A NOTE ON ENDPOINT BOCHNER–RIESZ ESTIMATES

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ABSTRACT. We revisit an ε-removal argument of Tao to obtain sharp $L^p \to L^r(L^p)$ estimates for sums of Bochner–Riesz bumps which are conditional on non-endpoint bounds for single scale bumps. These can be used to obtain sharp conditional sparse bounds for Bochner–Riesz multipliers at the critical index, refining the conditional weak-type (p,p) estimates of Tao.

1. INTRODUCTION

Let Ω be a convex open subset of $\mathbb{R}^d$, $d \ge 2$, containing the origin. We assume that Ω has C^∞-boundary with non-vanishing Gaussian curvature. Let

$$\rho(\xi) = \inf\{t > 0 : \xi/t \in \Omega\}$$

be the Minkowski functional of Ω. Then $\rho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and ρ is homogeneous of degree 1, $\rho(\xi) > 0$ for $\xi \ne 0$ and $\rho(\xi) = 1$ on the boundary $\partial\Omega$. Given $\lambda > 0$, consider the Bochner–Riesz type operator

$$\mathcal{R}^\lambda := (1 - \rho(D))_+^\lambda.$$

The critical index for $L^p \to L^r$ boundedness is defined by

$$\lambda(r) = d\Big(\frac{1}{r} - \frac{1}{2}\Big) - \frac{1}{2}.$$

In this note we establish $L^p \to L^r(L^p)$ vector-valued inequalities for Bochner–Riesz bumps, and acting on families of functions $\{f_Q\}$ indexed by dyadic cubes $\mathfrak{D}$. We denote by $\mathfrak{D}_j$ the dyadic cubes of of sidelength 2^j.

For $M \ge 1$ define $\mathcal{Y}_M$ as the class of all C^M functions χ supported on $(\frac12, 2)$ so that $\|\chi\|_{C^M} = \sum_{\nu=0}^M \|\chi^{(\nu)}\|_\infty \le 1$.

Definition 1.1. *For $1 \le p \le r < \infty$ let $\mathrm{VBR}(p,r)$ denote the following statement. There exists $M > 0$ such that for all collections of functions χ_j in $\mathcal{Y}_M$, the inequality*

$$\Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} \chi_j(2^j(1-\rho(D)))\Big[\sum_{Q\in\mathfrak{D}_j} f_Q\Big]\Big\|_{L^r(\mathbb{R}^d)} \le C\Big(\sum_Q |Q| \|f_Q\|^r_{L^p(\mathbb{R}^d)}\Big)^{1/r} \tag{1.1}$$

holds for all families $\{f_Q\}_{Q\in\mathfrak{D}}$ of L^p functions f_Q, with $\mathrm{supp}\,(f_Q) \subseteq Q$.

The statement $\mathrm{VBR}(p,r)$ plays a significant role in [1] which deals with essentially sharp sparse domination results for the operator $\mathcal{R}^{\lambda(p)}$ in the sense that such sparse bounds follow from $\mathrm{VBR}(p,r)$. Satisfactory $\mathrm{VBR}(p,r)$ bounds are known for

Date: September 22, 2023.
2020 *Mathematics Subject Classification.* 42B15, 42B20, 42B25.
Key words and phrases. Bochner-Riesz means, Weak-type estimates.

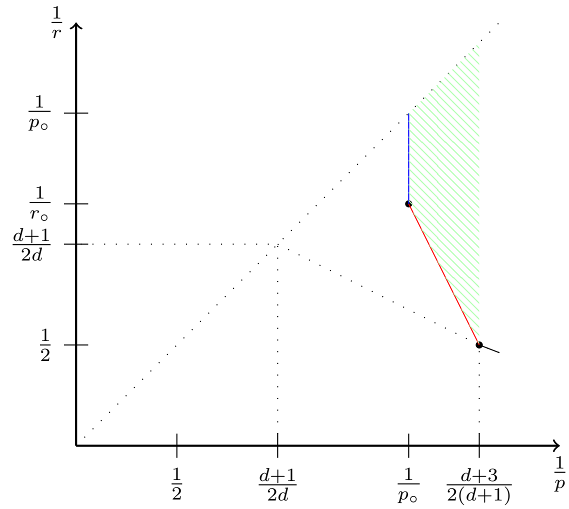

FIGURE 1. The conclusion of Theorem 1.2 holds in the interior of the green region. The red line corresponds to the line $\frac{1}{r} = \frac{1}{r_*(p,p_\circ,r_\circ)}$.

example in two dimensions for p, r in the range $1 \le p < 4/3$, $p \le r < \min\{p'/3, 2\}$ and in higher dimensions for $p \le \frac{2(d+1)}{d+3}$ and $r = 2$ (the Stein-Tomas range), see [13, 14, 16]. Familiar necessary conditions based on Knapp examples show that we need to have $r_\circ \le \frac{d-1}{d+1}p'_\circ$; and thus for $p_\circ \ge \frac{2(d+1)}{d+3}$ we must have $r_\circ \le 2$. Our result will involve the exponent $r_*(p, p_\circ, r_\circ)$ obtained by interpolation of the pairs $(p_\circ, r_\circ)$ and the Stein-Tomas pair $(\frac{2(d+1)}{d+3}, 2)$; the desired vector-valued inequalities for the latter follow from well-known arguments. It is given by

$$\frac{1}{r_*(p,p_\circ,r_\circ)} := \frac{\frac{1}{r_\circ}\big(\frac{d+3}{2(d+1)} - \frac{1}{p}\big) + \frac{1}{2}\big(\frac{1}{p} - \frac{1}{p_\circ}\big)}{\frac{d+3}{2(d+1)} - \frac{1}{p_\circ}}. \tag{1.2}$$

Theorem 1.2. *Let $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ and $r_\circ \in [p_\circ, \frac{d-1}{d+1}p'_\circ]$. Assume that $\mathcal{R}^\lambda$ maps $L^{p_\circ}(\mathbb{R}^d)$ to $L^{r_\circ}(\mathbb{R}^d)$ for all $\lambda > \lambda(r_\circ)$. Let $\frac{2(d+1)}{d+3} \le p < p_\circ$. Then* VBR$(p, r)$ *holds for $p \le r < r_*(p, p_\circ, r_\circ)$*

It is useful to note that $r_*(p, p_\circ, r_\circ) \to \frac{d-1}{d+1}p'$ as $r_\circ \nearrow \frac{d-1}{d+1}p'_\circ$. This implies that if we have the non-endpoint Bochner-Riesz $L^{p_\circ} \to L^{r_\circ}$ assumption for some $p_\circ \in [\frac{2(d+1)}{d+3}, \frac{2d}{d+1})$ and all $r \in [p_\circ, \frac{d-1}{d+1}p'_\circ)$ then the conclusion of VBR(p, r) holds for the full non-endpoint range $\frac{2(d+1)}{d+3} < p < p_\circ$ and $r \in [p, \frac{d-1}{d+1}p')$.

Theorem 1.2 corresponds to an off-diagonal version of a theorem of Tao [16] in which the $r_\circ = p_\circ$ version was obtained. The purpose of the resulting VBR(p, r) estimate in [16] was to prove conditional weak type (p, p) bounds for $\mathcal{R}^{\lambda(p)}$ and strong type results for a class of related multipliers such as $(1-\rho)^\lambda_+(1-\log(1-\rho))^{-\gamma}$, based on reductions in [3, 2, 13, 16]. These reductions also work in the off-diagonal case and yield the following endpoint multiplier theorems (we will not provide more details).

Corollary 1.3. *Let $p_\circ, r_\circ$ be as in Theorem 1.2. Assume $1 \le p < p_\circ$, $p \le r < r_*(p, p_\circ, r_\circ)$, $r \le 2$, $r \le \sigma \le \infty$. Then, for sequences $a = \{a_j\}_{j=1}^\infty \in \ell^\sigma$ we have the inequality*

$$\Big\| \sum_{j>0} a_j 2^{-j\lambda(q)} \chi_j(2^j(1-\rho(D)))f \Big\|_{L^{r,\sigma}} \lesssim \|a\|_{\ell^\sigma} \|f\|_{L^p} \tag{1.3}$$

The proof of Theorem 1.2 for $r > p$ is a re-elaboration of that of Tao for $r = p$. We claim no originality but provide full details of the argument in view of the applicability in [1] and also in view of Tao's question [16] concerning the possibility of ε-removal results for $L^p \to L^r$ bounds. As Tao remarks, such bounds would be especially interesting for the critical line $\frac{1}{r_{\mathrm{crit}}(p)} = \frac{d+1}{d-1}(1 - \frac{1}{p})$. For applicability in [1] we only need to address the case $p < r < r_*(p, p_\circ, r_\circ)$; the latter condition becomes $p < r < r_{\mathrm{crit}}(p)$ if we assume $\mathrm{VBR}(p_\circ, r_\circ)$ for all $r_\circ = \frac{d-1}{d+1}p_\circ' - \varepsilon$ and $\varepsilon \to 0$.

1.1. *Notation.* We list some frequently used notation.

- *Families of dyadic cubes.* We let $\mathfrak{D}$ be a fixed dyadic lattice, which may or may not satisfy the assumptions in the setup by Lerner-Nazarov [10] (this requirement is only important when considering sparse bounds as in [1]). Let $\mathfrak{D}_j$ denote the subset of cubes in $\mathfrak{D}$ of sidelength 2^j. Cubes in $\mathfrak{D}_j$ are assumed to be half open, i.e. of the form $\prod_{i=1}^d [a_i, a_i + 2^j)$ for suitable $a \in \mathbb{R}^d$. We use $\mathfrak{Q}$ for general subcollections of $\mathfrak{D}$, and let $\mathfrak{Q}_j$ be the cubes in $\mathfrak{Q}$ which are of sidelength 2^j. The sidelength of a dyadic cube Q is denoted by $2^{L(Q)}$ with $L(Q) \in \mathbb{Z}$.
- *Constants.* Given a list of objects L and real numbers A, $B \ge 0$, we write $A \lesssim_L B$ or $B \gtrsim_L A$ to indicate $A \le C_L B$ for some constant C_L which depends only items in the list L. We write $A \sim_L B$ to indicate $A \lesssim_L B$ and $B \lesssim_L A$.
- *Normalized bump functions.* Throughout the paper we shall fix a number $N \ge d+1$, and set $\mathcal{Y} = \mathcal{Y}_{d+1+N}$. The functions χ_j are throughout assumed to belong to $\mathcal{Y}$.
- *Multiplier notation.* For $m \in L^\infty(\mathbb{R}^d)$ we define the multiplier operator $m(D)$ which acts initially on Schwartz functions by

$$m(D)f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x, \xi\rangle} m(\xi) \widehat{f}(\xi) \, \mathrm{d}\xi.$$

Structure of this note. In §2 we provide some single scale estimates for suitable frequency or spatially localized bumps associated to Bochner–Riesz multipliers, as well as a vector-valued Stein–Tomas type estimate. In §3 we present some L^2-estimates based on a finer localization that will feature in the proof of Theorem 1.2. In §4 we formulate a discrete variant of Theorem 1.2. In §5 we recall a stopping time lemma due to Tao that features in many ε-removal arguments. In §6, in which we give an $L^p \to L^r$ variant of an argument by Tao that deduces Fourier restriction estimates from non-endpoint Bochner-Riesz assumptions. We present the proof of Proposition 4.2 in §7, which in turn implies that of Theorem 1.2.

Acknowledgements. This research was supported through the program Oberwolfach Research Fellows by Mathematisches Forschungsinstitut Oberwolfach in 2023. The authors were supported in part by National Science Foundation grants DMS-1954479 (D.B.), DMS-2154835 (J.R.), DMS-2054220 (A.S.), and by the AEI grants RYC2020-029151-I and PID2022-140977NA-I00 (D.B.).

2. Single-scale estimates

In this section we provide estimates for suitable frequency and spatially localised Bochner–Riesz bumps. Before going into details, we make a couple of observations regarding the function ρ that will be useful in upcoming arguments. First, we can use polar coordinates for the distance function ρ and write $\xi = \varrho\xi'$ with $\xi' \in \partial\Omega = \{\xi : \rho(\xi) = 1\}$, and

$$\mathrm{d}\xi = \varrho^{d-1}\,\mathrm{d}\varrho\,\mathrm{d}\mu(\xi') \quad \text{where } \mathrm{d}\mu(\xi') = \frac{|\langle \xi', \nabla\rho(\xi')\rangle|}{|\nabla\rho(\xi')|}\,\mathrm{d}\sigma(\xi'). \tag{2.1}$$

Second, by the homogeneity and positivity of ρ there are constants $c_0 < 1$ and $C_0 > 2$ such that

$$c_0|\xi| \le \rho(\xi), \quad |\nabla\rho(\xi)| \le C_0 \tag{2.2}$$

for all $\xi \in \mathbb{R}^d$.

2.1. *Fractional derivatives and subordination formula.* Given $L^p \to L^r$ bounds for $(1-\rho(D))_+^\lambda$ one can derive analogous estimates for the Fourier multiplier operators $\chi_j(2^j(1-\rho(D)))$ and their spatially localized versions using the subordination formula [18]

$$h_j(\varrho) = \frac{1}{\Gamma(\lambda+1)} \int_0^\infty (s-\varrho)_+^\lambda h_j^{(\lambda+1)}(s)\,\mathrm{d}s. \tag{2.3}$$

Here, for smooth h compactly supported in $(0,\infty)$, $h^{(a)}$ for $a \in (0,\infty)\setminus\mathbb{N}$ refers to a fractional derivative for functions on $(0,\infty)$ introduced in [5]. More precisely, when $a \in (0,1)$ one defines

$$h^{(a)}(\varrho) = \frac{-1}{\Gamma(1-a)} \lim_{u\to\infty} \frac{d}{d\varrho} \int_\varrho^u (s-\varrho)^{-a} h(s)\,\mathrm{d}s,$$

and for $a \in (0,\infty)\setminus\mathbb{N}$, $a > 1$ one defines inductively $h^{(a)} = \frac{d}{d\rho} h^{(a-1)}$. This leads to the formula

$$\widehat{h^{(a)}}(\tau) = (-i\tau)^a \widehat{h}(\tau) = (\cos\tfrac{\pi a}{2} - i\,\mathrm{sign}(\tau)\sin\tfrac{\pi a}{2})|\tau|^a \widehat{h}(\tau).$$

Note that $h^{(a)}$ coincides with $(-1)^a$ times the ordinary derivative of order a when a is a positive integer. The following lemma will be relevant in the aforementioned transference of bounds.

Lemma 2.1. *For $\lambda > -1$ we have the inequality*

$$\int_0^\infty t^{d(\frac1p - \frac1r)+\lambda} |h_j^{(\lambda+1)}(t)|\,\mathrm{d}t \lesssim 2^{j\lambda}. \tag{2.4}$$

Proof. Observe that h_j is supported in $I_j := [1-2^{-j+1}, 1-2^{-j-1}]$. If $\lambda+1$ is an integer we have $|h^{(\lambda+1)}(s)| \lesssim 2^{j(\lambda+1)} \mathbb{1}_{I_j}(s)$ and the asserted inequality is immediate.

Assume that $\kappa < \lambda + 1 < \kappa + 1$ for $\kappa \in \mathbb{N}_0$. By the definition of fractional derivative

$$h_j^{(\lambda+1)}(\varrho) = 0 \quad \text{for } \varrho > 1 - 2^{j-1}. \tag{2.5}$$

We claim that

$$|h_j^{(\lambda+1)}(\varrho)| \lesssim \frac{2^{j(\lambda+1)}}{1 + (2^j(1-\varrho))^{\lambda+2}}, \quad 0 \le \varrho \le 1\,. \tag{2.6}$$

The inequality (2.4) is now immediate from (2.5) and (2.6).

To show (2.6), let $a = \lambda + 1 - \kappa \in (0,1)$ and observe that by integration by parts we have the formulas

$$\begin{aligned} h_j^{(\lambda+1)}(\varrho) &= c_{\kappa,1}\Big(\frac{d}{d\varrho}\Big)^{(\kappa+1)} \int_\varrho^\infty (s-\varrho)^{-a} \chi_j(2^j(1-s))\,\mathrm{d}s && \text{(2.7a)}\\ &= c_{\kappa,2}\Big(\frac{d}{d\varrho}\Big)^{(\kappa+1)} \int_\varrho^\infty (s-\varrho)^{\kappa+2-a} 2^{j(\kappa+2)} \chi_j^{(\kappa+2)}(2^j(1-s))\,\mathrm{d}s \\ &= c_{\kappa,3} 2^{j(\kappa+2)} \int_\varrho^\infty (s-\varrho)^{1-a} \chi_j^{(\kappa+2)}(2^j(1-s))\,\mathrm{d}s. && \text{(2.7b)} \end{aligned}$$

From (2.7b) we get that $|h_j^{(\lambda+1)}(\varrho)| \lesssim 2^{j(\kappa+a)}$ for $\varrho > 1 - 2^{-j+2}$ which gives (2.6) in this range.

Next assume $\varrho < 1 - 2^{-j+2}$. We can now differentiate under the integral sign directly in (2.7a) and use

$$h^{(\lambda+1)}(\varrho) = c_{\kappa,4} \int_\rho^\infty (s-\varrho)^{-a-\kappa-1} \chi_j(2^j(1-s))\,\mathrm{d}s.$$

We can estimate this integral by $2^{-j}(1-\varrho)^{-a-\kappa-1}$ and since $a + \kappa + 1 = \lambda + 1$ we obtain (2.6) also for $\varrho < 1 - 2^{-j+2}$. □

2.2. *Spatial localizations and single scale-estimates.* Let ϕ_0 be a $C_c^\infty(\mathbb{R}^d)$ function supported in $\{x : |x| < 1\}$ such that $\phi_0(x) = 1$ for $|x| \le 1/2$. For $j > 0$ define

$$\begin{aligned} \phi_{j,0}(x) &:= \phi_0(2^{-j}x) \\ \phi_{j,n}(x) &:= \phi_0(2^{-j-n}x) - \phi_0(2^{-j-n+1}x) \quad \text{for } n \ge 1. \end{aligned} \tag{2.8}$$

For any $j > 0$, define with $\chi_j \in \mathcal{Y}$,

$$m_j(\xi) := h_j(\rho(\xi)), \quad \text{where } h_j(\varrho) := \chi_j(2^j(1-\varrho)), \tag{2.9}$$

and let, for any $n \ge 0$,

$$m_{j,n} := m_j * \widehat{\phi_{j,n}}. \tag{2.10}$$

In forthcoming arguments, we will use the estimate

(2.11)

$$|m_{j,n}(\xi)| \lesssim_N \sum_{k=1}^N \int \int_0^1 \tfrac{(1-s)^{N-1}}{(N-1)!} 2^{jN} |\eta|^N |\widehat{\phi_{j,n}}(\eta)| |\chi_j^{(k)}(2^j(1-\rho(\xi - s\eta)))|\,\mathrm{d}s\,\mathrm{d}\eta$$

for all $n > 0$. Note that this follows from the vanishing moments of $\widehat{\phi_{j,n}}$ and Taylor's formula for $m_j(\xi-\eta)$, which together with the multidimensional Faà di Bruno formula allow to write

(2.12)
$$\begin{aligned} m_{j,n}(\xi) &= \int\int_0^1 \tfrac{(1-s)^{N-1}}{(N-1)!}\langle -\eta, \nabla\rangle^N \big[\chi_j(2^j(1-\rho(\xi-s\eta)))\big]\widehat{\phi_{j,n}}(\eta)\,\mathrm{d}s\,\mathrm{d}\eta \\ &= \sum_{k=1}^N \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=N} 2^{jk}\int_0^1\int b_k(s,\xi,\eta)\eta^\alpha\widehat{\phi_{j,n}}(\eta)\chi_j^{(k)}(2^j(1-\rho(\xi-s\eta)))\,\mathrm{d}\eta\,\mathrm{d}s \end{aligned}$$

for $b_k\in C^\infty$. From (2.11) we also obtain the pointwise estimate

$$|m_{j,n}(\xi)| \lesssim_{N_1} 2^{-nN}(1+2^j|1-\rho(\xi)|)^{-N_1} \tag{2.13}$$

for all $n>0$, where $N_1>0$ is arbitrary. This inequality also extends to the case $n=0$ by a straightforward convolution inequality.

One can transfer bounds for $(1-\rho(D))_+^\lambda$ to bounds on m_j and $m_{j,n}$ through the following lemma.

Lemma 2.2. *Let $1\le p\le r\le\infty$ and assume that $(1-\rho(D))_+^\lambda$ is bounded from $L^p(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$. Let m_j be as in (2.9), with $\chi_j\in\mathcal{Y}_{d+1}$. Then we have*

$$\|m_j(D)\|_{L^p\to L^r}\lesssim 2^{j\lambda}. \tag{2.14}$$

If $\chi_j\in\mathcal{Y}$ then

$$\|m_{j,n}(D)\|_{L^p\to L^r}\lesssim 2^{-nN}2^{j\lambda}, \qquad n\ge 0. \tag{2.15}$$

Moreover, if $p\le q\le r$,

$$\|m_j(D)\|_{L^p\to L^q}\lesssim 2^{j(\lambda+d(\frac1q-\frac1r))}, \tag{2.16a}$$

$$\|m_{j,n}(D)\|_{L^p\to L^q}\lesssim 2^{-nN}2^{j(\lambda+d(\frac1q-\frac1r))}, \qquad n\ge 0. \tag{2.16b}$$

Proof. Since ρ is homogeneous of degree 1 we get using (2.3) for $m_j=h_j\circ\rho$,

$$\begin{aligned} \|m_j(D)\|_{L^p\to L^r} &\le \int_0^\infty s^\lambda|h_j^{(\lambda+1)}(s)|\,\|(1-\rho(D)/s)_+^\lambda\|_{L^p\to L^r}\,\mathrm{d}s \\ &= \|(1-\rho(D))_+^\lambda\|_{L^p\to L^r}\int_0^\infty s^{d(\frac1p-\frac1r)+\lambda}|h_j^{(\lambda+1)}(s)|\,\mathrm{d}s\lesssim 2^{j\lambda} \end{aligned} \tag{2.17}$$

where in the last inequality we have used the hypothesis and Lemma 2.1. Similarly, for $n=0$ we obtain

$$\|m_{j,0}(D)\|_{L^p\to L^r}\lesssim\int|\widehat{\phi_{j,0}}(\eta)|\|m_j(D-\eta)\|_{L^p\to L^r}\,\mathrm{d}\eta\lesssim 2^{j\lambda}$$

where we have used the modulation invariance of the operator norm.

For $n>0$ we use (2.11) to obtain that $\|m_{j,n}(D)\|_{L^p\to L^r}$ is bounded by a constant times

$$\sum_{k=1}^N\int\int_0^1\tfrac{(1-s)^{N-1}}{(N-1)!}2^{jN}|\eta|^N|\widehat{\phi_{j,n}}(\eta)|\big\|\chi_j^{(k)}(2^j(1-\rho(D-s\eta))\big\|_{L^p\to L^r}\,\mathrm{d}s\,\mathrm{d}\eta.$$

Since the functions $\chi_j^{(k)}$ are by assumption fixed multiples of $\mathcal{Y}_{d+1}$ functions and $\int 2^{jN}|\eta|^N|\widehat{\phi_{j,n}}(\eta)|\,\mathrm{d}\eta = O(2^{-nN})$, we get (2.15) by the modulation invariance of the multiplier norms.

Finally, note that the convolution kernel of $m_{j,n}$ is supported on a set of diameter $O(2^{j+n})$. Therefore, using (2.15) and Hölder's inequality,

$$\begin{aligned}\|m_{j,n}(D)f\|_q &= \Big\| \sum_{Q\in\mathfrak{D}_{j+n}} m_{j,n}(D)[f\mathbb{1}_Q]\Big\|_q \lesssim \Big(\sum_{Q\in\mathfrak{D}_{j+n}} \|m_{j,n}(D)[f\mathbb{1}_Q]\|_q^q\Big)^{1/q} \\ &\lesssim \Big(\sum_{Q\in\mathfrak{D}_{j+n}} \big[\|m_{j,n}(D)[f\mathbb{1}_Q]\|_r 2^{(j+n)d(\frac{1}{q}-\frac{1}{r})}\big]^q\Big)^{1/q} \\ &\lesssim \Big(\sum_{Q\in\mathfrak{D}_{j+n}} \big[2^{-nN}2^{j\lambda}\|f\mathbb{1}_Q\|_p 2^{(j+n)d(\frac{1}{q}-\frac{1}{r})}\big]^q\Big)^{1/q} \\ &\lesssim 2^{-n(N-d(\frac{1}{q}-\frac{1}{r}))}2^{j(\lambda+d(\frac{1}{q}-\frac{1}{r}))}\|f\|_p\end{aligned}$$

which is (2.16b). Inequality (2.16a) follows after summing in $n \geq 0$. □

It is well-known by the work of Fefferman and Stein [8] that $\mathcal{R}^\lambda$ maps $L^{\frac{2(d+1)}{d+3}}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ for all $\lambda > -1/2$. Taking this into account we note in the following corollary that boundedness of the Bochner–Riesz operator $\mathcal{R}^\lambda$ for a specific pair of exponents $(p_\circ, r_\circ)$ implies $L^p \to L^r$ bounds for the region in Figure 1.

Corollary 2.3. *Let* $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ *and* $r_\circ \in [p_\circ, \frac{d-1}{d+1}p_\circ']$. *Assume that* $\mathcal{R}^\lambda$ *maps* $L^{p_\circ}(\mathbb{R}^d)$ *to* $L^{r_\circ}(\mathbb{R}^d)$ *for all* $\lambda > \lambda(r_\circ)$. *Let* $\frac{2(d+1)}{d+3} \leq p_1 < p_\circ$. *Then for all* $\varepsilon > 0$, *the inequalities*

$$\begin{aligned}&\|m_j(D)\|_{L^{p_1}\to L^r} \lesssim_\varepsilon 2^{j(\varepsilon+\lambda(r))} \\ &\|m_{j,n}(D)\|_{L^{p_1}\to L^r} \lesssim_\varepsilon 2^{-nN}2^{j(\varepsilon+\lambda(r))}, \qquad n \geq 0\end{aligned}$$

hold for all $p_1 \leq r \leq r_*(p_1, p_\circ, r_\circ)$. *Moreover, in this range,* $\mathcal{R}^\lambda$ *maps* L^{p_1} *to* L^r *for* $\lambda > \lambda(r)$.

Proof. By Lemma 2.2 it suffices to prove this for $r = r_*(p_1, p_\circ, r_\circ)$. By the same lemma and the boundedness assumption on the Bochner-Riesz operator we have

$$\|m_{j,n}(D)\|_{L^{p_\circ}\to L^{r_\circ}} \lesssim_\varepsilon 2^{-nN}2^{j(\varepsilon+\lambda(r_\circ))}.$$

On the other hand, by the same lemma and the aforementioned $L^{\frac{2(d+1)}{d+3}}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ boundedness,

$$\|m_{j,n}(D)\|_{L^{\frac{2(d+1)}{d+3}}\to L^2} \lesssim_\varepsilon 2^{-nN}2^{-j/2+j\varepsilon}.$$

Interpolating these two inequalities yields the assertion on the $L^{p_1} \to L^{r_*(p_1,p_\circ,r_\circ)}$ operator norm of $m_{j,n}(D)$ and summing in n yields the corresponding assertion for $m_j(D)$. The implication on $\mathcal{R}^\lambda$ follows by the standard decomposition of $(1-\rho(D))_+^\lambda$ as a sum of operators of type $m_j(D)$ and an L^1 bounded operator. □

Remark 2.4. A standard argument using the Stein–Tomas theorem [7, 8] reveals that that the $L^p \to L^2$ operator norm of $m_j(D)$ is $O(2^{-j/2})$ for $1 \le p \le \frac{2(d+1)}{d+3}$, without any ε-loss. Using the orthogonality of the m_j, then running the decomposition $m_j = \sum_{n=0}^{\infty} m_{j,n}$, and using the support assumptions of $m_{j,n}$ we can use this to upgrade the bounds for $m_j(D)$ when $r = 2$ in Corollary 2.3 to the estimates

$$\Big\| \sum_{j>0} \sum_{Q\in\mathfrak{D}_j} 2^{j/2} m_j(D)[f_Q \mathbb{1}_Q] \Big\|_2 \lesssim \Big(\sum_{Q\in\mathfrak{D}} \|f_Q\|_p^2 \Big)^{\frac12} \tag{2.18}$$

which correspond to VBR$(p, 2)$ for $1 \le p \le \frac{2(d+1)}{d+3}$.

2.3. *A kernel estimate.* We finish this section with a pointwise bound for the kernels associated to the multipliers $|m_{j,n}|^2$. This will be used in T^*T arguments in Section 3, and the proof is based on stationary phase calculations.

Lemma 2.5. *For $j > 0$, $n \ge 0$ let $\kappa_{j,n} = \mathcal{F}^{-1}[|m_{j,n}|^2]$. Then*

$$\sup_{x\in\mathbb{R}^d} (1 + |x|)^{\frac{d-1}{2}} |\kappa_{j,n}(x)| \lesssim 2^{-2nN} 2^{-j}. \tag{2.19}$$

Proof. We give the proof assuming $n > 0$; a small modification yields the cases $n = 0$. Using (2.13) it is straightforward to see that

$$|\kappa_{j,n}(x)| \lesssim 2^{-j} 2^{-2nN}; \tag{2.20}$$

we use this for $|x| \lesssim C_0$ (where C_0 is as in (2.2)). Note that $\kappa_{j,n}(x) = 0$ if $|x| \ge 2^{j+n+2}$.

Now assume $C_0 \le |x| \le 2^{j+n+2}$. We use formula (2.12) for $m_{j,n}$ and its complex conjugate. We then write

$$\kappa_{j,n} = \kappa_{j,n,0} + \kappa_{j,n,\complement}$$

where

$$\kappa_{j,n,0}(x) = (2\pi)^{-d} \int e^{i\langle x,\xi\rangle} \iiiint\limits_{[0,1]^2\times\mathcal{U}} \frac{((1-s_1)(1-s_2))^{N-1}}{((N-1)!)^2} \widehat{\phi_{j,n}}(v) \overline{\widehat{\phi_{j,n}}(w)} \times$$
$$\langle -v, \nabla\rangle^N \big[\chi_j(2^j(1-\rho(\xi - s_1 v)))\big] \overline{\langle -w, \nabla\rangle^N \big[\chi_j(2^j(1-\rho(\xi - s_2 w)))\big]}\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}\xi$$

where the set $\mathcal{U} \equiv \mathcal{U}(x, j, n)$ is defined by

$$\mathcal{U} = \{(v, w) : |v| < \tfrac{1}{8C_0} 2^{-j-n/2} |x|^{1/4}, |w| < \tfrac{1}{8C_0} 2^{-j-n/2} |x|^{1/4}\}.$$

The term $\kappa_{j,n,\complement}(x)$ is the analogous expression where the region $\mathcal{U}$ is replaced by $\mathcal{U}^\complement = \mathbb{R}^{2d} \setminus \mathcal{U}$. We first analyze the terms $\kappa_{j,n,\complement}(x)$. We first note that for all $v, w \in \mathbb{R}^d$, $(s_1, s_2) \in [0, 1]^2$,

$$\mathrm{meas}(\{\xi : \max\{|\rho(\xi - s_1 v) - 1|, |\rho(\xi - s_2 v) - 1|\} \le 2^{-j+1}\}) \lesssim 2^{-j}$$

and interchange the order of integration to apply the integral in ξ first. For $|x| \gg 1$

$$\int_{|v|\ge 2^{-j-n/2}|x|^{1/4}} |v|^N |\widehat{\phi_{j+n}}(v)|\, \mathrm{d}\eta \lesssim_{N_2} 2^{(j+n)(d-N_2)} \int_{|\eta|\ge 2^{-j-n/2}|x|^{1/4}} |v|^{N-N_2}\, \mathrm{d}v$$
$$\lesssim 2^{-jN} 2^{-n\frac{N_2-d}{2}} |x|^{\frac{N+d-N_2}{4}}$$

provided we take $N_2 > N + d$. The same consideration applies to the w-integral. We use this in conjunction with the second part of the formula (2.12) we get for all $N_1 \in \mathbb{N}$,

$$|\kappa_{j,n,\complement}(x)| \lesssim_{N_1} 2^{-j}2^{-nN_1}|x|^{-N_1}, \quad |x| \geq C_0. \tag{2.21}$$

We now turn to the main term $\kappa_{j,n,0}$. By (2.12), $\kappa_{j,n,0}$ is a linear combination of terms of the form

$$\iiiint_{\mathcal{U}} 2^{j(k_1+k_2)} v^\alpha w^\beta \widehat{\phi_{j,n}}(v)\overline{\widehat{\phi_{j,n}}(w)}\mathcal{J}_{k_1,k_2}(x, s_1, s_2, v, w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}s_1\,\mathrm{d}s_2$$

where $\alpha, \beta \in \mathbb{N}_0^d$, $|\alpha| = |\beta| = N$, $1 \leq k_1, k_2 \leq N$, and $\mathcal{J}_{k_1,k_2}$ is given by

$$\begin{aligned} &\mathcal{J}_{k_1,k_2}(x, s_1, s_2, v, w) = \\ &\int e^{i\langle x,\xi\rangle} b_{k_1,k_2}(\xi, s_1, s_2, v, w)\chi_j^{(k_1)}(2^j(1-\rho(\xi - s_1 v))\overline{\chi_j^{(k_2)}(2^j(1-\rho(\xi - s_2 w))}\,\mathrm{d}\xi\,, \end{aligned} \tag{2.22}$$

with $b_{k_1,k_2} \in C^\infty$.

We now use polar coordinates $\xi = \rho(\xi)\Xi^\xi$ where $\Xi^\xi \in \partial\Omega$, with the intent to apply the method of stationary phase in the variables parametrizing $\partial\Omega$. Some care is needed since these variables show up in the rough terms $\chi_j^{(k_i)}(2^j(1-\rho(\xi - s_i v)))$, and for an application of the method of stationary phase we need that the amplitude behaves reasonably well under differentiation.

Let $|v| \leq (4C_0)^{-1}$. Then $|\rho(\xi - sv) - \rho(\xi)| \leq C_0|v| \leq 1/4$ and we have

$$\begin{aligned} \xi - sv &= \rho(\xi - sv)\Xi^{\xi - sv} \\ &= \rho_{s,v}(\rho(\xi), \Xi^\xi)\,\Xi_{s,v}(\rho(\xi), \Xi^\xi) \end{aligned} \tag{2.23}$$

where

$$\rho_{s,0}(\varrho, \Xi) = \varrho, \quad \Xi_{s,0}(\varrho, \Xi) = \Xi \tag{2.24}$$

and

$$(\varrho, \Xi) \mapsto \big(\rho_{s,v}(\varrho, \Xi), \Xi_{s,v}(\varrho, \Xi)\big)$$

is a diffeomorphism that maps $(\frac{1}{2}, 2) \times \partial\Omega$ into an open set containing $(\frac{3}{4}, \frac{7}{4}) \times \partial\Omega$ and contained in $(\frac{1}{4}, \frac{9}{4}) \times \partial\Omega$. For $(v, w) \in \mathcal{U}$ we have $|v| < (8C_0)^{-1}2^{-j-n/2}|x|^{1/4} \leq (8C_0)^{-1}2^{-j3/4-n/4+1/2} \leq (4C_0)^{-1}$ and if $1/4 < \varrho < 3$ then also $\rho_{s_1,v}(\varrho, \Xi) \approx 1$ and $\rho_{s_2,w}(\varrho, \Xi) \approx 1$.

Using ρ-polar coordinates we write $\mathcal{J}_{k_1,k_2}$ in (2.22) as

$$\begin{aligned} \mathcal{J}_{k_1,k_2}(x, s_1, s_2, v, w) \;=\; &\int_\varrho \int_{\partial\Omega} e^{i\langle x,\varrho\Xi\rangle}\beta_{s_1,s_2,v,w}(\varrho, \Xi) \times \\ &\chi_j^{(k_1)}(2^j(1-\rho_{s_1,v}(\varrho, \Xi)))\overline{\chi_j^{(k_2)}(2^j(1-\rho_{s_2,w}(\varrho, \Xi)))}\,\mathrm{d}\mu(\Xi)\,\mathrm{d}\varrho. \end{aligned}$$

For a coordinate patch on $\partial\Omega$ with parametrization $y \mapsto \Xi(y)$ we observe that the y-derivatives of

$$y \mapsto \chi_j^{(k)}(2^j(1-\rho_{s_1,v}(\varrho, \Xi(y)))) \tag{2.25}$$

vanish for $v = 0$, by (2.24). Hence the y-derivatives of order L of the function in (2.25) are $O(1 + (2^j|v|)^L)$. By the assumption $|v| \lesssim 2^{-j-n/2}|x|^{1/4}$, this is $O(|x|^{L/4})$.

The same applies to the entire amplitude of the y integral. By the inequalities $|v|, |w| \lesssim 2^{-j-n/2}|x|^{1/4}$ we see that the derivatives of total order L in Ξ are $O(|x|^{L/4})$. Since the oscillation parameter is $|x|$, we are still able to use the method of stationary phase to see that the inner Ξ-integral is $O(|x|^{-\frac{d-1}{2}})$, uniformly in ϱ and $(s_1, s_2) \in [0,1]$, $(v,w) \in \mathcal{U}$. For each $(s_1, s_2, v, w) \in [0,1]^2 \times \mathcal{U}$, the ρ integration is over a set of measure $O(2^{-j})$ and we obtain for $C_0 \le |x| \le 2^{j+n+2}$

$$|\mathcal{J}_{k_1,k_2}(x, s_1, s_2, v, w)| \lesssim 2^{-j}|x|^{-(d-1)/2}.$$

Finally the v, w integrations give a bound of $O(2^{-nN})$ each and we arrive at the estimate

$$|\kappa_{j,n,0}(x)| \lesssim 2^{-2Nn}2^{-j}|x|^{-\frac{d-1}{2}} \text{ for } C_0 \le |x| \le 2^{j+n+2}. \tag{2.26}$$

We finish by combining (2.20), (2.21) and (2.26). □

3. Auxiliary L^2 bounds and finer localizations

Let η be a real valued non-negative Schwartz function such that $\widehat{\eta}$ has compact support in $\{\xi : |\xi| \le 2\}$ and such that $\eta(x) \ge 1$ for $\max_{1 \le i \le d} |x_i| \le 2$. Let B be a cube of sidelength $R = R_B > 1$ and center x_B and define

$$\eta_B(x) = \eta\Big(\frac{x - x_B}{R_B}\Big). \tag{3.1}$$

Let $j > 0$ and m_j be as in (2.9). For $Q \in \mathfrak{D}_j$, let $\mathcal{B}_Q$ a family of pairwise disjoint subcubes of Q of sidelength R. From Plancherel's theorem and the decay properties of η, it is easy to see that for $R \le 2^j$ and functions $\{f_{Q,B}\}_{Q \in \mathfrak{D}_j, B \in \mathcal{B}_Q}$

$$\Big\| m_j(D)\big[\sum_{Q \in \mathfrak{D}_j} \sum_{B \in \mathcal{B}_Q} \eta_B f_{Q,B}\big]\Big\|_2 \lesssim \Big(\sum_{Q,B} \|f_{Q,B}\|_2^2\Big)^{1/2}. \tag{3.2}$$

A key insight used in Tao's work [16] is that certain standard L^2 bounds can be improved under the assumptions that the families $\mathcal{B}_Q$ are sufficiently separated. We start with a definition.

Definition 3.1. *Let $R \ge 1$ and $S \ge 3R$. A family $\mathcal{B}$ of axis-parallel cubes of sidelength R is S-separated if $\operatorname{dist}(x_B, x_{B'}) > S$ for all $B, B' \in \mathcal{B}$, $B \ne B'$, where x_B denotes the center of B.*

Proposition 3.2. *Let $j > 0$, $R \le 2^j$. For each $Q \in \mathfrak{D}_j$ let $S_Q \ge 3R$ and $\mathcal{B}_Q$ be a finite family of S_Q-separated cubes of sidelength R intersecting Q. Then*

$$\Big\| m_j(D)\big[\sum_{Q \in \mathfrak{D}_j} \sum_{B \in \mathcal{B}_Q} \eta_B f_{Q,B}\big]\Big\|_2 \lesssim (2^{-j}R)^{1/2} \sup_Q \big(1 + S_Q^{-\frac{d-1}{2}} R^{d-1} \#\mathcal{B}_Q\big)^{1/2} \Big(\sum_{Q,B} \|f_{Q,B}\|_2^2\Big)^{1/2} \tag{3.3}$$

and

$$\Big\| m_j(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_B f_{Q,B}\big]\Big\|_1 \lesssim 2^{j(d-1)/2}R^{1/2}\sup_Q\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{1/2}\sum_Q\Big(\sum_B\|f_{Q,B}\|_2^2\Big)^{1/2} \tag{3.4}$$

for all families of functions $\{f_{Q,B}\}$.

Clearly (3.3) is a significant improvement over (3.2) if S_Q is large enough for all Q, specifically if $S_Q > (R^{d-1}\#\mathcal{B}_Q)^{2/(d-1)}$.

In the proof of Proposition 3.2, we will work with variants of m_j and η_B which are localized in space. We may decompose $m_j = \sum_{n_1=0}^\infty m_{j,n_1}$ with m_{j,n_1} as in (2.10). We also decompose η_B using a decomposition analogous to (2.8). Define

$$\begin{aligned}\eta_{B,0}(x) &:= \phi_0(\tfrac{x-x_B}{R_B})\eta_B(x)\\ \eta_{B,n}(x) &:= \big(\phi_0(\tfrac{x-x_B}{2^nR_B})-\phi_0(\tfrac{x-x_B}{2^{n-1}R_B})\big)\eta_B(x) \quad \text{for } n\ge 1,\end{aligned} \tag{3.5}$$

so that $\eta_B = \sum_{n_2=0}^\infty \eta_{B,n_2}$. The key estimate towards establishing Proposition 3.2 is the following lemma (in which the constant $N\ge d+1$ is as in the notation section).

Lemma 3.3. *Let* $j>0$, $R\le 2^j$, $Q\in\mathfrak{D}_j$, $S_Q\ge 3R$ *and* $\mathcal{B}_Q$ *be a finite family of* S_Q*-separated cubes of sidelength* R. *Then*

$$\Big\| m_{j,n_1}(D)\big[\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_B\big]\Big\|_2 \lesssim 2^{-n_1N}2^{-n_2N_2}(2^{-j}R)^{1/2}\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{1/2}\Big(\sum_B\|f_B\|_2^2\Big)^{1/2} \tag{3.6}$$

for all $N_2>0$ *and all families of functions* $\{f_B\}$ *indexed in* $\mathcal{B}_Q$.

Proof. We first give the proof under the stronger separation condition

$$B,B'\in\mathcal{B}_Q,\ B\ne B' \implies \operatorname{dist}(2^{n_2+2}B,2^{n_2+2}B') > S_Q\,. \tag{3.7}$$

We write

$$\Big\| m_{j,n_1}(D)\big[\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_B\big]\Big\|_2^2 = \sum_{B,B'\in\mathcal{B}_Q}\langle T_{B,B'}f_B,f_{B'}\rangle$$

where $T_{B,B'}$ is defined by

$$T_{B,B'}f(x) = \overline{\eta_{B',n_2}(x)}|m_{j,n_1}|^2(D)[\eta_{B,n_2}f](x)$$

and has Schwartz kernel

$$K_{B,B'}(x,y) = \overline{\eta_{B',n_2}(x)}\mathcal{F}^{-1}[|m_{j,n_1}|^2](x-y)\eta_{B,n_2}(y).$$

Note that if $x,y\in\operatorname{supp}(\eta_{B,n_2})$ then $\phi_0(\frac{x-y}{2^{n_2+2}R})=1$ and thus for the case $B'=B$

$$K_{B,B}(x,y) = \overline{\eta_{B,n_2}(x)}\mathcal{F}^{-1}[a_{j,B,n_1,n_2}](x-y)\eta_{B,n_2}(y)$$

where

$$a_{j,B,n_1,n_2}(\xi) = \int |m_{j,n_1}(\xi-v)|^2(2^{n_2+2}R)^d\widehat{\phi_0}(2^{n_2+2}Rv)\,\mathrm{d}v.$$

Since the operator $f \mapsto \eta_{B,n} f$ is bounded on L^2 with operator norm $O(2^{-n(N_2+d)})$ for all $N_2 \in \mathbb{N}$, then $\|T_{B,B}\|_{L^2 \to L^2} \lesssim_{N_2} 2^{-2n_2(N_2+d)} \|a_{j,B,n_1,n_2}\|_\infty$. By (2.13)

$$|a_{j,B,n_1,n_2}(\xi)| \lesssim_{N_2,N_3} 2^{-2n_1 N} \sup_\xi \int \frac{(2^{n_2+2}R)^d |\widehat{\phi}_0(2^{n_2+2}Rv)|}{(1+2^j|1-\rho(\xi-v)|)^{N_3}} \mathrm{d}v$$

for any $N_3 \geq 0$. A computation shows that the integral is $\lesssim \min\{1, 2^{-j+n_2}R\}$. Hence we obtain

$$\|T_{B,B}\|_{L^2 \to L^2} \lesssim_{N,N_2} 2^{-2n_1 N - 2n_2(N_2+d)} \min\{1, 2^{-j+n_2}R\}. \tag{3.8}$$

We now consider the case $B \neq B'$; recall from (3.7) that $\mathrm{dist}(B, B') \geq 2^{n_2+10} S_Q$. We use Lemma 2.5 and (3.7) together with the pointwise bound for η_{B,n_2} to get

$$|K_{B,B'}(x,y)| \lesssim 2^{-2n_1 N} 2^{-2n_2(N_2+d)} 2^{-j} |S_Q|^{-\frac{d-1}{2}} \mathbb{1}_{2^{n_2+2}B}(x) \mathbb{1}_{2^{n_2+2}B'}(y)$$

for all $N_2 > 0$. By Schur's test, for $B \neq B'$

$$\|T_{B,B'}\|_2 \lesssim 2^{-2n_1 N} 2^{-2n_2 N_2 - n_2 d} R^d 2^{-j} |S_Q|^{-\frac{d-1}{2}}. \tag{3.9}$$

Now we estimate the left-hand side of (3.6) by

$$\sum_{B \in \mathcal{B}_Q} |\langle T_{B,B} f_B, f_B \rangle| + \sum_{\substack{B,B' \in \mathcal{B}_Q \\ B \neq B}} |\langle T_{B,B'} f_B, f_{B'} \rangle| = I + II.$$

The diagonal terms are estimated using Cauchy-Schwarz and (3.8), and we get

$$I \lesssim_{N,N_2} 2^{-2n_1 N - 2n_2 N_2} \min\{1, 2^{-j+n_2}R\} \sum_{B \in \mathcal{B}_Q} \|f_B\|_2^2.$$

Moreover, using instead (3.9) we obtain for the off-diagonal terms that

$$\begin{aligned} II &\lesssim 2^{-n_2(2N_2+d)} 2^{-n_1 N} R^d 2^{-j} |S_Q|^{-\frac{d-1}{2}} \sum_{B \in \mathcal{B}_Q} \|f_B\|_2 \sum_{B' \in \mathcal{B}_Q} \|f_{B'}\|_2 \\ &\lesssim 2^{-n_2(2N_2+d)} 2^{-n_1 N} R^d 2^{-j} |S_Q|^{-\frac{d-1}{2}} \#\mathcal{B}_Q \sum_{B \in \mathcal{B}_Q} \|f_B\|_2^2. \end{aligned}$$

Thus, under the additional assumption (3.7), we have estimated the left-hand side of (3.6) by $2^{-n_2 d}$ times the right-hand side of (3.6). For the general case note that each S_Q-separated family of cubes of sidelength $R < 3S_Q$ can be split into $O(2^{n_2 d})$ sub-families, each of them $S_Q 2^{n_2+10}$-separated. Applying Minkowski's inequality we lose a factor of $O(2^{n_2 d})$ which leads to (3.6). □

Now we are in position of proving Proposition 3.2.

Proof of Proposition 3.2. We first consider (3.3) and estimate the left-hand side by

$$\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \Big\| m_{j,n_1}(D) \big[\sum_{Q \in \mathfrak{D}_j} \sum_{B \in \mathcal{B}_Q} \eta_{B,n_2} f_{Q,B} \big] \Big\|_2.$$

Now fix j, n_1, n_2, R and set

$$U \equiv U_{j,n_1,n_2,R} = \max\{2^{j+n_1+10}, 2^{n_2+10}R\}. \tag{3.10}$$

Let $\mathfrak{Q}_j \subset \mathfrak{D}_j$ be a family of U-separated 2^j-cubes. We use the localization properties of $\mathcal{F}^{-1}[m_{j,n_1}]$ and η_{B,n_2} followed by Lemma 3.3 to obtain

$$\Big\|m_{j,n_1}(D)\big[\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_2 \lesssim \Big(\sum_{Q\in\mathfrak{Q}_j}\Big\|m_{j,n_1}(D)\big[\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_2^2\Big)^{1/2}$$
$$\lesssim_{N_2}\Big(\sum_{Q\in\mathfrak{Q}_j}2^{-2n_1N}2^{-2n_2N_2}2^{-j}R\,\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)\sum_{B\in\mathcal{B}_Q}\|f_{Q,B}\|_2^2\Big)^{1/2}$$
$$\lesssim 2^{-n_1N}2^{-n_2N_2}(2^{-j}R)^{1/2}\sup_{Q\in\mathfrak{Q}_j}\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{1/2}\Big(\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q}\|f_{Q,B}\|_2^2\Big)^{1/2}.$$

We can write $\mathfrak{D}_j$ as a union of $O((2^{-j}U)^d)$ U-separated families of 2^j-cubes. By an application of the Minkowski and Cauchy-Schwarz inequalities we lose a factor of $O((2^{-j}U)^{d/2})=O(2^{(n_1+n_2)d/2})$ and obtain

$$\Big\|m_{j,n_1}(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_2 \lesssim 2^{-n_1(N-\frac{d}{2})-n_2(N_2-\frac{d}{2})}(2^{-j}R)^{\frac12}\quad\times$$
$$\sup_{Q\in\mathfrak{D}_j}\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{\frac12}\Big(\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\|f_{Q,B}\|_2^2\Big)^{\frac12}. \tag{3.11}$$

Since $N_1>d/2$, $N_2>d/2$, we may sum in n_1, n_2 and obtain (3.3).

We now turn to (3.4) and estimate the left-hand side by

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Big\|m_{j,n_1}(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_1.$$

Fix j,n_1,n_2,R and let U be as in (3.10). We now tile $\mathbb{R}^d$ by dyadic cubes of sidelength $\approx U$. Then

$$\Big\|m_{j,n_1}(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_1 \lesssim \sum_{\square}\sum_{\substack{Q\in\mathfrak{D}_j\\ Q\cap\square\neq\emptyset}}U^{d/2}\Big\|m_{j,n_1}(D)\big[\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_2$$
$$\lesssim 2^{-n_1(N-\frac{d}{2})-n_2(N_2-\frac{d}{2})}2^{j\frac{d-1}{2}}R^{\frac12}\sup_{Q\in\mathfrak{D}_j}\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{\frac12}\sum_{Q\in\mathfrak{D}_j}\Big(\sum_{B\in\mathcal{B}_Q}\|f_{Q,B}\|_2^2\Big)^{\frac12}.$$

Here we have used the Cauchy-Schwarz inequality and the localization properties in the first inequality, and an application of the estimate (3.3) in the last inequality. Summing in n_1,n_2 yields (3.4). □

4. Discretization

Given $\mathfrak{z}\in\mathbb{Z}^d$, let $q_{\mathfrak{z}}$ denote the unique dyadic cube in $\mathfrak{Q}_0$ containing $\mathfrak{z}$. Let $\{F_{Q,\mathfrak{z}}\}$ be a collection of C^∞ functions parametrized by $(Q,\mathfrak{z})\in\mathfrak{D}\times\mathbb{Z}^d$ satisfying

$$\operatorname{supp}F_{Q,\mathfrak{z}}\subseteq 2q_{\mathfrak{z}}\cap Q\qquad\text{and}\qquad\sup_{Q\in\mathfrak{D}}\sup_{\mathfrak{z}\in\mathbb{Z}^d}\|F_{Q,\mathfrak{z}}\|_\infty\le 1. \tag{4.1}$$

We start with a reformulation of Theorem 1.2.

Theorem 4.1. *Let* $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ *and* $r_\circ \in [p_\circ, \frac{d-1}{d+1}p_\circ']$. *Assume that* $\mathcal{R}^\lambda$ *maps* $L^{p_\circ}(\mathbb{R}^d)$ *to* $L^{r_\circ}(\mathbb{R}^d)$ *for all* $\lambda > \lambda(r_\circ)$. *Let* $\frac{2(d+1)}{d+3} < p < p_\circ$, $p \le r < r_*(p, p_\circ, r_\circ)$. *Then the inequality*

$$(4.2)\quad \Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \sum_{\mathfrak{z}\in Q\cap\mathbb{Z}^d} \gamma(Q,\mathfrak{z}) F_{Q,\mathfrak{z}} \big] \Big\|_r \lesssim_{p,r} \Big(\sum_{Q\in\mathfrak{D}} |Q| \Big(\sum_{\mathfrak{z}} |\gamma(Q,\mathfrak{z})|^p \Big)^{r/p} \Big)^{1/r}$$

holds for all functions $\gamma : \mathfrak{D} \times \mathbb{Z}^d \to \mathbb{C}$ *and all* $F_{Q,\mathfrak{z}}$ *satisfying* (4.1).

Proof of Theorem 1.2 (assuming Theorem 4.1). Let u be a Schwartz function such that $\widehat{u}$ is compactly supported and $\widehat{u}(\xi) = 1$ if $\rho(\xi) < 2$; then clearly $m_j = m_j \widehat{u}$. Let $\Phi \in C_c^\infty(\mathbb{R}^d)$ be supported in $\{x : |x| < 1/2\}$ and such that $\widehat{\Phi}(\xi) \ge 1/2$ on the support of $\widehat{u}$. Observe that $\widehat{u}/\widehat{\Phi}$ is a Schwartz function and therefore convolution with its inverse Fourier transform is bounded on L^r. Thus it suffices to prove

$$(4.3)\quad \Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \Phi * (f_Q \mathbb{1}_Q) \big] \Big\|_r \lesssim \Big(\sum_{Q\in\mathfrak{D}} |Q| \|f_Q\|_p^r \Big)^{1/r}.$$

Now, given $Q \in \mathfrak{D}$, $\mathfrak{z} \in \mathbb{Z}^d$, let

$$(4.4)\quad F_{Q,\mathfrak{z}}(x) = \frac{\Phi * [f_Q \mathbb{1}_{q_\mathfrak{z}\cap Q}](x)}{\|\Phi\|_{p'} \|f_Q \mathbb{1}_{q_\mathfrak{z}\cap Q}\|_p} \quad \text{if } \|f_Q \mathbb{1}_{q_\mathfrak{z}\cap Q}\|_p \ne 0$$

and $F_{Q,\mathfrak{z}} = 0$ otherwise. Notice that $F_{Q,\mathfrak{z}}$ is supported in the double of $q_\mathfrak{z}$ and that $\|F_{Q,\mathfrak{z}}\|_\infty \le 1$ by Hölder's inequality. Let

$$\gamma(Q,\mathfrak{z}) = \|f_Q \mathbb{1}_{q_\mathfrak{z}\cap Q}\|_p.$$

Then the left-hand side of (4.3) becomes

$$\|\Phi\|_{p'} \Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \sum_{\mathfrak{z}\in\mathbb{Z}^d} \gamma(Q,\mathfrak{z}) F_{Q,\mathfrak{z}} \big] \Big\|_r$$

which by assumption is $\lesssim \big(\sum_{Q\in\mathfrak{D}} |Q| \big(\sum_{\mathfrak{z}} |\gamma(Q,\mathfrak{z})|^p \big)^{r/p} \big)^{1/r}$. By definition of γ this gives the right-hand side of (4.3) and thus the estimate VBR(p,r) claimed in Theorem 1.2. □

We next reduce the main estimate for the proof of Theorem 4.1 to the situation where for each Q the function $\mathfrak{z} \to \gamma(Q,\mathfrak{z})$ is replaced by the characteristic function of a finite set $\mathcal{E}_Q \subset \mathbb{Z}^d \cap Q$.

Proposition 4.2. *Let* $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ *and* $r_\circ \in [p_\circ, \frac{d-1}{d+1}p_\circ']$. *Assume that* $\mathcal{R}^\lambda$ *maps* $L^{p_\circ}(\mathbb{R}^d)$ *to* $L^{r_\circ}(\mathbb{R}^d)$ *for all* $\lambda > \lambda(r_\circ)$. *Fix* $\mathcal{M} \ge 2$ *and* $\delta > 0$. *Let* $\frac{2(d+1)}{d+3} < p < p_\circ$ *and* $p \le r < r_*(p, p_\circ, r_\circ)$. *Then the inequality*

$$(4.5)\quad \Big\| \sum_{j>0} \sum_{\substack{Q\in\mathfrak{D}_j \\ \#\mathcal{E}_Q \le \mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} 2^{j\frac{d+1}{2}} \beta_Q m_j(D) F_{Q,\mathfrak{z}} \Big\|_{L^r(\mathbb{R}^d)} \lesssim_\delta \mathcal{M}^\delta \Big(\sum_{Q\in\mathfrak{D}} |Q| |\beta_Q|^r (\#\mathcal{E}_Q)^{\frac{r}{p_1}} \Big)^{\frac{1}{r}}$$

holds for all $p \le p_1 < p_\circ$*, all subsets* $\mathcal{E}_Q \subseteq \mathbb{Z}^d$*, all real-valued coefficients* β_Q *and all families of functions* $F_{Q,\mathfrak{z}}$ *satisfying* (4.1)*.*

Proof of Theorem 4.1 (assuming Proposition 4.2). Fix $p, p_\circ, r, r_\circ$ as in the assumptions. If $p_1 > p$ observe that $r_*(p_1, p_\circ, r_\circ) < r_*(p, p_\circ, r_\circ)$ and $r_*(p_1, p_\circ, r_\circ) \to r_*(p, p_\circ, r_\circ)$ as $p_1 \to p$. Thus we can choose p_1 with $p < p_1 < p_\circ$ such that $r < r_*(p_1, p_\circ, r_\circ)$.

Let $F_{Q,\mathfrak{z}}$ be functions satisfying (4.1) and consider a function $\gamma : \mathfrak{D} \times \mathbb{Z}^d \to \mathbb{C}$. Define $\gamma_Q(\mathfrak{z}) := \gamma(Q, \mathfrak{z})$. Without loss of generality we may assume that $\|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}$ is finite (otherwise there is nothing to prove). Clearly $|\gamma(Q,\mathfrak{z})| \le \|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}$ and therefore we can decompose $\gamma_Q \mathbb{1}_Q = \sum_{k\ge 0} \gamma_Q \mathbb{1}_{\mathcal{E}_Q^k}$ where

$$\mathcal{E}_Q^k := \{\mathfrak{z} \in \mathbb{Z}^d \cap Q : 2^{-(k+1)/p} \|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)} < |\gamma(Q,\mathfrak{z})| \le 2^{-k/p} \|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}\}.$$

For each Q we apply Chebyshev's inequality to get $\#\mathcal{E}_Q^k \le 2^{k+1}$. Let

$$\beta_Q^k := 2^{-k/p} \|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}, \qquad F_{Q,\mathfrak{z}}^k := \frac{\gamma(Q,\mathfrak{z})}{\beta_Q^k} F_{Q,\mathfrak{z}} \mathbb{1}_{\mathcal{E}_Q^k}(\mathfrak{z}).$$

Then for each k the family of functions $F_{Q,\mathfrak{z}}^k$ continues to satisfy (4.1). Hence, by (4.5) with exponents (p_1, r), with $\mathcal{M} = 2^{k+1}$ and $\delta < \frac{1}{2}(\frac{1}{p} - \frac{1}{p_1})$ we obtain

$$\begin{aligned}
&\Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \sum_{\mathfrak{z}\in Q\cap\mathbb{Z}^d} \gamma(Q,\mathfrak{z}) F_{Q,\mathfrak{z}} \big] \Big\|_r \\
&\lesssim \sum_{k\ge 0} \Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \beta_Q^k \sum_{\mathfrak{z}\in\mathcal{E}_Q^k} F_{Q,\mathfrak{z}}^k \big] \Big\|_r \\
&\lesssim C(\delta, p_1) \sum_{k\ge 0} 2^{(k+1)\delta} \Big(\sum_{Q\in\mathfrak{D}} |Q| |\beta_Q^k|^r (\#\mathcal{E}_Q^k)^{\frac{r}{p_1}} \Big)^{\frac{1}{r}} \\
&\lesssim C(\delta, p_1) \sum_{k\ge 0} 2^{(k+1)\delta} \Big(\sum_{Q\in\mathfrak{D}} |Q| \big[2^{-\frac{k}{p}} \|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)} \big]^r 2^{(k+1)\frac{r}{p_1}} \Big)^{\frac{1}{r}}
\end{aligned}$$

and thus we get

$$\begin{aligned}
&\Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \sum_{\mathfrak{z}\in Q\cap\mathbb{Z}^d} \gamma(Q,\mathfrak{z}) F_{Q,\mathfrak{z}} \big] \Big\|_r \\
&\lesssim C(\delta, p_1) \sum_{k\ge 0} 2^{k\delta} 2^{-k(\frac{1}{p} - \frac{1}{p_1})} \Big(\sum_{Q\in\mathfrak{D}} |Q| \|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}^r \Big)^{\frac{1}{r}} \lesssim \tfrac{C(\delta,p_1)}{p_1 - p} \Big(\sum_{Q\in\mathfrak{D}} |Q| \|\gamma_Q\|_p^r \Big)^{\frac{1}{r}}.
\end{aligned}$$

Hence (4.2) is established. □

5. A decomposition lemma

The following lemma is a discretized version of the stopping time argument by Tao [16, Lemma 4.3].

Lemma 5.1. *Let* $\mathcal{E}$ *be a finite subset of* $\mathbb{Z}^d$ *and* $V \in \mathbb{N}$*. Let* $\{L_k\}_{k=0}^V$ *be a sequence of integers such that* $L_0 = 0$ *and* $L_k > L_{k-1}$ *for* $k = 1, \dots, V$*. Then for each*

$0 \le k \le V-1$ there exist indexing sets A_k satisfying

$$\#A_k \le 2^d (\#\mathcal{E})^{1/V} \tag{5.1}$$

and families $\{\mathcal{B}_{k,\alpha}\}_{\alpha \in A_k}$ with the following properties:

(i) $\mathcal{B}_{k,\alpha}$ is a collection of dyadic cubes in $\mathfrak{D}_{L_k}$.

(ii) Any two different cubes in $\mathcal{B}_{k,\alpha}$ have mutual distance at least $2^{L_{k+1}}$.

(iii) For each k, $\alpha \in A_k$, $B \in \mathcal{B}_{k,\alpha}$ there exists non-empty subsets $\mathcal{E}_{k,B} \subset \mathcal{E} \cap B$ such that

$$\mathcal{E} = \bigcup_{k=0}^{V-1} \bigcup_{\alpha \in A_k} \bigcup_{B \in \mathcal{B}_{k,\alpha}} \mathcal{E}_{k,B}\,.$$

Proof. For each $\nu \in \{0,1\}^d$ and each nonnegative integer L we denote by $\mathfrak{D}_{L,\nu}$ the collection of dyadic cubes $\prod_{i=1}^d [n_i 2^L, (n_i+1)2^L)$ where $n_i = \nu_i \mod 2$ for $i = 1, \dots d$. Notice that for fixed ν two different cubes in $\mathfrak{D}_{L,\nu}$ have mutual distance at least 2^L. For each $\mathfrak{z} \in \mathbb{Z}^d$ and each nonnegative integer L, let $B(\mathfrak{z}, L)$ be the unique cube in $\mathfrak{D}_L$ containing $\mathfrak{z}$.

For each $\mathfrak{z} \in \mathcal{E}$, let $\kappa(\mathfrak{z}) \in [1, N] \cap \mathbb{N}$ denote the least positive integer such that

$$\#(\mathcal{E} \cap B(\mathfrak{z}, L_{\kappa(\mathfrak{z})})) \le (\#\mathcal{E})^{\kappa(\mathfrak{z})/V}.$$

For $k = 0, \dots, V-1$, let $\mathcal{E}_k := \{\mathfrak{z} \in \mathcal{E} : \kappa(\mathfrak{z}) = k+1\}$. Clearly $\mathcal{E} = \bigcup_{k=0}^{V-1} \mathcal{E}_k$.

Let $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ be the collection of cubes in $\mathfrak{D}_{L_k}$ that contain a point in $\mathcal{E}_k$. Each cube in $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ is contained in a unique cube in $\mathfrak{D}_{L_{k+1}}$. For each $\nu \in \{0,1\}^d$ denote by $\mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k)$ be the family of dyadic cubes B' in $\mathfrak{D}_{L_{k+1},\nu}$ which contain a point in $\mathcal{E}_k$; hence each such B' also contains a cube $B \in \mathfrak{D}_{L_k}(\mathcal{E}_k)$.

For $B' \in \mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k)$, enumerate the cubes in $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ that are contained in B' by

$$B_\ell(B'), \quad \text{with} \quad \ell = 1, \dots, n(B').$$

By the definition of the stopping time $\kappa(\mathfrak{z}) = k+1$ for $\mathfrak{z} \in \mathcal{E}_k$ we have

$$\#(\mathcal{E} \cap B_\ell(B')) \ge (\#\mathcal{E})^{k/V}, \quad \#(\mathcal{E} \cap B') \le (\#\mathcal{E})^{(k+1)/V},$$

and since the cubes $B_\ell(B')$ are disjoint this implies $n(B') \le (\#\mathcal{E})^{1/V}$ for all cubes $B' \in \mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k)$.

Now for fixed k, $\nu \in \{0,1\}^d$ and $1 \le \ell \le (\#\mathcal{E})^{1/V}$ let

$$\mathcal{B}_{k,(\nu,\ell)} = \{B_\ell(B') : \, B' \in \mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k),\, n(B') \ge \ell\}$$

and

$$A_k = \{\alpha \equiv (\nu, \ell) : \nu \in \{0,1\}^d,\, 1 \le \ell \le (\#\mathcal{E})^{1/V},\, \mathcal{B}_{k,(\nu,\ell)} \ne \emptyset\}.$$

Then clearly $\#A_k \le 2^d (\#\mathcal{E})^{1/V}$. If $\mathcal{B}_{k,(\nu,\ell)}$ is not empty then it consists of cubes in $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ which are $2^{L_{k+1}}$-separated, and we get properties (i) and (ii). Moreover for every cube B in $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ there is a unique ℓ, ν and $B' \in \mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k)$ such that $B = B_\ell(B')$. This implies

$$\mathcal{E}_k = \bigcup_{(\nu,\ell) \in A_k} \bigcup_{B \in \mathcal{B}_{k,(\nu,\ell)}} (\mathcal{E}_k \cap B)\,.$$

Recall that $\mathcal{E} = \bigcup_{k=0}^{V-1} \mathcal{E}_k$ is a disjoint union, and setting $\mathcal{E}_{k,B} = \mathcal{E}_k \cap B$ property (iii) follows. □

6. A Fourier restriction bound

For the proof of Proposition 4.5 we shall use a Fourier restriction bound when j in the sum in (4.5) is very large. We show that such a Fourier restriction bound is implied by the non-endpoint Bochner-Riesz assumption in Theorem 1.2.

Proposition 6.1. *Let $1 < p \le r \le 2$, $\alpha > 0$. Suppose that $\mathcal{R}^{\lambda(r)+\alpha}$ maps L^p to L^r. Then for all $R \ge 2$, the inequality*

$$\|\widehat{f}|_{\partial\Omega}\|_{L^r(\partial\Omega)} \lesssim R^{2\alpha}\|f\|_{L^p(\mathbb{R}^d)} \tag{6.1}$$

holds for all f supported in a cube of sidelength R.

Remark 6.2. Under the assumption that $\|\widehat{f}|_{\partial\Omega}\|_{L^{r_\circ}(\partial\Omega)} \lesssim R^{2\alpha}\|f\|_{L^{p_\circ}(\mathbb{R}^d)}$ holds for some $p_\circ > \frac{2(d+1)}{d+3}$, $r_\circ \le \frac{d-1}{d+1}p_\circ'$ and all $\alpha > 0$ one can also show a full Fourier restriction result, i.e. for $p < p_\circ$, $r < r_*(p, p_\circ, r_\circ)$ the operator $f \mapsto \widehat{f}|_{\partial\Omega}$ maps $L^p(\mathbb{R}^d)$ to $L^r(\partial\Omega)$. This is accomplished by an adaptation of Tao's ε-removal argument in [17] (also based on the previous stopping time argument). This upgrade is not needed here.

Proof of Proposition 6.1. The proof is an adaptation of Tao's argument in [17, Thm. 1.1]. For the sake of completeness we provide the details. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ supported in $\{x : 1/4 < |x| < 4\}$ and define for large A

$$T_A f(x) = \int e^{iA\rho^*(x-y)}\eta(x-y)f(y)\,\mathrm{d}y. \tag{6.2}$$

We shall first show that under the $L^p \to L^r$ boundedness assumption on $\mathcal{R}^{\lambda(r)+a}$ we have for large A

$$\|T_A\|_{L^p\to L^r} \lesssim A^{\alpha-d/p'}. \tag{6.3}$$

Let $\varepsilon > 0$ so that for every $\xi_0 \in \partial\Omega$ the portion of the boundary in $B_{8\varepsilon}(\xi_0)$ can be parametrized by a regular parametrization $y \mapsto \Xi(y)$ (with y in an open set in $\mathbb{R}^{d-1}$). Note that by the assumption of convexity with nonvanishing curvature there is for every $x \ne 0$ a unique $\xi(x) \in \partial\Omega$ such that $x/|x|$ is the outer unit normal to $\partial\Omega$ at $\xi(x)$; moreover we may choose a $\delta > 0$ such that $|\xi(x) - \xi(\widetilde{x})| < \varepsilon$ for all $x, \widetilde{x}$ with $1/4 \le |x|, |\tilde{x}| \le 4$ and $|x - \tilde{x}| \le 4\delta$. We may now construct a finite family of C_c^∞ functions η_ν such that $\sum_\nu \eta_\nu = \eta$, where η_ν is supported in a ball $B_\delta(x_\nu)$. Let $T_{\nu,a}$ be defined as in (6.2) but with η_ν in place of η; it then suffices to prove $\|T_{\nu,A}\|_{L^p\to L^r} = O(A^{\alpha-d/p'})$.

Let $w_\nu \in C_c^\infty$ be supported in $B_{2\varepsilon}(\xi_\nu)$ such that $w_\nu(\xi) = 1$ for $\xi \in B_\varepsilon(\xi_\nu)$. Let $\mathfrak{S}_\nu = \{x \ne 0 : \xi(\frac{x}{|x|}) \in B_\varepsilon(\xi_\nu)\}$ and let $y \mapsto \Xi(y)$ be a regular parametrization of $\partial\Omega \cap B_{8\varepsilon}(\xi_\nu)$. By the assumption on $\mathcal{R}^{\lambda(r)+\alpha}$, the operator $w_\nu(D)\mathcal{R}^{\lambda(r)+\alpha}$ is $L^p \to L^r$ bounded. Let K_ν be its convolution kernel. For $x \in \mathfrak{S}_\nu$ we can express $K_\nu(x)$ using ρ-polar coordinates (see the proof of Lemma 2.5) by

$$K_\nu(x) = (2\pi)^{-d}\int_0^1 (1-\varrho)^\lambda \varrho^{d-1} \int w_\nu(\varrho\Xi(y))e^{i\varrho\langle x,\Xi(y)\rangle}\langle \Xi(y), \mathfrak{n}(\Xi(y))\rangle\,\mathrm{d}y\,\mathrm{d}\varrho.$$

We use the assumption that $\partial\Omega$ has positive Gaussian curvature. By a standard asymptotic expansion based on stationary phase calculations in the y-variable and of asymptotic expansions involving Fourier transforms of $\chi(\varrho)(1-\varrho)_+^\lambda$ ([6, §2.8], [12], [15, §VIII]), with $\lambda = \lambda(r) + \alpha$, we see that there is a constant $A_\nu \gg 1$ such that

$$K_\nu(x) = b(x)|x|^{-\lambda(r)-a-1-\frac{d-1}{2}} e^{i\rho^*(x)} \quad \text{for } x \in \mathfrak{S}_\nu^\infty := \mathfrak{S}_\nu \cap \{x : |x| \geq A_\nu\} \tag{6.4}$$

where b is a standard symbol of order 0, with

$$0 < c_1 \leq |b(x)| \leq C_1 \quad \text{if } x \in \mathfrak{S}_\nu^\infty \ .$$

See e.g. [12]. Let $Y(x)$ be the unique critical point for which $\nabla_y \langle x, \Xi(y)\rangle = 0$. Note that x is perpendicular to the tangent space $T_{\Xi(Y(x))}$. Then the phase function ρ^* is given by

$$\rho^*(x) = \langle x, \Xi(Y(x))\rangle = \sup_{\xi:\rho(\xi)\leq 1} \langle x, \xi\rangle. \tag{6.5}$$

It turns out that ρ^* is smooth, homogeneous of degree 1 and that the level sets of ρ^* are strictly convex hypersurfaces with nonvanishing curvature (see [11], [9, §5.1] for these calculations and more background on convex bodies). By Euler's homogeneity relation we have $\rho^*(x) = \langle x, \nabla\rho^*(x)\rangle$ and thus $\langle x, \Xi(Y(x))\rangle = \langle x, \nabla\rho^*(x)\rangle$. From the strict convexity property we get

$$\nabla\rho^*(x) = \Xi(Y(x)). \tag{6.6}$$

We now choose $A \geq 8A_\nu$ and define

$$u_{\nu,A}(x) := \frac{\eta_\nu(x)|x|^{d+1+\lambda(r)+a}}{b(Ax)};$$

we verify that in view of the symbol and nonvanishing properties of b the functions $u_{\nu,A}$ form a bounded family of C_c^∞-functions. Note that the functions $\eta_\nu(A^{-1}\cdot)$ are supported in $\mathfrak{S}_\nu^\infty$ and that from (6.4) we get for all $x \in \mathbb{R}^d$

$$u_{\nu,A}(A^{-1}x)K_\nu(x) = A^{-\frac{d+1}{2}-\lambda(r)-\alpha}\eta_\nu(A^{-1}x)e^{i\rho^*(x)}.$$

Clearly $\|\widehat{u_{\nu,A}(A^{-1}\cdot)}\|_1 = O(1)$ and therefore the convolution operator with convolution kernel $u_{\nu,A}(A^{-1}x)K_\nu(x)$ is $L^p \to L^r$ bounded with operator norm uniform in A. Denote the operator with convolution kernel $A^{-\frac{d+1}{2}-\lambda(r)-\alpha}A^d\eta_\nu(x)e^{i\rho^*(Ax)}$ by $O_{\nu,A}$; then by scaling we see that $O_{\nu,A}$ has $L^p \to L^r$ operator norm $O(A^{d/p-d/r})$. Since $\rho^*(x)$ is homogeneous of degree 1 we get

$$\|T_{\nu,A}\|_{L^p\to L^r} = A^{\frac{d+1}{2}+\lambda(r)+\alpha}A^{-d}\|O_{\nu,A}\|_{L^p\to L^r} \lesssim A^{\alpha-\frac{d}{p'}}$$

and (6.3) follows by summing in ν, provided that $A \geq \max_\nu 8A_\nu$.

We now turn to the Fourier restriction operator. By (6.6) $\partial\Omega$ is described by $\theta \mapsto \nabla\rho^*(\theta)$ for $x \in S^{d-1}$, and we have

$$\Big(\int_{\partial\Omega} |\widehat{f}(\xi)|^r \,\mathrm{d}\sigma(\xi)\Big)^{\frac{1}{r}} \lesssim \Big(\int_{S^{d-1}} |\widehat{f}(\nabla\rho^*(\theta))|^r \,\mathrm{d}\theta\Big)^{\frac{1}{r}} \lesssim \Big(\int_{1\leq|x|\leq 2} |\widehat{f}(\nabla\rho^*(x))|^r \,\mathrm{d}x\Big)^{\frac{1}{r}}$$

as $\nabla\rho^*$ is homogeneous of degree 0. Our goal is therefore to show, for $R \gg 2\sqrt{d}$, the estimate

$$\Big(\int_{1\le|x|\le2} |\widehat{f}(\nabla\rho^*(x))|^r\,\mathrm{d}x\Big)^{1/r} \lesssim R^{2\alpha}\|f\|_p \quad \text{if } \operatorname{supp}(f)\subset Q_R; \tag{6.7}$$

here Q_R is the cube of sidelength R centered at the origin.

We may choose η in the definition of T_A above so that $\eta(w)=1$ if $1/2\le|w|\le3$, in particular $\eta(x-y)=1$ for $|y|\le1/2$ and $1\le|x|\le2$. Then (6.3) yields

$$\Big(\int_{1\le|x|\le2} |T_Ag(x)|^r\Big)^{1/r} \lesssim A^{\alpha-d/p'}\|g\|_p \quad \text{if } \operatorname{supp}(g)\subseteq\{y:|y|\le\tfrac12\}. \tag{6.8}$$

Changing variables in the oscillatory integral, we get

$$\begin{aligned}&\Big(\int_{1\le|x|\le2} |\widehat{f}(\nabla\rho^*(x))|^r\,\mathrm{d}x\Big)^{1/r}\\ &\lesssim \Big(\int_{1\le|x|\le2}\Big|\int_{|y|_\infty\le R^{-1}} e^{-i\langle R^2y,\nabla\rho^*(x)\rangle}R^{2d}f(R^2y)\,\mathrm{d}y\Big|^r\,\mathrm{d}x\Big)^{1/r}\end{aligned} \tag{6.9}$$

where $|y|_\infty=\max_{1\le i\le d}|y_i|$. We rewrite the phase function using Taylor's formula:

$$\langle y,\nabla\rho^*(x)\rangle = \rho^*(x-y)-\rho^*(x)-\langle y,\mathcal{H}(x,y)y\rangle$$

with $\mathcal{H}(x,y)=\int_0^1(1-s)\nabla^2\rho^*(x-sy)\,\mathrm{d}s$ where $\nabla^2\rho^*$ denotes the matrix of second derivatives of ρ^*. We then see that the right-hand side of (6.9) is estimated by

$$\Big(\int_{1\le|x|\le2}\Big|\int_{|y|_\infty\le R^{-1}} e^{iR^2\rho^*(x-y)}e^{i\langle Ry,\mathcal{H}(x,y)Ry\rangle}R^{2d}f(R^2y)\,\mathrm{d}y\Big|^r\,\mathrm{d}x\Big)^{1/r}.$$

For $|x|\le2$, $|w|_\infty\le1$ expand

$$e^{i\langle w,\mathcal{H}(x,\frac{w}{R})w\rangle} = \sum_{(n_1,n_2)\in\mathbb{Z}^d\times\mathbb{Z}^d} c_{n_1,n_2}(R)e^{i\langle n_1,x\rangle}e^{i\langle n_2,w\rangle}$$

with $|c_{n_1,n_2}(R)|\le C(1+|n_1|+|n_2|)^{-10d}$ and C independent of R; this is applied for $w=Ry$. After taking out the sum in (n_1,n_2) by Minkowski's inequality, we can apply (6.8) with $A=R^2$, since $f(R^2\cdot)$ is supported in $\{|y|_\infty\le R^{-1}\}$ and $R>2\sqrt{d}$. We obtain

$$\Big(\int_{1\le|x|\le2}|\widehat{f}\circ\nabla\rho^*(x)|^r dx\Big)^{1/r} \lesssim \sum_{(n_1,n_2)\in\mathbb{Z}^d\times\mathbb{Z}^d}|c_{n_1,n_2}|(R^2)^{\alpha-d/p'}\|R^{2d}f(R^2\cdot)e^{i\langle n_2,R\cdot\rangle}\|_p$$

which is $\lesssim R^{2\alpha}\|f\|_p$, yielding (6.7). □

7. Proof of Proposition 4.2

Let p, r be as in the statement of Proposition 4.2, i.e. $\frac{2(d+1)}{d+3}<p<p_\circ$, $p\le r< r_*(p,,p_\circ,r_\circ)$. Let $p\le p_1<p_\circ$ and $r_1:=\max\{r,p_1\}$.

Let $\delta>0$, $\mathcal{M}\ge2$. For the proof we will make the choice

$$V\equiv V(\delta)=\lceil 2/\delta\rceil,\qquad H\equiv H(\delta)=V2^{V+1}. \tag{7.1}$$

We estimate the L^r norm of

$$\sum_{j>0}\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q} 2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}.$$

We will separately consider the terms with $2^j\le(2\mathcal{M})^H$ and $2^j>(2\mathcal{M})^H$. The estimate for the terms with $2^j\le(2\mathcal{M})^H$ will rely on the Bochner-Riesz hypothesis in Theorem 1.2, which combined with Corollary 2.3 yields

$$\big\|2^{j\frac{d+1}{2}}m_{j,n}(D)\big\|_{L^{p_1}\to L^{r_1}}\lesssim_\varepsilon 2^{-nN}2^{j(\frac{d}{r_1}+\varepsilon)} \tag{7.2}$$

for all $n\ge 0$ and $\varepsilon>0$ and, moreover, that $\mathcal{R}^\lambda$ maps L^{p_1} to L^{r_1} for all $\lambda>\lambda(r_1)$. The estimate for the terms with $2^j>(2\mathcal{M})^H$ will rely on the $L^{p_1}\to L^{r_1}$ Fourier restriction estimate implied by the just mentioned Bochner–Riesz bound via Proposition 6.1.

We first estimate the terms with $2^j\le(2\mathcal{M})^H$ and prove

$$\Big\|\sum_{\substack{j>0:\\ 2^j\le(2\mathcal{M})^H}}\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)} \lesssim_\delta \mathcal{M}^\delta\Big(\sum_{Q\in\mathfrak{D}}|Q||\beta_Q|^r(\#\mathcal{E}_Q)^{r/p_1}\Big)^{1/r}. \tag{7.3}$$

To prove (7.3) it suffices to establish that for fixed $j>0$ with $2^j\le(2\mathcal{M})^H$

$$\Big\|\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)}\lesssim_\delta \mathcal{M}^{\delta/2}\Big(\sum_{Q\in\mathfrak{D}}|Q||\beta_Q|^r(\#\mathcal{E}_Q)^{r/p_1}\Big)^{1/r}; \tag{7.4}$$

then (7.3) follows from the triangle inequality summing over all $j>0$ with $2^j\le(2\mathcal{M})^H$. Write $m_j=\sum_{n=0}^\infty m_{j,n}$ where $m_{j,n}$ are as in (2.10). By the support properties of $\mathcal{F}^{-1}[m_{j,n}]$ and the triangle inequality, we have for fixed $j>0$ (with $2^j\le(2\mathcal{M})^H$)

$$\begin{aligned}&\Big\|2^{j(d+1)/2}m_j(D)\big[\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}\beta_Q F_{Q,\mathfrak{z}}\big]\Big\|_r\\ &\lesssim\sum_{n=0}^\infty\Big(\sum_{Q'\in\mathfrak{D}_{j+n}}\Big\|2^{j(d+1)/2}m_{j,n}(D)\big[\sum_{\substack{Q\in\mathfrak{Q}_j:Q\subset Q'\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}\beta_Q F_{Q,\mathfrak{z}}\big]\Big\|_r^r\Big)^{1/r}.\end{aligned}$$

By the support properties, Hölder's inequality, (7.2) with $\varepsilon = \frac{\delta}{2H}$ and (4.1), we have we then have that for each $n \geq 0$ and $Q' \in \mathfrak{D}_{j+n}$

$$\begin{aligned}
&\Big\| 2^{j(d+1)/2} m_{j,n}(D) \big[\sum_{\substack{Q\in\mathfrak{D}_j:Q\subset Q'\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} \beta_Q F_{Q,\mathfrak{z}} \big] \Big\|_r \\
&\lesssim 2^{(j+n)d(\frac{1}{r}-\frac{1}{r_1})} \Big\| 2^{j(d+1)/2} m_{j,n}(D) \big[\sum_{\substack{Q\in\mathfrak{D}_j:Q\subset Q'\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} \beta_Q F_{Q,\mathfrak{z}} \big] \Big\|_{r_1} \\
&\lesssim_\delta 2^{(j+n)d(\frac{1}{r}-\frac{1}{r_1})} 2^{-nN} 2^{j(\frac{d}{r_1}+\frac{\delta}{2H})} \Big\| \sum_{\substack{Q\in\mathfrak{D}_j:Q\subset Q'\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} \beta_Q F_{Q,\mathfrak{z}} \Big\|_{p_1} \\
&\lesssim 2^{-n(N-d(\frac{1}{r}-\frac{1}{r_1}))} 2^{j(\frac{d}{r}+\frac{\delta}{2H})} \Big(\sum_{\substack{Q\in\mathfrak{D}_j:Q\subset Q'\\ \#\mathcal{E}_Q\le\mathcal{M}}} |\beta_Q|^{p_1} (\#\mathcal{E}_Q) \Big)^{1/p_1}.
\end{aligned}$$

Furthermore, observe that

$$\Big(\sum_{\substack{Q\in\mathfrak{D}_j:Q\subset Q'\\ \#\mathcal{E}_Q\le\mathcal{M}}} |\beta_Q|^{p_1} (\#\mathcal{E}_Q) \Big)^{1/p_1} \lesssim \max\{1, 2^{nd(\frac{1}{p_1}-\frac{1}{r})}\} \Big(\sum_{\substack{Q\in\mathfrak{D}_j:Q\subset Q'\\ \#\mathcal{E}_Q\le\mathcal{M}}} |\beta_Q|^{r} (\#\mathcal{E}_Q)^{r/p_1} \Big)^{1/r};$$

this follows for $r \geq p_1$ by Hölder's inequality and for $r < p_1$ by the embedding $\ell^r \subset \ell^{p_1}$. We can combine the above observations to obtain

$$\begin{aligned}
&\Big\| 2^{j(d+1)/2} m_j(D) \big[\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} \beta_Q F_{Q,\mathfrak{z}} \big] \Big\|_r \\
&\lesssim_\delta \sum_{n=0}^{\infty} 2^{-n(N+\frac{d}{r_1})} \max\{2^{n\frac{d}{p_1}}, 2^{n\frac{d}{r}}\} \mathcal{M}^{\delta/2} \Big(\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} |Q||\beta_Q|^r (\#\mathcal{E}_Q)^{r/p_1} \Big)^{1/r}
\end{aligned}$$

where we have used that $2^j \leq (2\mathcal{M})^H$. Since $N > \max\{d/p_1, d/r\} - d/r_1$, we immediately obtain (7.4). Thus (7.3) is established.

We now address the terms with $2^j > (2\mathcal{M})^H$ and prove

$$\begin{aligned}
\text{(7.5)} \quad &\Big\| \sum_{\substack{j>0:\\ 2^j>(2M)^H}} \sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} 2^{j\frac{d+1}{2}} \beta_Q m_j(D) F_{Q,\mathfrak{z}} \Big\|_{L^r(\mathbb{R}^d)} \\
&\qquad\lesssim_\delta \mathcal{M}^\delta \Big(\sum_{Q\in\mathfrak{D}} |Q||\beta_Q|^r (\#\mathcal{E}_Q)^{r/p_1} \Big)^{1/r}.
\end{aligned}$$

To show (7.5) we use Lemma 5.1 for each non-empty set $\mathcal{E}_Q$ with $Q \in \mathfrak{D}_j$ satisfying

$$\#\mathcal{E}_Q \leq \mathcal{M} < 2^{j/H-1};$$

specifically we apply it with $V = \lceil 2/\delta \rceil$, and the integer sequence $L_0 < \cdots < L_V$ defined by

$$L_0 = 0, \quad 2L_k + \tfrac{2}{d-1}\log_2(\#\mathcal{E}_Q) < L_{k+1} \leq 2L_k + \tfrac{2}{d-1}\log_2(\#\mathcal{E}_Q) + 1\,.$$

We then write

$$\mathcal{E}_Q = \bigcup_{k=0}^{V-1} \bigcup_{\alpha\in A_Q^k} \bigcup_{B\in\mathcal{B}_Q^{k,\alpha}} \mathcal{E}_{Q,B}^k \tag{7.6}$$

where A_Q^k is an indexing set of cardinality

$$\#A_Q^k \le 2^d(\#\mathcal{E}_Q)^{1/V} \le 2^d\mathcal{M}^{\delta/2} \tag{7.7}$$

and each $\mathcal{B}_Q^{k,\alpha}$ is a family of cubes of sidelength 2^{L_k}, with each pair of them having distance at least $2^{L_{k+1}}$. It will be crucial to bound 2^{L_k} by a suitable power of $\mathcal{M}$; note that for $k \ge 1$

$$L_k \le \sum_{\ell=0}^{k} 2^\ell + \frac{\log_2(\#\mathcal{E}_Q)}{d-1}\sum_{\ell=1}^{k} 2^\ell \tag{7.8}$$

as one may check by induction from the definition. Hence, for $k = 1, \dots, V-1$

$$2^{L_k} \le 2^{2^{k+1}-1}\big(\#\mathcal{E}_Q\big)^{\frac{2^{k+1}-2}{d-1}} \le (2\mathcal{M})^{2^{k+1}} \le (2\mathcal{M})^{2^V} = (2\mathcal{M})^{2^{\lceil\frac{2}{\delta}\rceil}} \tag{7.9}$$

and thus, we have

$$2^{L_k} \le (2\mathcal{M})^{2^V} \le 2^{j2^V/H} \le 2^{\frac{j}{2V}} \le 2^{j\delta/4} \quad \text{provided that } 2^j \ge (2\mathcal{M})^H. \tag{7.10}$$

Also, for each α, k and $Q \in \mathfrak{D}_j$, the cubes in $\mathcal{B}_Q^{k,\alpha}$ are $2^{L_{k+1}}$-separated, in particular, $2^{2L_k}(\#\mathcal{E}_Q)^{\frac{2}{d-1}}$-separated.

We will show that

$$\begin{aligned}\Big\| \sum_{\substack{j\ge 0\\ 2^j>(2\mathcal{M})^H}} \sum_{\substack{Q\in\mathfrak{Q}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{B\in\mathcal{B}_Q^{k,\alpha(Q)}} \sum_{\mathfrak{z}\in\mathcal{E}_{k,Q,B}} 2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)} \\ \lesssim \mathcal{M}^{\delta/2}\Big(\sum_{Q\in\mathfrak{Q}} |Q||\beta_Q|^r\Big(\sum_{B\in\mathcal{B}_Q^{k,\alpha(Q)}} \#\mathcal{E}_{Q,B}^k\Big)^{r/p_1}\Big)^{1/r}\end{aligned} \tag{7.11}$$

uniformly in $0 \le k < V$, in all subcollections $\mathfrak{Q} \subset \mathfrak{D}$, and all mappings $Q \mapsto \alpha(Q)$ where $\alpha(Q)$ is an index in A_Q^k. We may then obtain (7.5) by the triangle inequality. Indeed, enumerate $A_Q^k = \{\alpha_1, \dots, \alpha_{\mathfrak{n}(k,Q)}\}$ where $\mathfrak{n}(k,Q) \le 2^d\mathcal{M}^{\delta/2}$. Then from (7.6) and (7.11)

$$\begin{aligned}
&\Big\|\sum_{\substack{j>0:\\ 2^j>(2\mathcal{M})^H}}\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q} 2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)}\\
&\le \sum_{k=0}^{V-1}\sum_{1\le i\le 2^d\mathcal{M}^{\delta/2}}\Big\|\sum_{\substack{j>0:\\ 2^j>(2\mathcal{M})^H}}\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}\\ \mathfrak{n}(k,Q)\ge i}}\sum_{B\in\mathcal{B}_Q^{k,\alpha_i}}\sum_{\mathfrak{z}\in\mathcal{E}^k_{Q,B}} 2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)}\\
&\lesssim \mathcal{M}^{\delta/2}\sum_{k=0}^{V-1}\sum_{1\le i\le 2^d\mathcal{M}^{\delta/2}}\Big(\sum_{\substack{Q\in\mathfrak{D}\\ \mathfrak{n}(k,Q)\ge i}}|Q||\beta_Q|^r\Big(\sum_{B\in\mathcal{B}_Q^{k,\alpha_i}}\#\mathcal{E}^k_{Q,B}\Big)^{r/p_1}\Big)^{1/r}\\
&\lesssim \mathcal{M}^{\delta/2}(V2^d\mathcal{M}^{\delta/2})\Big(\sum_{Q\in\mathfrak{D}}|Q||\beta_Q|^r(\#\mathcal{E}_Q)^{r/p_1}\Big)^{1/r}
\end{aligned}$$

and since $V2^d\mathcal{M}^{\delta/2}\lesssim_\delta \mathcal{M}^{\delta/2}$ we get (7.5).

It remains to show (7.11). For this we need an auxiliary lemma. For a dyadic cube B with sidelength R_B recall the definition of η_B in (3.1) and note that $\widehat{\eta}_B$ is supported in $\{\xi : |\xi|\le 2/R_B\}$.

Lemma 7.1. *Let $L\in\mathbb{N}$ such that $2^L>8C_0$. For every $j\ge 2L$ let $\mathfrak{Q}_j$ be a collection of dyadic cubes of sidelength 2^j. For every $Q\in\mathfrak{Q}_j$ let $\mathcal{B}_Q$ be a family of dyadic subcubes Q, of sidelength 2^L. Let $S_Q\ge 2^{2L}(\#\mathcal{B}_Q)^{\frac{2}{d-1}}$ and assume that $\mathcal{B}_Q$ is S_Q-separated for all $Q\in\mathfrak{Q}_j$, $j\ge 2L$. Then for all $1\le q_2\le q_1\le 2$, the inequality*

$$\Big\|\sum_{j\ge 2L}\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q}2^{j\frac{d+1}{2}}\mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_{q_2} \lesssim 2^{L/q_1}\Big(\sum_j\sum_{Q\in\mathfrak{Q}_j}|Q|\Big(\sum_{B\in\mathfrak{D}_L}\|G_{Q,B}\|_{q_1}^{q_1}\Big)^{q_2/q_1}\Big)^{1/q_2} \tag{7.12}$$

holds for all functions $G_{Q,B}$ indexed by $\mathfrak{Q}\times\mathfrak{D}_L$.

We first show how the proof of (7.11) is concluded, assuming the lemma. Define, for $B\in\mathcal{B}_Q^{k,\alpha(Q)}$

$$f_{Q,B}(x)=\frac{\beta_Q}{\eta_B(x)}\sum_{\mathfrak{z}\in\mathcal{E}^k_{Q,B}}F_{Q,\mathfrak{z}}(x). \tag{7.13}$$

Define the ρ-annulus

$$\mathcal{A}_{L_k}=\{\xi: 1-4C_0 2^{-L_k}\le\rho(\xi)\le 1+4C_0 2^{-L_k}\}.$$

We claim that

$$m_j(\widehat{\eta_B} * \widehat{f_{Q,B}}) = m_j\big(\widehat{\eta_B} * (\mathbb{1}_{\mathcal{A}_{L_k}}\widehat{f_{Q,B}})\big) \tag{7.14}$$

whenever $j\ge L_k$; this condition is certainly guaranteed in our situation by (7.10).

To see this, first note that by the mean value theorem and by (2.2) we have $|\rho(\xi+h)-\rho(\xi)|\le C_0|h|$. Hence if $\rho(\xi)\le 1-4C_0 2^{-L_k}$ and $|h|\le 2^{1-L_k}$ then

$\rho(\xi+h) \le 1-2C_0 2^{-L_k}$. Likewise if $\rho(\xi) \ge 1+4C_0 2^{-L_k}$ and $|h| \le 2^{1-L_k}$ then $\rho(\xi+h) \ge 1+2C_0 2^{-L_k}$. The support of $\widehat{\eta}_B$ is in $\{h : |h| \le 2^{1-L_k}\}$ and the support of m_j is contained in $\{\xi : 1-2^{-j+1} < \rho(\xi) < 1-2^{-j-1}\}$. Since $C_0 \ge 2$ we see that for $j \ge L_k$ the supports of m_j and $\widehat{\eta}_B * (\mathbb{1}_{\mathcal{A}^{\complement}_{L_k}} \widehat{f_{Q,B}})$ are disjoint, from which (7.14) follows.

Hence

$$\sum_{\mathfrak{z}\in\mathcal{E}^k_{Q,B}} \beta_Q m_j(D) F_{Q,\mathfrak{z}} = \mathcal{F}^{-1}\big[m_j \widehat{\eta_B f_{Q,B}}\big] = \mathcal{F}^{-1}\big[m_j\big(\widehat{\eta_B} * (\mathbb{1}_{\mathcal{A}_{L_k}} \widehat{f_{Q,B}})\big)\big]$$

$$= \mathcal{F}^{-1}\big[m_j\big(\widehat{\eta_B} * G_{Q,B}\big)\big] \quad \text{where } G_{Q,B} = \mathbb{1}_{\mathcal{A}_{L_k}} \widehat{f_{Q,B}}$$

provided that $Q \in \mathfrak{D}_j$, $j \ge L_k$, $B \in \mathcal{B}^{k,\alpha(Q)}_Q$; otherwise $G_{Q,B} = 0$. Apply Lemma 7.1 with $(q_1, q_2) = (r_1, r)$. This yields

$$\Big\| \sum_{\substack{j\ge 0\\ 2^j > (2\mathcal{M})^H}} \sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q \le \mathcal{M}}} \sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q} \sum_{\mathfrak{z}\in\mathcal{E}^k_{Q,B}} 2^{j\frac{d+1}{2}} \beta_Q m_j(D) F_{Q,\mathfrak{z}} \Big\|_{L^r(\mathbb{R}^d)}$$

$$\lesssim 2^{L_k/r_1} \Big(\sum_{j \ge L_k} \sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q \le \mathcal{M}}} |Q| \Big(\sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q} \int_{\mathcal{A}_{L_k}} |\widehat{f_{Q,B}}(\xi)|^{r_1} \, \mathrm{d}\xi \Big)^{\frac{r}{r_1}} \Big)^{1/r}. \tag{7.15}$$

Using ρ-polar coordinates as in (2.1) we get

$$\int_{\mathcal{A}_{L_k}} |\widehat{f}_{Q,B}(\xi)|^{r_1} \, \mathrm{d}\xi \;\le\; \int_{1-C_0 2^{2-L_k}}^{1+C_0 2^{2-L_k}} \int_{\partial\Omega} |\widehat{f_{Q,B}}(\varrho\xi')|^{r_1} \, \mathrm{d}\mu(\xi') \varrho^{d-1} \, \mathrm{d}\varrho.$$

By Proposition 6.1 with parameters (p_1, r_1), and since $f_{Q,B}$ is supported in B we have for every $\varepsilon_1 > 0$,

$$\lesssim_{\varepsilon_1} 2^{L_k \varepsilon_1} \int_{1-C_0 2^{2-L_k}}^{1+C_0 2^{2-L_k}} \big\| \varrho^{-d} f_{Q,B}(\varrho^{-1}\cdot) \big\|_{p_1}^{r_1} \varrho^{d-1} \, \mathrm{d}\varrho$$

$$\lesssim_{\varepsilon_1} 2^{L_k \varepsilon_1} \|f_{Q,B}\|_{p_1}^{r_1} \int_{1-C_0 2^{2-L_k}}^{1+C_0 2^{2-L_k}} \varrho^{(\frac{d}{r_1}-d) r_1 + d - 1} \, \mathrm{d}\varrho \lesssim 2^{L_k \varepsilon_1} 2^{-L_k} \|f_{Q,B}\|_{p_1}^{r_1}. \tag{7.16}$$

Since $\eta_B(x) \ge 1$ on B we can bound $\|f_{Q,B}\|_{p_1} \lesssim \beta_Q (\#\mathcal{E}^k_{Q,B})^{1/p_1}$ using the properties (4.1). We apply this with

$$\varepsilon_1 < \delta 2^{-\lceil \frac{2}{\delta} \rceil - 1}$$

which implies $2^{L_k \varepsilon_1 / r_1} \le 2^{L_k \varepsilon_1} \le \mathcal{M}^{\delta/2}$ by (7.9). Use this in (7.16) and plug it into (7.15) to obtain that the left-hand side of (7.15) is dominated by

$$\mathcal{M}^{\delta/2} \Big(\sum_{j \ge L_k} \sum_{Q\in\mathfrak{D}_j} |Q| |\beta_Q|^r \Big(\sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q} (\#\mathcal{E}^k_{Q,B})^{r_1/p_1} \Big)^{r/r_1} \Big)^{1/r}.$$

Since $r_1 \ge p_1$ we can use the embedding $\ell^1 \hookrightarrow \ell^{r_1/p_1}$ so that

$$\sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q} (\#\mathcal{E}^k_{Q,B})^{r_1/p_1} \le \Big(\sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q} \#\mathcal{E}^k_{Q,B} \Big)^{r_1/p_1} \le (\#\mathcal{E}_Q)^{r_1/p_1}$$

and hence we get (7.11). □

Finally, we give the proof of Lemma 7.1.

Proof of Lemma 7.1. The proof follows by interpolation between the cases

(i) $q_1 = q_2 = 2$;
(ii) $q_1 = 2$, $q_2 = 1$;
(iii) $q_1 = q_2 = 1$.

As before, we use $m_j(D)\big(\eta_B \mathcal{F}^{-1}[G_{Q,B}]\big) = \mathcal{F}^{-1}\big[m_j(\widehat{\eta_B} * G_{Q,B})\big]$.

We start with (i). From (3.3) in Proposition 3.2 with $f_{Q,B} = \mathcal{F}^{-1}[G_{Q,B}]$ we get using the condition on the S_Q and Plancherel's theorem

$$2^{j/2}\Big\| \sum_{Q\in\mathfrak{Q}_j} \sum_{B\in\mathcal{B}_Q} \mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_2 \lesssim 2^{L/2}\Big(\sum_{Q\in\mathfrak{Q}_j} \sum_{B\in\mathcal{B}_Q} \|G_{Q,B}\|_2^2\Big)^{1/2}. \tag{7.17}$$

Since the supports of m_j have bounded overlap we get

$$\begin{aligned}
&\Big\| \sum_{j\geq 2L} \sum_{Q\in\mathfrak{Q}_j} \sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}} \mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_2 \\
&\lesssim \Big(\sum_{j\geq 2L} \Big\| \sum_{Q\in\mathfrak{Q}_j} \sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}} \mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_2^2\Big)^{1/2} \\
&\lesssim 2^{L/2}\Big(\sum_{j} \sum_{Q\in\mathfrak{Q}_j} |Q| \sum_{B\in\mathfrak{D}_L} \|G_{Q,B}\|_2^2\Big)^{1/2}
\end{aligned}$$

which is the case $q_1 = q_2 = 2$ of Lemma 7.1.

The case (ii) follows in a similar way, using (3.4) in Proposition 3.2 and the triangle inequality. Indeed, one has

$$\begin{aligned}
&\Big\| \sum_{j\geq 2L} \sum_{Q\in\mathfrak{Q}_j} \sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}} \mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_1 \\
&\lesssim \sum_{j\geq 2L} \Big\| \sum_{Q\in\mathfrak{Q}_j} \sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}} \mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_2 \\
&\lesssim 2^{L/2} \sum_{j\geq 2L} \sum_{Q\in\mathfrak{Q}_j} \Big(\sum_{B\in\mathfrak{D}_L} \|G_{Q,B}\|_2^2\Big)^{1/2},
\end{aligned}$$

as desired.

Finally, we prove (iii), that is,

$$\Big\| \sum_{j\geq 2L} \sum_{Q\in\mathfrak{Q}_j} \sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}} \mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_1 \lesssim 2^L \sum_{j} \sum_{Q\in\mathfrak{Q}_j} |Q| \sum_{B\in\mathfrak{D}_L} \|G_{Q,B}\|_1. \tag{7.18}$$

We decompose in a familiar way (see e.g. [4])

$$m_j = \sum_{\nu\in\mathfrak{A}_j} m_j^\nu$$

where m_j^ν is supported in a $C2^{-j/2} \times \cdots \times C2^{-j/2} \times C2^{-j}$ box $R_{j,\nu}$ with long sides tangential to $\partial\Omega$ at some point, and the boxes have bounded overlap. We have

$\|\mathcal{F}^{-1}[m_j^\nu]\|_1 = O(1)$ and $\#\mathfrak{A}_j \lesssim 2^{j(d-1)/2}$. Moreover, denote by $D(\omega, 2^{1-L})$ the ball of radius 2^{1-L} centered at ω and define

$$\mathfrak{A}_j(\omega) := \{\nu \in \mathfrak{A}_j : R_{j,\nu} \cap D(\omega, 2^{1-L}) \neq \emptyset\}.$$

Then for $L \le j/2$ we have $\#\mathfrak{A}_j(\omega) \lesssim 2^{(-L+\frac{j}{2})(d-1)}$ and this bound is uniform in ω. We estimate

$$\|\mathcal{F}^{-1}[m_j(\widehat{\eta_B} * G_{Q,B})]\|_1 \le \int |G_{Q,B}(\omega)| \int \left|\mathcal{F}^{-1}\left[m_j(\cdot)\widehat{\eta_B}(\cdot - \omega)\right](x)\right| \mathrm{d}x \, \mathrm{d}\omega$$

and get for fixed ω

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left[m_j(\cdot)\widehat{\eta_B}(\cdot-\omega)\right]\right\|_1 &\lesssim 2^{Ld} \sum_{\nu \in \mathfrak{A}_j(\omega)} \|\mathcal{F}^{-1}[m_j^\nu]\|_1 \|\mathcal{F}^{-1}[\widehat{\eta}(\tfrac{\cdot-\omega}{2^{-L}})]\|_1 \\ &\lesssim 2^{Ld} \#\mathfrak{A}_j(\omega) \lesssim 2^L 2^{j(d-1)/2}.\end{aligned}$$

Consequently,

$$\left\|2^{j(d+1)/2}\mathcal{F}^{-1}[m_j(\widehat{\eta_B} * G_{Q,B})]\right\|_1 \lesssim 2^L 2^{jd} \|G_{Q,B}\|_1$$

and (7.18) follows. This concludes the proof. □

References

[1] David Beltran, Joris Roos, and Andreas Seeger, *Bochner-Riesz operators at the critical index: Weighted and sparse bounds*, posted on arXiv.org, 2023.

[2] Michael Christ, *Weak type endpoint bounds for Bochner-Riesz multipliers*, Rev. Mat. Iberoamericana **3** (1987), no. 1, 25–31.

[3] ———, *Weak type* $(1,1)$ *bounds for rough operators*, Ann. of Math. (2) **128** (1988), no. 1, 19–42.

[4] Antonio Córdoba, *A note on Bochner-Riesz operators*, Duke Math. J. **46** (1979), no. 3, 505–511.

[5] J. Cossar, *A theorem on Cesàro summability*, J. London Math. Soc. **16** (1941), 56–68.

[6] A. Erdélyi, *Asymptotic expansions*, Dover Publications, Inc., New York, 1956.

[7] Charles Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.

[8] ———, *A note on spherical summation multipliers*, Israel J. Math. **15** (1973), 44–52.

[9] Ekkehard Krätzel, *Analytische Funktionen in der Zahlentheorie*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 139, B. G. Teubner, Stuttgart, 2000.

[10] Andrei K. Lerner and Fedor Nazarov, *Intuitive dyadic calculus: the basics*, Expo. Math. **37** (2019), no. 3, 225–265.

[11] Wolfgang Müller, *On the average order of the lattice rest of a convex body*, Acta Arith. **80** (1997), no. 1, 89–100.

[12] Burton Randol, *The asymptotic behavior of a Fourier transform and the localization property for eigenfunction expansions for some partial differential operators*, Trans. Amer. Math. Soc. **168** (1972), 265–271.

[13] Andreas Seeger, *Endpoint estimates for multiplier transformations on compact manifolds*, Indiana Univ. Math. J. **40** (1991), no. 2, 471–533.

[14] ———, *Endpoint inequalities for Bochner-Riesz multipliers in the plane*, Pacific J. Math. **174** (1996), no. 2, 543–553.

[15] Elias M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, 1993.

[16] Terence Tao, *The weak-type endpoint Bochner-Riesz conjecture and related topics*, Indiana Univ. Math. J. **47** (1998), no. 3, 1097–1124.

[17] ______, *The Bochner-Riesz conjecture implies the restriction conjecture*, Duke Math. J. **96** (1999), no. 2, 363–375.

[18] Walter Trebels, *Multipliers for (C, α)-bounded Fourier expansions in Banach spaces and approximation theory*, Lecture Notes in Mathematics, vol. 329, Springer-Verlag, Berlin-New York, 1973.

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