BOCHNER-RIESZ MEANS AT THE CRITICAL INDEX: WEIGHTED AND SPARSE BOUNDS

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Abstract. We consider Bochner-Riesz means on weighted L^p spaces, at the critical index $\lambda(p) = d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. For every A_1-weight we obtain an extension of Vargas' weak type $(1,1)$ inequality in some range of $p > 1$. To prove this result we establish new endpoint results for sparse domination. These are almost optimal in dimension $d = 2$; partial results as well as conditional results are proved in higher dimensions. For the means of index $\lambda_* = \frac{d-1}{2d+2}$ we prove fully optimal sparse bounds.

1. Introduction

Let Ω be a convex open subset of $\mathbb{R}^d$, $d \geq 2$, containing the origin. We assume that Ω has C^∞-boundary with nonvanishing Gaussian curvature. Let

$$\rho(\xi) := \inf\{t > 0 : \xi/t \in \Omega\}$$

be the Minkowski functional of Ω. Then $\rho \in C^\infty(\mathbb{R}^d \setminus \{0\})$, ρ is homogeneous of degree 1, $\rho(\xi) > 0$ for $\xi \neq 0$ and $\rho(\xi) = 1$ on the boundary $\partial\Omega$. Let $a > 0$. Given $\lambda > 0$, we define the *Riesz means* of index λ of the inverse Fourier integral by

$$\mathcal{R}^\lambda_{a,t} f(x) := \frac{1}{(2\pi)^d} \int \Big(1 - \frac{\rho(\xi)^a}{t^a}\Big)^\lambda_+ \widehat{f}(\xi) e^{i\langle x,\xi\rangle}\, \mathrm{d}\xi,$$

where $\widehat{f}(\xi) = \int f(y) e^{-i\langle y,\xi\rangle}\, \mathrm{d}y$ denotes the Fourier transform of a Schwartz function f on $\mathbb{R}^d$ and $s_+ := \max\{s, 0\}$. The case of $\Omega = \{\xi : |\xi| \leq 1\}$ yields $\rho(\xi) = |\xi|$; in this case the means with $a = 1$ are the *classical radial Riesz means* of index λ while the case $a = 2$ corresponds to the *Bochner-Riesz means* of index λ.

Given $1 \leq p < \frac{2d}{d+1}$, the value

$$\lambda(p) := d\Big(\frac{1}{p} - \frac{1}{2}\Big) - \frac{1}{2} \tag{1.1}$$

is referred to as the *critical index*, and it is conjectured that in this range the operators $\mathcal{R}^{\lambda(p)}_{a,t}$ are of weak type (p,p). The case $p = 1$, corresponding to the index $\lambda(1) = \frac{d-1}{2}$, was first proved by Christ [9] and later substantially extended by Vargas [35] who proved an $L^1(w) \to L^{1,\infty}(w)$ result for all A_1 weights w, that is, for all $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying the pointwise inequality $Mw \lesssim w$, where M denotes the Hardy-Littlewood maximal operator. Sharp weak type endpoint results for $p > 1$

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were proved by Christ [8] in the range $1 < p < \frac{2(d+1)}{d+3}$, by Tao for $p = \frac{2(d+1)}{d+3}$, and complete results in two dimensions were obtained by one of the authors in [30]. Later, Tao [34] showed that for $1 < p < \frac{2d}{d+1}$ the weak type endpoint estimates follow from the corresponding strong type results for all $\lambda > \lambda(p)$. For $d = 2$ these are well-known and due to Carleson-Sjölin [6], allowing to recover the weak-type results from [30]. In higher dimensions many sharp partial results for the strong type estimate have been proved; see [14, 25, 26, 37, 17] and the references in those papers.

The goal of this paper is to establish new estimates for the operators $\mathcal{R}^{\lambda(p)}_{a,t}$ whenever $1 < p < \frac{2d}{d+1}$.

1.1. *Weighted estimates.* We will be concerned with weights in the Muckenhoupt A_s classes and the reverse Hölder classes RH_σ; see §9 for the precise definitions. By testing against Schwartz functions it is easy to see that for $p < \frac{2d}{d+1}$ the operators $\mathcal{R}^{\lambda(p)}_{a,t}$ fail to satisfy weighted weak-type (p, p) estimates for the power weights $|x|^\varepsilon$ for any $\varepsilon > 0$. This rules out, in particular, the Muckenhoupt A_s classes for any $s > 1$ (which can also be ruled out by the weak-type version of Rubio de Francia's extrapolation theorem [28]). However, it is natural to ask whether the $L^1(w) \to L^{1,\infty}(w)$ estimate for A_1 weights w has an extension for the critical $\lambda(p)$ and some $p > 1$, and what the p-range of this extension is. We give an affirmative answer to the first part of this question.

Theorem 1.1. *Let $a > 0$. For every $w \in A_1$ there exists an exponent $p_1(w) > 1$ such that the operators $\mathcal{R}^{\lambda(p)}_{a,t}$ are bounded from $L^p(w)$ to $L^{p,\infty}(w)$ for $1 \le p < p_1(w)$, uniformly in $t > 0$. Moreover, $\lim_{t\to\infty} \|\mathcal{R}^{\lambda(p)}_{a,t} f - f\|_{L^{p,\infty}(w)} = 0$ for all $f \in L^p(w)$.*

The case $p = 1$ in Theorem 1.1 is Vargas' result [35]; our contribution here corresponds to $p > 1$.

In order to prove Theorem 1.1, we establish new sparse domination results for Bochner-Riesz means at the critical index, which will be presented in §1.2. These can be combined with a result of Frey and Nieraeth [16] to yield that, under the assumptions of the Bochner–Riesz conjecture in d dimensions, the operators $\mathcal{R}^{\lambda(p)}_{a,t}$ map $L^p(w)$ to $L^{p,\infty}(w)$ for $w \in A_1 \cap \mathrm{RH}_\sigma$ and $p < 1 + \frac{d-1}{d+1}(1 - \frac{1}{\sigma})$. This holds unconditionally if $d = 2$ or if $d \ge 3$ and σ belongs to a suitable range that includes $[1, \frac{d+3}{2}]$: see Section 9. Theorem 1.1 will be a consequence of this, using the standard fact that every A_1 weight belongs to RH_σ for some $\sigma > 1$.

It does not seem to be known whether $p < 1 + \frac{d-1}{d+1}(1 - \frac{1}{\sigma})$ is the sharp p-range in terms of the reverse Hölder exponent σ in the $L^p(w) \to L^{p,\infty}(w)$ estimates. It would be interesting to investigate relevant examples.

1.2. *Sparse bounds.* Let $\mathfrak{D}$ denote a *dyadic lattice* in the sense of the monograph by Lerner and Nazarov [27, §2]. For a locally integrable function f, a cube $Q \in \mathfrak{D}$ and $1 \le p < \infty$, let $\langle f\rangle_{Q,p} = (|Q|^{-1}\int_Q |f(y)|^p\,\mathrm{d}y)^{1/p}$. Given $0 < \gamma < 1$, the collection $\mathfrak{S} \in \mathfrak{D}$ is called *γ-sparse* if for every $Q \in \mathfrak{S}$ there is a measurable subset $E_Q \subset Q$ so that $|E_Q| \ge \gamma|Q|$ and $\{E_Q : Q \in \mathfrak{S}\}$ is a collection of pairwise disjoint sets. Let $1 \le p, q < \infty$. For a γ-sparse family $\mathfrak{S}$ of cubes we define a sparse form $\Lambda^{\mathfrak{S}}_{p,q}$ and a corresponding maximal form $\Lambda^*_{p,q}$ by

$$\Lambda^{\mathfrak{S}}_{p,q}(f_1, f_2) = \sum_{Q\in\mathfrak{S}} |Q|\langle f_1\rangle_{Q,p}\langle f_2\rangle_{Q,q}, \tag{1.2}$$

$$\Lambda^*_{p,q}(f_1, f_2) = \sup_{\mathfrak{S}:\gamma\text{-sparse}} \Lambda^{\mathfrak{S}}_{p,q}(f_1, f_2), \tag{1.3}$$

where the sup is taken over all γ-sparse families (which are allowed to be subcollections of different dyadic lattices). These definitions are of interest in the range $p \le q < p'$. A linear operator $T : C^\infty_c(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ satisfies a (p, q) sparse bound if for all $f_1,\ f_2 \in C^\infty_c$ the inequality

$$|\langle Tf_1, f_2\rangle| \le C\Lambda^*_{p,q}(f_1, f_2) \tag{1.4}$$

holds with some constant C independent of $f_1,\ f_2$. In this case, we say that T belongs to the space $\mathrm{Sp}_\gamma(p, q; \mathbb{R}^d)$ and we denote by $\|T\|_{\mathrm{Sp}_\gamma(p,q;\mathbb{R}^d)}$ the best constant in (1.4). The space $\mathrm{Sp}_\gamma(p, q; \mathbb{R}^d)$ does not depend on γ (*cf.* [27]), so we usually keep γ fixed and drop the subscript γ. If the dimension is clear from the context we will also drop the mention of $\mathbb{R}^d$.

Given $0 < \lambda \le \frac{d-1}{2}$, let $\triangle_d(\lambda)$ denote the trapezoid with corners

$$\begin{aligned} &P_1 = (\tfrac{2\lambda+d+1}{2d}, \tfrac{d-2\lambda-1}{2d}), && P_2 = (\tfrac{2\lambda+d+1}{2d}, \tfrac{d-1}{2d} + \tfrac{\lambda(d+1)}{d(d-1)}),\\ &P_3 = (\tfrac{d-1}{2d} + \tfrac{\lambda(d+1)}{d(d-1)}, \tfrac{2\lambda+d+1}{2d}), && P_4 = (\tfrac{d-2\lambda-1}{2d}, \tfrac{2\lambda+d+1}{2d}). \end{aligned} \tag{1.5}$$

One might conjecture that sparse bounds for $\mathcal{R}^\lambda_{a,t}$ and $\lambda > 0$ hold for all $(\frac{1}{p}, \frac{1}{q}) \in \triangle_d(\lambda)$. This would be a strengthening of the Lebesgue mapping properties of $\mathcal{R}^\lambda_{a,t}$; thus, one typically aims to only obtain the sparse improvement for values of $\lambda > 0$ for which the Bochner-Riesz conjectured has been verified. It was observed in [4, 24] that for $(\frac{1}{p}, \frac{1}{q})$ in the interior of the trapezoid, (p, q)-sparse bounds for $\mathcal{R}^\lambda_{a,t}$ can be obtained via a single-scale analysis, with affirmative results depending on the partial knowledge on the Bochner-Riesz conjecture. Henceforth we will focus on the endpoint cases in which $(\frac{1}{p}, \frac{1}{q})$ belongs to the boundary of $\triangle_d(\lambda)$. Furthermore, since sparse bounds are scale-invariant we will consider the case $t = 1$, and write $\mathcal{R}^\lambda_a = \mathcal{R}^\lambda_{a,1}$.

The sharpness of the region $\triangle_d(\lambda)$ was first observed in [4], and can also be deduced from general necessary conditions for sparse domination (*cf.* [2, Prop.1.9]). The numerology of (1.5) at P_2 is related to the conjectured $L^p \to L^r$ bounds for Fourier multiplier operators with radial bumps on thin annuli (see (2.4) below), which have as necessary condition $\frac{1}{r} \ge \frac{d+1}{d-1}(1 - \frac{1}{p})$ from Knapp examples. Note that $P_2 = (\frac{1}{p_2}, \frac{1}{q_2})$ in (1.5) satisfies $1 - \frac{1}{q_2} = \frac{d+1}{d-1}(1 - \frac{1}{p_2})$ and that the vertical line segment P_1P_2 corresponds to the critical case where $\lambda = \lambda(p)$.

Almost sharp results at the critical line P_1P_2 were obtained in the case $\lambda = \frac{d-1}{2}$ (that is, $p = 1$) by Conde-Alonso–Culiuc–Di Plinio–Ou [13]; namely they proved a $(1, q)$ sparse bound for all $q > 1$. Partial results on the line P_1P_2 were obtained in two dimensions by Kesler and Lacey [22] whenever $0 < \lambda < 1/2$. At the critical $p_\lambda = \frac{4}{3+2\lambda}$, they showed a $\mathrm{Sp}(p_\lambda, q; \mathbb{R}^2)$ bound for $q > 4$, thereby strengthening the weak type (p_λ, p_λ) inequality in [30]. They posed as an open question whether

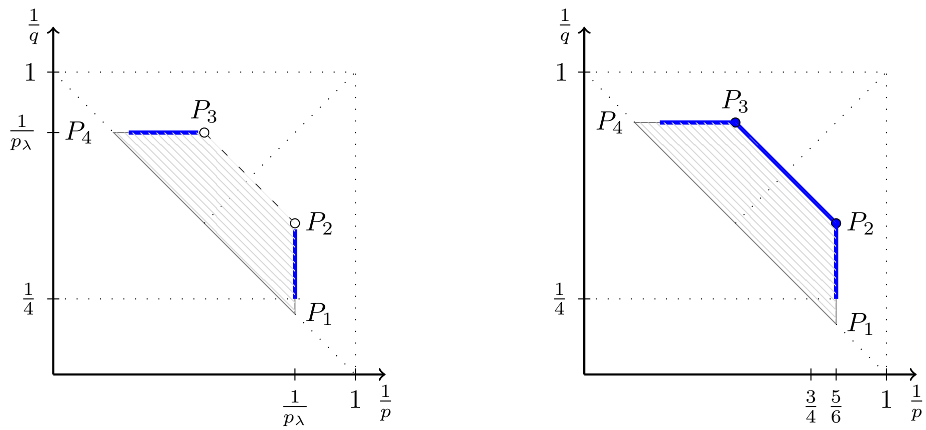

FIGURE 1. Sparse bounds for Riesz means $\mathcal{R}^\lambda_{a,t}$ in $\mathbb{R}^2$ for any $0 < \lambda < 1/2$ on the left, and for the special case $\lambda = 1/6$ on the right. The blue boundary segments correspond to the new content of Theorems 1.2 and 1.5, resp. Similar figures hold for $d \geq 3$ for a restricted range of λ; see Remarks after Theorem 1.3, and Theorem 1.5.

$\mathrm{Sp}(p_\lambda, q; \mathbb{R}^2)$ bounds hold in the range $\frac{4}{1+6\lambda} < q \leq 4$. Here we answer this question affirmatively, so that by duality we obtain a positive result for the full interior of the sides (P_1P_2) and (P_3P_4) in $\triangle_2(\lambda)$. It remains open what happens on the top side $\overline{P_2P_3}$ of $\triangle_2(\lambda)$, except for the special case $\lambda_* = 1/6$ covered in Theorem 1.5 below.

Theorem 1.2. *Let $d = 2$, $a > 0$. For $0 < \lambda < 1/2$, let $p_\lambda = \frac{4}{3+2\lambda}$. Then we have*

$$\|\mathcal{R}^\lambda_a\|_{\mathrm{Sp}(p_\lambda,q;\mathbb{R}^2)} < \infty, \qquad \textit{for } q > \tfrac{4}{1+6\lambda}.$$

In higher dimensions, we obtain similar optimal results but only for a partial range of λ away from 0. This is natural in view of the currently incomplete knowledge on $L^p \to L^r$ bounds for Bochner-Riesz type operators. It will be convenient to formulate the sparse bounds conditional on off-diagonal Lebesgue space estimates for the Bochner-Riesz operators $\mathcal{R}^\lambda_1$ (and unconditional for the Stein–Tomas exponent and some range beyond).

Theorem 1.3. *Let $d \geq 2$, $a > 0$ and $\frac{2(d+1)}{d+3} \leq p_\circ < \frac{2d}{d+1}$. Assume that for all $r_\circ \in [p_\circ, \frac{d-1}{d+1}p_\circ')$ the operator $\mathcal{R}^\lambda_1$ maps $L^{p_\circ}(\mathbb{R}^d) \to L^{r_\circ}(\mathbb{R}^d)$ for all $\lambda > \lambda(r_\circ)$. Then $\mathcal{R}^{\lambda(p)}_a \in \mathrm{Sp}(p,q)$ for $1 \leq p < p_\circ$ and $q > q_{opt} := \frac{(d-1)p}{d+1-2p}$.*

Several remarks are in order.

Remark 1.4. (i) The condition $q > q_{opt}$ is equivalent to saying that for the value $\lambda = \lambda(p)$, sparse bounds hold on the critical vertical line segment P_1P_2, except at the point P_2.

(ii) Theorem 1.2 is an immediate corollary of Theorem 1.3 due to the resolution of the Bochner–Riesz conjecture in 2 dimensions.

(iii) For $d \geq 3$, we are seeking to show that

$$\mathcal{R}_a^\lambda \in \mathrm{Sp}(p_\lambda, q), \quad p_\lambda = \tfrac{2d}{d+1+2\lambda}, \quad q > \tfrac{2(d-1)d}{2\lambda(d+1)+(d-1)^2}, \tag{1.6}$$

in a large range of λ. Since $\lambda(p_\lambda) = \lambda$ this corresponds to bounds on the critical endpoint segment P_1P_2, and the range of q is optimal up to P_2. By Theorem 1.3 this can be achieved if we have a non-endpoint Bochner-Riesz $L^{p_\circ} \to L^{r_\circ}$ bound for some $p_\circ > p_\lambda$ and all $r_\circ \in [p_\circ, \frac{d-1}{d+1}p'_\circ)$. Instances for which this Bochner-Riesz hypothesis is known (and therefore our theorem is unconditional) are:

- The Stein–Tomas [14] exponent $p_\circ = \frac{2(d+1)}{d+3}$. This leads to (1.6) for $\frac{d-1}{2(d+1)} < \lambda < \frac{d-1}{2}$.
- The so-called bilinear Fourier restriction exponent, that is, for $p_\circ < \frac{2(d+2)}{d+4}$, proven in [7] for $\rho(\xi) = |\xi|$. This leads to (1.6) for $\frac{d-2}{2(d+2)} < \lambda < \frac{d-1}{2}$.
- The exponents obtained through multilinear restriction: $p_\circ < \frac{2(d^2+3d-2)}{d^2+5d-2}$ for even $d \geq 4$, and $p_\circ < \frac{2(d^2+4d-1)}{d^2+6d+1}$ for odd $d \geq 5$, proven in [23] via the oscillatory integral estimates in [18]; these exponents correspond to the dual exponents to $q_\circ$ in [23, (1.15)]. This extends (1.6) to a range of λ's smaller than $\frac{d-2}{2(d+2)}$.

(iv) Key to Theorem 1.3 is Theorem 2.3, which replaces the non-endpoint Bochner–Riesz boundedness assumption by an endpoint variant for certain vector-valued functions, labelled $\mathrm{VBR}(p, r)$ in Definition 2.2. For further details see §2.

(v) Theorem 1.3 follows from a more general result that only imposes the $L^{p_\circ} \to L^{r_\circ}$ non-endpoint inequalities for the Bochner-Riesz operator in Theorem 1.3 for a specific $r_\circ$ (instead of the almost optimal range of $r_\circ$). Such a theorem is formulated as Theorem 2.1 below.

In Theorems 1.2 and 1.3 it remains open whether the $\mathrm{Sp}(p_\lambda, q_{opt,\lambda})$ bound holds with $q_{opt,\lambda} := \frac{2(d-1)d}{2\lambda(d+1)+(d-1)^2}$, that is, at the endpoint P_2. We can prove this when the Bochner-Riesz index is equal to $\lambda_* = \frac{d-1}{2(d+1)}$; in this instance $q_{opt,\lambda} = 2$. This corresponds to the endpoint in the Stein-Tomas restriction theorem and gives us added flexibility to use L^2 methods. We also obtain the corresponding sparse bounds on the full top side $\overline{P_2P_3}$, thereby proving the optimal sparse bounds in the closed trapezoid $\triangle_d(\lambda_*)$, for this special case.

Theorem 1.5. *Let $d \geq 2$, $a > 0$. Let $\lambda_* = \frac{d-1}{2(d+1)}$ and $(\frac{1}{p}, \frac{1}{q}) \in \triangle_d(\lambda_*)$. Then $\mathcal{R}_a^{\lambda_*} \in \mathrm{Sp}(p, q; \mathbb{R}^d)$.*

The main novelty of this paper is the introduction of a refined decomposition of the Riesz means $\mathcal{R}_{a,t}^\lambda$ which has improved kernel localization properties in the spirit of Christ [9] but still retains good Fourier support properties. This allows to combine the two existing sparse endpoint approaches for $\mathcal{R}_{a,t}^\lambda$, that is, the $p = 1$ result of [13], and the partial two-dimensional result for $p > 1$ of [22]. When $q = 2$ one can further exploit the Fourier orthogonality properties of the decomposition to obtain Theorem 1.5.

Notation. We list some frequently used notation.

- *Families of dyadic cubes.* We let $\mathfrak{D}$ be a dyadic lattice of cubes in the sense of Lerner and Nazarov [27]. We use $\mathfrak{Q}$ for general subcollections of $\mathfrak{D}$. We use the notation $\mathfrak{W}$ if such a subcollection is obtained by a Whitney decomposition of an open set with certain quantitative properties. We use $\mathfrak{S}$ for sparse families of dyadic cubes. The sidelength of a dyadic cube Q is denoted by $2^{L(Q)}$ with $L(Q) \in \mathbb{Z}$. For a collection $\mathfrak{Q}$ of cubes we denote by $\mathfrak{Q}_j$ the collection of cubes in $\mathfrak{Q}$ with sidelength 2^j. Similarly $\mathfrak{Q}_{\geq j}$ denotes the cubes $Q \in \mathfrak{Q}$ with $L(Q) \geq 2^j$. Analogously we define $\mathfrak{Q}_{\leq j}, \mathfrak{Q}_{>j}, \mathfrak{Q}_{<j}$.
- *Normalized bump functions.* For $M \geq 1$ let $\mathcal{Y}_M$ be the class of all C^M functions χ supported on $(\frac{1}{2}, 2)$ such that $\|\chi\|_{C^M} := \sum_{\nu=0}^{M} \|\chi^{(\nu)}\|_\infty \leq 1$.
- *Riesz multipliers.* We write $h_\lambda(\varrho) = \chi(\varrho)(1-\varrho)_+^\lambda$ with $\chi \in C_c^\infty((1/2, 2))$ and $\chi(\varrho) = 1$ near $\varrho = 1$ (see (3.1) below). The decomposition $h_\lambda = \sum_{\ell=0}^\infty h_{\lambda,\ell}$ is defined in (3.9).

Outline of the paper. In §2 we formulate refined versions of Theorem 1.3 involving Bochner–Riesz type inequalities for certain vector-valued functions and discuss how Theorems 1.2 and 1.3 follow from them. In §3 we introduce a crucial decomposition of the Riesz multipliers. In §4 we shall state the main technical estimates used in the sparse domination argument, with a key result (Theorem 4.2) proved in §5. Theorem 2.3, which is the main black-box sparse domination result, is proved in §6. The endpoint sparse domination results for the Riesz means at the index $\lambda = \frac{d-1}{2d+2}$ (Theorem 1.5) are treated in §7 and §8. Some consequences for weak type inequalities with weights, including the proof of Theorem 1.1, are discussed in §9.

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2. A refined version of Theorem 1.3 and Bochner-Riesz type bounds for vector-valued functions

We next formulate a more refined version of Theorem 1.3 which only involves a Bochner-Riesz non-endpoint $L^{p_\circ} \to L^{r_\circ}$ assumption for a specific value of $r_\circ$, as opposed to all values of $r_\circ \in [p_\circ, \frac{d-1}{d+1}p_\circ')$.

Let $d \geq 2$, $\frac{2(d+1)}{d+3} \leq p_\circ < \frac{2d}{d+1}$, and $p_\circ \leq r_\circ \leq \frac{d-1}{d+1}p_\circ'$. Define the exponent $r_*(p, p_\circ, r_\circ)$ and its dual $q_*(p, p_\circ, r_\circ)$ by

(2.1)

$$1 - \frac{1}{q_*(p, p_\circ, r_\circ)} = \frac{1}{r_*(p, p_\circ, r_\circ)} := \begin{cases} \frac{\frac{1}{r_0}(\frac{d+3}{2(d+1)} - \frac{1}{p}) + \frac{1}{2}(\frac{1}{p} - \frac{1}{p_\circ})}{\frac{d+3}{2(d+1)} - \frac{1}{p_\circ}} & \text{if } \frac{2(d+1)}{d+3} \leq p < p_\circ, \\ \frac{d+1}{d-1}(1 - \frac{1}{p}) & \text{if } 1 \leq p \leq \frac{2(d+1)}{d+3}. \end{cases}$$

These are motivated by interpolation numerology between the pairs $(\frac{1}{p_\circ}, \frac{1}{r_\circ})$ and $(\frac{d+3}{2(d+1)}, \frac{1}{2})$ when $\frac{2(d+1)}{d+3} < p < p_\circ$. Moreover, $q_*(p, p_\circ, r_\circ) = \frac{(d-1)p}{d+1-2p}$ when $p \leq \frac{2(d+1)}{d+3}$

and

$$\lim_{r_\circ\to\frac{d-1}{d+1}p'_\circ} r_*(p,p_\circ,r_\circ)=\tfrac{d-1}{d+1}p'\,,\qquad \lim_{r_\circ\to\frac{d-1}{d+1}p'_\circ} q_*(p,p_\circ,r_\circ)=\tfrac{(d-1)p}{d+1-2p} \tag{2.2}$$

for all $p<p_\circ$. The refinement of Theorem 1.3 is as follows.

Theorem 2.1. *Let $d\ge 2$, $a>0$, $\frac{2(d+1)}{d+3}\le p_\circ<\frac{2d}{d+1}$ and $p_\circ\le r_\circ\le \frac{d-1}{d+1}p'_\circ$. Assume that the operator $\mathcal{R}_1^\lambda$ maps $L^{p_\circ}(\mathbb{R}^d)$ to $L^{r_\circ}(\mathbb{R}^d)$ for all $\lambda>\lambda(r_\circ)$. Then $\mathcal{R}_a^{\lambda(p)}\in \mathrm{Sp}(p,q)$ for all $1\le p<p_\circ$ and $q>q_*(p,p_\circ,r_\circ)$.*

Note that Theorem 1.3 follows from Theorem 2.1 by using the second limiting relation in (2.2).

2.1. *Auxiliary inequalities on vector-valued functions.* The proof of Theorem 2.1 relies on certain inequalities for families of operators of Bochner-Riesz type acting on certain vector-valued L^p spaces, depending on admissible parameters p and r. We give a formal statement in the following definition; the set of normalized bump functions $\mathcal{Y}_M$ is defined in the notation section above.

Definition 2.2. *Let $1\le p\le r<\infty$. Let $\mathrm{VBR}(p,r)$ denote the following statement. There is $M>0$ such that for all collections χ_j of functions in $\mathcal{Y}_M$ the inequality*

$$\Big\|\sum_{j>0}2^{j\frac{d+1}{2}}\chi_j(2^j(1-\rho(D)))\big[\sum_{Q\in\mathfrak{D}_j}f_Q\big]\Big\|_{L^r(\mathbb{R}^d)}\le C_{p,r,d}\Big(\sum_{Q\in\mathfrak{D}}|Q|\|f_Q\|^r_{L^p(\mathbb{R}^d)}\Big)^{1/r} \tag{2.3}$$

holds for all families $\{f_Q\}_{Q\in\mathfrak{D}}$ of L^p functions f_Q with $\mathrm{supp}\,(f_Q)\subset\overline{Q}$.

Applying (2.3) to a family of cubes of a fixed sidelength 2^j shows that $\mathrm{VBR}(p,r)$ yields the multiplier bound for a single bump $\chi\in\mathcal{Y}_M$

$$\|\chi(2^j(1-\rho(D)))\|_{L^p\to L^r}=O(2^{j\lambda(r)}) \tag{2.4}$$

which is conjectured to hold for $1\le p<\frac{2d}{d+1}$, $p\le r\le\frac{d-1}{d+1}p'$; recall $\lambda(r)=d(\frac1r-\frac12)-\frac12$. The inequalities (2.3) are a multi-scale version of (2.4).

The main technical result that is used to prove essentially sharp sparse bounds for $\mathcal{R}_a^{\lambda(p)}$ reduces the conclusion of sparse bounds to estimates of VBR-type.

Theorem 2.3. *Let $d\ge 2$, $a>0$, $\frac{2(d+1)}{d+3}\le p_\circ<\frac{2d}{d+1}$ and $p_\circ\le r_\circ\le\frac{d-1}{d+1}p'_\circ$. Assume that $\mathrm{VBR}(p,r)$ holds for all $p\in[\frac{2(d+1)}{d+3},p_\circ)$ and $r\in[p,r_*(p,p_\circ,r_\circ))$. Then $\mathcal{R}_a^{\lambda(p)}\in\mathrm{Sp}(p,q)$ for $1\le p<p_\circ$, $q>q_*(p,p_\circ,r_\circ)$.*

The proof of Theorem 2.3 will be given in §§4–6. The conclusion of Theorem 2.3 also holds with $p=\frac{2(d+1)}{d+3}$ and $q=2$; this is the statement of Theorem 1.5, which is proved in §§7–8.

2.2. *Instances in which* $\mathrm{VBR}(p,r)$ *holds and relation with Theorem 2.1.* In order to fill Theorem 2.3 with content we first gather known results regarding $\mathrm{VBR}_d(p,r)$. The following results are available in the literature.

(i) For $d\ge2$, $\mathrm{VBR}(p,r)$ holds for $1\le p\le\frac{2(d+1)}{d+3}$, and $p\le r\le 2$.
(ii) For $d=2$, $\mathrm{VBR}(p,r)$ holds for $1\le p<4/3$, $p\le r<\min\{p'/3,2\}$.

(iii) Suppose that $1 < p_\circ < \frac{2d}{d+1}$ and suppose that $\mathcal{R}_1^\lambda$ is bounded on $L^{p_\circ}$ for all $\lambda > \lambda(p_\circ)$. Then $\mathrm{VBR}(p,p)$ holds for $1 \le p < p_\circ$.

Part (i) of this statement for $r = 2$ is just Lemma 3.7 and it is a standard consequence of the L^2-restriction theorem. The statement for $p \le r < 2$ is in [29], in the slightly more general setup for spectral multipliers on compact manifolds. Part (ii) for $p = r$ is an immediate consequence of a vector-valued inequality in [30], the general case follows by interpolating with the result in part (i). The conditional result in part (iii) was proved by Tao in his paper [34] on weak type (p,p) estimates for Bochner-Riesz means.

The bounds (i)-(iii) can be combined with Theorem 2.3 to deduce endpoint sparse bounds for $\mathcal{R}_a^\lambda$.

(i') The VBR inequalities in two dimensions stated in (ii) yield Theorem 1.2 (without passing through Theorem 1.3).

(ii') The VBR inequalities in the Stein-Tomas-range in (i) yield (1.6) for $\frac{d-1}{2(d+1)} < \lambda < \frac{d-1}{2}$ (that is, the conclusion of Theorem 1.3 if one inputs $p_\circ = \frac{2(d+1)}{d+3}$).

(iii') The $\mathrm{VBR}(p,p)$ bounds by Tao in (iii) for $\frac{2(d+1)}{d+3} < p < p_\circ$ yield some endpoint (p,q)-sparse bounds on a portion of the segment P_1P_2. However, this does not yet lead to close to optimal bounds for q in the sparse bounds. This phenomenon also occurs in the work by Kesler and Lacey [22] in two dimensions who essentially work with a $\mathrm{VBR}(p,p)$ input bound from [30].

In order to effectively prove sparse bounds in the whole (open) segment P_1P_2 beyond the Stein-Tomas range one needs to obtain an off-diagonal version of Tao's theorem. Tao [34, p. 1111] raises this question on whether there are such $L^p \to L^r$ versions of his theorem. Away from the critical line $r = \frac{d-1}{d+1}p'$ such versions can be obtained by using modifications of his proof which relies on ε-removal arguments. The interested reader can find the details in [3].

Theorem 2.4 ([3, Theorem 1.2]). *Let $d \ge 2$, $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ and $p_\circ \le r_\circ \le \frac{d-1}{d+1}p_\circ'$. Assume that the operator $\mathcal{R}_1^\lambda$ maps $L^{p_\circ} \to L^{r_\circ}$ for all $\lambda > \lambda(r_\circ)$. Then $\mathrm{VBR}(p,r)$ holds for $\frac{2(d+1)}{d+3} \le p < p_\circ$, $p \le r < r_*(p,p_\circ,r_\circ)$.*

It is clear that Theorem 2.1 is now a consequence of Theorems 2.3 and 2.4, which in turn implies Theorem 1.3.

3. Decompositions of Riesz means

We introduce a decomposition of the Riesz multipliers which has strong localization properties on both the kernel and the multiplier side and will play a crucial role in the estimates needed to establish the sparse domination results. We remark that rudimentary versions of this decomposition already featuring variants of condition (3.2) below go back to [8] and [33]. However, these have weaker conclusions that we found to be insufficient for our arguments in the proof of Theorem 2.3.

We start with some basic reductions. Let $\widetilde{\chi} \in C^\infty$ be supported in $(1/2, 2)$ such that $\widetilde{\chi}(\varrho) = 1$ in a neighborhood of 1. We note that for all $1 \le p < \infty$, a standard sparse $\mathrm{Sp}(p,p)$ bound holds for the Fourier multiplier operator with multiplier $(1 - \widetilde{\chi}(\rho(\xi))(1 - \rho(\xi)^a)_+^\lambda$. Indeed, note that for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \ge 1$ we have

$$\left|\partial_\xi^\alpha\left[(1-\widetilde{\chi}(\rho(\xi))(1-\rho(\xi)^a)_+^\lambda\right]\right| \lesssim_\alpha (1+|\xi|^{a-|\alpha|}+|\xi|^{1-|\alpha|})$$

which together with the support property implies a kernel estimate $O((1+|x|)^{-d-\varepsilon})$ with $\varepsilon<\min\{1,a\}$ for the underlying kernel. We therefore focus on the essential contribution, corresponding to the multiplier $\widetilde{\chi}(\rho(\xi))(1-\rho(\xi)^a)_+^\lambda$. We also note that we can assume without loss of generality that $a=1$. This is because

$$\chi_{a,\lambda}(\varrho)=\widetilde{\chi}(\varrho)\frac{(1-\varrho^a)^\lambda}{(1-\varrho)^\lambda} \tag{3.1a}$$

is smooth near $\varrho=1$ and thus it suffices to just consider the multiplier $h_\lambda(\rho(\xi))$ with

$$h_\lambda(\varrho)=\chi_{a,\lambda}(\varrho)(1-\varrho)_+^\lambda, \tag{3.1b}$$

where for fixed a, the family $\{\chi_{a,\lambda}:|\lambda|\le d\}$ is a bounded collection of C_c^∞ functions supported in $(\frac12,2)$. We shall write $\chi\equiv\chi_{a,\lambda}$ in what follows.

The following lemmas will be useful in further splitting the multiplier h_λ.

Lemma 3.1. *Let $\lambda>0$, $N_\circ\in\mathbb{N}$. There exists an even $C_c^\infty(\mathbb{R})$ function $\Phi_\circ$ such that $\Phi_\circ(s)=1$ for $|s|\le 1/2$ and $\Phi_\circ(s)=0$ for $|s|\ge 1$ and, in addition,*

$$\int_0^\infty \varrho^\lambda\big(\tfrac{d}{d\varrho}\big)^j\widehat{\Phi_\circ}(\varrho)\,\mathrm{d}\varrho=0 \quad \textit{for}\;\; j=0,1,\dots,N_\circ,\; j\ne\lambda. \tag{3.2}$$

Proof. We consider the interval $I=[-7/4,-5/4]$ and $L^2(I)$ with the usual scalar product. Let $\mathbb{V}$ be the span of the functions $s\mapsto|s|^{-\lambda+j}\mathbb{1}_{[-7/4,-5/4]}$ where $j=0,\dots,N_\circ$ with $j\ne\lambda$. We pick $u\in L^2$ supported on I such that

$$\int_I u(s)\,\mathrm{d}s=1$$

and such that $u\in\mathbb{V}^\perp$; that is, we have $\int_I u(s)|s|^{j-\lambda}\,\mathrm{d}s=0$ for integers $0\le j\le N_\circ$ with $j\ne\lambda$. Note that also $\int_{-\infty}^0 u(s/t)|s|^{j-\lambda}\,\mathrm{d}s=0$ for those j and all $t>0$. This suggests that in order to regularize u we should work with a multiplicative convolution. Let $0<\varepsilon<1/8$ and $w\in C_c^\infty$ supported in $(1-\varepsilon,1+\varepsilon)$ with $\int w(x)\,\mathrm{d}x=1$. Define for $x<0$

$$U(x)=\int_0^\infty u\Big(\frac{x}{t}\Big)w(t)\frac{\mathrm{d}t}{t}=\int_0^\infty u(-t)w\Big(\frac{-x}{t}\Big)\frac{\mathrm{d}t}{t}$$

and set, for $x>0$, $U(x)=-U(-x)$, and $U(0)=0$. In view of the support properties of u and w, we see that U is an odd C_c^∞ function supported in $(-2,-1)\cup(1,2)$ and we have

$$\int_{-\infty}^0 U(s)|s|^{j-\lambda}\,\mathrm{d}s=\int_{1-\varepsilon}^{1+\varepsilon} w(t)t^{j-\lambda}\,\mathrm{d}t\int_I u(s)|s|^{j-\lambda}\,\mathrm{d}s=0 \tag{3.3}$$

for all $j\in\{0,1,\dots,N_\circ\}\backslash\{\lambda\}$, since $I\subseteq[-2/t,-1/t]$ for $t\in(1-\varepsilon,1+\varepsilon)$. Similarly, for $-1\le x\le 1$,

$$\int_{-\infty}^x U(s)\,\mathrm{d}s=\int_{-\infty}^{-1}U(s)\,\mathrm{d}s=\int_{1-\varepsilon}^{1+\varepsilon}w(t)\,\mathrm{d}t\int_I u(s)\,\mathrm{d}s=1.$$

We now define

$$\Phi_\circ(x) := \int_{-\infty}^{x} U(s)\,\mathrm{d}s.$$

From the above calculations, we obtain that $\Phi_\circ$ is an even C_c^∞ function supported in $(-2,2)$ such that $\Phi_\circ(x) = 1$ for $|x| \le 1$ and

$$\int_1^2 \Phi_\circ'(x) x^{j-\lambda}\,\mathrm{d}x = 0, \quad j \in \{0,1,\dots,N_\circ\} \setminus \{\lambda\}. \tag{3.4}$$

We will next show that (3.4) implies (3.2).

Recall that for $\lambda > -1$ the distributional Fourier transform of $\varrho_+^\lambda/\Gamma(\lambda+1)$ is the distribution $e^{-i\pi(\lambda+1)/2}(\xi - i0)^{-\lambda-1}$; see for example [20, p.167]. This means that for Schwartz functions ϕ we have

$$\int_0^\infty \widehat{\phi}(\varrho) \frac{\varrho^\lambda}{\Gamma(\lambda+1)}\,\mathrm{d}\varrho = e^{-i\pi(\lambda+1)/2} \lim_{y\to 0+} \int_{-\infty}^{\infty} \phi(x)(x-iy)^{-\lambda-1}\,\mathrm{d}x \tag{3.5}$$

and the limit exists (*cf.* [20, Thm 3.1.11]); moreover the tempered distribution $(x-i0)^{-\lambda-1}$ is identified with the function $x^{-\lambda-1}$ in $(0,\infty)$. The previous display gives

$$\int_0^\infty \varrho^\lambda \big(\frac{d}{d\varrho}\big)^j \widehat{\Phi_\circ}(\varrho)\,\mathrm{d}\varrho = \frac{\Gamma(\lambda+1)}{e^{i\pi(\lambda+1)/2}} \lim_{y\to 0+} \int_{-\infty}^{\infty} (-ix)^j \Phi_\circ(x)(x-iy)^{-\lambda-1}\,\mathrm{d}x.$$

In view of the existence of the boundary value distribution $(x-i0)^{-\lambda-1}$ it is immediate that for $j \ge 1$

$$\lim_{y\to 0+} \int_{-\infty}^{\infty} (-i)^j \big(x^j - (x-iy)^j\big) \Phi_\circ(x)(x-iy)^{-\lambda-1}\,\mathrm{d}x = 0;$$

indeed the integral can be written as $\sum_{k=1}^{j} y^k \int \phi_k(x)(x-iy)^{-\lambda-1}\,\mathrm{d}x$ with suitable test functions ϕ_k. Therefore we get

$$\int_0^\infty \varrho^\lambda \big(\frac{d}{d\varrho}\big)^j \widehat{\Phi_\circ}(\varrho)\,\mathrm{d}\varrho = \frac{(-i)^j\Gamma(\lambda+1)}{e^{i\pi(\lambda+1)/2}} \lim_{y\to 0+} \int_{-\infty}^{\infty} \Phi_\circ(x)(x-iy)^{j-\lambda-1}\,\mathrm{d}x.$$

Integrating by parts and using $j \ne \lambda$ we also get for fixed $y > 0$

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi_\circ(x)(x-iy)^{j-\lambda-1}\,\mathrm{d}x &= -\frac{1}{j-\lambda} \int_{-\infty}^{\infty} \Phi_\circ'(x)(x-iy)^{j-\lambda}\,\mathrm{d}x \\ &= -\frac{1}{j-\lambda}\Big(\int_1^2 \Phi_\circ'(x)(x-iy)^{j-\lambda}\,\mathrm{d}x + \int_{-2}^{-1} \Phi_\circ'(x)(x-iy)^{j-\lambda}\,\mathrm{d}x\Big).\end{aligned}$$

For $j \ne \lambda$ the boundary value distribution $(x-i0)^{j-\lambda}$ is identified with the function $x^{j-\lambda}$ on $(0,\infty)$ and with the function $(e^{-i\pi}|x|)^{j-\lambda}$ on $(-\infty,0)$. Also recall that $\Phi_\circ' \equiv U$ is odd. Combining the above observations we obtain after taking the limit,

$$\int_0^\infty \varrho^\lambda \big(\frac{d}{d\varrho}\big)^j \widehat{\Phi}(\varrho)\,\mathrm{d}\varrho = \frac{\Gamma(\lambda+1)}{e^{i\pi(\lambda+1)/2}} \frac{(-1)^{j+1}}{j-\lambda} (1 - e^{i\pi(j-\lambda)}) \int_1^2 \Phi_\circ'(x) x^{j-\lambda}\,\mathrm{d}x$$

and (3.2) follows from (3.4). □

The condition (3.2) fails when $j = \lambda$. In this case we have instead

Lemma 3.2. *For all even Schwartz functions ϕ, and $j = 0, 1, 2, 3, \dots$*

$$\int_0^\infty \varrho^j \Big(\frac{d}{d\varrho}\Big)^j \widehat{\phi}(\varrho)\,\mathrm{d}\varrho = (-1)^j \pi j! \phi(0). \tag{3.6}$$

Proof. A j-fold integration by parts yields that

$$\int_0^\infty \varrho^j \Big(\frac{d}{d\varrho}\Big)^j \widehat{\phi}(\varrho)\,\mathrm{d}\varrho = (-1)^j j! \int_0^\infty \widehat{\phi}(\varrho)\,\mathrm{d}\varrho = \frac{(-1)^j}{2} j! \int_{-\infty}^\infty \widehat{\phi}(\varrho)\,\mathrm{d}\varrho,$$

where the second identity follows since $\widehat{\phi}$ is even. The claim now follows from the Fourier inversion formula. □

As an immediate consequence of Lemma 3.1, and in the case of integer λ also Lemma 3.2, we obtain

Corollary 3.3. *Let $\lambda > 0$, $N_\circ \in \mathbb{N}$. Let $\Phi_\circ$ be as in Lemma 3.1 and let*

$$\Psi(x) = \Phi_\circ(x/2) - \Phi_\circ(x). \tag{3.7}$$

Then Ψ is an even $C_c^\infty(\mathbb{R})$ function such that $\Psi(s) = 0$ for $|s| \le 1/2$ and $\Psi(s) = 0$ for $|s| \ge 2$ and such that

$$\int_0^\infty \varrho^\lambda \big(\tfrac{d}{d\varrho}\big)^j \widehat{\Psi}(\varrho)\,\mathrm{d}\varrho = 0, \quad j = 0, 1, \dots, N_\circ. \tag{3.8}$$

We now decompose $\mathcal{F}_{\mathbb{R}}^{-1}[(1-\varrho)_+^\lambda]$ dyadically, using the functions $\Phi_\circ$, and dilates of Ψ as in (3.7). In the following definition (and then throughout the paper) we will assume that $N_\circ$ in (3.2) satisfies $N_\circ > d$. We get

$$h_\lambda = \sum_{\ell=0}^\infty h_{\lambda,\ell} \quad \text{with} \tag{3.9a}$$

$$h_{\lambda,0}(\varrho) = \frac{\chi(\varrho)}{2\pi} \int_{-\infty}^\infty (1-u)_+^\lambda \widehat{\Phi_\circ}(\varrho - u)\,\mathrm{d}u, \tag{3.9b}$$

$$h_{\lambda,\ell}(\varrho) = \frac{\chi(\varrho)}{2\pi} \int_{-\infty}^\infty (1-u)_+^\lambda 2^{\ell-1} \widehat{\Psi}(2^{\ell-1}(\varrho - u))\,\mathrm{d}u, \quad \ell > 0. \tag{3.9c}$$

Lemma 3.4. *For all $N_1 \in \mathbb{N}$ and for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le N_1$*

$$|\partial_\xi^\alpha [h_{\lambda,\ell} \circ \rho](\xi)| \le C_{N_1,\alpha} 2^{-\ell(\lambda - |\alpha|)} (1 + 2^\ell |1 - \rho(\xi)|)^{-N_1}. \tag{3.10}$$

Let $\ell \ge 0$, $N_\circ$ as in Corollary 3.3 and $|\alpha| \le N_\circ$. Then

$$|\partial_\xi^\alpha [h_{\lambda,\ell} \circ \rho](\xi)| \lesssim 2^{-\ell(\lambda - |\alpha|)} |2^\ell (1 - \rho(\xi))|^{N_\circ + 1 - |\alpha|} \ \textit{if}\ |1 - \rho(\xi)| \le 2^{-\ell}. \tag{3.11}$$

Proof. Repeated integration by parts yields (3.10). We use Corollary 3.3 and Taylor's theorem to compute for $\ell \ge 1$

$$\begin{aligned} h_{\lambda,\ell}(\varrho) &= \chi(\varrho)\frac{1}{2\pi}\int_{-\infty}^{1}(1-u)^{\lambda}2^{\ell}\widehat{\Psi}(2^{\ell}(\varrho-u))\,\mathrm{d}u \\ &= \frac{\chi(\varrho)}{2\pi}\int_{-\infty}^{1}(1-u)^{\lambda}2^{\ell}\big[\widehat{\Psi}(2^{\ell}(\varrho-u)) - \sum_{j=0}^{N_\circ}\frac{(2^{\ell}(\varrho-1))^{j}}{j!}\widehat{\Psi}^{(j)}(2^{\ell}(1-u))\big]\,\mathrm{d}u \\ &= (2^{\ell}(\varrho-1))^{N_\circ+1}\frac{\chi(\varrho)}{2\pi}\times \\ &\qquad \int_0^1\frac{(1-\sigma)^{N_\circ}}{N_\circ!}\int_{-\infty}^{1}(1-u)^{\lambda}2^{\ell}\widehat{\Psi}^{(N_\circ+1)}(2^{\ell}(1-u+\sigma(\varrho-1)))\,\mathrm{d}u\,\mathrm{d}\sigma \end{aligned}$$

which in turn gives $|h_{\lambda,\ell}(\varrho)| \lesssim 2^{-\ell\lambda}|2^{\ell}(1-\varrho)|^{N_\circ+1}$ for $|1-\varrho| \le 2^{-\ell}$. Thus, setting $\varrho = \rho(\xi)$, (3.11) follows for $\alpha = 0$. A similar calculation follows for $\partial_\varrho^j h_{\lambda,\ell}$ and then (3.11) for higher derivatives follows by applications of the multivariate Leibniz rule and the Faà di Bruno's formula. □

Since $\nabla\rho$ is homogeneous of order zero and since $\nabla\rho$ does not vanish on $\partial\Omega$ there are two positive constants c_0, C_0 such that $C_0 \ge 1$ and

$$c_0 < |\nabla\rho(\xi)| \le C_0 \text{ for all } \xi \ne 0. \tag{3.12a}$$

Later in the paper it will also be useful to fix a positive integer $n_\circ$ such that

$$2^{d+4}C_0 < 2^{n_\circ}. \tag{3.12b}$$

We next study the properties of the convolution kernels

$$K_{\lambda,\ell}(x) := \mathcal{F}^{-1}[h_{\lambda,\ell}\circ\rho](x), \qquad \ell \ge 0.$$

The next lemma shows that for $\ell > 0$, the kernels $K_{\lambda,\ell}$ are essentially supported in $\{x : c_0 2^{\ell-2} \le |x| \le C_0 2^{\ell+2}\}$. No curvature assumption is necessary here.

Lemma 3.5. *For all $N \in \mathbb{N}$,*

$$|K_{\lambda,\ell}(x)| \lesssim_N \begin{cases} |x|^{-N} & \text{for } |x| \ge 2^{\ell+2}C_0 \\ 2^{-\ell N} & \text{for } |x| \le 2^{\ell-2}c_0. \end{cases}$$

Proof. The statement for $\ell = 0$ follows from integration by parts. We thus assume $\ell \ge 1$ in what follows. We use the definition $h_{\lambda,\ell}\circ\rho$ to write

$$h_{\lambda,\ell}(\rho(\xi)) = m_{\lambda,\ell,1}(\xi) + m_{\lambda,\ell,2}(\xi)$$

where $m_{\lambda,\ell,1}(\xi) = \chi(\rho(\xi))\frac{1}{2\pi}\iint \chi_1(u)(1-u)_+^{\lambda}\Psi(2^{-\ell}s)e^{is(\rho(\xi)-u)}\,\mathrm{d}s\,\mathrm{d}u$. We have

$$\mathcal{F}^{-1}[m_{\lambda,\ell,1}](x) = \frac{1}{(2\pi)^{d+1}}\iint \chi_1(u)(1-u)_+^{\lambda}\Psi(2^{-\ell}s)e^{-isu}\int \chi(\rho(\xi))e^{is\rho(\xi)+i\langle x,\xi\rangle}\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}u. \tag{3.13}$$

Analyzing the gradient of the phase function in the inner ξ integral we get for $2^{\ell-1} < s < 2^{\ell+1}$

$$|s\nabla\rho(\xi) + x| \ge \begin{cases} |x|/2 & \text{for } |x| \ge 2^{\ell+2}C_0 \\ 2^{-\ell-2}c_0 & \text{for } |x| \le 2^{\ell-2}c_0 \end{cases}$$

and an integration by parts in ξ shows that for $2^{\ell-1} < s < 2^{\ell+1}$

$$\Big|\int \chi(\rho(\xi))e^{is\rho(\xi)+i\langle x,\xi\rangle}\,\mathrm{d}\xi\Big| \lesssim_N \begin{cases} |x|^{-N-1} & \text{for } |x| \ge 2^{\ell+2}C_0 \\ 2^{-\ell(N+1)} & \text{for } |x| \le 2^{\ell-2}c_0 \end{cases} \tag{3.14}$$

which after a trivial integration in s and u implies the desired bounds on $\mathcal{F}^{-1}[m_{\lambda,\ell,1}](x)$.

We now examine $\mathcal{F}^{-1}[m_{\lambda,\ell,2}]$ where $m_{\lambda,\ell,2}(\xi) = h_{\lambda,\ell}(\rho(\xi)) - m_{\lambda,\ell,1}(\xi)$. The definition of of $m_{\lambda,\ell,2}$ involves a one-dimensional Fourier transform of $u \mapsto (1-\chi_1(u))(1-u)_+^\lambda$, where the latter is supported in $(-\infty, 4/5)$. We perform a dyadic decomposition in the negative u variables. Let $\eta_0 \in C_c^\infty(\mathbb{R})$ such that η_0 is supported in $(-5/6, 5/6)$ and $\eta_0(u) = 1$ for $u \in (-4/5, 4/5)$ and let, for $k \ge 1$, $\eta_k(u) = \eta_0(2^{-k}u) - \eta_0(2^{1-k}u)$. We then have $m_{\lambda,\ell,2} = \sum_{k=0}^\infty m_{\lambda,\ell,2,k}$ where by integration by parts for all $N_1 \ge 0$,

$$\begin{aligned} m_{\lambda,\ell,2,k}(\xi) &= \frac{\chi(\rho(\xi))}{2\pi}\iint \eta_k(u)(1-\chi_1(u))(1-u)_+^\lambda \Psi(2^{-\ell}s)e^{is(\rho(\xi)-u)}\,\mathrm{d}s\,\mathrm{d}u \\ &= \frac{\chi(\rho(\xi))}{2\pi}\iint \partial_u^{N_1}\big[\eta_k(u)(1-\chi_1(u))(1-u)_+^\lambda\big]\frac{\Psi(2^{-\ell}s)}{(is)^{N_1}}e^{is(\rho(\xi)-u)}\,\mathrm{d}s\,\mathrm{d}u \end{aligned}$$

and the sum in k converges rapidly in view of the estimate

$$|m_{\lambda,\ell,2,k}(\xi)| \lesssim 2^{k(\lambda-N_1)}2^{\ell(1-N_1)}.$$

Note that because of the cutoff $\chi(\rho(\xi))$ the same bound immediately holds for $\|\mathcal{F}^{-1}[m_{\lambda,\ell,2,k}]\|_\infty$. We will apply this with $N_1 \gg N+\lambda$. After summing in k we get a satisfactory bound for $|x| \le C_0 2^{\ell+3}$. For $|x| \ge C_0 2^{\ell+2}$ we again integrate by parts in ξ (*cf.* (3.14)) and obtain the bound $|\mathcal{F}^{-1}[m_{\lambda,\ell,2,k}](x)| \lesssim 2^{k(\lambda-N_1)}2^{\ell(1-N_1)}(2^\ell|x|)^{-N-1}$ which again can be summed in k. Altogether we get

$$|\mathcal{F}^{-1}[m_{\lambda,\ell,2}](x)| \lesssim_N 2^{-\ell N}(1+2^{-\ell}|x|)^{-N} \tag{3.15}$$

for all $x \in \mathbb{R}^d$, which completes the proof. □

We get sharp estimates for the region $|x| \approx 2^\ell$ since $\partial\Omega$ has nonvanishing Gaussian curvature everywhere.

Lemma 3.6. $\|K_{\lambda,\ell}\|_\infty \lesssim 2^{-\ell(\lambda+\frac{d+1}{2})}$.

Proof. By Lemma 3.5 it suffices to prove the bound for $|x| \approx 2^\ell$. We write the Fourier integral in ρ-polar coordinates $\xi = \varrho\xi'$ with $\xi' \in \partial\Omega$, $\mathrm{d}\mu(\xi') = \langle \mathfrak{n}(\xi'), \xi'\rangle d\sigma(\xi')$, where $\mathfrak{n}$ is the outer normal at $\xi' \in \partial\Omega$. We obtain

$$\begin{aligned} (2\pi)^{d+1}K_{\lambda,\ell}(x) &= \iint (1-u)_+^\lambda \Psi(2^{-\ell}s)e^{-isu}\iint \varrho^{d-1}\chi(\varrho)e^{is\varrho+i\langle x,\varrho\xi'\rangle}\,\mathrm{d}\mu(\xi')\,\mathrm{d}\varrho\,\mathrm{d}s\,\mathrm{d}u \\ &= c\int \Psi(2^{-\ell}s)s^{-\lambda-1}\int \varrho^{d-1}\chi(\varrho)e^{is(\varrho-1)}\int_{\partial\Omega} e^{i\varrho\langle x,\xi'\rangle}\,\mathrm{d}\mu(\xi')\,\mathrm{d}\varrho\,\mathrm{d}s. \end{aligned}$$

Since $\partial\Omega$ has nonvanishing Gaussian curvature, the inner integral can be written, by the method of stationary phase, as a sum of two expressions of the form $c_\pm e^{i\varrho\langle x,\xi'_\pm(x)\rangle}a_\pm(\varrho,x)$ where $a_\pm$ are smooth and, together with their derivatives, satisfy the bound $O((1+|x|)^{-\frac{d-1}{2}})$. The points $\xi'_\pm(x)$ are the two unique points on $\partial\Omega$ where x is normal to $\partial\Omega$. Subsequent integration by parts in ρ yields for $|s| \approx |x| \approx 2^\ell$

$$\Big| \int \varrho^{d-1}\chi(\varrho) e^{is(\varrho-1)} \int_{\partial\Omega} e^{i\varrho\langle x,\xi'\rangle}\, \mathrm{d}\mu(\xi')\, \mathrm{d}\varrho \Big| \lesssim 2^{-\ell(d+1)/2}$$

and after integrating in s obtain the asserted bound. □

Stein–Tomas type estimates. For the proof of Theorem 1.5 we need the following consequences of the Stein–Tomas restriction theorem. Note that (3.18) corresponds to the VBR$(p, 2)$ condition mentioned in (i), §2.2.

Lemma 3.7. *Let $1 \le p \le \frac{2(d+1)}{d+3}$ and let M be an integer with $M > d(\frac{1}{p} - \frac12)$. Let $s \mapsto \vartheta_j(s)$ satisfy, for $\nu = 0, 1, \dots, M$,*

$$\Big| \Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^{\nu} (\vartheta_j(s)) \Big| \le (1 + |s|)^{-M} . \tag{3.16}$$

Let $m_j(\xi) = \vartheta_j(2^j(1 - \rho(\xi)))$. Then for each $j \ge 0$

$$\Big\| \sum_{Q\in\mathfrak{D}_j} 2^{j\frac{d+1}{2}} m_j(D)[f_Q \mathbb{1}_Q] \Big\|_2 \lesssim \Big(\sum_{Q\in\mathfrak{D}_j} |Q| \|f_Q\|_p^2 \Big)^{\frac12}. \tag{3.17}$$

If, in addition, the functions ϑ_j are supported in $(1/4, 4)$ then

$$\Big\| \sum_{j\ge0} \sum_{Q\in\mathfrak{D}_j} 2^{j\frac{d+1}{2}} m_j(D)[f_Q \mathbb{1}_Q] \Big\|_2 \lesssim \Big(\sum_{Q\in\mathfrak{D}} |Q| \|f_Q\|_p^2 \Big)^{\frac12}. \tag{3.18}$$

We omit the proof; it relies on a standard argument by Fefferman and Stein [15], with a refinement in [29].

4. The main estimates

At the heart of the matter of the proof of Theorem 2.3 lie certain estimates in Proposition 4.4 below in terms of collections of functions stemming from the Calderón–Zygmund decomposition. To prove these estimates it is convenient to introduce a family of bilinear operators which allow an abstract formulation that is a priori unrelated to the Calderón–Zygmund decomposition.

In the following let $\mathfrak{Q} \subset \mathfrak{D}_{\ge0}$. On the set $\mathfrak{Q}$ we will consider the atomic measure given by $\mu(\{Q\}) = |Q|$; i.e. for each subset $\mathfrak{E} \subset \mathfrak{Q}$ we have

$$\mu(\mathfrak{E}) = \sum_{j\ge0} 2^{jd} \#\mathfrak{E}_j \tag{4.1}$$

where again $\mathfrak{E}_j$ is the subset of $\mathfrak{E}$ consisting of cubes of sidelength 2^j. This choice of measure is natural since in the special case where $\mathfrak{E}$ is a *disjoint* collection of dyadic cubes $\mu(\mathfrak{E})$ is just the Lebesgue measure of the union of the Q in $\mathfrak{E}$. Fix λ and let $h_{\lambda,\ell}$ be defined as in (3.9). Set

$$A_{\lambda,\ell} f = 2^{\ell(\lambda + \frac{d+1}{2})} h_{\lambda,\ell}(\rho(D)) f. \tag{4.2}$$

For $\mathfrak{Q} \subset \mathfrak{D}$ and functions $\beta : \mathfrak{Q} \to \mathbb{C}$ we denote by $\ell^r(\mathfrak{Q}, \mu)$ the space of all β such that $\|\beta\|_{\ell^r(\mu)} = (\sum_{Q\in\mathfrak{Q}} |\beta(Q)|^r |Q|)^{1/r}$ and by $\ell^{r,1}(\mathfrak{Q}, \mu)$ the corresponding Lorentz space. We also consider families of $L^p(\mathbb{R}^d)$ functions $F = \{F_Q\}_{Q\in\mathfrak{Q}}$ and set

$\|F\|_{\ell^\infty(L^p)} = \sup_{Q\in\mathfrak{Q}} \|F_Q\|_p$. For any integer $s \geq 0$, define the bilinear operators $\Xi_{s,\mathfrak{Q}}$ acting on $\ell^\infty(L^p(\mathbb{R}^d)) \times \ell^{r,1}(\mu)$ by

$$\Xi_{s,\mathfrak{Q}}[F,\beta] := \sum_{\ell\geq s} \sum_{Q\in\mathfrak{Q}_{\ell-s}} \beta(Q) A_{\lambda,\ell}[F_Q \mathbb{1}_Q]. \tag{4.3}$$

The definition of $\Xi_{s,\mathfrak{Q}}$ depends on λ via $A_{\lambda,\ell}$ in (4.2) but the operators $A_{\lambda,\ell}$ satisfy bounds that are uniform in λ when λ varies over a compact set, and this will also hold for the $\Xi_{s,\mathfrak{Q}}$. In analogy with the setup in [19], the normalization chosen in (4.2) is advantageous for standard interpolation arguments. To shorten the notation we use the following.

Definition 4.1. *Let* $\mathrm{Hyp}(p_\circ, r_\circ)$ *denote the statement that* $\mathrm{VBR}(p,r)$ *holds for all* $p \in [\frac{2(d+1)}{d+3}, p_\circ)$ *and* $r \in [p, r_*(p, p_\circ, r_\circ)$.

Note that these correspond exactly to the hypothesis in Theorem 2.3.

Theorem 4.2. *Let* $d \geq 2$, $\frac{2(d+1)}{d+3} \leq p_\circ < \frac{2d}{d+1}$, $p_\circ \leq r_\circ \leq \frac{d-1}{d+1} p_\circ'$ *and assume that* $\mathrm{Hyp}(p_\circ, r_\circ)$ *holds. Then for* $1 \leq p < p_\circ$ *and* $p < r < r_*(p, p_\circ, r_\circ)$ *there is* $\varepsilon = \varepsilon(p,r) > 0$ *such that for all* $s \geq 0$ *and collections of disjoint cubes* $\mathfrak{Q} \subset \mathfrak{D}_{\geq 0}$,

$$\|\Xi_{s,\mathfrak{Q}}[F,\beta]\|_{L^r} \lesssim 2^{s(\frac{d}{p}-\varepsilon)} \|\beta\|_{\ell^{r,1}(\mu)} \|F\|_{\ell^\infty(L^p)}.$$

We will prove Theorem 4.2 in §5. It will be convenient to also state a straightforward variant with larger cubes in $\mathfrak{Q}_{\ell+n}$, which is implied by Theorem 4.2.

Corollary 4.3. *Assume the assumptions of Theorem 4.2 and let for* $n \geq 0$,

$$\Xi_{-n,\mathfrak{Q}}[F,\beta] := \sum_{\ell\geq 0} \sum_{Q\in\mathfrak{Q}_{\ell+n}} \beta(Q) A_{\lambda,\ell}[F_Q \mathbb{1}_Q].$$

Then for all $n \geq 0$, *and collections of disjoint cubes* $\mathfrak{Q} \subset \mathfrak{D}_{\geq 0}$,

$$\|\Xi_{-n,\mathfrak{Q}}[F,\beta]\|_r \lesssim \|\beta\|_{\ell^{r,1}(\mu)} \|F\|_{\ell^\infty(L^p)}.$$

Proof. We apply Theorem 4.2 for $s = 0$. Indeed let for each cube $Q \in \mathfrak{D}$ denote by $R^n(Q)$ the unique cube in $\mathfrak{D}_{L(Q)+n}$ which contains Q. Let $\widetilde{\mathfrak{Q}}$ be the collection of all cubes $Q \in \mathfrak{D}_{\geq 0}$ such that $R^n(Q) \in \mathfrak{Q}$. If the cubes in $\mathfrak{Q}$ are disjoint then the cubes in $\widetilde{\mathfrak{Q}}$ are also disjoint. For $Q \in \widetilde{\mathfrak{Q}}$ we set $\widetilde{\beta}(Q) = \beta(R^n(Q))$ and $\widetilde{F}_Q = F_{R^n(Q)}$. Then $\Xi_{-n,\mathfrak{Q}}(F,\beta) = \Xi_{0,\widetilde{\mathfrak{Q}}}(\widetilde{F},\widetilde{\beta})$, $\|\widetilde{F}\|_{\ell^\infty(\widetilde{\mathfrak{Q}},L^p)} = \|F\|_{\ell^\infty(\mathfrak{Q},L^p)}$ and

$$\|\widetilde{\beta}\|^r_{\ell^r(\mu,\widetilde{\mathfrak{Q}})} = \sum_{\ell\geq 0} \sum_{Q\in\widetilde{\mathfrak{Q}}_\ell} |Q||\widetilde{\beta}(Q)|^r = \sum_{\ell\geq 0} \sum_{Q'\in\mathfrak{Q}_{\ell+n}} \sum_{\substack{Q\in\mathfrak{D}_\ell \\ Q\subset Q'}} |Q||\beta(Q')|^r = \|\beta\|^r_{\ell^r(\mu,\mathfrak{Q})}.$$

The corollary now follows applying Theorem 4.2 to $\Xi_{0,\widetilde{\mathfrak{Q}}}[\widetilde{F},\widetilde{\beta}]$. □

The main motivation for Theorem 4.2 is its applications to the action of Bochner-Riesz type operators on the collection of functions in a Calderón–Zygmund decomposition.

Proposition 4.4. *Let $d \ge 2$, $\frac{2(d+1)}{d+3} \le p_\circ < \frac{2d}{d+1}$, $p_\circ \le r_\circ \le \frac{d-1}{d+1} p'_\circ$ and assume that* $\mathrm{Hyp}(p_\circ, r_\circ)$ *holds. Let $1 \le p < p_\circ$ and $p < r < r_*(p, p_\circ, r_\circ)$. Let $\mathfrak{Q} \subset \mathfrak{D}_{\ge 0}$ be a collection of disjoint cubes, $\alpha > 0$ and $\{f_Q\}_{Q \in \mathfrak{Q}}$ functions with* $\mathrm{supp}\,(f_Q) \subset \overline{Q}$ *and*

$$\int_Q |f_Q|^p \le \alpha^p |Q| \qquad \text{for all } Q \in \mathfrak{Q}. \tag{4.4}$$

Then there exists an $\varepsilon = \varepsilon(p, r) > 0$ such that for all $s \ge 0$, the inequality

$$\Big\| \sum_{\ell \ge s} u_\ell h_{\lambda(p),\ell}(\rho(D)) \big[\sum_{Q \in \mathfrak{Q}_{\ell - s}} f_Q \big] \Big\|_r^r \lesssim \|u\|_{\ell^\infty}^r 2^{-\varepsilon s r} \alpha^{r-p} \sum_{Q \in \mathfrak{Q}} \|f_Q\|_p^p \tag{4.5}$$

holds for all sequences of complex numbers $u = \{u_\ell\}_{\ell=0}^\infty$. Moreover, for all $n \ge 0$,

$$\Big\| \sum_{\ell \ge 0} u_\ell h_{\lambda(p),\ell}(\rho(D)) \big[\sum_{Q \in \mathfrak{Q}_{\ell + n}} f_Q \big] \Big\|_r^r \lesssim \|u\|_{\ell^\infty}^r 2^{ndr/p} \alpha^{r-p} \sum_{Q \in \mathfrak{Q}} \|f_Q\|_p^p. \tag{4.6}$$

Proof (assuming Theorem 4.2). Note $h_{\lambda(p),\ell}(\rho(D)) = 2^{-\ell d/p} A_{\lambda(p),\ell}$. For $Q \in \mathfrak{Q}$ set

$$F_Q(x) = \begin{cases} f_Q/\|f_Q\|_p, & \text{if } \|f_Q\|_p \ne 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta(Q) = u_{L(Q)+s} 2^{-L(Q)d/p} \|f_Q\|_p.$$

Then we get, with $\Xi_{s,\mathfrak{Q}}$ as in (4.3),

$$\sum_{\ell \ge s} u_\ell h_{\lambda(p),\ell}(\rho(D)) \big[\sum_{Q \in \mathfrak{Q}_{\ell - s}} f_Q \big] = 2^{-sd/p} \Xi_{s,\mathfrak{Q}}[F, \beta]$$

where we have of course used that $\ell = L(Q) + s$ for $Q \in \mathfrak{Q}_{\ell - s}$. Applying Theorem 4.2 and the normalization $\|F\|_{\ell^\infty(L^p)} \le 1$ the left-hand side of (4.5) is dominated by $[C 2^{-\varepsilon s} \|\beta\|_{\ell^{r,1}(\mu)}]^r$. For $p < r < \infty$ the space $\ell^{r,1}$ is the real interpolation space $[\ell^\infty, \ell^p]_{\vartheta,1}$ with $\vartheta = p/r$ and therefore

$$\begin{aligned} \|\beta\|_{\ell^{r,1}(\mathfrak{Q},\mu)} &\lesssim \|\beta\|_{\ell^\infty(\mathfrak{Q},\mu)}^{1-\frac{p}{r}} \|\beta\|_{\ell^p(\mathfrak{Q},\mu)}^{\frac{p}{r}} \\ &\lesssim \|u\|_\infty \Big(\sup_Q \Big(\frac{1}{|Q|} \int_Q |f_Q|^p \Big)^{1/p} \Big)^{1-\frac{p}{r}} \Big(\sum_{Q \in \mathfrak{Q}} \|f_Q\|_p^p \Big)^{1/r} \\ &\lesssim \|u\|_\infty \, \alpha^{1-\frac{p}{r}} \Big(\sum_{Q \in \mathfrak{Q}} \|f_Q\|_p^p \Big)^{1/r} \end{aligned}$$

using the assumption (4.4). This establishes (4.5) and (4.6) is obtained in the same way, using Corollary 4.3. □

5. Proof of Theorem 4.2

5.1. *Reduction to a linear operator.* With $\mathfrak{Q}, \mathfrak{Q}_j$ as above and $A_{\lambda,\ell}$ defined as in (4.2), let

$$\mathcal{A}_{s,\mathfrak{Q}} F := \sum_{\ell \ge s} \sum_{Q \in \mathfrak{Q}_{\ell - s}} A_{\lambda,\ell}[F_Q \mathbb{1}_Q]. \tag{5.1}$$

We will show that Theorem 4.2 is a consequence of the following.

Theorem 5.1. *Let $d \geq 2$, $\frac{2(d+1)}{d+3} \leq p_\circ < \frac{2d}{d+1}$, $p_\circ \leq r_\circ \leq \frac{d-1}{d+1}p_\circ'$ and assume that* $\mathrm{Hyp}(p_\circ, r_\circ)$ *holds. Then for $1 \leq p < p_\circ$, $p < r < r_*(p, p_\circ, r_\circ)$ there is $\varepsilon = \varepsilon(p,r) > 0$ such that for all $s \geq 0$, and all collections of disjoint cubes $\mathfrak{Q} \subset \mathfrak{D}_{\geq 0}$*

$$\|\mathcal{A}_{s,\mathfrak{Q}} F\|_r \lesssim 2^{s(\frac{d}{p}-\varepsilon)} \mu(\mathfrak{Q})^{1/r} \|F\|_{\ell^\infty(L^p)}. \tag{5.2}$$

Proof of Theorem 4.2 assuming Theorem 5.1. For completeness we include the standard argument (*cf.* [32, Ch. V.3]). Let β^* be the nonincreasing rearrangement of β. We may decompose $\beta = \sum_{k\in\mathbb{Z}} \beta^k$ where $\beta^k(Q) = \beta(Q)\mathbb{1}_{\mathfrak{E}^k}(Q)$ and $\mathfrak{E}^k = \{Q \in \mathfrak{Q} : \beta^*(2^{k+1}) < |\beta(Q)| \leq \beta^*(2^k)\}$. Observe that $\mu(\mathfrak{E}^k) \leq 2^{k+1}$, which also shows that the $\mathfrak{E}^k$ are finite sets. We have $\|\Xi_{s,\mathfrak{Q}}[F,\beta]\|_r \lesssim \sum_k \|\Xi_{s,\mathfrak{Q}}[F,\beta^k]\|_r$ which is written as

$$\sum_k \beta^*(2^k) \|\Xi_{s,\mathfrak{Q}}[F, \tfrac{\beta \mathbb{1}_{\mathfrak{E}^k}}{\beta^*(2^k)}]\|_r = \sum_k \beta^*(2^k) \|\mathcal{A}_{s,\mathfrak{E}^k} F^k\|_r$$

where the function G^k is defined by $F^k_Q(x) = F_Q(x)\mathbb{1}_{\mathfrak{E}^k}(Q)\frac{\beta(Q)}{\beta^*(2^k)}$. Note that $|F^k_Q(x)| \leq |F_Q(x)|$. Applying Theorem 5.1 to $\mathcal{A}_{s,\mathfrak{E}^k} F^k$ we get

$$\sum_k \beta^*(2^k) \|\mathcal{A}_{s,\mathfrak{E}^k}[F^k]\|_r \lesssim 2^{s(\frac{d}{p}-\varepsilon)} \sum_k \beta^*(2^k) \mu(\mathfrak{E}^k)^{1/r} \|F^k\|_{\ell^\infty(L^p)}$$

and since $\mu(\mathfrak{E}^k)^{1/r} \lesssim 2^{k/r}$ and $\|F^k\|_{\ell^\infty(L^p)} \leq \|F\|_{\ell^\infty(L^p)}$ we see that the right-hand side is $\lesssim 2^{s(\frac{d}{p}-\varepsilon)} \|\beta\|_{\ell^{r,1}} \|F\|_{\ell^\infty(L^p)}$, as desired. □

The key to prove Theorem 5.1 are the following propositions.

Proposition 5.2. *Let $d \geq 2$, $\frac{2(d+1)}{d+3} \leq p_\circ < \frac{2d}{d+1}$, $p_\circ \leq r_\circ \leq \frac{d-1}{d+1}p_\circ'$ and assume that* $\mathrm{Hyp}(p_\circ, r_\circ)$ *holds. Then for $\frac{2(d+1)}{d+3} \leq p < p_\circ$, $p \leq r < r_*(p, p_\circ, r_\circ)$ and all $s \geq 0$, and collections of disjoint cubes $\mathfrak{Q} \subset \mathfrak{D}_{\geq 0}$,*

$$\|\mathcal{A}_{s,\mathfrak{Q}} F\|_r \lesssim 2^{s\frac{d}{p}} \mu(\mathfrak{Q})^{\frac{1}{r}} \|F\|_{\ell^\infty(L^p)}. \tag{5.3}$$

Proposition 5.3. *Let $d \geq 2$. For all $1 < r < \infty$ there exists $\varepsilon(r) > 0$ such that for all $s \geq 0$, and collections of disjoint cubes $\mathfrak{Q} \subset \mathfrak{D}_{\geq 0}$,*

$$\|\mathcal{A}_{s,\mathfrak{Q}} F\|_r \lesssim 2^{s(d-\varepsilon(r))} \mu(\mathfrak{Q})^{\frac{1}{r}} \|F\|_{\ell^\infty(L^1)}. \tag{5.4}$$

We note that Proposition 5.2 is essentially a re-statement of $\mathrm{Hyp}(p_\circ, r_\circ)$, and Proposition 5.3 is an improvement over the trivial

$$\|\mathcal{A}_{s,\mathfrak{Q}} F\|_1 \lesssim 2^{sd} \mu(\mathfrak{Q}) \|F\|_{\ell^\infty(L^1)} \tag{5.5}$$

which follows since the $L^1 \to L^1$ operator norm of $2^{\ell(\lambda+\frac{d+1}{2})} h_{\lambda,\ell}(\rho(D))$ is $O(2^{\ell d})$. They will be proven in §5.2 and §5.3 respectively.

Theorem 5.1 now follows by a standard complex interpolation argument based on the interpolation formula

$$[\ell^\infty(L^{u_0}), \ell^\infty(L^{u_1})]_\vartheta = \ell^\infty(L^u)$$

with $(1-\vartheta)/u_0 + \vartheta/u_1 = 1/u$ which holds if the ℓ^∞ norms are taken on a finite set (as in our applications).[1] We first interpolate (5.3) for $p = \frac{2(d+1)}{d+3}$ and (5.4) to

[1]One has to use the second $[\cdot,\cdot]^\theta$ method by Calderón in the general case.

obtain

(5.6) $\|\mathcal{A}_{s,\mathfrak{Q}}F\|_v \lesssim 2^{s(\frac{d}{u}-\varepsilon(u,v))}\mu(\mathfrak{Q})^{\frac{1}{v}}\|F\|_{\ell^\infty(L^u)}$, for $1<u<\frac{2(d+1)}{d+3}$, $u<v<\frac{d-1}{d+1}u'$

for some $\varepsilon(u,v)>0$. Now fix p and r with $\frac{2(d+1)}{d+3}\le p<p_\circ$ and $p<r<r_*(p,p_\circ,r_\circ)$. We can then find pairs (u,v) such that $1<u<\frac{2(d+1)}{d+3}$, $u<v<\frac{d-1}{d+1}u'$ and (p_1,r_1) such that $\frac{2(d+1)}{d+3}\le p_1<p_\circ$, $p_1\le r_1<r_*(p_1,p_\circ,r_\circ)$, with $(\frac1p,\frac1r)$ in the open line segment connecting $(\frac1u,\frac1v)$ with $(\frac{1}{p_1},\frac{1}{r_1})$, i.e. $(\frac1p,\frac1r)=(1-\vartheta)(\frac{1}{p_1},\frac{1}{r_1})+\vartheta(\frac1u,\frac1v)$ for some $\vartheta\in(0,1)$.[2] Now interpolate (5.3) for $(\frac{1}{p_1},\frac{1}{r_1})$ with (5.6) for $(\frac1u,\frac1v)$. We then obtain (5.2) for the pair $(\frac1p,\frac1r)$ with $\varepsilon(p,r):=\vartheta\varepsilon(u,v)>0$.

5.2. *Proof of Proposition 5.2.* As mentioned above, this is essentially a reformulation of $\mathrm{Hyp}(p_\circ,r_\circ)$ in which one replaced the normalized bumps $\chi(2^\ell(1-\varrho))$ by $2^{\ell\lambda}h_{\ell,\lambda}(\varrho)$. The technical lemma that takes care of it is the following.

Lemma 5.4. *Let $d\ge2$, $1\le p\le r<\infty$ and assume that* VBR(p,r) *holds. Then for all $s\ge0$,*

$$\Big\|\sum_{\ell\ge s}A_{\lambda,\ell}\big[\sum_{Q\in\mathfrak{D}_{\ell-s}}f_{\ell,Q}\mathbb{1}_Q\big]\Big\|_r\lesssim 2^{s\frac{d}{p}}\Big(\sum_\ell\sum_Q|Q|\|f_{\ell,Q}\|_p^r\Big)^{\frac1r}. \tag{5.7}$$

If $f_{\ell,Q}=F_Q$ for $Q\in\mathfrak{Q}_{\ell-s}$ (and 0 otherwise) then the right-hand side in (5.7) is clearly bounded by $2^{s\frac{d}{p}}\mu(\mathfrak{Q})^{\frac1r}\|F\|_{\ell^\infty(L^p)}$, implying thus Proposition 5.2.

Proof. We first examine the case $s=0$. Let $\eta\in C_c^\infty$ be supported in $(1/2,2)$ such that $\sum_{k\in\mathbb{Z}}\eta(2^ku)=1$ for $u>0$. In view of the support of $h_{\lambda,\ell}$, decompose the convolution kernel of $A_{\lambda,\ell}$ using

$$2^{\ell(\lambda+\frac{d+1}{2})}h_{\lambda,\ell}=\sum_{0\le m_1<\ell}\theta_{\lambda,\ell,m_1}+\sum_{m_2>0}\widetilde{\theta}_{\lambda,\ell,m_2} \tag{5.8a}$$

where

$$\theta_{\lambda,\ell,m_1}(\varrho)=2^{\ell(\lambda+\frac{d+1}{2})}h_{\lambda,\ell}(\varrho)\eta(2^{\ell-m_1}(1-\varrho)), \tag{5.8b}$$

$$\widetilde{\theta}_{\lambda,\ell,m_2}(\varrho)=2^{\ell(\lambda+\frac{d+1}{2})}h_{\lambda,\ell}(\varrho)\eta(2^{\ell+m_2}(1-\varrho)). \tag{5.8c}$$

This decomposition is done to exploit the hypothesis VBR(p,r), since the $\theta_{\lambda,\ell,m_1}$ and $\widetilde{\theta}_{\lambda,\ell,m_2}$ are now compactly supported. Our goal is to show the inequalities

(5.9)
$$\Big\|\sum_{\ell>m_1}\sum_{Q\in\mathfrak{D}_\ell}\theta_{\lambda,\ell,m_1}(\rho(D))[f_{\ell,Q}\mathbb{1}_Q]\Big\|_r\lesssim_N 2^{-m_1(N+\lambda(r))}\Big(\sum_\ell\sum_{Q\in\mathfrak{D}_\ell}2^{\ell d}\|f_{\ell,Q}\|_p^r\Big)^{1/r}$$

$$\Big\|\sum_{\ell>0}\sum_{Q\in\mathfrak{D}_\ell}\widetilde{\theta}_{\lambda,\ell,m_2}(\rho(D))[g_{\ell,Q}\mathbb{1}_Q]\Big\|_r\lesssim 2^{m_2(\lambda(p)-N_\circ)}\Big(\sum_\ell\sum_{Q\in\mathfrak{D}_\ell}2^{\ell d}\|f_{\ell,Q}\|_p^r\Big)^{1/r}. \tag{5.10}$$

Combining the estimates (5.9) and (5.10) (and recalling that $N_\circ>\lambda(p)$) yields (5.7) for $s=0$.

[2] Here (p_1,r_1) should be thought of being sufficiently close to $(p_\circ,r_\circ)$; note that $r_*(p_\circ,p_\circ,r_\circ)=r_\circ$.

Using Lemma 3.4 we can write

$$\theta_{\lambda,\ell,m_1}(\varrho) = c_{N,\lambda} 2^{-m_1 N} 2^{\ell\frac{d+1}{2}} \chi_{\lambda,\ell,m_1}(2^{\ell-m_1}(1-\varrho)), \tag{5.11a}$$

$$\widetilde{\theta}_{\lambda,\ell,m_2}(\varrho) = \widetilde{c}_{N_\circ,\lambda} 2^{-m_2 N_\circ} 2^{\ell\frac{d+1}{2}} \widetilde{\chi}_{\lambda,\ell,m_2}(2^{\ell+m_2}(1-\varrho)), \tag{5.11b}$$

for suitable χ_{λ,ℓ,m_1}, $\widetilde{\chi}_{\lambda,\ell,m_2} \in \mathcal{Y}_M$.

For each $R' \in \mathfrak{D}_{\ell-m_1}$ we let $Q(R')$ be the unique $Q \in \mathfrak{D}_\ell$ that contains R'. Writing $\mathbb{1}_Q = \sum_{R'\subset Q} \mathbb{1}_{R'}$ we then have

$$\Big\| \sum_{\ell>m_1} \sum_{Q\in\mathfrak{D}_\ell} \theta_{\lambda,\ell,m_1}(\rho(D))[f_{\ell,Q}\mathbb{1}_Q] \Big\|_r = c_{N,\lambda} 2^{-m_1(N-\frac{d+1}{2})} \times \\ \Big\| \sum_{\ell>m_1} \sum_{Q\in\mathfrak{D}_\ell} \sum_{R'\in\mathfrak{D}_{\ell-m_1}} 2^{(\ell-m_1)\frac{d+1}{2}} \chi_{\lambda,\ell,m_1}(2^{\ell-m_1}(1-\rho(D)))[f_{\ell-m_1,Q(R')}\mathbb{1}_{R'}] \Big\|_r$$

and we can use the hypothesis VBR(p,r) to bound the expression on the right-hand side by a constant times

$$2^{-m_1(N-\frac{d+1}{2})} \Big(\sum_{\ell>m_1} \sum_{Q\in\mathfrak{D}_\ell} \sum_{\substack{R'\in\mathfrak{D}_{\ell-m_1} \\ R'\subset Q}} \big[2^{(\ell-m_1)d/r} \| f_{\ell,Q}\mathbb{1}_{R'}\|_p\big]^r \Big)^{1/r} \\ \lesssim 2^{-m_1(N+\frac{d}{r}-\frac{d+1}{2})} \Big(\sum_{\ell} \sum_{Q\in\mathfrak{D}_\ell} \big[2^{\ell d/r} \| f_{\ell,Q}\mathbb{1}_Q\|_p\big]^r \Big)^{1/r}$$

where we have used $r \ge p$, $\sum_{R'\in\mathfrak{D}_{\ell-m_1}} \|f_{\ell,Q}\mathbb{1}_{R'}\|_p^r \le \|f_{\ell,Q}\|_p^r$ for all $Q \in \mathfrak{D}_\ell$. This finishes the proof of (5.9).

We now turn to the proof of (5.10). To apply Lemma 3.4 we label $j = \ell + m_2$, and set, for $R' \in \mathfrak{D}_j$,

$$g^{m_2}_{j,R'} = \sum_{Q\in\mathfrak{D}_{j-m_2}: Q\subset R'} f_{j-m_2,Q}\mathbb{1}_Q.$$

Then

$$\Big\| \sum_{\ell>0} \sum_{Q\in\mathfrak{D}_\ell} \widetilde{\theta}_{\lambda,\ell,m_2}(\rho(D))[f_{\ell,Q}\mathbb{1}_Q] \Big\|_r \\ = 2^{-m_2(N_\circ+\frac{d+1}{2})} \Big\| \sum_{j>m_2} \sum_{R'\in\mathfrak{D}_j} 2^{j\frac{d+1}{2}} \widetilde{\chi}_{\lambda,j-m_2,m_2}(2^j(1-\rho(D)))g^{m_2}_{j,Q} \Big\|_r \\ \lesssim 2^{-m_2(N_\circ+\frac{d+1}{2})} \Big(\sum_{j>m_2} \sum_{R'\in\mathfrak{D}_j} \big[2^{jd} \big\| g^{m_2}_{j,R'} \big\|_p\big]^r \Big)^{\frac{1}{r}}$$

where we applied the hypothesis VBR(p,r) to get the bound in the third line. Now for $j > m_2$,

$$\Big(\sum_{R'\in\mathfrak{D}_j} 2^{jd}\|g^{m_2}_{j,R'}\|_p^r\Big)^{\frac1r} = \Big(\sum_{R'\in\mathfrak{D}_j} 2^{jd}\Big(\sum_{\substack{Q\in\mathfrak{D}_{j-m_2}\\ Q\subset R'}} \|f_{j-m_2,Q}\|_p^p\Big)^{\frac rp}\Big)^{\frac1r}$$

$$\lesssim 2^{m_2d/p}\Big(2^{(j-m_2)d}\sum_{Q\in\mathfrak{D}_{j-m_2}} \|f_{j-m_2,Q}\|_p^r\Big)^{\frac1r},$$

by Hölder's inequality in the inner Q-sum. Combining the above we get (5.10).

Finally we consider the case $s > 0$. Define for $R' \in \mathfrak{D}_\ell$, $F_{\ell,R'} = \sum_{Q\subset R'} f_{\ell-s,Q}$ where the sum is taken over the cubes in $\mathfrak{D}_{\ell-s}$ which are subcubes of R'. Note that

$$\|F_{\ell,R'}\|_p = \Big(\sum_{\substack{Q\in\mathfrak{D}_{\ell-s}\\ Q\subset R'}} \|f_{\ell-s,Q}\|_p^p\Big)^{\frac1p} \le 2^{sd(\frac1p-\frac1r)}\Big(\sum_{\substack{Q\in\mathfrak{D}_{\ell-s}\\ Q\subset R'}} \|f_{\ell-s,Q}\|_p^r\Big)^{\frac1r}.$$

Applying the result for $s = 0$ proved above to the family of functions $\{F_{\ell,R'}\}$ we get that the left-hand side of (5.7) is dominated by a constant times

$$\Big(\sum_{\ell\ge s}\sum_{R'\in\mathfrak{D}_\ell} 2^{\ell d}\|F_{\ell,R'}\|_p^r\Big)^{1/r} \lesssim 2^{sd/p}\Big(\sum_{\ell\ge s}\sum_{Q\in\mathfrak{D}_{\ell-s}} 2^{(\ell-s)d}\|f_{\ell-s,Q}\|_p^r\Big)^{1/r}$$

and we get (5.7) for all $s \ge 0$. □

5.3. *Proof of Proposition 5.3.* It follows from the inequalities

$$\|\mathcal{A}_{s,\mathfrak{Q}}F\|_2 \lesssim 2^{s(3d+1)/4}\mu(\mathfrak{Q})^{1/2}\|F\|_{\ell^\infty(L^1)} \tag{5.12}$$

$$\|\mathcal{A}_{s,\mathfrak{Q}}F\|_{r_1} \lesssim 2^{sd}\mu(\mathfrak{Q})^{1/r_1}\|F\|_{\ell^\infty(L^1)}, \quad 1 \le r_1 < \infty. \tag{5.13}$$

Indeed, if $2 \le r < \infty$ we choose $r_1 > r$ large in (5.13) and obtain (5.4) by taking a mean of (5.12) and (5.13). Similarly (but less interesting for our purpose) one gets (5.4) for $1 < r \le 2$ by taking a mean of (5.12) and (5.5). We note that our argument for (5.12) does not use the disjointness property of the family of cubes $\mathfrak{Q}$, but the argument for (5.13) strongly relies on it.

5.3.1. The case $p = 1$, $r = 2$: proof of (5.12). We will first formulate a version of (5.12) for linear combinations of radial bump multipliers $\chi(2^\ell(1-\rho))$, and then subsequently replace the radial bumps by the multipliers $2^{\ell\lambda}h_{\lambda,\ell}\circ\rho$ to get (5.12).

Lemma 5.5. *Let $d \ge 2$ and $\{\chi_j\}_j \subseteq \mathcal{Y}_M$ for large $M \gg 10d$. For all $s \ge 0$,*

$$\Big\|\sum_{j\ge 2s} 2^{j\frac{d+1}{2}}\chi_j(2^j(1-\rho(D)))\big[\sum_{Q\in\mathfrak{Q}_{j-s}} F_Q\mathbb{1}_Q\big]\Big\|_2 \lesssim 2^{s\frac{d+1}{2}}\mu(\mathfrak{Q})^{\frac12}\|F\|_{\ell^\infty(L^1)} \tag{5.14}$$

holds for all finite $\mathfrak{Q} \subset \mathfrak{D}_{\ge 0}$. Moreover, for $j \ge 0$, $0 < L \le j/2$

$$\Big\|2^{j\frac{d+1}{2}}\chi_j(2^j(1-\rho(D)))\big[\sum_{Q\in\mathfrak{Q}_L} F_Q\mathbb{1}_Q\big]\Big\|_2 \lesssim 2^{j\frac{3d+1}{4}-Ld}\mu(\mathfrak{Q})^{\frac12}\|F\|_{\ell^\infty(L^1)}. \tag{5.15}$$

An immediate corollary (unifying and slightly weakening (5.14), (5.15)) is

Corollary 5.6. *For $\kappa \ge 0$ we have*

$$\Big\| \sum_{j\ge\kappa} 2^{j\frac{d+1}{2}} \chi_j(2^j(1-\rho(D)))\big[\sum_{Q\in\mathfrak{Q}_{j-\kappa}} F_Q \mathbb{1}_Q\big]\Big\|_2 \lesssim 2^{\kappa\frac{3d+1}{4}} \mu(\mathfrak{Q})^{\frac12} \|F\|_{\ell^\infty(L^1)}. \tag{5.16}$$

Proof. The term $\sum_{j\ge 2\kappa}$ is handled using (5.14) which gives the better L^2-bound $2^{\kappa\frac{d+1}{2}}\mu(\mathfrak{Q})^{1/2}$. For $\sum_{\kappa\le j<2\kappa}$ we apply (5.15) with $L = j-\kappa \in [0, j/2]$. By Minkowski's inequality the resulting L^2 bound is $\sum_{\kappa\le j\le 2\kappa} 2^{\kappa d - j\frac{d-1}{4}}\mu(\mathfrak{Q})^{\frac12}\|F\|_{\ell^\infty(L^1)} \lesssim 2^{\kappa\frac{3d+1}{4}}\mu(\mathfrak{Q})^{\frac12}\|F\|_{\ell^\infty(L^1)}$. This gives (5.16). □

Proof of Lemma 5.5. Assume, without loss of generality, that $\|F\|_{\ell^\infty(L^1)} \le 1$. We use arguments by Christ–Sogge [10, 11]; these do not require a curvature assumption on $\partial\Omega$. One can decompose

$$\chi_j(2^j(1-\rho(\xi))) = \sum_\nu \chi_{j,\nu}(\xi) \tag{5.17}$$

where the sum in ν is extended over an index set $\mathcal{I}_j$ of cardinality $O(2^{j(d-1)/2})$. Each multiplier $\chi_{j,\nu}$ is supported in a $(2^{-j}, 2^{-j/2}, \dots, 2^{-j/2})$ box essentially tangential to $\partial\Omega$. Moreover, the supports of $\chi_{j,\nu}$ have bounded overlap in the sense that $\sum_{j,\nu}|\chi_{j,\nu}(\xi)| \lesssim 1$, and we have the kernel estimates

$$\begin{aligned}(5.18)\quad &|\mathcal{F}^{-1}[\chi_{j,\nu}](x)| + |\mathcal{F}^{-1}[|\chi_{j,\nu}|^2](x)| \\ &\qquad\lesssim K_{j,\nu}(x) := 2^{-j(d+1)/2}(1+2^{-j}|\langle x, e_{j,\nu}\rangle|)^{-N_1}(1+2^{-j/2}|P^\perp_{j,\nu}(x)|)^{-N_2};\end{aligned}$$

here $e_{j,\nu}$ is a unit vector orthogonal to the surface $\partial\Omega$ on a point in $\operatorname{supp}(\chi_{j,\nu})$ and $P^\perp_{j,\nu}$ is the orthogonal projection to the hyperplane orthogonal to $e_{j,\nu}$ and $N_1, N_2 \le M$ (and by choosing M large enough we may assume that $N_1 > 1$, $N_2 > d-1$).

By orthogonality (due to the bounded overlap condition) we have

$$\begin{aligned}(5.19)\quad &\Big\|\sum_{j>2s} 2^{j\frac{d+1}{2}}\chi_j(2^j(1-\rho(D)))\big[\sum_{Q\in\mathfrak{Q}_{j-s}} F_Q\big]\Big\|_2 \\ &\qquad\lesssim \Big(\sum_{j>2s} 2^{j(d+1)} \sum_{\nu\in\mathcal{I}_j}\Big\|\chi_{j,\nu}(D)\big[\sum_{Q\in\mathfrak{Q}_{j-s}} F_Q\big]\Big\|_2^2\Big)^{1/2}\end{aligned}$$

and, similarly for every j, $L \le j/2$,

$$\begin{aligned}(5.20)\quad &\Big\|2^{j\frac{d+1}{2}}\chi_j(2^j(1-\rho(D)))\big[\sum_{Q\in\mathfrak{Q}_L} F_Q\big]\Big\|_2^2 \\ &\qquad\lesssim 2^{j(d+1)/2}\Big(\sum_{\nu\in\mathcal{I}_j}\Big\|\chi_{j,\nu}(D)\big[\sum_{Q\in\mathfrak{Q}_L} F_Q\big]\Big\|_2^2\Big)^{1/2}.\end{aligned}$$

We claim that for fixed $j \ge 0$, $\nu \in \mathcal{I}_j$

$$\Big\|\chi_{j,\nu}(D)\big[\sum_{Q\in\mathfrak{Q}_{j-s}} F_Q\big]\Big\|_2 \lesssim 2^{s\frac{d+1}{2} - j\frac{3d+1}{4}}\mu(\mathfrak{Q}_{j-s})^{\frac12}, \qquad s \le j/2, \tag{5.21}$$

$$\Big\|\chi_{j,\nu}(D)\big[\sum_{Q\in\mathfrak{Q}_L} F_Q\big]\Big\|_2 \lesssim 2^{-Ld}\mu(\mathfrak{Q}_L)^{\frac12}, \qquad L \le j/2 \tag{5.22}$$

and then the inequality (5.14) follows from (5.19) and (5.21), together with the bound $\#\mathcal{I}_j \lesssim 2^{j(d-1)/2}$ and $\sum_j \mu(\mathfrak{Q}_{j-s}) \le \mu(\mathfrak{Q})$. Likewise (5.15) follows from (5.20) and (5.22).

It remains to prove (5.21) and (5.22). For (5.21) we use $\|F\|_{\ell^\infty(L^1)} \le 1$ and write

$$\Big\|\chi_{j,\nu}(D)\big[\sum_{Q\in\mathfrak{Q}_{j-s}} F_Q\big]\Big\|_2^2 = \iint \mathcal{F}^{-1}[|\chi_{j,\nu}|^2](x-y) \sum_{Q\in\mathfrak{Q}_{j-s}} F_Q(y)\,\mathrm{d}y \sum_{Q\in\mathfrak{Q}_{j-s}} \overline{F_Q(x)}\,\mathrm{d}x$$
$$\lesssim \#\mathfrak{Q}_{j-s} \sup_x \int K_{j,\nu}(x-y) \sum_{Q\in\mathfrak{Q}_{j-s}} |F_Q(y)|\,\mathrm{d}y$$

with $K_{j,\nu}$ as in (5.18). For $x\in\mathbb{R}^d$ and $n_1,n_2>0$, define the regions

$$\begin{aligned}
\mathcal{R}^{0,0}_{j,\nu,s}(x) &:= \{y\in\mathbb{R}^d : |\langle x-y, e_{j,\nu}\rangle| \le 2^j,\ |P^\perp_{j,\nu}(x-y)| \le 2^{j-s}\},\\
\mathcal{R}^{n_1,0}_{j,\nu,s}(x) &:= \{y\in\mathbb{R}^d : 2^{j+n_1-1} \le |\langle x-y, e_{j,\nu}\rangle| \le 2^{j+n_1},\ |P^\perp_{j,\nu}(x-y)| \le 2^{j-s}\},\\
\mathcal{R}^{0,n_2}_{j,\nu,s}(x) &:= \{y\in\mathbb{R}^d : |\langle x-y, e_{j,\nu}\rangle| \le 2^j,\ 2^{j-s+n_2-1} \le |P^\perp_{j,\nu}(x-y)| \le 2^{j-s+n_2}\},\\
\mathcal{R}^{n_1,n_2}_{j,\nu,s}(x) &:= \{y\in\mathbb{R}^d : 2^{j+n_1-1} \le |\langle x-y, e_{j,\nu}\rangle| \le 2^{j+n_1},\\
&\qquad\qquad 2^{j-s+n_2-1} \le |P^\perp_{j,\nu}(x-y)| \le 2^{j-s+n_2}\}.
\end{aligned}$$

Observe that $\#\mathfrak{Q}_{j-s} \lesssim 2^{(s-j)d}\mu(\mathfrak{Q}_{j-s})$. Moreover, for $s\le j/2$ we have for all $n_1,n_2\ge 0$

$$\sup_x \sup_{y\in\mathcal{R}^{n_1,n_2}_{j,\nu,s}(x)} |K_{j,\nu}(x-y)| \le C_{N_1,N_2} 2^{-j\frac{d+1}{2}} 2^{-n_1N_1-(n_2+\frac{j}{2}-s)N_2}, \tag{5.23}$$

$$\sup_x \#\{Q\in\mathfrak{Q}_{j-s} : Q\cap\mathcal{R}^{n_1,n_2}_{j,\nu,s}(x)\neq\emptyset\} \lesssim 2^{s+n_1+n_2(d-1)}. \tag{5.24}$$

Combining these observations and summing in $n_1,n_2\ge 0$ yields (5.21).

In order to prove (5.22) we argue similarly. For $L<j/2$ we get as above

$$\Big\|\chi_{j,\nu}(D)\big[\sum_{Q\in\mathfrak{Q}_L} F_Q\big]\Big\|_2^2 \lesssim \#\mathfrak{Q}_L \sup_x \int K_{j,\nu}(x-y) \sum_{Q\in\mathfrak{Q}_L} |F_Q(y)|\,\mathrm{d}y.$$

Now use $\#\mathfrak{Q}_L \lesssim 2^{-Ld}\mu(\mathfrak{Q}_L)$, (5.23) with $s=j/2$ and the estimate

$$\sup_x \#\{Q\in\mathfrak{Q}_L : Q\cap\mathcal{R}^{n_1,n_2}_{j,\nu,j/2}(x)\neq\emptyset\} \lesssim 2^{-Ld}2^{j\frac{d+1}{2}}2^{s+n_1+n_2(d-1)}.$$

This leads to (5.22). □

We next show how to replace the normalized bumps in Corollary 5.6 by $2^{\ell\lambda}h_{\ell,\lambda}(\rho)$ to obtain (5.12). The argument is very similar to that in Lemma 5.4.

Proof of (5.12). Assume, without loss of generality, that $\|F\|_{\ell^\infty(L^1)}\le 1$. We decompose as in (5.8) and write $2^{\ell(\lambda+\frac{d-1}{2})}h_{\lambda,\ell}(\varrho)$ as

$$C_M 2^{\ell\frac{d+1}{2}}\Big[\sum_{m_1\le\ell} 2^{-m_1N}\chi_{\lambda,\ell,m_1}(2^{\ell-m_1}(1-\varrho)) + \sum_{m_2>0} 2^{-m_2N_\circ}\widetilde{\chi}_{\lambda,\ell,m_2}(2^{\ell+m_2}(1-\varrho))\Big]$$

with $\chi_{\lambda,\ell,m_1}, \widetilde{\chi}_{\lambda,\ell,m_2} \in \mathcal{Y}_M$ and $M > 100d$. We then bound $\|\mathcal{A}_{s,\mathfrak{Q}}F\|_2$ by a constant times

$$(5.25)\quad \sum_{m_1>s} 2^{-m_1(N-\frac{d+1}{2})} I_{m_1} + \sum_{m_1=0}^{s} 2^{-m_1(N-\frac{d+1}{2})} II_{m_1} + \sum_{m_2\ge 0} 2^{-m_2(\frac{d+1}{2}+N_\circ)} III_{m_2}$$

where

$$I_{m_1} = \Big\| \sum_{\ell\ge m_1} \sum_{R\in\mathfrak{D}_{\ell-m_1}} \sum_{Q\in\mathfrak{Q}_{\ell-s}} 2^{(\ell-m_1)\frac{d+1}{2}} \chi_{\lambda,\ell,m_1}(2^{\ell-m_1}(1-\rho(D)))[F_Q \mathbb{1}_{R\cap Q}]\Big\|_2,$$

$$II_{m_1} = \Big\| \sum_{\ell\ge s} 2^{(\ell-m_1)\frac{d+1}{2}} \chi_{\lambda,\ell,m_1}(2^{\ell-m_1}(1-\rho(D)))[\sum_{Q\in\mathfrak{Q}_{\ell-s}} F_Q \mathbb{1}_Q]\Big\|_2,$$

$$III_{m_2} = \Big\| \sum_{\ell\ge s} 2^{(\ell+m_2)\frac{d+1}{2}} \widetilde{\chi}_{\lambda,\ell,m_2}(2^{\ell+m_2}(1-\rho(D)))[\sum_{Q\in\mathfrak{Q}_{\ell-s}} F_Q \mathbb{1}_Q]\Big\|_2.$$

Let $m_1 > s$ and $Q^{m_1-s}(R)$ be the unique dyadic cube with sidelength $2^{L(R)+m_1-s}$ containing R. Let $\mathfrak{R}^{m_1-s}(\mathfrak{Q})$ be the family of all $R \in \mathfrak{D}$ such that $L(R) \ge 0$ and such that $Q^{m_1-s}(R)$ belongs to $\mathfrak{Q}$. Parametrizing $j = \ell - m_1$ the term I_{m_1} can be rewritten as

$$\|\sum_{j\ge 0} \sum_{R\in\mathfrak{R}_j^{m_1-s}(\mathfrak{Q})} 2^{j\frac{d+1}{2}} \chi_{\lambda,j+m_1,m_1}(2^j(1-\rho(D)))[f_R \mathbb{1}_R]\|_2, \text{ with } f_R := F_{Q^{m_1-s}(R)}.$$

We now apply Corollary 5.6 with $\kappa = 0$ and note that $\mu(\mathfrak{R}_j^{m_1-s}(\mathfrak{Q})) = \mu(\mathfrak{Q}_{j+m_1})$. Since $\|f_R\|_1 \le 1$, we obtain $I_{m_1} \lesssim \mu(\mathfrak{Q})^{1/2}$ and thus the first term on the right-hand side of (5.25) is bounded by $C\mu(\mathfrak{Q})^{1/2}$, which is a better bound.

For the terms II_{m_1} we have $s \ge m_1$. Changing the summation variable to $j = \ell - m_1$ one can apply Corollary 5.6 with $\kappa = s - m_1$ to get $II_{m_1} \lesssim 2^{(s-m_1)\frac{3d+1}{4}} \mu(\mathfrak{Q})^{1/2}$ Similarly for III_{m_2}, changing the summation variable to $j = \ell + m_2$ we see that Corollary 5.6 with $\kappa = s + m_2$ yields the bound $III_{m_2} \lesssim 2^{(s+m_2)\frac{3d+1}{4}} \mu(\mathfrak{Q})^{1/2}$. After summing we bound the second and third terms on the right-hand side of (5.25) both by $C2^{s(3d+1)/4}\mu(\mathfrak{Q})^{1/2}$. □

5.3.2. The case $p = 1$, $r > 2$: proof of (5.13). Since the inequality has already been proved (in fact improved) for $r \le 2$ we focus on the case for large r, and by interpolation it suffices to assume that $r > 2$ is an integer. We now rely on the kernel estimates in §3. We prove straightforward size estimates which are close to an argument used by Conde-Alonso, Culiuc, Di Plinio, and Ou [13] in the analysis of rough singular integral operators.

We let $K_\ell = \mathcal{F}^{-1}[2^{\ell(\frac{d+1}{2}+\lambda)} h_{\lambda,\ell} \circ \rho]$, the convolution kernel of the operator $A_{\lambda,\ell}$. From Lemmas 3.5 and 3.6 we have the kernel estimates $|K_\ell(x)| \lesssim 1$ for $|x| \approx 2^\ell$, $2^{-\ell d}|K_\ell(x)| \lesssim c_N 2^{-\ell N}$ for $|x| \le c_\circ 2^\ell$, and $2^{-\ell d}|K_\ell(x)| \lesssim \widetilde{c}_N |x|^{-N}$ for $|x| > C_\circ 2^\ell$. We shall only use the slightly weaker bound

$$(5.26)\quad 2^{-\ell d}|K_\ell(x)| \lesssim \sum_{n\ge 0} 2^{-n(N-d)} H_{\ell,n}(x) \qquad \text{with } H_{\ell,n}(x) = 2^{-(\ell+n)d} \mathbb{1}_{\{|x|\le 2^{\ell+n}\}},$$

where $N > d$. These favorable L^∞ bounds are crucial for our argument; if we were to replace $2^{\ell\lambda} h_{\ell,\lambda}(\varrho)$ with $\chi(2^\ell(1-\varrho))$ for generic $\chi \in \mathcal{Y}_M$ they would no longer hold. Assume, without loss of generality, that $\|F\|_{\ell^\infty(L^1)} \le 1$. By (5.26)

$$\|\mathcal{A}_{s,\mathfrak{Q}}F\|_r \lesssim \sum_{n=0}^\infty 2^{-n(N-d)} \Big\| \sum_{\ell>s} 2^{\ell d} H_{\ell,n} * \sum_{Q\in\mathfrak{Q}_{\ell-s}} |F_Q|\mathbb{1}_Q \Big\|_r .$$

Setting $G_L := 2^{Ld}\sum_{Q\in\mathfrak{Q}_L}|F_Q|\mathbb{1}_Q$ the inequality $\|\mathcal{A}_{s,\mathfrak{Q}}F\|_r \lesssim 2^{sd}\mu(\mathfrak{Q})^{1/r}$ follows from the bound

$$\Big\| \sum_{\ell>s} H_{\ell,n} * G_{\ell-s} \Big\|_r^r \lesssim_r \mu(\mathfrak{Q}). \tag{5.27}$$

Since r is an integer we have that the left-hand side in (5.27) is bounded by

$$Cr! \sum_{\ell_1\ge\ell_2\ge\dots\ge\ell_r} \int_{\substack{(y^1,\dots,y^r)\\ \in(\mathbb{R}^d)^r}} \int \prod_{i=1}^r \Big[H_{\ell_i,n}(x-y^i) G_{\ell_i-s}(y^i) \Big] \,\mathrm{d}x\,\mathrm{d}(y^1,\dots,y^r). \tag{5.28}$$

Observe that if there is an x such that $\prod_{i=1}^r H_{\ell_i,n}(x-y^i) \ne 0$ then we have $|y^i - y^{i+1}| \le 2^{\ell_i+n+1}$ for $i=1,\dots,r-1$. In this situation we also have the identity $H_{\ell_i,n}(x-y^i) = H_{\ell_i,n}(\frac{y^i-y^{i+1}}{2})$ for $1\le i\le r-1$; in addition, $\int H_{\ell_r,n}(x-y^r)\,\mathrm{d}x \lesssim 1$. We use these pointwise estimates and integrate in x first to bound (5.28) by a constant times

$$\int_{(\mathbb{R}^d)^r} \sum_{\ell_1} G_{\ell_1-s}(y^1) \prod_{i=1}^{r-1} \Big[\sum_{\ell_{i+1}=0}^{\ell_i} H_{\ell_i,n}(\tfrac{y^i-y^{i+1}}{2}) G_{\ell_{i+1}-s}(y^{i+1}) \Big] \,\mathrm{d}y^r \dots \mathrm{d}y^1. \tag{5.29}$$

For fixed y^i, with $1\le i\le r-1$ we have

$$\begin{aligned}
&\int_{y^{i+1}\in\mathbb{R}^d} \sum_{\ell_{i+1}=0}^{\ell_i} H_{\ell_i,n}(\tfrac{y^i-y^{i+1}}{2}) G_{\ell_{i+1}-s}(y^{i+1})\,\mathrm{d}y^{i+1} \\
&\lesssim 2^{-(\ell_i+n)d} \int_{|y^{i+1}-y^i|\le 2^{\ell_i+n+1}} \sum_{\ell_{i+1}=0}^{\ell_i} 2^{(\ell_{i+1}-s)d} \sum_{Q\in\mathfrak{Q}_{\ell_{i+1}-s}} |F_Q(y^{i+1})\mathbb{1}_Q(y^{i+1})|\,\mathrm{d}y^{i+1} \\
&\lesssim 2^{-(\ell_i+n)d} \sum_{\ell_{i+1}=0}^{\ell_i} \sum_{\substack{Q\in\mathfrak{Q}_{\ell_{i+1}-s}\\ \mathrm{dist}(Q,y^i)\le 2^{\ell_i+n+1}}} |Q| \int |F_Q(y^{i+1})|\,\mathrm{d}y^{i+1} \lesssim 1
\end{aligned}$$

where we used that the cubes in $\mathfrak{Q}$ are disjoint and the F_Q have normalized L^1 norm. Thus integrating in (5.29) first in y^r, then in y^{r-1}, and so on, we obtain

$$\begin{aligned}
\Big\| \sum_{\ell>s} H_{\ell,n} * G_{\ell-s} \Big\|_r^r &\lesssim_r \int_{y^1\in\mathbb{R}^d} \sum_{\ell_1} G_{\ell_1-s}(y^1)\,\mathrm{d}y^1 \\
&\lesssim_r \sum_{\ell_1} \sum_{Q\in\mathfrak{Q}_{\ell_1-s}} |Q|\|F_Q\|_1 \lesssim \mu(\mathfrak{Q})
\end{aligned}$$

and (5.27) is proved. This finishes the proof of (5.13). □

6. Sparse domination, Part I

Here we prove Theorem 2.3. Without loss of generality, we will assume that $q < p'$; the less interesting sparse bound (p, q_1) with $q_1 \geq p'$ would be implied by Hölder's inequality from any (p,q) with $q < p'$. In what follows we assume that $n_\circ$ is a fixed positive integer as in (3.12b); in particular, this implies $n_\circ \geq 5$. Implicit constants are allowed to depend on $n_\circ$. Define the modified functionals (allowing averages over triple cubes for the functions $|f_2|^q$)

$$\widetilde{\Lambda}^{\mathfrak{S}}_{p,q}(f_1, f_2) = \sum_{Q\in\mathfrak{S}} |Q| \langle f_1\rangle_{Q,p} \langle f_2\rangle_{3Q,q}, \tag{6.1}$$

$$\Lambda^{**}_{p,q}(f_1, f_2) = \sup_{\substack{\mathfrak{S}\subset\mathfrak{D}:\\ \mathfrak{S}:\gamma\text{-sparse}}} \widetilde{\Lambda}^{\mathfrak{S}}_{p,q}(f_1, f_2). \tag{6.2}$$

We use, for a cube $S \in \mathfrak{D}$, the notation $\Lambda^{S,**}_{p,q}(f_1, f_2)$ if we require that all the sparse families featuring in the sup consist of cubes contained in S. Recall the definition of $q_*(p, p_\circ, r_\circ)$ in (2.1). Fix p and let

$$T_\ell = h_{\lambda(p),\ell}(\rho(D)). \tag{6.3}$$

Definition 6.1. *Let $d \geq 2$, $\frac{2(d+1)}{d+3} \leq p_\circ < \frac{2d}{d+1}$, $p_\circ \leq r_\circ \leq \frac{d-1}{d+1}p'_\circ$ and assume that* $\mathrm{Hyp}(p_\circ, r_\circ)$ *holds. Let $1 \leq p < p_\circ$ and $q_*(p, p_\circ, r_\circ) < q < p'$. For $\mathfrak{n} = 0, 1, 2, \dots$ let $\mathbb{U}(\mathfrak{n}) \equiv \mathbb{U}_{p,q}(\mathfrak{n})$ be the smallest constant U such that for all bounded measurable functions f_1, f_2 with compact support and for all $S \in \mathfrak{D}$ with $n_\circ \leq L(S) \leq n_\circ + \mathfrak{n}$,*

$$\Big|\Big\langle \sum_{\ell\leq L(S)-n_\circ} T_\ell[f_1 \mathbb{1}_S], f_2 \mathbb{1}_{3S}\Big\rangle\Big| \leq U \Lambda^{S,**}_{p,q}(f_1, f_2).$$

The convolution kernels of T_ℓ are Schwartz functions and therefore it is immediate that $\mathbb{U}_{p,q}(\mathfrak{n})$ are finite for all $q \leq p'$. Our main task will be to prove for that $\sup_{\mathfrak{n}} \mathbb{U}_{p,q}(\mathfrak{n}) < \infty$ for p and q as above. This will be done by induction, by proving that there is a constant C such that for $\mathfrak{n} \geq 1$

$$\mathbb{U}(\mathfrak{n}) \leq \max\{\mathbb{U}(\mathfrak{n} - 1), C\}. \tag{6.4}$$

The main iteration step in the sparse domination argument has the same form as in [22].

Proposition 6.2. *Let $\frac{2(d+1)}{d+3} \leq p_\circ < \frac{2d}{d+1}$, $p_\circ \leq r_\circ \leq \frac{d-1}{d+1}p'_\circ$ and assume that* $\mathrm{Hyp}(p_\circ, r_\circ)$ *holds. Let $1 \leq p < p_\circ$ and $q_*(p, p_\circ, r_\circ) < q < p'$. Then there is a constant $C > 0$ such that for every $S \in \mathfrak{D}_{>0}$ and every bounded $f_1 : S \to \mathbb{C}$, $f_2 : 3S \to \mathbb{C}$, there is a collection $\mathfrak{W}$ of disjoint dyadic subcubes of S with the properties*

$$\Big|\bigcup_{Q\in\mathfrak{W}} Q\Big| \leq (1-\gamma)\,|S|, \tag{6.5}$$

(6.6)
$$\Big|\sum_{\ell=0}^{L(S)-n_\circ} \langle T_\ell f_1, f_2\rangle\Big| \leq C|S|\langle f_1\rangle_{S,p}\langle f_2\rangle_{3S,q} + \sum_{Q\in\mathfrak{W}_{\geq n_\circ}} \Big|\sum_{\ell=0}^{L(Q)-n_\circ} \langle T_\ell[f_1\mathbb{1}_Q], f_2\mathbb{1}_{3Q}\rangle\Big|.$$

Proof of Theorem 2.3, given Proposition 6.2. In order to prove (6.4) we fix $\mathfrak{n} \geq 1$ and let $S \in \mathfrak{D}_{n_\circ+\mathfrak{n}}$. Let $\epsilon > 0$. Let $\mathfrak{W}$ be the family of dyadic subcubes guaranteed by Proposition 6.2 such that (6.6) holds. Note that $n_\circ \leq L(Q) < n_\circ + \mathfrak{n}$ for all $Q \in \mathfrak{W}_{\geq n_\circ}$. Therefore, by the induction hypothesis, for each $Q \in \mathfrak{W}_{\geq n_\circ}$ there is a γ-sparse family $\mathfrak{S}_Q$ of dyadic subcubes of Q such that

$$\Big| \sum_{\ell=0}^{L(Q)-n_\circ} \langle T_\ell[f_1 \mathbb{1}_Q], f_2 \mathbb{1}_{3Q}\rangle \Big| \leq \mathbb{U}(\mathfrak{n}-1)\widetilde{\Lambda}^{\mathfrak{S}_Q}_{p,q}(f_1,f_2) + \epsilon.$$

Setting $E_S := S \setminus \bigcup_{Q\in\mathfrak{W}} Q$, the collection $\mathfrak{S} = \{S\} \cup \bigcup_{Q\in\mathfrak{W}} \mathfrak{S}_Q$ is a γ-sparse family of dyadic subcubes of S and we have

$$|S|\langle f_1\rangle_{S,p}\langle f_2\rangle_{3S,q} + \sum_{Q\in\mathfrak{W}} \widetilde{\Lambda}^{\mathfrak{S}_Q}_{p,q}(f_1,f_2) \leq \widetilde{\Lambda}^{\mathfrak{S}}_{p,q}(f_1,f_2).$$

Since $\epsilon > 0$ was arbitrary we deduce (6.4).

Finally, if f_1, f_2 are compactly supported L^∞-functions we choose N so that $2^{n_\circ+10}\mathrm{supp}\,(f_1), 2^{n_\circ+10}\mathrm{supp}\,(f_2) \subset [-N,N]^d$. By the properties of the Lerner–Nazarov [27] dyadic lattice $\mathfrak{D}$ and by Lemma 3.5, there is a cube $S \in \mathfrak{D}$ which contains $[-N,N]^d$ such that $|\langle T_\ell f_1, f_2\rangle| = |\langle T_\ell[f_1\mathbb{1}_S], f_2\mathbb{1}_{3S}\rangle| \lesssim \epsilon 2^{-\ell}$ for sufficiently large ℓ. Since $\epsilon > 0$ is arbitrary, this together with the main estimate (6.4), noting from (3.9a) that $h_\lambda = \sum_{\ell=0}^\infty h_{\lambda,\ell}$, yields the bound

$$|\langle h_{\lambda(p)}(\rho(D))f_1, f_2\rangle| \lesssim \Lambda^{**}(f_1,f_2).$$

A well-known argument relying on the three lattice theorem in [27] allows to replace Λ^{**} by the more standard maximal sparse form Λ^* (see e.g. [1, Ch.4.2] for details). Since $\mathcal{R}_a^{\lambda(p)} - h_{\lambda(p)}(\rho(D))$ satisfies a standard $\mathrm{Sp}(p,p)$ bound for all $p \geq 1$ (see the beginning of §3) we obtain the desired $\mathrm{Sp}(p,q)$ bound for $\mathcal{R}_a^{\lambda(p)}$. □

Proof of Proposition 6.2. Let $\alpha = \langle f_1\rangle_{S,p}$ and $\Omega = \{x : M_{HL}(|f_1|^p) \geq \frac{100d}{1-\gamma}\alpha^p\}$, where M_{HL} denotes the Hardy–Littlewood maximal function. Let $\mathfrak{W}$ be the collection of Whitney cubes of Ω satisfying that $\Omega = \bigcup_{Q\in\mathfrak{W}} Q$ and

$$\mathrm{diam}(Q) \leq \mathrm{dist}(Q,\Omega^{\complement}) \leq 4\,\mathrm{diam}(Q) \tag{6.7}$$

for all $Q \in \mathfrak{W}$; see [31, Ch. VI.1]. Since $|\Omega| \leq (1-\gamma)|S|$, condition (6.5) follows.

Define next $g = f_1\mathbb{1}_{\Omega^c}$ and $b_Q = f_1\mathbb{1}_Q$ for each $Q \in \mathfrak{W}$. By the standard Calderón–Zygmund properties,

$$\|g\|_\infty \lesssim \alpha \qquad \text{and} \qquad \int_Q |b_Q|^p \leq \alpha^p|Q|.$$

Let

$$B_0 = \sum_{Q\in\mathfrak{W}_{\leq 0}} b_Q, \qquad B_j = \sum_{Q\in\mathfrak{W}_j} b_Q, \quad j > 0.$$

With these definitions we have $\int_Q |B_j|^p \lesssim \alpha^p|Q|$ whenever Q is a dyadic cube with $L(Q) \geq j \geq 0$; note that we also have $\int_Q |B_j| \leq \alpha|Q|$.

Let $\mathcal{T}^S = \sum_{\ell=0}^{L(S)-n_\circ} T_\ell$. Then $|\langle \mathcal{T}^S f_1, f_2\rangle| \le I + II + III$, where

$$I = |\langle \mathcal{T}^S g, f_2\rangle|, \qquad II = \Big|\Big\langle \sum_{\ell=0}^{L(S)-n_\circ} T_\ell[\sum_{0\le j<\ell+n_\circ} B_j], f_2\Big\rangle\Big|,$$

$$III = \Big|\Big\langle \sum_{\ell=0}^{L(S)-n_\circ} T_\ell[\sum_{j\ge \ell+n_\circ} B_j], f_2\Big\rangle\Big|.$$

Estimation of I. Hyp$(p_\circ, r_\circ)$ implies VBR(p,p) and together with Lemma 5.4 this implies $\|T_\ell\|_{L^p\to L^p} = O(1)$. Since $\|h_{\lambda(p),\ell}\|_\infty = O(2^{-\lambda(p)})$ and $|\frac1p - \frac12| > |\frac1q - \frac12|$ we get $\|T_\ell\|_{L^q\to L^q} \lesssim 2^{-\varepsilon\ell}$ for some $\varepsilon > 0$, by interpolation. We can also apply this for the adjoint operators; indeed $T_\ell^*(h_{\lambda_*}(\rho(D))^* = h_{\lambda,*}(\widetilde{\rho}(D))$ where $\widetilde{\rho}$ is the Minkowski functional of $-\Omega$. Hence $\|T_\ell^*\|_{L^q\to L^q} \lesssim 2^{-\varepsilon\ell}$. Therefore

$$I = |\langle g\mathbb{1}_S, (\mathcal{T}^S)^* f_2\rangle| \lesssim \alpha \int_S |(\mathcal{T}^S)^* f_2| \lesssim \alpha |S|^{1-1/q} \sum_{\ell=0}^\infty \|T_\ell^* f_2\|_q$$

$$\lesssim \alpha |S|^{1-1/q} \Big(\int_{3S} |f_2|^q\Big)^{1/q} \lesssim |S| \langle f_1\rangle_{S,p} \langle f_2\rangle_{3S,q}. \tag{6.8}$$

Estimation of II. We estimate $II \le \sum_{s\ge -n_\circ} II_s$ where (with $s \wedge 0 := \max\{s, 0\}$)

$$II_s = \Big|\Big\langle \sum_{\ell=s\wedge 0}^{L(S)-n_\circ} T_\ell B_{\ell-s}, f_2\Big\rangle\Big|.$$

First assume $s > 0$. We use Proposition 4.4 with $r = q'$ and $\mathfrak{Q} = \mathfrak{W}_{\ge 1}$. Then

$$\Big\| \sum_{\ell=s+1}^{L(S)-n_\circ} T_\ell B_{\ell-s}\Big\|_{q'}^{q'} \lesssim 2^{-s\epsilon q'} \alpha^{q'-p} \sum_{Q\in\mathfrak{W}} \|b_Q\|_p^p$$
$$\lesssim 2^{-s\epsilon q'} \alpha^{q'-p} \sum_{Q\in\mathfrak{W}} \alpha^p |Q| \lesssim 2^{-s\varepsilon q'} \alpha^{q'} |S|.$$

For the term with $\ell = s$ we use Proposition 4.4 with $r = q'$ and $\mathfrak{Q} = \mathfrak{D}_0$; note that for $R \in \mathfrak{D}_0$ and $f_R := \sum_{Q\in\mathfrak{W}, Q\subset R} b_Q$ we have $\int_R |f_R|^p \lesssim \alpha^p |R|$. In (4.5) we set $u_\ell = 1$ if $\ell = s$ and $u_\ell = 0$ for $\ell \ne s$ and obtain

$$\|T_s B_0\|_{q'}^{q'} \lesssim 2^{-s\epsilon q'} \alpha^{q'-p} \sum_{R\in\mathfrak{D}_0} \Big\| \sum_{\substack{Q\in\mathfrak{W}\\ Q\subset R}} b_Q \Big\|_p^p \lesssim 2^{-s\epsilon q'} \alpha^{q'-p} \sum_{Q\in\mathfrak{W}} \|b_Q\|_p^p$$
$$\lesssim 2^{-s\epsilon q'} \alpha^{q'-p} \sum_{Q\in\mathfrak{W}} \alpha^p |Q| \lesssim 2^{-s\varepsilon q'} \alpha^{q'} |S|.$$

Finally, the terms with $-n_\circ \le s \le 0$ are treated by part (ii) of Proposition 4.4 with $n \le n_\circ$ (so that the polynomial growth of the constant in n is irrelevant). We get

$$\Big\| \sum_{\ell=0}^{L(S)-n_\circ} T_\ell B_{\ell-s}\Big\|_{q'}^{q'} \lesssim_{n_\circ} \alpha^{q'} |S|.$$

Combining these estimates we get for all $s \ge -n_\circ$

$$II_s \le \Big\| \sum_{\ell=s\wedge 0}^{L(S)-n_\circ} T_\ell B_{\ell-s} \Big\|_{q'} \|f_2\|_q \lesssim \min\{2^{-s\varepsilon}, 1\}\alpha |S|^{1-\frac{1}{q}} \Big(\int_{3S} |f_2|^q \Big)^{1/q}$$
$$\lesssim 2^{-s\varepsilon} |S| \langle f_1 \rangle_{S,p} \langle f_2 \rangle_{3S,q} \tag{6.9}$$

by the definition $\alpha = \langle f_1 \rangle_{S,p}$. After summing in $s \ge -n_\circ$ we obtain

$$II \lesssim |S| \langle f_1 \rangle_{S,p} \langle f_2 \rangle_{3S,q}. \tag{6.10}$$

Estimation of III. Write $III = |\sum_{j\ge n_\circ} \sum_{\ell=0}^{j-n_\circ} \langle T_\ell [\sum_{Q\in\mathfrak{W}_j} b_Q], f_2 \rangle|$ and estimate $III \le III_{\rm main} + III_{\rm err}$ where

$$\begin{aligned} III_{\rm main} &= \sum_{Q\in\mathfrak{W}} \Big| \Big\langle \sum_{\ell=0}^{L(Q)-n_\circ} T_\ell [f_1 \mathbb{1}_Q], f_2 \mathbb{1}_{3Q} \Big\rangle \Big|, \\ III_{\rm err} &= \sum_{Q\in\mathfrak{W}} \Big| \Big\langle \sum_{\ell=0}^{L(Q)-n_\circ} T_\ell [f_1 \mathbb{1}_Q], f_2 \mathbb{1}_{(3Q)^\complement} \Big\rangle \Big|. \end{aligned} \tag{6.11}$$

Note that $III_{\rm main}$ is the last term in (6.6). By the estimations for I and II we are done if we prove the stronger estimate

$$III_{\rm err} \lesssim |S| \langle f_1 \rangle_{S,p} \langle f_2 \rangle_{3S,1} \tag{6.12}$$

since by Hölder's inequality $\langle f_2 \rangle_{3S,1} \lesssim \langle f_2 \rangle_{3S,q}$.

To see (6.12) we use Lemma 3.5 and estimate $III_{\rm err}$ by

$$\begin{aligned} &\sum_{Q\in\mathfrak{W}} \int_Q |f_1(y)| \int_{(3Q)^\complement \cap 3S} |x-y|^{-N} |f_2(x)| \,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \sum_{m=n_\circ}^\infty \sum_{Q\in\mathfrak{W}_m} \alpha |Q| \sum_{n=1}^\infty 2^{-(n+m)N} \int_{2^{n+1}Q} |f_2| \\ &\lesssim \alpha \sum_{m,n} 2^{-(m+n)(N-d)} \sum_{R\in\mathfrak{D}_{m+n}} \int_{3R} |f_2| \lesssim \alpha \int_{3S} |f_2| \end{aligned}$$

which gives (6.12). □

7. Auxiliary estimates for the proof of Theorem 1.5

We consider multiplier transformations acting on families of functions $F = \{f_Q\}$, with $f_Q : \mathbb{R}^d \to \mathbb{C}$ in L^p, indexed by cubes $Q \in \mathfrak{D}_{\ge 0}$. These functions are assumed to belong to weighted $\ell^r(L^p)$ spaces $\mathcal{V}_{p,r}$ of vector-valued functions, with norm

$$\|F\|_{\mathcal{V}_{p,r}} = \Big(\sum_{j=0}^\infty \sum_{Q\in\mathfrak{D}_j} [2^{-jd(\frac{1}{p}-\frac{1}{r})} \|f_Q\|_p]^r \Big)^{1/r}. \tag{7.1}$$

In the present paper we take $r = 2$. The following result is equivalent with VBR($\frac{2(d+1)}{d+3}, 2$). Fix $\lambda_* = \frac{d-1}{2(d+1)}$ as in Theorem 1.5 and let

$$T_{\lambda_*,\ell} f = h_{\lambda_*,\ell}(\rho(D)) f. \tag{7.2}$$

Lemma 7.1. *Let* $p = \frac{2(d+1)}{d+3}$. *Then for* $0 \le \nu \le 4n_\circ$,

$$\Big\| \sum_{\ell=0}^{\infty} u_\ell T_{\lambda_*,\ell}\big[\sum_{j=0}^{\ell+\nu} \sum_{Q\in\mathfrak{D}_j} f_Q\big]\Big\|_2 \lesssim \|u\|_\infty \|F\|_{\mathcal{V}_{p,2}} \tag{7.3}$$

for any bounded sequence $u = \{u_\ell\}_{\ell=0}^{\infty}$. *Moreover,*

$$\Big(\sum_{\ell=0}^{\infty} \Big\| T_{\lambda_*,\ell}\big[\sum_{j=0}^{\ell+\nu} \sum_{Q\in\mathfrak{D}_j} f_Q\big]\Big\|_2^2\Big)^{1/2} \lesssim \|F\|_{\mathcal{V}_{p,2}}. \tag{7.4}$$

Proof. The inequality (7.4) is a formal consequence of (7.3); this can be seen by taking $u_\ell = r_\ell(t)$ where the r_ℓ are the Rademacher functions, and then averaging in t. Following an idea in [33] we may split, for any choice of j,

$$T_{\lambda_*,\ell} = 2^{-j\lambda_*}\omega_{\ell,j}(\rho(D))\vartheta_j(\rho(D)),$$

where

$$\omega_{\ell,j}(\varrho) = 2^{(j-\ell)\lambda_*}\frac{2^{\ell\lambda_*}h_{\lambda_*,\ell}(\varrho)}{\vartheta_j(\varrho)}, \qquad \text{with } \vartheta_j(\varrho) = (1+2^{2j}(1-\varrho)^2)^{-d}.$$

We change variables $\ell = j+n$. Using the estimates in Lemma 3.4 with large N_1 we obtain for each $n \ge -\nu$ and for $\varrho = \rho(\xi) \in \operatorname{supp}(\chi)$,

$$|\omega_{j+n,j}(\varrho)| \lesssim 2^{-n\lambda_*}\frac{\min\{(2^{(j+n)}|1-\varrho|)^{N_\circ+1}, (1+2^{j+n}|1-\varrho|)^{-N_1}\}}{(1+2^{2j}|1-\varrho|^2)^{-d}}$$

and from this

$$\sup_\xi \sum_{j\ge 0\wedge -n} |\omega_{j+n,j}(\rho(\xi))| \le C2^{-n\lambda_*}$$

with C independent of n (only dependent of $|\nu| \lesssim n_\circ$ which is fixed). Hence with $g_j := \sum_{Q\in\mathfrak{D}_j} f_Q$

$$\begin{aligned}\Big\|\sum_{\ell=0}^{\infty} u_\ell T_{\lambda_*,\ell}\sum_{j=0}^{\ell+n_\circ} g_j\Big\|_2 &= \sum_{n=-n_\circ}^{\infty}\Big\|\sum_{j\ge 0\wedge -n} 2^{-j\lambda_*}u_{j+n}\omega_{j+n,j}(\rho(D))\vartheta_j(\rho(D))g_j\Big\|_2\\ &\lesssim \sum_{n=-n_\circ}^{\infty} 2^{-n\lambda_*}\Big(\sum_{j\ge 0\wedge -n}|u_{j+n}|^2\big\|2^{-j\lambda_*}\vartheta_j(\rho(D))g_j\big\|_2^2\Big)^{1/2}. \end{aligned}\tag{7.5}$$

For $j \ge 0$ we have by (3.17)

$$\big\|2^{-j\lambda_*}\vartheta_j(\rho(D))g_j\|_2 \lesssim \Big(\sum_{Q\in\mathfrak{D}_j} 2^{-jd(\frac{2}{p}-1)}\|f_Q\|_p^2\Big)^{\frac12}$$

and (7.3) follows by combining the two previous displays. □

In order to prove Theorem 1.5 for $(\frac1p,\frac1q)$ on the open edge connecting $(\frac{d+3}{2(d+1)},\frac12)$ with $(\frac12,\frac{d+3}{2(d+1)})$ we need a refined bilinear variant of Lemma 7.1.

Let $u \in \ell^\infty(\mathbb{N}_0)$, $F_i = \{f_{i,Q}\}_{Q\in\mathfrak{D}_{\ge 0}}$, $i = 1,2$, $\ell, j_1, j_2 \ge 0$, $\lambda_* = \frac{d-1}{2(d+1)}$ and $\mathfrak{Q}, \mathfrak{Q}' \subset \mathfrak{D}_{\ge 0}$. Define

$$\Gamma^{\ell,j_1,j_2}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2) := \Big\langle T_{\lambda_*,\ell}\big[\sum_{Q\in\mathfrak{Q}_{j_1}} f_{1,Q}\mathbb{1}_Q\big], \sum_{Q'\in\mathfrak{Q}'_{j_2}} f_{2,Q'}\mathbb{1}_{Q'}\Big\rangle. \tag{7.6}$$

Next define a family of bilinear forms, depending on parameters $0 \le \nu_1, \nu_2 \le 4n_\circ$ by

$$\mathit{\Gamma}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u) := \sum_{\ell\ge 0} u_\ell \sum_{0\le j_1<\ell+\nu_1} \sum_{0\le j_2\le \ell+\nu_2} \Gamma^{\ell,j_1,j_2}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2). \tag{7.7}$$

Proposition 7.2. *Let $\mathfrak{Q} \subset \mathfrak{D}_{\ge 0}$, $\mathfrak{Q}' \subset \mathfrak{D}_{\ge 0}$ each be disjoint families of cubes such that*

$$\operatorname{dist}(Q,Q') > \tfrac{1}{2}\operatorname{diam}(Q') \tag{7.8}$$

for all $(Q,Q') \in \mathfrak{Q}\times\mathfrak{Q}'$ satisfying $L(Q') \ge L(Q)+4$. Then for $(\frac1p,\frac1q)$ on the closed edge connecting $(\frac{d+3}{2(d+1)},\frac12)$ with $(\frac12,\frac{d+3}{2(d+1)})$,

$$\big|\mathit{\Gamma}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)\big| \lesssim \|u\|_\infty \|F_1\|_{\mathcal{V}_{p,2}} \|F_2\|_{\mathcal{V}_{q,2}}. \tag{7.9}$$

We need an auxiliary lemma that states that under the separation condition (7.8) (which is common in Whitney type decompositions) the terms Γ_{ℓ,j_1,j_2} are negligible when $j_1 \le \ell \ll j_2$. Here we have essentially no restrictions on p, q, r.

Lemma 7.3. *Let $\mathfrak{Q} \subset \mathfrak{D}_{\ge 0}$, $\mathfrak{Q}' \subset \mathfrak{D}_{\ge 0}$ for which the separation condition (7.8) is satisfied. Let $1 \le r < \infty$, $1 \le p, q \le \infty$. Then for any $N \ge 0$, $0 \le j_1 \le \ell + 2n_\circ$, $j_2 > \ell + 3n_\circ$,*

$$\big|\Gamma^{\ell,j_1,j_2}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2)\big| \lesssim 2^{-j_2 N} \|F_1\|_{\mathcal{V}_{p,r}} \|F_2\|_{\mathcal{V}_{q,r'}}. \tag{7.10}$$

Proof. Note that if $j_1 \le \ell + 2n_\circ$, $j_2 > \ell + 3n_\circ$, $Q \in \mathfrak{Q}_{j_1}$, $Q' \in \mathfrak{Q}'_{j_2}$ then $j_2 \ge 4 + j_1$ and thus (7.8) holds by assumption. This separation condition implies $\operatorname{dist}(2^{\ell+n_\circ}Q, Q') \ge \sqrt{d}(2^{j_2-1} - 2^{\ell+n_\circ+1}) \ge 2^{j_2-2} \ge 2^{\ell+2n_\circ}$ so that the first estimate in Lemma 3.5 applies. That is, if $K_{\lambda^*,\ell} = \mathcal{F}^{-1}[h_{\lambda_*,\ell}\circ\rho]$ and $x \in Q'$, $y \in Q$ then $|K_{\lambda^*,\ell}(x-y)| \lesssim_{N_1} |x-y|^{-N_1}$, and $|x-y| \approx \operatorname{dist}(Q,Q')$. This yields for fixed j_1, j_2 the bound

$$|\Gamma^{\ell,j_1,j_2}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2)| \lesssim \sum_{m=0}^{\infty} 2^{-(j_2+m)N_1} \sum_{Q'\in\mathfrak{Q}'_{j_2}} \mathcal{I}^m_{j_1,j_2,Q'}, \tag{7.11}$$

$$\text{with } \mathcal{I}^m_{j_1,j_2,Q'} = \int_{Q'} |f_{2,Q'}(x)| \sum_{\substack{Q\in\mathfrak{Q}_{j_1}: 2^{j_2+m-2}\le \\ \operatorname{dist}(Q,Q')<2^{j_2+m+2}}} \int_Q |f_{1,Q}(y)|\,\mathrm{d}y\,\mathrm{d}x.$$

Now,

$$\begin{aligned}
\sum_{Q'\in\mathfrak{Q}'_{j_2}} \mathcal{I}^m_{j_1,j_2,Q'} &\lesssim 2^{j_2\frac{d}{q'}} \sum_{Q'\in\mathfrak{Q}'_{j_2}} \|f_{2,Q'}\|_q \sum_{Q\in\mathfrak{Q}_{j_1}:\mathrm{dist}(Q,Q')\approx 2^{j_2+m}} 2^{j_1\frac{d}{p'}}\|f_{1,Q}\|_p \\
&\lesssim 2^{j_2\frac{d}{q'}} \sum_{Q'\in\mathfrak{Q}'_{j_2}} \|f_{2,Q'}\|_q \Big(\sum_{Q\in\mathfrak{Q}_{j_1}:\mathrm{dist}(Q,Q')\approx 2^{j_2+m}} 2^{j_1\frac{rd}{p'}}\|f_{1,Q}\|_p^r\Big)^{1/r} 2^{(j_2+m-j_1)\frac{d}{r'}} \\
&\lesssim 2^{m\frac{d}{r'}} 2^{j_2 d(2-\frac1r-\frac1q)+j_1 d(\frac1r-\frac1p)} \Big(\sum_{Q'\in\mathfrak{Q}'_{j_2}} \|f_{2,Q'}\|_q^{r'}\Big)^{\frac{1}{r'}} \Big(\sum_{\substack{Q\in\mathfrak{Q}_{j_1},Q'\in\mathfrak{Q}'_{j_2}\\ \mathrm{dist}(Q,Q')\approx 2^{j_2+m}}} \|f_{1,Q}\|_p^r\Big)^{\frac1r} \\
&\lesssim 2^{(j_2+m)d}\Big(\sum_{Q\in\mathfrak{Q}_{j_1}} \big[2^{-j_1 d(\frac1p-\frac1r)}\|f_{1,Q}\|_p\big]^r\Big)^{\frac1r} \Big(\sum_{Q\in\mathfrak{Q}'_{j_2}} \big[2^{-j_2 d(\frac1q-\frac1{r'})}\|f_{2,Q}\|_q\big]^{r'}\Big)^{\frac1{r'}}
\end{aligned}$$

where in the last inequality we have used that for any $Q\in\mathfrak{Q}_{j_1}$ we have

$$\#\{Q'\in\mathfrak{Q}'_{j_2} : \mathrm{dist}(Q,Q')\approx 2^{j_2+m}\}\approx 2^{md}.$$

The claimed bound now follows immediately from (7.11) with $N_1 > N+d$. □

Proof of Proposition 7.2. First consider the case $p=p_1=\frac{2(d+1)}{d+3}$; $q=q_1=2$. We let $G^\ell=\sum_{0\le j_2\le \ell+\nu_2}\sum_{Q'\in\mathfrak{Q}'_{j_2}} f_{2,Q'}\mathbb{1}_{Q'}$, and observe that

$$\varGamma_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)=\sum_{\ell\ge 0} u_\ell\Big\langle T_{\lambda_*,\ell}\big[\sum_{\substack{0\le j_1<\ell+\nu_1,\\ Q\in\mathfrak{Q}_{j_1}}} f_{1,Q}\mathbb{1}_Q\big], G^\ell\Big\rangle. \tag{7.12}$$

Split

$$G^\ell = G - M_\ell - E_\ell, \quad \text{with} \quad G=\sum_{Q'\in\mathfrak{Q}'_{\ge 0}} f_{2,Q'}\mathbb{1}_{Q'}, \tag{7.13}$$

$$M_\ell=\sum_{j_2=\ell+\nu_2+1}^{\ell+4n_\circ}\sum_{Q'\in\mathfrak{Q}'_{j_2}} f_{2,Q'}\mathbb{1}_{Q'},\quad E_\ell=\sum_{Q'\in\mathfrak{Q}'_{>\ell+4n_\circ}} f_{2,Q'}\mathbb{1}_{Q'}.$$

We then have

$$\varGamma_{\mathfrak{Q},\mathfrak{Q}'}=\varGamma^{\mathrm{full}}_{\mathfrak{Q},\mathfrak{Q}'}-\varGamma^{\mathrm{mid}}_{\mathfrak{Q},\mathfrak{Q}'}-\varGamma^{\mathrm{err}}_{\mathfrak{Q},\mathfrak{Q}'}$$

where $\varGamma^{\mathrm{full}}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)$, $\varGamma^{\mathrm{mid}}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)$ and $\varGamma^{\mathrm{err}}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)$ are defined as in (7.12) but with G^ℓ replaced by G, M_ℓ and E_ℓ, respectively. By Lemma 7.1 and the Cauchy-Schwarz inequality we see that

$$\begin{aligned}
\varGamma^{\mathrm{full}}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u) &= \Big|\sum_{\ell=0}^{\infty} u_\ell\Big\langle T_{\lambda_*,\ell}\big[\sum_{\substack{0\le j_1<\ell+\nu_1,\\ Q\in\mathfrak{Q}_{j_1}}} f_{1,Q}\mathbb{1}_Q\big], G\Big\rangle\Big| \\
&\lesssim \|u\|_\infty \|F_1\|_{\mathcal{V}_{p_1,2}}\|G\|_2 \lesssim \|u\|_\infty\|F_1\|_{\mathcal{V}_{p_1,2}}\|F_2\|_{\mathcal{V}_{2,2}}
\end{aligned}$$

where the last inequality uses the disjointness of the cubes in $\mathfrak{Q}'$. Next we apply the Cauchy-Schwarz inequality with respect to x and ℓ and get

$$\begin{aligned}\big|\varGamma^{\mathrm{mid}}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)\big| &= \Big|\sum_{\ell=0}^{\infty} u_\ell \Big\langle T_{\lambda_*,\ell}\big[\sum_{\substack{0\le j_1<\ell+\nu_1,\\ Q\in\mathfrak{Q}_{j_1}}} f_{1,Q}\mathbb{1}_Q\big], M_\ell\Big\rangle\Big| \\ &\lesssim \Big(\sum_{\ell=0}^{\infty}|u_\ell|^2\Big\|T_{\lambda_*,\ell}\big[\sum_{\substack{0\le j_1<\ell+\nu_1,\\ Q\in\mathfrak{Q}_{j_1}}} f_{1,Q}\mathbb{1}_Q\big]\Big\|_2^2\Big)^{1/2}\Big(\sum_{\ell=0}^{\infty}\|M_\ell\|_2^2\Big)^{1/2} \\ &\lesssim \|u\|_\infty\|F_1\|_{\mathcal{V}_{p_1,2}}\|F_2\|_{\mathcal{V}_{2,2}}\end{aligned}$$

where in the last inequality we have used Lemma 7.1 and the square-function estimate $(\sum_{\ell=0}^{\infty}\|M_\ell\|_2^2)^{1/2}\lesssim\|F_2\|_{\mathcal{V}_{2,2}}$. For the terms involving the E_ℓ we use Lemma 7.3 and obtain

$$\big|\varGamma^{\mathrm{err}}_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)\big| = \Big|\sum_{\ell=0}^{\infty} u_\ell \Big\langle T_{\lambda_*,\ell}\big[\sum_{\substack{0<j_1<\ell+\nu_1,\\ Q\in\mathfrak{Q}_{j_1}}} f_{1,Q}\mathbb{1}_Q\big], E_\ell\Big\rangle\Big|$$

$$\lesssim \sum_{\ell=0}^{\infty}\sum_{0<j_1<\ell+\nu_1}\sum_{j_2\ge\ell+4n_\circ} 2^{-j_2N}\|u\|_\infty\|F_1\|_{\mathcal{V}_{p_1,2}}\|F_2\|_{\mathcal{V}_{2,2}} \lesssim \|u\|_\infty\|F_1\|_{\mathcal{V}_{p_1,2}}\|F_2\|_{\mathcal{V}_{2,2}}.$$

Combining the estimates for $\varGamma^{\mathrm{full}}_{\mathfrak{Q},\mathfrak{Q}'}$, $\varGamma^{\mathrm{mid}}_{\mathfrak{Q},\mathfrak{Q}'}$ and $\varGamma^{\mathrm{err}}_{\mathfrak{Q},\mathfrak{Q}'}$ we get

$$\big|\varGamma_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)\big| \lesssim \|u\|_\infty\|F_1\|_{\mathcal{V}_{p_1,2}}\|F_2\|_{\mathcal{V}_{2,2}}. \tag{7.14}$$

Next observe that

$$\varGamma_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)=\widetilde{\varGamma}_{\mathfrak{Q}',\mathfrak{Q}}(F_2,F_1,u),$$

where $\widetilde{\varGamma}$ is the bilinear form associated with the domain $-\Omega$. Hence we get

$$\big|\varGamma_{\mathfrak{Q},\mathfrak{Q}'}(F_1,F_2,u)\big| \lesssim \|u\|_\infty\|F_1\|_{\mathcal{V}_{2,2}}\|F_2\|_{\mathcal{V}_{p_1,2}}. \tag{7.15}$$

It is straightforward to show the interpolation formula $[\mathcal{V}_{q_1,2},\mathcal{V}_{q_2,2}]_\theta=\mathcal{V}_{q,2}$ for the Calderón complex interpolation spaces with $(1-\theta)/q_1+\theta/q_2=1/q$, $1\le q_1,q_2<\infty$. Thus the assertion (7.9) follows by complex interpolation of (7.14) and (7.15). □

8. Sparse domination, Part II

We now prove Theorem 1.5. We have $\lambda_*=\frac{d-1}{2(d+1)}$ and it only remains to prove the $\mathrm{Sp}(p,q)$ bound for $(\frac1p,\frac1q)$ on the closed line segment connecting $(\frac{d+3}{2(d+1)},\frac12)$ and $(\frac12,\frac{d+3}{2(d+1)})$; note that the points on this segment satisfy $\frac1p+\frac1q=\frac{d+2}{d+1}$. As the point $(\frac{d+2}{2(d+1)},\frac{d+2}{2(d+1)})$ is the center of this line segment we may, by symmetry of sparse bounds, assume that $\frac1p\ge\frac{d+2}{2(d+1)}$. As before $T_{\lambda_*,\ell}=h_{\lambda_*,\ell}(\rho(D))$ as in (7.2).

Setting up an induction argument as in §6 one reduces the proof of the sparse bound to the following proposition which contains the main iteration step.

Proposition 8.1. *Let $\frac{2(d+1)}{d+3}\le p\le\frac{2(d+1)}{d+2}$, $\frac1q=\frac{d+2}{d+1}-\frac1p$. Then there is a constant $C>0$ such that for every $S\in\mathfrak{D}_{>0}$ and bounded $f_1:S\to\mathbb{C}$, $f_2:3S\to\mathbb{C}$, there is* *a collection* $\mathfrak{W}$ *of disjoint dyadic subcubes of* S *with the properties*

$$\Big|\bigcup_{Q\in\mathfrak{W}} Q\Big| \le (1-\gamma)\,|S|, \tag{8.1}$$

$$\Big|\sum_{\ell=0}^{L(S)-n_\circ}\langle T_{\lambda_*,\ell} f_1, f_2\rangle\Big| \le C|S|\langle f_1\rangle_{S,p}\langle f_2\rangle_{3S,q} + \sum_{Q\in\mathfrak{W}_{\ge n_\circ}}\Big|\sum_{\ell=0}^{L(Q)-n_\circ}\langle T_{\lambda_*,\ell}[f_1\mathbb{1}_Q], f_2\mathbb{1}_{3Q}\rangle\Big|. \tag{8.2}$$

Proof. Let $\alpha_1 = \langle f_1\rangle_{S,p}$, $\alpha_2 = \langle f_2\rangle_{3S,q}$ and let $\Omega = \Omega_1\cup\Omega_2$ where

$$\Omega_1 = \{x : M_{HL}(|f_1|^p) \ge \tfrac{100d}{1-\gamma}\alpha_1^p\}, \quad \Omega_2 = \{x\in 3S : M_{HL}(|f_2|^q) \ge \tfrac{100d}{1-\gamma}\alpha_2^q\}.$$

Let $\mathfrak{W}$ consist of the subcubes of S which are Whitney cubes of Ω. Since $|\Omega| \le (1-\gamma)|S|$, (8.1) immediately follows. Observe that the pair of collections $(\mathfrak{W},\mathfrak{W})$ satisfies the separation condition (7.8). Indeed, let $Q, Q'\in\mathfrak{W}$ such that $L(Q') \ge L(Q)+4$, i.e. $\mathrm{diam}(Q') \ge 16\,\mathrm{diam}(Q)$. There is $x\in\Omega^{\complement}$ such that $\mathrm{dist}(x,Q)\le 4\,\mathrm{diam}(Q) \le \frac14\,\mathrm{diam}(Q')$ and therefore

$$\mathrm{dist}(Q,Q') \ge \mathrm{dist}(Q',x) - 4\,\mathrm{diam}(Q) \ge \mathrm{diam}(Q') - 4\,\mathrm{diam}(Q) \ge \tfrac34\,\mathrm{diam}(Q'),$$

from which (7.8) holds. Below we will also use that if $\mathfrak{Q}_0$ is the collection of $Q\in\mathfrak{D}_0$ such that Q contains a cube in $\mathfrak{W}$ then the pair $(\mathfrak{Q},\mathfrak{Q}') := (\mathfrak{Q}_0,\mathfrak{W}_{>0})$ also satisfies (7.8). This is shown by a similar argument. Namely if $Q'\in\mathfrak{W}_{>0}$, with $L(Q')\ge 4$ and $Q\in\mathfrak{Q}_0$, $\widetilde{Q}\in\mathfrak{W}$ with $\widetilde{Q}\subseteq Q$ then by the above argument $\mathrm{dist}(\widetilde{Q},Q') \ge \frac34\,\mathrm{diam}(Q')$ and thus $\mathrm{dist}(Q,Q') \ge \frac34\,\mathrm{diam}(Q') - \mathrm{diam}(Q) \ge (\frac34 - \frac1{16})\,\mathrm{diam}(Q')$.

Define $g_i = f_i\mathbb{1}_{\Omega^{\complement}}$ and $b_{i,Q} = f_i\mathbb{1}_Q$, for $i=1,2$. Then

$$\|g_i\|_\infty \lesssim \alpha_i, \qquad \int_Q |b_{1,Q}|^p \le \alpha_1^p|Q|, \qquad \int_Q |b_{2,Q}|^q \le \alpha_2^q|Q|.$$

For $j>0$, $i=1,2$, let

$$B_{i,0} = \sum_{Q\in\mathfrak{W}_{\le 0}} b_{i,Q}, \qquad B_{i,j} = \sum_{Q\in\mathfrak{W}_j} b_{i,Q}.$$

Setting again $\mathcal{T}^S = \sum_{\ell=0}^{L(S)-n_\circ} T_{\lambda_*,\ell}$ we have

$$|\langle\mathcal{T}^S f_1, f_2\rangle| \le I + II + III$$

where

$$I = \big|\langle\mathcal{T}^S g_1, f_2\rangle\big|, \quad II = \Big|\Big\langle\sum_{\ell=0}^{L(S)-n_\circ} T_{\lambda_*,\ell}[\sum_{j=0}^{\ell+n_\circ} B_{1,j}], f_2\Big\rangle\Big|$$

and

$$III = \Big|\Big\langle\sum_{\ell=0}^{L(S)-n_\circ} T_{\lambda_*,\ell}[\sum_{j>\ell+n_\circ} B_{1,j}], f_2\Big\rangle\Big|.$$

Below it will be advantageous to also use the definitions, for $Q \in \mathfrak{D}_{\geq 0}$ and $i = 1, 2$,

$$B_i^Q = \begin{cases} b_{i,Q} & \text{if } Q \in \mathfrak{W}_{>0}, \\ 0 & \text{if } Q \notin \mathfrak{W},\ L(Q) > 0, \\ \sum_{\substack{Q' \in \mathfrak{W} \\ Q' \subset Q}} b_{i,Q'} & \text{if } L(Q) = 0\,. \end{cases} \tag{8.3}$$

With these definitions we have $\int |B_i^Q(x)|^p \, \mathrm{d}x \lesssim \alpha_i^p |Q|$, for $i = 1, 2$ and any $Q \in \mathfrak{D}_{\geq 0}$.

8.1. *Estimation of the terms I and III.* We get $|I| \lesssim |S|\alpha_1\alpha_2$ by exactly the same argument as used in (6.8) (replacing α there by α_1).

Regarding III, the estimation is identical to the estimation of III in the proof of Proposition 6.2. We bound $III \leq III_{\text{main}} + III_{\text{err}}$ with the definition of these terms in (6.11); the main term matches the second term on the right-hand side of (8.2) and the error term is as before estimated by $|S| \langle f_1 \rangle_{S,p} \langle f_2 \rangle_{3S,q}$ using Lemma 3.5.

8.2. *Estimation of II, in the case $p = \frac{2(d+1)}{d+3}$.* This is very similar to the bound for the term II in the proof of Proposition 6.2, except that now we use the improved bound of Lemma 7.1 for $q = 2$. By Lemma 7.1,

$$\Big\| \sum_{\ell=0}^{L(S)-n_\circ} T_{\lambda_*,\ell} \sum_{j=1}^{\ell+n_\circ} B_{1,j} \Big\|_2 \lesssim \Big(\sum_{j \geq 1} \sum_{Q \in \mathfrak{W}_j} 2^{-2jd(\frac{1}{p}-\frac{1}{2})} \|B_{1,j}\|_p^2 \Big)^{\frac{1}{2}}$$

$$\lesssim \alpha_1^{1-\frac{p}{2}} \Big(\sum_{Q \in \mathfrak{W}_{>0}} \|B_1^Q\|_p^p \Big)^{\frac{1}{2}} \lesssim \alpha_1^{1-\frac{p}{2}} \Big(\alpha_1^p \sum_{Q \in \mathfrak{W}_{>0}} |Q| \Big)^{\frac{1}{2}} \lesssim \alpha_1 |S|^{1/2} \lesssim |S|^{1/2} \langle f_1 \rangle_{S,p}.$$

Moreover,

$$\Big\| \sum_{\ell=0}^{L(S)-n_\circ} T_{\lambda_*,\ell} B_{1,0} \Big\|_2 \lesssim \Big(\sum_{Q \in \mathfrak{D}_0} \|B_1^Q\|_p^2 \Big)^{\frac{1}{2}} \lesssim \alpha_1^{1-\frac{p}{2}} \Big(\sum_{Q \in \mathfrak{D}_0} \|B_1^Q\|_p^p \Big)^{\frac{1}{2}}$$

$$\lesssim \alpha_1^{1-\frac{p}{2}} \Big(\alpha_1^p \sum_{Q' \in \mathfrak{D}_0} \sum_{\substack{Q \in \mathfrak{W} \\ Q \subset Q'}} |Q| \Big)^{\frac{1}{2}} \lesssim \alpha_1 |S|^{1/2} \lesssim |S|^{1/2} \langle f_1 \rangle_{S,p}.$$

Combining these two estimates and applying the Cauchy-Schwarz inequality we obtain

$$II \leq \Big\| \sum_{\ell=0}^{L(S)-n_\circ} T_{\lambda_*,\ell} \sum_{j=0}^{\ell+n_\circ} B_{1,j} \Big\|_2 \Big(\int_{3S} |f_2|^2 \Big)^{1/2} \lesssim |S| \langle f_1 \rangle_{S,p} \langle f_2 \rangle_{3S,2}.$$

8.3. *Estimation of II, in the case $\frac{2(d+1)}{d+3} < p \leq \frac{2(d+1)}{d+2}$.* We now split II further as $II \leq II_1 + II_2 + II_3$ where

$$II_1 = \Big| \Big\langle \sum_{\ell=0}^{L(S)-n_\circ} T_{\lambda_*,\ell} [\sum_{j=0}^{\ell+n_\circ} B_{1,j}], g_2 \Big\rangle \Big|, \quad II_2 = \Big| \sum_{\ell=0}^{L(S)-n_\circ} \Big\langle T_{\lambda_*,\ell} [\sum_{j=0}^{\ell+n_\circ} B_{1,j}], \sum_{j'=0}^{\ell+4n_\circ} B_{2,j'} \Big\rangle \Big|,$$

$$II_3 = \Big| \sum_{\ell=0}^{L(S)-n_\circ} \Big\langle T_{\lambda_*,\ell} [\sum_{j=0}^{\ell+n_\circ} B_{1,j}], \sum_{j' > \ell+4n_\circ} B_{2,j'} \Big\rangle \Big|.$$

8.3.1. Estimation of II_1. Since now in the given range we have $\|T_{\lambda_*,\ell}\|_{L^p\to L^p} \lesssim 2^{-\ell\varepsilon(p)}$ we obtain by Hölder's inequality

$$II_1 \le \sum_{\ell=0}^{L(S)-n_\circ} 2^{-\ell\varepsilon(p)}\Big\|\sum_{j=0}^{\ell+n_\circ} B_{1,j}\Big\|_p \|g_2\|_{p'} \lesssim \Big(\int_S |f_1|^p\Big)^{1/p}\Big(\int_{3S}|g_2|^{p'}\Big)^{1/p'}$$
$$\lesssim |S|^{1/p}\langle f_1\rangle_{S,p}|S|^{1/p'}\alpha_2 \lesssim |S|\langle f_1\rangle_{S,p}\langle f_2\rangle_{3S,q}.$$

8.3.2. Estimation of II_2. Now $\frac{1}{q}=\frac{d+2}{d+1}-\frac{1}{p}$. We split $II_2 \le \sum_{i=1}^4 II_{2,i}$ where

$$II_{2,1} = \Big|\sum_{\ell=0}^{L(S)-n_\circ}\Big\langle T_{\lambda_*,\ell}[\sum_{j=1}^{\ell+n_\circ} B_{1,j}], \sum_{j'=1}^{\ell+4n_\circ} B_{2,j'}\Big\rangle\Big|,\ II_{2,2} = \Big|\sum_{\ell=0}^{L(S)-n_\circ}\Big\langle T_{\lambda_*,\ell}[B_{1,0}], \sum_{j'=1}^{\ell+4n_\circ} B_{2,j'}\Big\rangle\Big|,$$
$$II_{2,3} = \Big|\sum_{\ell=0}^{L(S)-n_\circ}\Big\langle T_{\lambda_*,\ell}[\sum_{j=1}^{\ell+n_\circ} B_{1,j}], B_{2,0}\Big\rangle\Big|,\quad II_{2,4} = \Big|\sum_{\ell=0}^{L(S)-n_\circ}\langle T_{\lambda_*,\ell}[B_{1,0}], B_{2,0}\rangle\Big|.$$

We first consider $II_{2,1}$ and apply Proposition 7.2, with $\nu_1 = n_\circ$ and $\nu_2 = 4n_\circ$, letting $\mathfrak{Q} = \mathfrak{Q}'$ be the family of all cubes in $\mathfrak{W}_{>0}$ so that the separation condition (7.8) is satisfied. We then obtain

$$II_{2,1} \lesssim \Big(\sum_{Q\in\mathfrak{W}} 2^{-2L(Q)d(\frac{1}{p}-\frac{1}{2})}\|b_{1,Q}\|_p^2\Big)^{1/2}\Big(\sum_{Q'\in\mathfrak{W}} 2^{-2L(Q')d(\frac{1}{q}-\frac{1}{2})}\|b_{2,Q'}\|_q^2\Big)^{\frac{1}{2}}$$

and write the right-hand side as $II_{2,1}(p)II_{2,1}(q)$. We have

$$II_{2,1}(p) \lesssim \Big(\sum_{Q\in\mathfrak{W}} 2^{-2L(Q)d(\frac{1}{p}-\frac{1}{2})}\|b_{1,Q}\|_p^2\Big)^{1/2}$$
$$\lesssim \Big(\sum_{Q\in\mathfrak{W}} 2^{-2L(Q)d(\frac{1}{p}-\frac{1}{2})}(\alpha_1^p|Q|)^{2/p}\Big)^{1/2} \lesssim \Big(\sum_{Q\in\mathfrak{W}}|Q|\Big)^{1/2}\alpha_1 \lesssim |S|^{1/2}\alpha_1.$$

In exactly the same way we obtain $II_{2,1}(q) \lesssim |S|^{1/2}\alpha_2$ and hence $II_{2,1} \lesssim |S|\alpha_1\alpha_2$.

The expressions $II_{2,2}$, $II_{2,3}$ and $II_{2,4}$ are bounded similarly. For $II_{2,2}$ we let $\mathfrak{Q}_0$ be the family of dyadic unit cubes Q with the property that Q contains a cube in $\mathfrak{W}$, and $\mathfrak{Q}' = \mathfrak{W}_{>0}$. As observed in our discussion at the beginning of the proof we have the separation condition (7.8) in this case. Applying Proposition 7.2 to $\Gamma_{\mathfrak{Q}_0,\mathfrak{W}_{>0}}$ we get

$$II_{2,2} \lesssim \Big(\sum_{Q\in\mathfrak{Q}_0}\Big\|\sum_{\substack{W\in\mathfrak{W}\\ W\subset Q}} b_{1,W}\Big\|_p^2\Big)^{\frac{1}{2}}\Big(\sum_{Q'\in\mathfrak{W}} 2^{-2L(Q')d(\frac{1}{q}-\frac{1}{2})}\|b_{2,Q'}\|_q^2\Big)^{\frac{1}{2}}$$

which we write as $\widetilde{II}_{2,2}(p) II_{2,2}(q)$. Note that $II_{2,2}(q) = II_{2,1}(q) \lesssim |S|^{1/2}\alpha_2$. Moreover,

$$\widetilde{II}_{2,2}(p) \lesssim \Big(\sum_{Q\in\mathfrak{D}_0} \Big(\sum_{\substack{W\in\mathfrak{W}\\ W\subset Q}} \|b_{1,W}\|_p^p \Big)^{\frac{2}{p}} \Big)^{\frac12}$$

$$\lesssim \alpha_1 \Big(\sum_{Q\in\mathfrak{D}_0} \Big(\sum_{\substack{W\in\mathfrak{W}\\ W\subset Q}} |W| \Big)^{\frac{2}{p}} \Big)^{\frac12} \lesssim \alpha_1 \Big(\sum_{Q\in\mathfrak{D}_0} \sum_{\substack{W\in\mathfrak{W}\\ W\subset Q}} |W| \Big)^{\frac12} \lesssim \alpha_1 |S|^{1/2}$$

and hence $II_{2,2} \lesssim \widetilde{II}_{2,2}(p) II_{2,2}(q) \lesssim \alpha_1\alpha_2|S|$. For $II_{2,3}$ we apply Proposition 7.2 with $\mathfrak{Q} = \mathfrak{W}_{>0}$ and with $\mathfrak{Q}'$ being the family of those $Q \in \mathfrak{D}_0$ which contain at least one cube in $\mathfrak{W}$. Likewise for $II_{2,4}$ we use Proposition 7.2 with the families $\mathfrak{Q}, \mathfrak{Q}'$ both consisting of those $Q \in \mathfrak{D}_0$ which contain at least one cube in $\mathfrak{W}$.

8.3.3. Estimation of II_3. Here we use Lemma 7.3 and the assumptions on p, q are irrelevant. We can write $II_3 \le II_{3,1} + II_{3,2}$ with

$$II_{3,1} = \Big| \sum_{\ell=0}^{L(S)-n_\circ} \sum_{j_1=1}^{\ell+n_\circ} \sum_{j_2>\ell+4n_\circ} \Gamma^{\ell,j_1,j_2}_{\mathfrak{Q},\mathfrak{Q}'}(B_1, B_2) \Big|$$

$$II_{3,2} = \Big| \sum_{\ell=0}^{L(S)-n_\circ} \sum_{j_2>\ell+4n_\circ} \Gamma^{\ell,0,j_2}_{\mathfrak{Q},\mathfrak{Q}'}(B_1, B_2) \Big|$$

where $\Gamma^{\ell,j_1,j_2}_{\mathfrak{Q},\mathfrak{Q}'}$ is as in (7.6). Then $|\Gamma^{\ell,j_1,j_2}_{\mathfrak{Q},\mathfrak{Q}'}(B_1, B_2)| \lesssim 2^{-j_2N}\|B_1\|_{\mathcal{V}_{p,2}}\|B_2\|_{\mathcal{V}_{q,2}}$ and since we trivially have $\sum_{\ell\ge0}\sum_{j_1=0}^{\ell+n_\circ}\sum_{j_2\ge\ell+4n_\circ} 2^{-j_2N} = O(1)$ we see that

$$II_{3,1} \lesssim \Big(\sum_{Q\in\mathfrak{W}_{>0}} 2^{-2L(Q)d(\frac1p-\frac12)}\|b_{1,Q}\|_p^2 \Big)^{1/2} \Big(\sum_{Q'\in\mathfrak{W}_{>0}} 2^{-2L(Q')d(\frac1q-\frac12)}\|b_{2,Q'}\|_q^2 \Big)^{\frac12}.$$

Arguing as for the term II_2, this immediately leads to $|II_{3,1}| \lesssim |S|\alpha_1\alpha_2$. Similarly

$$|II_{3,2}| \lesssim \Big(\sum_{Q\in\mathfrak{D}_0} \Big\| \sum_{\substack{W\in\mathfrak{W}\\ W\subset Q}} b_{1,W} \Big\|_p^2 \Big)^{1/2} \Big(\sum_{Q'\in\mathfrak{W}_{>0}} 2^{-2L(Q')d(\frac1q-\frac12)}\|b_{2,Q'}\|_q^2 \Big)^{\frac12}$$

and arguing as in the estimation of $II_{2,2}$ we obtain $II_{3,2} \lesssim |S|\alpha_1\alpha_2$. □

An open problem. It remains open whether for any $\lambda \in (0, \frac{d-1}{2}) \setminus \{\frac{d-1}{2(d+1)}\}$ the sharp $\mathrm{Sp}(p_\lambda, q_\lambda)$ bound with $p_\lambda = \frac{2d}{d+1+2\lambda}$ and $\frac{1}{q_\lambda} = \frac{d+1}{(d-1)p_\lambda} - \frac{2}{d-1}$ holds (and then also the sparse bounds at the top of the trapezoid $\triangle_d(\lambda)$). If in the analysis for the terms II above we replace the Cauchy-Schwarz inequality by Hölder's inequality we see that we would need a sharp version of Lemma 7.1 with $\mathcal{V}^{\mathfrak{Q}}_{p_\lambda,q'_\lambda} \to L^{q'_\lambda}$-boundedness for a disjoint family $\mathfrak{Q}$ of dyadic cubes, where $\mathcal{V}^{\mathfrak{Q}}_{p,r}$ denotes the closed subspace of $\mathcal{V}_{p,r}$ consisting of all $F = \{f_Q\} \in \mathcal{V}_{p,r}$ such that $f_Q = 0$ for $Q \notin \mathfrak{Q}$. The latter is analogous to verifying an endpoint version of $\mathrm{VBR}(p, r)$ where one allows $\frac1r = \frac{d+1}{d-1}(1-\frac1p)$ and assumes that the $f_{j,R}$ are zero if $R \notin \mathfrak{Q}$. We do not know whether such endpoint inequalities hold for $r \neq 2$.

9. Consequences for weak type inequalities with A_1 weights

We record some consequences of our sparse domination results on new weak type weighted inequalities for $\mathcal{R}_a^\lambda$ when $\lambda < \frac{d-1}{2}$. Frey and Nieraeth [16] (extending earlier results in [5]) formulated general theorems about weak type weighted inequalities for operators in $\mathrm{Sp}(p,q)$, satisfying certain A_1 and reverse Hölder conditions. Recall the definitions of the A_1, RH_σ characteristics for a nonnegative measurable function, i.e. a weight w:

$$(9.1)\qquad \begin{aligned} [w]_{A_1} &= \sup_B \Big(\fint w(x)\,\mathrm{d}x\Big)\Big(\operatorname{ess\,inf}_{x\in B} w(x)\Big)^{-1} \\ [w]_{\mathrm{RH}_\sigma} &= \sup_B \Big(\fint_B w(x)^\sigma\,\mathrm{d}x\Big)^{\frac{1}{\sigma}}\Big(\fint_B w(x)\,\mathrm{d}x\Big)^{-1} \end{aligned}$$

if $\sigma \in (1,\infty)$. The relevant class here is $A_1 \cap \mathrm{RH}_\sigma$, for which both characteristics are finite; we recall that w belongs to this class if and only if $w^\sigma \in A_1$ [21]. By [16, Theorem 1.4] operators in $\mathrm{Sp}(p,q)$ map $L^p(w)$ to $L^{p,\infty}(w)$ provided that $w \in A_1 \cap \mathrm{RH}_\sigma$ with $\sigma = (q'/p)' = \frac{q}{q+p-pq}$. More specifically,

$$(9.2)\qquad \|T\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim \|T\|_{\mathrm{Sp}(p,q)} [w^{(q'/p)'}]_{A_\infty}^{1+\frac{1}{p}} [w]_{A_1}^{\frac{1}{p}} [w]_{\mathrm{RH}_{(q'/p)'}}^{\frac{1}{p}},$$

where $[v]_{A_\infty} := \sup_B (v(B))^{-1} \int_B M[v\mathbb{1}_B](x)\,\mathrm{d}x$ is Wilson's A_∞-constant [36], in which the supremum is taken over all balls. For convergence results it is important to note that the A_1, A_∞ and RH_σ characteristics satisfy translation and dilation invariance properties, in the sense that the characteristics for $w(\cdot - h)$ and $t^d w(t\cdot)$ are the same as the corresponding characteristics for w. In two dimensions Kesler and Lacey use (9.2) to obtain the weighted weak type (p_λ, p_λ) inequalities for $\mathcal{R}_a^\lambda$ when $w \in A_1 \cap \mathrm{RH}_\sigma$ and $\sigma > \frac{4}{4-3p_\lambda} = \frac{3+2\lambda}{2\lambda}$. Using Theorems 1.2 and 1.5 we can lower the reverse Hölder exponent by a factor of 4.

Corollary 9.1. *Let $d=2$, $a>0$, $0<\lambda<1/2$, $p_\lambda = \frac{4}{3+2\lambda}$, $\sigma_\circ(\lambda) = \frac{3+2\lambda}{8\lambda}$. Assume that $\sigma > \sigma_\circ(\lambda)$ if $\lambda \in (0,\frac{1}{2}) \setminus \{\frac{1}{6}\}$ and $\sigma \geq \sigma_\circ(\lambda) = \frac{5}{2}$ if $\lambda = \frac{1}{6}$. Then for all $w \in A_1 \cap \mathrm{RH}_\sigma$,*

$$\mathcal{R}_{a,t}^\lambda : L^{p_\lambda}(w,\mathbb{R}^2) \to L^{p_\lambda,\infty}(w,\mathbb{R}^2)$$

with operator norms uniform in t.

Note that when $\lambda \to 1/2$ the reverse Hölder exponent tends to 1 which is to be expected since no reverse Hölder condition is needed in Vargas' result [35] for $p=1$, $\lambda = 1/2$. Similar results can be formulated in higher dimensions for $\sigma_\circ(\lambda) = \frac{(d-1)(d+1+2\lambda)}{4d\lambda}$ and a partial range of λ, depending on the knowledge of sharp $L^p \to L^r$ for the Bochner–Riesz operator. In particular, in view of Remark 1.4, this currently holds for $\frac{d-1}{2(d+1)} \leq \lambda < \frac{d-1}{2}$, which suffices to establish Theorem 1.1.

9.1. *Proof of Theorem 1.1.* We only prove the case $p>1$ since $p=1$ is Vargas' result [35]. Let $\sigma_* = \sigma_\circ(\frac{d-1}{2(d+1)}) = \frac{d+3}{2} > 1$. It is well known [12] that every A_1 weight belongs to $\mathrm{RH}_{\sigma(w)}$ for some $\sigma(w) > 1$; without loss of generality we can assume $1 < \sigma(w) < \sigma_*$. Let $p_1(w) := 1 + \frac{d-1}{d+1}(1 - \frac{1}{\sigma(w)})$ and $1 < p < p_1(w)$. By the preceding discussion, we have that $\mathcal{R}_{a,t}^{\lambda(p)}$ maps $L^p(w) \to L^{p,\infty}(w)$ provided

(i) $\sigma(w) > \sigma_\circ(\lambda(p))$;
(ii) $\frac{d-1}{2(d+1)} < \lambda(p) < \frac{d-1}{2}$.

On the one hand, the condition $\sigma(w) > \sigma_\circ(\lambda(p))$ can be quickly seen to be equivalent to $\frac{d^2-1}{4d\sigma(w)-2(d-1)} < \lambda(p)$, which in turn is equivalent to the condition $p < \frac{d-1}{d+1}(\frac{2d}{d-1} - \frac{1}{\sigma(w)}) = p_1(w)$, which holds by assumption.

On the other hand, since $\sigma_\circ(\lambda(p)) < \sigma(w) < \sigma_* = \sigma_\circ(\frac{d-1}{2(d+1)})$ and $\sigma_\circ(\lambda)$ decreases as a function of λ, we have $\lambda(p) > \frac{d-1}{2(d+1)}$. Moreover, since $p > 1$ we have $\sigma(\lambda(p)) > \sigma(\lambda(1)) = \sigma(\frac{d-1}{2})$, which implies $\lambda(p) < \frac{d-1}{2}$, concluding the proof of (ii).

By the above-mentioned invariance properties and (9.2), the operator norms are uniform in t. Moreover, since the usual approximation of the identity results with L^1 kernels hold in $L^p(w)$ with A_1 weights one can use routine arguments to see that $\lim_{t\to\infty} \mathcal{R}^{\lambda(p)}_{a,t} f = f$ in the $L^{p,\infty}(w)$ norm, for all $f \in L^p(w)$. □

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