) = 0, \tag{3.2}$$

for t near 1, for $\underline{x}$ near ω° and for $u'' = (u_{r+1}, \dots, u_k)$ near $0 \in \mathbb{R}^{k-r}$. This is possible since $\underline{x} - t\Gamma(u)|_{(\underline{x},t,u)=(\omega^\circ,1,0)} = 0$ and

$$\frac{\partial}{\partial u_i}\big(\underline{x} - t\Gamma(u)\big)|_{u=0} = -t\frac{\partial \Gamma}{\partial u_i}(0), \quad i = 1, \dots, r$$

are linearly independent. As a consequence we solve (3.2) by $u' = h(\underline{x}, t, u'')$ for $(\underline{x}, t, u')$ near $(\omega^\circ, 1, 0)$. Introducing coordinates $u' = h(\underline{x}, t, u'') + \beta'$ for small β' there is a small $\varepsilon > 0$, with $\varepsilon < \varepsilon_0$, such that for $\delta \ll \varepsilon$ and positive constants c_1, c_2 the measure of the set $\{u \in \mathbb{R}^k : c_1\delta/2 \le |u' - h(\underline{x}, t, u'')| \le c_1\delta, |u''| \le c_2\varepsilon\}$ is bounded below by $c\varepsilon^{k-r}\delta^r$, for $|\underline{x} - \omega^\circ| \le \varepsilon$, $|t-1| \le \varepsilon$. Consequently, the surface measure of

$$\mathcal{W}_\delta(\underline{x}, t) = \big\{\omega \in \widetilde{\Sigma} : \frac{\delta}{2} < |\Pi(\underline{x} - t\omega)| \le \delta, |\omega - \omega^\circ| \le \varepsilon\big\} \tag{3.3}$$

satisfies

$$\inf\big\{\sigma\big(\mathcal{W}_\delta(\underline{x}, t)\big) :\ |\underline{x} - \omega^\circ| \le \varepsilon, |t-1| \le \varepsilon\big\} \ \ge\ c_\varepsilon \delta^r. \tag{3.4}$$

We choose ε so small such that $\chi(\underline{x}) \ne 0$ for $|\underline{x} - \omega^\circ| \le \varepsilon$.

In what follows we fix $\varepsilon > 0$ such that

$$\varepsilon \ll \tfrac{1}{4}(1 + |\lambda| + \|J\| + |\omega^\circ| + \|J\||\omega^\circ|)^{-1},$$

and work with a parameter $\delta \ll \varepsilon$. Define

$$R_\delta = \{(\underline{y}, y_{d+1}) : \frac{\delta}{2} \le |\Pi(\underline{y})| \le \delta,\ |y_{d+1}| \le \varepsilon^{-1}\delta,\ |\underline{y}| \le 1\},$$
$$g_\delta = \mathbb{1}_{R_\delta}.$$

We test the maximal operator on g_δ.

Let $m_* = \max\{|\lambda^\intercal \underline{x}| : |\underline{x} - \omega^\circ| \le \varepsilon\}$. Then $m_* = |\lambda^\intercal \omega^\circ| + \varepsilon|\lambda|$ which lies between $\varepsilon|\lambda|$ and $(|\omega^\circ| + 2\varepsilon)|\lambda|$. Let

$$V_\varepsilon = \{\underline{x} \in \mathbb{R}^d : |\underline{x} - \omega^\circ| \le \varepsilon,\ m_*/2 \le |\lambda^\intercal \underline{x}| \le m_*\}.$$

Then $|V_\varepsilon| > 0$. Let

$$U_\varepsilon = \Big\{(\underline{x}, x_{d+1}) : \underline{x} \in V_\varepsilon,\ 1 - \frac{\varepsilon}{2} < \frac{x_{d+1}}{|\lambda^\intercal \underline{x}|} < 1 + \frac{\varepsilon}{2}\Big\}.$$

Observe

$$|U_\varepsilon| = \int_{V_\varepsilon} \varepsilon|\lambda^\intercal \underline{x}| d\underline{x} \ge \frac{m_*\varepsilon|V_\varepsilon|}{2} > 0.$$

We wish to derive a lower bound for $A_t f(x)$, for $x \in U_\varepsilon$ and suitable $t = t_x \in (1-\varepsilon, 1+\varepsilon)$. Observe that

$$x_{d+1} - t^2\lambda^\intercal\omega - t\underline{x}^\intercal J\omega = x_{d+1} - t\lambda^\intercal\underline{x} + (t\lambda - J\underline{x})^\intercal(\underline{x} - t\omega).$$

It is natural to choose

$$t_x = \frac{x_{d+1}}{\lambda^\intercal \underline{x}}$$

which for $x \in U_\varepsilon$ lies in $(1-\varepsilon, 1+\varepsilon)$ and gives

$$\begin{aligned} x_{d+1} - t_x^2 \lambda^\intercal \omega - t_x \underline{x}^\intercal J\omega &= (t_x\lambda - J\underline{x})^\intercal (\underline{x} - t_x\omega) \\ &= (t_x\lambda - J\underline{x})^\intercal \Pi(\underline{x} - t_x\omega). \end{aligned}$$

Note that $|t_x\lambda - J\underline{x}| \le (1+\varepsilon)|\lambda| + \|J\|(|\omega^\circ| + \varepsilon) \le \varepsilon^{-1}$. For $\omega \in \mathcal{W}_\delta(\underline{x}, t_x)$ we get $\frac{\delta}{2} \le |\Pi(\underline{x} - t_x\omega)| \le \delta$ and hence

$$|x_{d+1} - t_x^2\lambda^\intercal\omega - t_x\underline{x}^\intercal J\omega| \le \varepsilon^{-1}|\Pi(\underline{x} - t_x\omega)| \le \varepsilon^{-1}\delta.$$

Also, $|\underline{x} - t_x\omega| \le |\underline{x} - \omega^\circ| + |\omega^\circ(1-t_x)| + t_x|\omega^\circ - \omega| \le \varepsilon + \varepsilon|\omega^\circ| + 2\varepsilon < 1$. We have thus shown that $(\underline{x} - t_x\omega, x_{d+1} - t_x^2\lambda^\intercal\omega - t_x\underline{x}^\intercal J\omega) \in R_\delta$ for $x \in U_\varepsilon$ and $\omega \in W_\delta(\underline{x}, t_x)$ and consequently we obtain

$$\mathfrak{M}_I g_\delta(x) \ge A_{t_x} g_\delta(x) \ge c\sigma(W_\delta(\underline{x}, t_x)) \ge c_\varepsilon c\delta^r \text{ for } x \in U_\varepsilon.$$

Hence $\|\mathfrak{M}_I g_\delta\|_{L^p} \gtrsim_\varepsilon \delta^r |U_\varepsilon|^{1/p}$ and since $\|g_\delta\|_p = |R_\delta|^{1/p} \lesssim_\varepsilon \delta^{\frac{r+1}{p}}$ we obtain

$$\frac{\|\mathfrak{M}_I g_\delta\|_{L^p(U_\varepsilon)}}{\|g_\delta\|_p} \gtrsim_\varepsilon \delta^{r-(r+1)/p}.$$

Letting $\delta \to 0$ shows that $\mathfrak{M}_I$ cannot be L^p-bounded for $p < \frac{r+1}{r}$.

Now let $p(r) = \frac{r+1}{r}$. To disprove $L^{p(r)}$-boundedness we define for $N \gg \log_2 \frac{1}{\varepsilon}$

$$F_N = \sum_{\log_2 \frac{1}{\varepsilon} < j \le N} 2^{jr} g_{2^{-j}}.$$

Since $\|g_{2^{-j}}\|_{p(r)} \lesssim c_\varepsilon 2^{-jr}$ and the sets $R_{2^{-j}}$ are disjoint we get

$$\|F_N\|_{p(r)} \lesssim N^{1/p(r)}.$$

On the other hand, the above lower bounds show $A_{t_x} F_N(x) \gtrsim N$ for $x \in U_\varepsilon$ and thus

$$\frac{\|\mathfrak{M}_I F_N\|_{p(r)}}{\|F_N\|_{p(r)}} \gtrsim N^{1/(r+1)}.$$

Letting $N \to \infty$ we see that $L^{p(r)}$-boundedness fails. □

4. Unboundedness of the maximal operator in the case $r = 1$

By the reduction in §2.3 we may assume $m = 1$. If $r = 1$ then $J = 0$. Let λ be a nonzero vector in $\mathbb{R}^d$. We prove a more general result, replacing the surface measure of Σ by a more general finite (positive) Borel measure μ and consider

$$\Gamma_t f(\underline{x}, x_{d+1}) = \int f(\underline{x} - t\underline{y}, x_{d+1} - t^2\lambda^\intercal\underline{y}) d\mu(\underline{y}).$$

Note that for the special case of μ as in Theorem 1.1 this is the case $J = 0$ in (3.1). $\Gamma_t f$ is well-defined for continuous functions f with compact support, and $\Gamma_t f$ is then a continuous function. For a compact interval $I \subset (0, \infty)$ the maximal function

$$M_I f(x) = \sup_{t \in I} |\Gamma_t f(x)|$$

is then well defined as a Borel measurable function in $\mathbb{R}^{d+1}$. We show that no nontrivial boundedness property holds for M_I if μ is not supported in the orthogonal complement of λ in $\mathbb{R}^d$. In particular this applies to prove part (ii) of Theorem 1.1.

Proposition 4.1. *Assume that $\lambda \neq 0$ and that μ is a finite positive Borel measure in $\mathbb{R}^d$, with the property that $\mu((\lambda^\perp)^\complement) > 0$. Suppose that there is a positive constant C such that the inequality*

$$\|M_I f\|_p \leq C \|f\|_p$$

holds for all characteristic functions of open sets with compact closure. Then $p = \infty$.

Proof. In what follows we will work with nonnegative functions throughout, so we may reduce the length of the interval for lower bounds. By parabolic scaling (§2.2) and possible shrinking the t-interval we may assume

$$I = [a, 1] \text{ where } a = \tan(\tfrac{\pi}{4}(1 - \tfrac{1}{N})),$$

for some $N \geq 6$.

Let R be a rotation in $\mathbb{R}^d$ such that $Re_d = \lambda/\|\lambda\|$. Let $\mu_R = \mu(R\cdot)$, formally defined by $\int u(\underline{y}) d\mu_R(\underline{y}) = \int u(R^{-1}\underline{z}) d\mu(\underline{z})$ for test functions u. Then if

$$\mathcal{A}_t f(x) := \int f(\underline{x} - t\underline{y}, x_{d+1} - t^2 e_d^\intercal \underline{y}) d\mu_R(\underline{y})$$

we have

$$\Gamma_t f(\underline{x}, x_{d+1}) = \mathcal{A}_t F_{R,\lambda}(R^{-1}\underline{x}, \|\lambda\|^{-1} x_{d+1}) \qquad \text{with} \quad F_{R,\lambda}(\underline{w}, w_{d+1}) = f(R\underline{w}, \|\lambda\| w_{d+1}).$$

We also note that the assumption of $\text{supp}(\mu)$ not contained in $\lambda^\perp$ is equivalent with $\text{supp}(\mu_R)$ not contained in $e_d^\perp$. Hence $f \mapsto \sup_{t \in [a,1]} |\Gamma_t f|$ is bounded on L^p if and only if $f \mapsto \sup_{t \in [a,1]} |\mathcal{A}_t f|$ is bounded on L^p.

Proof of unboundedness for $p < \infty$. We split variables in $\mathbb{R}^d$ as $\underline{y} = (y', y_d)$. Since the measure μ_R is not supported in $e_d^\perp$ there exists a bounded open subset $W \subset \{\underline{y} \in \mathbb{R}^d : y_d \neq 0\}$ such that $\mu_R(W) > c > 0$. Let $L > 0$ be such that $|y'| \leq L$ for all $y = (y', y_d) \in W$. When deriving lower bounds on nonnegative functions we will replace μ_R by $\chi \mu_R$ where χ is continuous and compactly supported in W and such that

$$\int \chi(\underline{w}) d\mu_R(\underline{w}) > c/2.$$

We use the Nikodym set in $\mathbb{R}^2$ to construct our counterexample. The original complicated construction by Nikodym appeared in [10]. Subsequent constructions are simpler and based on the Perron tree constructions used for the Kakeya set, see de Guzmán's book [3]. According to [3, Theorem 3.4] there exists a measurable set $F_0 \subset [-1,1]^2$ of full Lebesgue measure $|F_0| = 4$ and a Lebesgue null set $E_0 \subset \mathbb{R}^2$ such that for each $w \in F_0$ there is a straight line

$$l(w) = w + \{r(\cos(\alpha(w)), \sin(\alpha(w))) : r \in \mathbb{R}\}$$

with $l(w) \setminus \{w\} \subset E_0$; here $\alpha(w) \in [0, \pi)$ and $w \mapsto \alpha(w)$ is a measurable function. Let $\widetilde{F} = \{w \in F_0 : |w| \le 1\}$.

By pigeonholing there exists $k \in \{0, 1, \dots, 4N-1\}$ so that

$$\widetilde{F}_k = \{w \in \widetilde{F} : \alpha(w) \in [\tfrac{k\pi}{4N}, \tfrac{(k+1)\pi}{4N})\}$$

satisfies

$$|\widetilde{F}_k| \ge |\widetilde{F}|/4N = \pi/4N.$$

Let ρ_β be the planar rotation $\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$ with $\beta = -\frac{k+1}{4N}\pi + \frac{\pi}{4}$. Let $F = \rho_\beta \widetilde{F}_k$ and $E = \rho_\beta E_0$.

Then $|F| \ge \pi/4N$ and E is a Lebesgue null set. By construction there is, for every $w \in F$, a line $\ell(w) = \{(r\cos\gamma(w), r\sin\gamma(w)) : r \in \mathbb{R}\}$ such that $\gamma(w) \in [(1 - \frac{1}{N})\frac{\pi}{4}, \frac{\pi}{4})$ and $\ell(w) \setminus \{w\} \subset E$.

For each $w \in F$ let $s(w)$ be the slope of the line $\ell(w)$ and then

$$(s(w), s(w)^2) = s(w)\sqrt{1 + s(w)^2}(\cos\gamma(w), \sin\gamma(w)),$$

and $\gamma(w) = \arctan s(w)$. In particular

$$\ell(w) \setminus \{w\} = w + \{(rs(w), rs(w)^2) : r \ne 0\}.$$

We return to the task of deriving lower bounds for the maximal function $\sup_{t \in [a,1]} |\mathcal{A}_t f|$. Let

$$\mathcal{E} = \{(y', y_d, y_{d+1}) \in \mathbb{R}^{d+1} : |y'| \le 2 + L, (y_d, y_{d+1}) \in E\}.$$

$\mathcal{E}$ is a Lebesgue null set and thus for small $\varepsilon > 0$ there is an open subset $\mathcal{V}_\varepsilon$ of measure $< \varepsilon$ with $\mathcal{V}_\varepsilon \supset \mathcal{E}$. Let $f_\varepsilon = \mathbb{1}_{\mathcal{V}_\varepsilon}$, then clearly $\|f_\varepsilon\|_p \le \varepsilon^{1/p}$.

Let

$$\mathcal{F} = \{(x', x_d, x_{d+1}) \in \mathbb{R}^{d+1} : |x'| \le 1, (x_d, x_{d+1}) \in F\}$$

so that $\mathcal{F}$ has positive Lebesgue measure. Let

$$t_x := s(x_d, x_{d+1}).$$

For $x \in \mathcal{F}$ we have

$$\mathcal{A}_{t_x} f_\varepsilon(x) \ge \int f_\varepsilon(x' - t_x y', x_d - t_x y_d, x_{d+1} - t_x^2 y_d)\chi(y', y_d)\, d\mu_R(\underline{y}).$$

On the support of χ we have $y_d \ne 0$. The vector

$$(x_d - t_x y_d, x_{d+1} - t_x^2 y_d)$$

thus belongs to $\ell(x_d, x_{d+1}) \setminus \{(x_d, x_{d+1})\} \subset E$ and since $|x' - t_x y'| \le 2 + L$ we get

$$\sup_{a \le t \le 1} \mathcal{A}_t f_\varepsilon(x) \ge \mathcal{A}_{t_x} f_\varepsilon(x) \ge \int \chi(y', y_d) d\mu_R(\underline{y}) > c/2$$

for all $\varepsilon > 0$. If M_I were bounded on L^p there would be a constant $C > 0$ such that $0 < |\mathcal{F}|^{1/p} c/2 \le C\varepsilon^{1/p}$ which for sufficiently small $\varepsilon > 0$ cannot hold if $p < \infty$. □

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