RESTRICTION OF FOURIER TRANSFORMS TO CURVES II: SOME CLASSES WITH VANISHING TORSION

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Abstract. We consider the Fourier restriction operators associated to certain degenerate curves in $\mathbb{R}^d$ for which the highest torsion vanishes. We prove estimates with respect to affine arclength and with respect to the Euclidean arclength measure on the curve. The estimates have certain uniform features, and the affine arclength results cover families of flat curves.

1. Introduction

We suppose that γ is a curve in $\mathbb{R}^d$ and consider the problem of obtaining $L^p \to L^q$ bounds for the restriction of the Fourier transform to γ. This problem has a long and interesting history which is described at length in [7] and [2]. Though we will not repeat much of that description here, we recall one of the main results from [2], concerning the moment curve $\gamma_0(t) = (t, t^2, \dots, t^d)$ in dimension $d \geq 3$. Write $p_d = \frac{d^2+d+2}{d^2+d}$. Then there is the restricted strong type inequality

$$\Big(\int_a^b |\widehat{f}(\gamma(t))|^{p_d} dt\Big)^{1/p_d} \leq C(\gamma)\, \|f\|_{L^{p_d,1}(\mathbb{R}^d)}, \tag{1.1}$$

for all Schwartz functions f on $\mathbb{R}^d$. The estimate (1.1) is, as described in [2], best possible and yields all other $L^p \to L^q$ restriction results for the moment curve γ_0 by interpolation with the trivial $L^1 \to L^\infty$ estimate. It is natural to wonder what happens to (1.1) when γ_0 is replaced by more general curves. If $\gamma : [a, b] \to \mathbb{R}^d$ is nondegenerate in the sense that for each $t \in [a, b]$ the derivatives $\gamma'(t), \gamma''(t), \dots, \gamma^{(d)}(t)$ are linearly independent, then the analogue of (1.1) is proved in [2]. But if one attempts to go further by dropping the hypothesis of nondegeneracy, it is easy to see that the exact analogues of (1.1) and its interpolants may fail. There are then two possibilities which have been considered in the literature. The first is to "dampen" the measure dt by introducing a weight $w(t)$ which is small where γ is degenerate, to replace dt with $w(t)\, dt$, and then to attempt to obtain

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the exact analogue of (1.1). The second approach is to retain dt for the reference measure and to see what changes must then be made in order to obtain sharp restriction results. In this paper we explore both approaches, but only for γ of the form

$$\gamma(t) = \Big(t, \frac{t^2}{2}, \dots, \frac{t^{d-1}}{(d-1)!}, \phi(t)\Big). \tag{1.2}$$

These curves are termed *simple* in [8] and are distinguished by the fact that only the highest torsion may vanish.

Concerning the first approach, it was observed in [8] that if γ is as in (1.2), then the correct weight $w(t)$ is given by

$$w(t) = |\phi^{(d)}(t)|^{\frac{2}{d(d+1)}}. \tag{1.3}$$

Then the measure $w(t)\,dt$ is, up to a constant depending only on the dimension, the affine arclength measure on γ. Here we have the following result.

Theorem 1.1. *Fix $d \ge 2$. Suppose $0 \le a < b \le \infty$ and let γ be of the form (1.2) where ϕ is a C^d function on (a,b) for which the derivatives $\phi', \dots, \phi^{(d)}$ are nonnegative and nondecreasing on (a,b) and for which $\phi^{(d)}$ satisfies the condition*

$$\Big(\prod_{j=1}^{d} \phi^{(d)}(s_j)\Big)^{1/d} \le A\, \phi^{(d)}\big(\tfrac{s_1+\cdots+s_d}{d}\big) \tag{1.4}$$

for all $s = (s_1, \dots, s_d)$ with $a < s_1 \le s_2 \le \cdots \le s_d < b$.

Suppose $1 \le P < \frac{d^2+d+2}{d^2+d}$, and $1 - \frac{1}{P} = \frac{2}{d(d+1)} \frac{1}{Q}$. Then there is $C(d,P) < \infty$ so that for all $g \in L^P(\mathbb{R}^d)$

$$\Big(\int_a^b |\widehat{g}(\gamma(t))|^Q w(t)\,dt\Big)^{1/Q} \le C(d,P)\, A^{1-1/P}\, \|g\|_{L^P(\mathbb{R}^d)}. \tag{1.5}$$

The proof of Theorem 1.1 is analogous to the proof of Theorem 1.3 in [2]. The range of indices in Theorem 1.1 is the range given by interpolating the $L^{p_d,1} \to L^{p_d}$ estimate (1.1) with the trivial $L^1 \to L^\infty$ estimate, and it would be interesting to know if the endpoint result (the exact analogue of (1.1)) holds for the curves of Theorem 1.1.

In the case $d = 2$ it follows from [12] that the conclusion of Theorem 1.1 holds with $A = 1$ (and without any additional hypotheses like (1.4)). For many interesting examples a slightly stronger condition holds where the arithmetic mean in the argument of $\phi^{(d)}$ on the right hand side of (1.4) is replaced by a geometric mean, *i.e.*,

$$\Big(\prod_{j=1}^{d} \phi^{(d)}(s_j)\Big)^{1/d} \le A\, \phi^{(d)}\big(\sqrt[d]{s_1 \cdots s_d}\big). \tag{1.6}$$

It is obvious that condition (1.6) holds for $\phi(t) = t^\beta$, $\beta \geq d$ on the interval $(0, \infty)$; in particular (1.6) is satisfied with $A = 1$. Moreover, if for $t \geq 0$ we define $\phi_0(t) = t^\beta$ for some $\beta > d$, and for $n \geq 1$,

$$\phi_n(t) = \int_0^t (t-u)^{d-1} \exp\Big(- \frac{1}{\phi_{n-1}^{(d)}(u)} \Big) du,$$

then ϕ_n satisfy (1.6) with $A = 1$ on $(0, \infty)$ (see §4.3). This yields a sequence of functions which are progressively flatter at the origin for which the restriction theorem holds uniformly (i.e., with constant depending only on the Lebesgue space indices). These two observations raise the interesting question of whether or not the hypothesis (1.6) in Theorem 1.1 can be dropped to yield, subject to ϕ's being sufficiently monotone, a uniform restriction theorem for the curves (1.2).

Regarding the second of the above-mentioned possibilities, keeping the measure dt, Drury and Marshall [9] proved sharp results for classes of finite type curves. Here we are aiming for a result for curves of the form (1.2) which is expressed in terms of a natural geometric condition and also has a certain uniform feature.

We will say that a set E in $\mathbb{R}^d$ is a *parallelepiped* if E is a translate of a set of the form $\{\sum_{j=1}^d t_j x_j : 0 \leq t_j \leq 1\}$ where the $x_j \in \mathbb{R}^d$ are linearly independent. Given γ we shall write λ_γ for the measure on γ given by

$$\langle d\lambda_\gamma, f \rangle = \int f(\gamma(t))dt.$$

We denote Lebesgue measure in $\mathbb{R}^d$ by m_d.

Theorem 1.2. *Suppose $-\infty < a < b < \infty$ and let γ be of the form* (1.2) *where ϕ is a C^d function on (a,b) for which the derivatives $\phi', \ldots, \phi^{(d)}$ are nonnegative and nondecreasing on (a,b). Suppose that $\alpha \in (0, \frac{2}{d(d+1)}]$ if $d \geq 3$ and that $\alpha \in (0, 1/3)$ if $d = 2$. Suppose also that the estimate*

$$\lambda_\gamma(E) \leq B\, m_d(E)^\alpha \tag{1.7}$$

holds for some $B > 0$ and for all parallelepipeds $E \subset \mathbb{R}^d$.

Then there is $C(d, \alpha) < \infty$ so that for all $g \in L^{1+\alpha,1}(\mathbb{R}^d)$

$$\Big(\int_a^b |\widehat{g}(\gamma(t))|^{1+\alpha} dt \Big)^{1/(1+\alpha)} \leq C(d,\alpha)\, B^{\frac{1}{1+\alpha}} \, \|g\|_{L^{1+\alpha,1}(\mathbb{R}^d)}. \tag{1.8}$$

On the other hand, if the estimate

$$\Big(\int_a^b |\widehat{g}(\gamma(t))|^Q dt \Big)^{1/Q} \leq c^{1/Q} \, \|g\|_{L^P(\mathbb{R}^d)} \tag{1.9}$$

holds for some P and Q satisfying $1 - \frac{1}{P} = \frac{\alpha}{Q}$, then (1.7) *holds for all parallelepipeds E with B replaced by $C(d,p)\, c$.*

The proof of Theorem 1.2 is analogous to the proof of (1.1) given in [2]. Interpolation of (1.8) with the trivial L^1 estimate yields the estimate (1.9) whenever $1 \le P < 1+\alpha$ and $1/P' = \alpha/Q$. It would be interesting to know whether in the generality of Theorem 1.2 the exponent $1+\alpha$ is sharp when $\alpha < 2/(d^2+d)$ or whether there is always $P(\alpha) > 1+\alpha$ such that (1.7) implies (1.9) whenever $1 \le p < P(\alpha)$ and $1/P' = \alpha/Q$. For many concrete examples such improvements can indeed be obtained by rescaling arguments from the nondegenerate case – for this and related observations see §7.

This paper: In order to prove Theorem 1.1 we shall use the method of offspring curves that originated in [6], and was further developed in [8], [9] and [2]. The crucial technical point is to give lower bounds for a certain Jacobian of a change of variable, estimate (2.4) below. The new features about Theorem 1.1 concern the verification of this estimate, and the technical details are contained in §2. The proof of Theorem 1.1 is then discussed in §3 (a reader not familiar with the method should start reading here). In §4 we discuss some examples to which Theorem 1.1 can be applied. Sections §5 and §6 contain the proof of Theorem 1.2. In §7 we show how Theorem 1.2 can be extended for some classes of examples.

2. The main technical estimate

In this section we assume that ϕ is defined on $[a,b]$, $0 \le a < b$ and assume that the derivatives of ϕ up to order d are positive and nondecreasing on (a,b).

We establish some notation. For a vector $x \in \mathbb{R}^d$ let $V_d(x)$ be the determinant of the $d \times d$ Vandermonde matrix:

$$V_d(x) = \prod_{1 \le i < j \le d} (x_j - x_i). \tag{2.1}$$

For $h = (h_1, \dots, h_{d-1}) \in (\mathbb{R}_+)^{d-1}$ define $\kappa(h) \in [0,\infty)^d$ by

$$\kappa_1(h) = 0, \quad \kappa_j(h) = h_1 + \cdots + h_{j-1}, \quad 2 \le j \le d$$

and put

$$v(h) \equiv v_d(h) = V_d(\kappa(h)).$$

If $\gamma : (a,b) \to \mathbb{R}^d$ and if $a < t < b - \kappa_d(h)$, we write

$$\Gamma(t,h) = \sum_{j=1}^{d} \gamma(t + \kappa_j(h)). \tag{2.2}$$

Following the terminology of Drury and Marshall [8] we call $\Gamma(\cdot, h)$, for fixed h, an *offspring curve* of γ.

Denote by $J_\phi(t,h)$ the Jacobi-determinant of the transformation $(t,h) \mapsto \Gamma(t,h)$; that is

$$J_\phi(t,h) = \det \begin{pmatrix} \frac{\partial \Gamma}{\partial t} & \frac{\partial \Gamma}{\partial h_1} & \cdots & \frac{\partial \Gamma}{\partial h_{d-1}} \end{pmatrix}. \tag{2.3}$$

As in [8] it will be crucial to verify the identity

$$|J_\phi(t,h)| \geq \sigma v(h)\Big(\prod_{i=1}^{d}\phi^{(d)}(t+\kappa_i(h))\Big)^{1/d}. \tag{2.4}$$

Here we prove

Proposition 2.1. *Let $0 \leq a < b \leq 1$. Suppose that $\phi^{(d)}$ is nonnegative and nondecreasing on (a,b), and that for any $a < s_1 \leq \cdots \leq s_d \leq b$, the condition* (1.4) *is satisfied. Then condition* (2.4) *holds with*

$$\sigma = c_0(d)A^{-1}$$

for all (t,h) such that $a \leq t \leq b$, $h \in (0,b)^{d-1}$, and $t+\kappa_d(h) \leq b$.

The proof of Proposition 2.1 uses the following technical lemma.

Lemma 2.2. *Fix $\lambda \in (0,1)$. Suppose*

$$a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_N < b_N.$$

Suppose that, for $m = 1,\dots,M$, l_m is a function of $t = (t_1,\dots,t_N)$ having one of the three following forms:

$$l_m(t) = \begin{cases} t_k - t_j & \textit{for some } \ 1 \leq j < k \leq N, \quad \textit{or} \\ d_j - t_j & \textit{for some } \ d_j \geq b_j, \quad \textit{or} \\ t_j - c_j & \textit{for some } \ c_j \leq a_j. \end{cases}$$

Suppose that $\lambda_j \in (0,1)$ and $\lambda_j \leq \lambda$, for $j = 1,\dots,N$. Let $\mathcal{R}_N(a,b,\lambda)$ be the region of all $t = (t_1,\dots,t_N) \in \mathbb{R}^N$ satisfying $(1-\lambda_j)a_j + \lambda_j b_j \leq t_j \leq b_j$ for $j = 1,\dots,N$. Then

$$\int_{\mathcal{R}_N(a,b,\lambda)} \prod_{m=1}^{M} l_m(t)\, dt_N \cdots dt_1 \geq C(M,\lambda)^N \int_{a_1}^{b_1}\cdots\int_{a_N}^{b_N}\prod_{m=1}^{M} l_m(t)\, dt_N\cdots dt_1.$$

Proof of Lemma 2.2. An easy induction argument shows that it is enough to prove the lemma when $N = 1$. A translation and then a scaling reduce that case to the inequality

$$\int_1^{1/\lambda}\prod_{m=1}^{M} l_m(t)\, dt \geq C(M,\lambda)\int_0^{1/\lambda}\prod_{m=1}^{M} l_m(t)\, dt \tag{2.5}$$

where

$$l_m(t) = \begin{cases} d_m - t & \text{for some } \ d_m \geq 1/\lambda, \quad \text{or} \\ t - c_m & \text{for some } \ c_m \leq 0. \end{cases}$$

It is clear that (2.5) is true when $M = 0$. So assume that (2.5) is true for $M-1$. Suppose first that at least one of the functions l_m is increasing, say $l_M(t) = t - c$. Then, by the inductive assumption,

$$\int_1^{1/\lambda} \prod_{m=1}^{M-1} l_m(t)\,dt \geq \frac{C(M-1,\lambda)}{1-C(M-1,\lambda)} \int_0^1 \prod_{m=1}^{M-1} l_m(t)\,dt.$$

Thus

$$\begin{aligned}\int_1^{1/\lambda} \prod_{m=1}^{M-1} l_m(t)(t-c)\,dt &\geq (1-c)\int_1^{1/\lambda} \prod_{m=1}^{M-1} l_m(t)\,dt \\ &\geq \frac{C(M-1,\lambda)}{1-C(M-1,\lambda)} \int_0^1 \prod_{m=1}^{M-1} l_m(t)(t-c)\,dt,\end{aligned}$$

and this is equivalent to (2.5) with $C(M-1,\lambda)$ instead of $C(M,\lambda)$. Therefore we can assume that $l_m(t) = d_m - t$ for all m. There are now two cases to consider. First suppose that one of the d_m's, say d_M, exceeds $2/\lambda$. Let $\tau = (1+1/\lambda)/2$. Then

$$\begin{aligned}\int_0^1 \prod_{m=1}^{M} l_m(t)\,dt &\leq d_M \int_0^1 \prod_{m=1}^{M-1} l_m(t)\,dt \\ &\leq d_M \frac{1-C(M-1,\lambda)}{C(M-1,\lambda)} \int_1^{1/\lambda} \prod_{m=1}^{M-1} l_m(t)\,dt.\end{aligned} \tag{2.6}$$

We further estimate

$$\begin{aligned}\int_1^{1/\lambda} \prod_{m=1}^{M-1} l_m(t)\,dt &\leq 2\int_1^{\tau} \prod_{m=1}^{M-1} l_m(t)\,dt \\ &\leq \frac{2}{d_M-\tau}\int_1^{\tau} \prod_{m=1}^{M} l_m(t)\,dt \\ &\leq \frac{2}{d_M-\tau}\int_1^{1/\lambda} \prod_{m=1}^{M} l_m(t)\,dt\end{aligned} \tag{2.7}$$

where the first inequality follows because $\prod_1^{M-1} l_m(t)dt$ is decreasing. Since $d_M \leq 2/\lambda$, we have $d_M/(d_M-\tau) \leq 2$. Combined with (2.6) and (2.7), this implies (2.5) if one of the d_m's exceeds $2/\lambda$. If, on the other hand, $d_m \leq 2/\lambda$ for all m, then the crude estimates

$$\int_0^{1/\lambda} \prod_{m=1}^{M} l_m(t)\,dt \leq \frac{2^M}{\lambda^{M+1}}$$

and

$$\int_1^{1/\lambda} \prod_{m=1}^{M} l_m(t)\,dt \geq \int_1^{1/\lambda} \Big(\frac{1}{\lambda} - t\Big)^M\,dt = \frac{(1/\lambda-1)^{M+1}}{M+1},$$

give (2.5) again and conclude the proof of Lemma 2.2. □

It will be useful to write the Jacobian $J_\phi(\cdot, h)$ as a convolution with a nonnegative function, depending on the parameter $h \in (\mathbb{R}_+)^{d-1}$.

To this end we define for $h_1 \geq 0$

$$\Psi_2(t; h_1) = \chi_{[0,h_1]}(t). \tag{2.8}$$

For $d \geq 3$ and $t \leq h_1 + \cdots + h_{d-1}$ we set

$$\begin{aligned}\mathcal{R}_{d-1}(t,h) = \big\{\sigma \in &\mathbb{R}^{d-1} : 0 \leq \sigma_1 \leq \min\{h_1, t\},\\ &h_1 + ... + h_{j-1} \leq \sigma_j \leq h_1 + ... + h_j, \quad j = 2, ..., d-2,\\ &\max\{h_1 + ... + h_{d-2}, t\} \leq \sigma_{d-1} \leq h_1 + ... + h_{d-1}\big\},\end{aligned}$$

and define recursively

(2.9)

$$\Psi_d(t; h_1, \dots, h_{d-1}) = \int_{\mathcal{R}_{d-1}(t,h)} \Psi_{d-1}(t - \sigma_1; \sigma_2, \dots, \sigma_{d-1}) d\sigma_1 \dots d\sigma_{d-1}$$

if $t \leq h_1 + \cdots + h_{d-1}$; we also set $\Psi_d(t; h) = 0$ if $t \geq h_1 + \cdots + h_{d-1}$.

Lemma 2.3. *Let Ψ_d be as in (2.8), (2.9) and let, for $s \in \mathbb{R}^d$ with $s_1 \leq s_2 \leq ... \leq s_d$, $\mathcal{J}_d(s_1, \dots, s_d; \phi)$ denote the determinant of the $d \times d$ matrix with columns $(1, s_j, \dots, \frac{s_j^{d-2}}{(d-2)!}, \phi'(s_j))^T$.*

Then

$$\mathcal{J}_d(s; \phi) = \int_{s_1}^{s_d} \Psi_d(u - s_1; s_2 - s_1, \dots, s_d - s_{d-1})\, \phi^{(d)}(u) du. \tag{2.10}$$

Proof. If $d = 2$ then the asserted formula holds since

$$\mathcal{J}_2(s_1, s_2; \phi) = \phi'(s_2) - \phi'(s_1) = \int_{s_1}^{s_2} \phi''(u) du$$

and $\Psi_2(u - s_1; s_2 - s_1) = \chi_{[s_1,s_2]}(u)$.

We now argue by induction and assume $d \geq 3$.

We first note by expanding $\partial_1 \dots \partial_{d-1} \mathcal{J}_d$ with respect to the last column that

$$\partial_{s_1} \dots \partial_{s_{d-1}} \mathcal{J}_d(s_1, \dots, s_d; \phi) = (-1)^{d+1} \mathcal{J}_{d-1}(s_1, \dots, s_{d-1}; \phi').$$

Next observe that $\mathcal{J}_d(s; \phi) = 0$ if $s_1 = s_2$ and that $\partial_1 \dots \partial_k \mathcal{J}_d(s; \phi) = 0$ if $s_{k+1} = s_{k+2}$ and $k \leq d - 2$. Thus we repeatedly integrate and see that

(2.11)

$$\begin{aligned}&\mathcal{J}_d(s; \phi)\\ &= (-1)^{d-1} \int_{s_1}^{s_2} \dots \int_{s_{d-1}}^{s_d} \partial_{s_1} \dots \partial_{s_{d-1}} \mathcal{J}_d(\sigma_1, \dots, \sigma_{d-1}, s_d; \phi)\, d\sigma_{d-1} \dots d\sigma_1\\ &= \int_{s_1}^{s_2} \dots \int_{s_{d-1}}^{s_d} \mathcal{J}_{d-1}(\sigma_1, \dots, \sigma_{d-1}; \phi')\, d\sigma_{d-1} \dots d\sigma_1.\end{aligned}$$

Thus by the induction hypothesis

$$\mathcal{J}_d(s;\phi) = \int_{s_1}^{s_2} \dots \int_{s_{d-1}}^{s_d} \int_{\sigma_1}^{\sigma_{d-1}} \phi^{(d)}(u) \quad \times \\ \Psi_{d-1}(u-\sigma_1; \sigma_2-\sigma_1, \dots, \sigma_{d-1}-\sigma_{d-2})\, du\, d\sigma_{d-1} \dots d\sigma_1 \tag{2.12}$$

and by Fubini's theorem this can be written in the form

$$\mathcal{J}_d(s;\phi) = \int_{s_1}^{s_d} \phi^{(d)}(u) \int_{\tau\in\Omega(u)} \Psi_{d-1}(u-\tau_1; \tau_2-\tau_1, \dots, \tau_{d-1}-\tau_{d-2}) d\tau du$$

where $\Omega(u)$ consists of those $\tau \in \mathbb{R}^{d-1}$ for which $s_i \le \tau_i \le s_{i+1}$, $i = 1, \dots d-1$ and $\tau_1 \le u \le \tau_{d-1}$.

We change variables $\tau_i = s_1 + \sigma_i$ for $i = 1, ..., d-1$, so that $\tau \in \Omega(u)$ corresponds to $\sigma \in \mathcal{R}_{d-1}(u-s_1, h)$ with $h_i = s_{i+1} - s_i$. Thus from the definition (2.9) we obtain

$$\mathcal{J}_d(s;\phi) = \int_{s_1}^{s_d} \Psi_d(u-s_1; s_2-s_1, \dots, s_d-s_{d-1})\, \phi^{(d)}(u) du$$

which yields the assertion. □

Lemma 2.4. *Let* Ψ_d *as in* (2.8), (2.9) *and let*

$$g_d(t,h) = t + \frac{1}{d}\sum_{i=1}^{d} \kappa_i(h).$$

Then Ψ_d *satisfies*

$$\int_{g_d(t,h)}^{t+\kappa_d(h)} \Psi_d(u-t;h) du \ge c(d) v(h) \tag{2.13}$$

where $c(d) > 0$.

Proof. First, in order to prepare for the proof of (2.13), we observe that (2.11) for the special case $\phi(s) = s^d/d!$ gives us the formula for the Vandermonde determinant $V_d(s) = \prod_{j=1}^{d-1}(j!)\mathcal{J}_d(s,\phi)$ in all dimensions namely

$$V_n(s_1, \dots, s_d) = (n-1)! \int_{s_1}^{s_2} \dots \int_{s_{n-1}}^{s_n} V_{n-1}(\sigma_1, \dots, \sigma_{n-1}) d\sigma_{n-1} \dots d\sigma_1. \tag{2.14}$$

We now use Lemma 2.2 to establish the following inequality for all $n \ge 2$. Suppose that $0 \le a_1 \le \dots \le a_n$ and let

$$\mathcal{U}_{n-1}(a) = \{x' = (x_1, \dots, x_{n-1}) \in (\mathbb{R}_+)^{n-1} : \frac{1}{n-1}\sum_{i=1}^{n-1} x_i \ge \frac{1}{n}\sum_{k=1}^{n} a_k\}.$$

Then

$$\int_{a_1}^{a_2} \dots \int_{a_{n-1}}^{a_n} V_{n-1}(x') \cdot \chi_{\mathcal{U}_{n-1}(a)}(x')\, dx' \;\ge\; C(n)\, V_n(a_1, \dots, a_n). \tag{2.15}$$

To check (2.15), note that if $\lambda_j = (n-j)/n$, then the left hand side of (2.15) certainly exceeds

$$\int_{\lambda_1 a_1 + (1-\lambda_1)a_2}^{a_2} \cdots \int_{\lambda_{n-1}a_{n-1}+(1-\lambda_{n-1})a_n}^{a_n} V_{n-1}(x)\, dx_{n-1} \cdots dx_1.$$

By Lemma 2.2 this expression is bounded below by a positive constant times the integral of V_{n-1} over the entire rectangle $\prod_{i=1}^{n-1}[a_i, a_{i+1}]$, and by (2.14), that integral is equal to $C(n)\, V_n(a_1, \dots, a_n)$.

We shall now prove (2.13). The case $d = 2$ is immediate since $\Psi_2(\cdot\,; h) = \chi_{[0,h_1]}$ and $v(h_1) = h_1$: we find that (2.13) holds with $c(2) = 1/2$. Now we argue by induction and assume that (2.13) holds if $d-1 \ge 2$. With $s_j = t + \kappa_j(h)$ we use (2.12) for a ϕ with $\phi^{(d)}(u) = 1$ for $u \ge \overline{s} = (s_1 + \cdots + s_d)/d$ and $\phi^{(d)}(u) = 0$ for $u < \overline{s}$. We thereby obtain

$$\begin{aligned}
&\int_{g_d(t,h)}^{t+\kappa_d(h)} \Psi_d(u-t;h)\, du = \mathcal{J}_d(s;\phi) \\
&= \int_{s_1}^{s_2} \cdots \int_{s_{d-1}}^{s_d} \int_{\sigma_1}^{\sigma_{d-1}} \chi_{\{u \ge \overline{s}\}}(u) \;\; \times \\
&\qquad \Psi_{d-1}(u-\sigma_1; \sigma_2 - \sigma_1, \dots, \sigma_{d-1} - \sigma_{d-2})\, du\, d\sigma_{d-1} \dots d\sigma_1 \\
&\ge \int_{\lambda_1 s_1 + (1-\lambda_1)s_2}^{s_2} \cdots \int_{\lambda_{d-1}s_{d-1}+(1-\lambda_{d-1})s_d}^{s_d} \int_{\sigma_1}^{\sigma_{d-1}} \chi_{\{u \ge \overline{\sigma}\}}(u) \times \\
&\qquad \Psi_{d-1}(u-\sigma_1; \sigma_2 - \sigma_1, \dots, \sigma_{d-1} - \sigma_{d-2})\, du\, d\sigma_{d-1} \cdots d\sigma_1,
\end{aligned}$$

where $\lambda_j = (d-j)/d$. Here the inequality follows because the conditions $\sigma_j \ge \lambda_j s_j + (1-\lambda_j)s_{j+1}$ and $u \ge \overline{\sigma} = (\sigma_1 + \cdots + \sigma_{d-1})/(d-1)$ together imply $u \ge \overline{s}$. It follows from the induction hypothesis that

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_{d-1}} \chi_{\{u \ge \overline{\sigma}\}}(u) \Psi_{d-1}(u-\sigma_1; \sigma_2 - \sigma_1, \dots, \sigma_{d-1} - \sigma_{d-2})\, du \\
&\qquad\qquad \ge c(d-1) V_{d-1}(\sigma_1, \sigma_2, \dots, \sigma_{d-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{g_d(t,h)}^{t+\kappa_d(h)} \Psi_d(u-t;h) du \ge c(d-1) \times \\
&\int_{\lambda_1 s_1 + (1-\lambda_1)s_2}^{s_2} \cdots \int_{\lambda_{d-1}s_{d-1}+(1-\lambda_{d-1})s_d}^{s_d} V_{d-1}(\sigma_1, \dots, \sigma_{d-1})\, d\sigma_1 \cdots d\sigma_{d-1}.
\end{aligned}$$

With Lemma 2.2 and (2.14), this yields (2.13). □

Proof of Proposition 2.1, conclusion. We first observe that

$$J_\phi(t,h) = \mathcal{J}_d(t, t+\kappa_2(h), \dots, t+\kappa_d(h); \phi). \tag{2.16}$$

Recall $g_d(t,h) := \sum_{i=1}^d (t+\kappa_i(h))/d$ so that $t \le g_d(t,h) \le t+\kappa_d(h)$. We apply (2.10), (2.13) to get

$$\begin{aligned} J_\phi(t,h) &\ge \int_{\bar{t}(t,h)}^{t+\kappa_d(h)} \Psi_d(u-t;h)\ \phi^{(d)}(u)\ du \\ &\ge \phi^{(d)}\big(g_d(t,h)\big) \int_{\bar{t}(t,h)}^{t+\kappa_d(h)} \Psi_d(u-t;h) du \\ &\ge c_d \phi^{(d)}(g_d(t,h))\, v(h) \ge c_d A^{-1} \Big(\prod_{j=1}^d \phi^{(d)}(t+\kappa_j(h))\Big)^{1/d} v(h) \end{aligned}$$

where we have used that $\phi^{(d)}$ is nonnegative and nondecreasing, and in the last estimate we have employed the hypothesized condition (1.4). $\square$

3. Proof of Theorem 1.1

We first note that ϕ satisfies condition (1.6) on $(0,b)$ if, and only if the function $s \mapsto \phi(bs)$ satisfies condition (1.6) on the interval $(0,1)$. The desired estimate is invariant under the change of variable

$$x \mapsto (b^{-1}x_1, b^{-2}x_2, \dots, b^{1-d}x_{d-1}, x_d)$$

and thus we may replace ϕ by $\phi(b\cdot)$. Thus we may and shall assume

$$b \le 1 \tag{3.1}$$

in what follows. We shall assume also that $\phi^{(d)}(t)$ is positive and nondecreasing in $[a,b]$ and it then suffices to prove the estimate (1.5) with the interval $(0,b)$ replaced with (a,b) and $b \le 1$.

Given Proposition 2.1 the argument is very similar to the argument in the proof of the result for monomial curves in [2], based substantially on previous ideas in papers by Christ [5], Drury [6] and Drury and Marshall [8], and the exposition will be somewhat sketchy. We aim for an estimation of an adjoint operator and thus will set $p = Q' = Q/(Q-1)$ and $q = P' = P/(P-1)$. Thus we fix $p < q_d = \frac{d^2+d+2}{2}$ and $q = \frac{d^2+d}{2}p' > q_d$. We shall now *assume* that the condition (2.4) is satisfied with a positive constant c_0, for all $(t,h) \in [0,1]^d$ such that $t + \kappa_d(h) \le 1$. Note that by Proposition 2.1 this assumption is implied by (1.6).

Definition. *Let $0 \le a < b \le 1$, $0 \le M < \infty$, $\sigma > 0$, and let $\mathcal{K}_{a,b,M}(\sigma)$ be the class of all real valued functions ϕ defined on $[a,b]$ for which*

(i) $\phi \in C^d([a,b])$, $\phi^{(d)}(t) \le M$ for all $t \in [a,b]$, $\phi, \phi', \dots, \phi^{(d)}$ are nonnegative on $[a,b]$, and

(ii) for all $h \in [0,1]^{d-1}$ with $\kappa_d(h) \le b-a$ the inequality

$$J_\phi(t,h) \ge \sigma v(h) \Big(\prod_{i=1}^d \phi^{(d)}(t+\kappa_i(h))\Big)^{1/d}$$

holds for all t such that $a \le t \le b - \kappa_d(h)$.

Let $R \ge 1$, $B_R = \{x \in \mathbb{R}^d : |x| \le R\}$, and define

$$\mathcal{A} \equiv \mathcal{A}(R, M, \mathfrak{c}) := \sup_{\sigma \le \mathfrak{c}} \frac{\sigma}{\mathfrak{c}} \quad \times \tag{3.2}$$

$$\sup_{\substack{\phi \in \mathcal{K}_{a,b,M}(\sigma) \\ 0 \le a < b \le 1}} \sup_{\substack{\|g\|_{L^{q'}(\mathbb{R}^d)} \le 1 \\ \mathrm{supp}(g) \subset B_R}} \Big(\int_a^b |\widehat{g}(t, \dots, \tfrac{t^{d-1}}{(d-1)!}, \phi(t))|^{p'} |\phi^{(d)}(t)|^{\frac{2}{d^2+d}} dt \Big)^{1/p'}.$$

Clearly $\mathcal{A}(R, M, \mathfrak{c})$ is finite for each R and M, indeed in view of $b \le 1$ we have $\mathcal{A}(R, M, \mathfrak{c}) \le C_d M^{1/p'} R^{d/q'}$. The theorem is proved if we can show that $\mathcal{A}$ only depends on $\mathfrak{c}, p, d$; in fact we will prove that

$$\mathcal{A}(R, M, \mathfrak{c}) \le C(p, d)\, \mathfrak{c}^{-1/q}. \tag{3.3}$$

The restriction inequality

$$\Big(\int_a^b |\widehat{g}(\gamma(t))|^{p'} w(t)\, dt \Big)^{1/p'} \le \frac{\mathfrak{c}}{\sigma} \mathcal{A}\, \|g\|_{L^{q'}(B_R)}$$

with $w = |\phi^{(d)}|^{2/[d(d+1)]}$ is equivalent to the inequality

$$\|Tf\|_{L^q(B_R)} \le \frac{\mathfrak{c}}{\sigma} \mathcal{A}\, \|f\|_{L^p(wdt)}, \tag{3.4}$$

where

$$Tf(x) = \int_a^b f(t) w(t) e^{-i\langle x, \gamma(t)\rangle} dt.$$

For fixed $h \in (\mathbb{R}_+)^{d-1}$ let

$$H(t, h) = \prod_{i=1}^d w(t + \kappa_i(h)). \tag{3.5}$$

With $I_h = (a, b - \kappa_d(h))$ and with the convention that $\int \cdots dt$ will mean $\int_{I_h} \cdots dt$, we write

$$S_h[F](x) = \int e^{-i\langle x, \Gamma(t,h)\rangle} F(t, h) H(t, h)\, dt.$$

We form d-fold products and, with the additional convention that h integrals are extended over the region where $\kappa_d(h) \le b$, write

$$\prod_{i=1}^d Tf_i = \sum_{\pi \in \mathfrak{S}_d} \int S_h[F^\pi] dh$$

where

$$F^\pi(h, t) = \prod_{i=1}^d f_{\pi(i)}(t + \kappa_i(h)).$$

The strategy in establishing (3.4) will be to estimate the $L^{q/d}(B_R)$ norm of $\prod_{i=1}^d Tf_i$ by estimating the $L^{q/d}(B_R)$ norms of $\int S_h[F^\pi]\, dh$.

Lemma 3.1. *For every h with $\kappa_d(h) \le b-a$ the inequality*

$$\big\|S_h[FH^{-\frac{d-1}{d}}]\big\|_{L^q(B_R)} \le d^{d/q'}\frac{\mathfrak{c}}{\sigma}\mathcal{A}(Rd^3, M, \sigma/d)\Big(\int |F(t,h)|^p H(t,h)^{1/d}dt\Big)^{1/p} \tag{3.6}$$

holds for $\phi \in \mathcal{K}_{a,b,M}(\sigma)$.

Proof. Set $\overline{h} = d^{-1}\sum_{k=1}^{d-1}(d-k)h_k$. A quick computation involving expansions of powers of t about the point $t+\overline{h}$ shows that

$$\Gamma(t,h) = \mathfrak{v}(h) + d\mathfrak{A}(h)\widetilde{\gamma}(t+\overline{h}, h) \tag{3.7}$$

where $\mathfrak{v}(h)$ is a vector in $\mathbb{R}^d$ with coordinates $\mathfrak{v}_k(h) = \sum_{\nu=1}^{d}(\kappa_\nu(h)-\overline{h})^k$ and $\mathfrak{v}_d(h)=0$, and $\mathfrak{A}(h)$ is a $d\times d$ matrix with

$$\mathfrak{A}_{ij}(h) = \begin{cases} 1, & i=j, \\ 0, & i>j \\ d^{-1}\sum_{\nu=1}^{d}\frac{(\kappa_\nu(h)-\overline{h})^{j-i}}{(j-i)!}, & i<j\le d-1 \\ 0, & i<d, j=d. \end{cases}$$

Finally $\widetilde{\gamma}(s,h) = (s,\dots,\frac{s^{d-1}}{(d-1)!}, \widetilde{\phi}(s;h))$ with

$$\widetilde{\phi}(s;h) = \frac{1}{d}\sum_{i=1}^{d}\phi(s-\overline{h}+\kappa_i(h)).$$

The function $\widetilde{\phi}$ and the curve $\widetilde{\gamma}(t,h)$ are defined on $[a(h), b(h)] \subset [0,1]$ where $a(h) = a+\overline{h}$ and $b(h) = b-\kappa_d(h)+\overline{h}$. It is now crucial to note that for $\phi\in\mathcal{K}_{a,b,M}(\sigma)$ and fixed h the offspring function $\widetilde{\phi}\equiv\widetilde{\phi}(\cdot;h)$ belongs to $\mathcal{K}_{a(h),b(h),M}(\sigma/d)$. This follows from (2.16), (2.10) for the function $\widetilde{\phi}$. Indeed the nonnegativity of Ψ_d imply that if $\widetilde{h}\in(\mathbb{R}_+)^{d-1}$ satisfies $\kappa_d(\widetilde{h}) \le b(h)-a(h)$, then

$$\begin{aligned} J_{\widetilde{\phi}(\cdot,h)}(t,\widetilde{h}) &= \int_t^{t+\kappa_d(\widetilde{h})} \Psi_d(u-t;\tilde{h})\frac{1}{d}\sum_{i=1}^{d}\phi^{(d)}(u-\overline{h}+\kappa_i(h))\,du \\ &\ge \frac{\sigma}{d}\,v(h)\,\sum_{i=1}^{d}\Big(\prod_{j=1}^{d}\phi^{(d)}(t-\overline{h}+\kappa_i(h)+\kappa_j(\tilde{h}))\Big)^{1/d} \\ &\ge \frac{\sigma}{d}\,v(h)\,\Big(\prod_{j=1}^{d}\phi^{(d)}(t-\overline{h}+\kappa_d(h)+\kappa_j(\tilde{h}))\Big)^{1/d} \\ &\ge \frac{\sigma}{d}\,v(h)\,\prod_{j=1}^{d}\Big(\frac{1}{d}\sum_{i=1}^{d}\phi^{(d)}\big(t-\overline{h}+\kappa_i(h)+\kappa_j(\tilde{h})\big)\Big)^{1/d}. \end{aligned}$$

Here the first inequality follows from (2.10) and $\phi \in \mathcal{K}_{a,b}(\sigma)$. The last inequality shows that $\tilde{\phi}(\cdot;h) \in \mathcal{K}_{a(h),b(h)}(\sigma)$; it follows from the fact that $\phi^{(d)}$ is nondecreasing.

Now let g_h be defined by $\widehat{g}_h(\xi) = \widehat{g}(\mathfrak{v}(h) + d\mathfrak{A}(h)\xi)$. Then because of the unimodularity of $\mathfrak{A}(h)$ we have $\|g_h\|_{q'} = d^{d/q'}\|g\|_{q'}$. Also if g is supported in B_R then g_h is supported in the ball of radius Rd^3 (observe that all the entries of $\mathfrak{A}(h)$ are at most d).

Comparing a geometric to an arithmetic mean we see that

$$\begin{aligned}
&\Big(\int_a^{b-\kappa_d(h)} \big|\widehat{g}(\Gamma(t,h))\big|^{p'} H(t,h)^{1/d} dt\Big)^{1/p'} \\
&\le \Big(\int_a^{b-\kappa_d(h)} \big|\widehat{g}(\Gamma(t,h))\big|^{p'} \Big(\frac{1}{d}\sum_{i=1}^{d} \phi^{(d)}(t+\kappa_i(h))\Big)^{2/(d^2+d)} dt\Big)^{1/p'} \\
&= \Big(\int_{a+\overline{h}}^{b-\kappa_d(h)+\overline{h}} \big|\widehat{g}_h(\widetilde{\gamma}(s,h))\big|^{p'} \big(\widetilde{\phi}^{(d)}(s;h)\big)^{2/(d^2+d)} ds\Big)^{1/p'} \\
&\le \frac{\mathfrak{c}/d}{\sigma/d}\,\mathcal{A}(Rd^3, M, \sigma/d)\|g_h\|_{q'} = \frac{\mathfrak{c}}{\sigma} d^{d/q'} \mathcal{A}(Rd^3, M, \sigma/d)\|g\|_{q'}.
\end{aligned}$$

By duality this also implies (3.6). □

We now proceed exactly as in the proof of Proposition 6.1 in [2]. We first have, by an application of Plancherel's theorem and the change of variable $(t,h) \mapsto \Gamma(t,h)$

$$\Big\| \int S_{R,h}[F]dh \Big\|_2 \le C\Big(\iint \big|F(t,h)H(t,h)J(t,h)^{-1/2}\big|^2 dt\, dh\Big)^{1/2}; \tag{3.8}$$

the change of variable can be justified as in [8], p. 549.

Replacing F with $FH^{(d-1)/d}$ in (3.6) and then integrating with respect to h now yields, according to Minkowski's inequality, the estimate

$$\begin{aligned}
&\Big\| \int S_h[F]dh \Big\|_{L^q(B_R)} \\
&\quad \le d^{d/q'}\, \mathfrak{c}\, \sigma^{-1}\, \mathcal{A}(Rd^3, M, \mathfrak{c}/d) \int \Big(\int |F(t,h)H(t,h)^{1-\frac{1}{d}+\frac{1}{dp}}|^p dt\Big)^{1/p} dh.
\end{aligned} \tag{3.9}$$

By analytic interpolation of (3.9) and (3.8) one obtains

$$\begin{aligned}
&\Big\| \int S_{R,h}[F]dh \Big\|_{L^s(B_R)} \le C\Big(\frac{\mathfrak{c}}{\sigma}\mathcal{A}(Rd^3, M, \sigma/d)\Big)^{1-\vartheta} \\
&\qquad \times \Big(\int \Big(\int \big|F(t,h)H(t,h)^{\eta} J(t,h)^{-\vartheta/2}\big|^{B(\vartheta)} dt\Big)^{A(\vartheta)/B(\vartheta)} dh\Big)^{1/A(\vartheta)}
\end{aligned} \tag{3.10}$$

where $0 \le \vartheta \le 1$ and A, B, s, η are defined by

$$\begin{aligned} &\frac{1}{A(\vartheta)} = 1 - \frac{\vartheta}{2}, && \frac{1}{B(\vartheta)} = \frac{1}{p} + \vartheta(\frac{1}{2} - \frac{1}{p}), \\ &\frac{1}{s(\vartheta)} = \frac{1-\vartheta}{q} + \frac{\vartheta}{2}, && \eta(\vartheta) = 1 - \frac{d+1}{2q}(1-\vartheta). \end{aligned} \tag{3.11}$$

Now let

$$\vartheta(p) = \frac{4(d-1)}{(d+1)dp' - 4} = \frac{2(d-1)}{q-2} \tag{3.12}$$

and let $A_p = A(\vartheta(p))$, $B_p = B(\vartheta(p))$, $s_p = s(\vartheta(p))$ and $\eta = \eta(\vartheta(p))$. Then $\eta_p - (d+1)\vartheta/4 = 1/p$ and $s_p = q/d = (d+1)p'/2$. As $\phi \in \mathcal{K}_{a,b,M}(\sigma)$ we may use the crucial inequality $J_\phi(t) \ge \sigma v(h) H^{(d+1)/2}(t,h)$ and obtain

$$\begin{aligned} \Big\| \int S_h[F] dh \Big\|_{L^{q/d}(B_R)} &\le C\sigma^{-\vartheta(p)/2} (\mathfrak{c}\,\sigma^{-1} \mathcal{A}(Rd^3, M, \sigma/d))^{1-\vartheta(p)} \times \\ \Big(\int \Big(\int \prod_{j=1}^{d} &\big| F(t,h) H(t,h)^{\eta_p - \frac{d+1}{4}\vartheta(p)} \big|^{B_p} dt \Big)^{A_p/B_p} v(h)^{1-A_p} dh \Big)^{1/A_p}. \end{aligned} \tag{3.13}$$

We are now in the position to apply an inequality by Drury and Marshall [8] for multilinear operators involving Vandermonde's determinant, see also [2] for an exposition. To state this let

$$\mathfrak{V}[f_1, \dots, f_d](t,h) := v(h)^{-1} \prod_{i=1}^{d} f_i(t + \kappa_i(h))$$

and $L^A_v(L^B)$ denote the weighted mixed norm space consisting of functions $(t,h) \mapsto G(t,h)$ with $\|G\|_{L^A_v(L^B)} = (\int \|G(\cdot,h)\|_B^A v(h) dh)^{1/A} < \infty$. One assumes that $1 < A < \frac{d+2}{d}$, $1 < A \le B < \frac{2A}{d+2-dA}$, and sets $\sigma = 2/(d + 2 - dA)$. For $l = 1, \dots, d$ let Q_l denote the point in $\mathbb{R}^d_+$ for which the j^{th} coordinate is $(\sigma A)^{-1}$, if $j \ne l$ and the l^{th} coordinate is B^{-1}. Let $\Sigma(A,B)$ be the $d-1$ dimensional closed convex hull of the points $Q_1, \dots, Q_d$. Then the inequality

$$\big\| \mathfrak{V}[f_1, \dots, f_d] \big\|_{L^A_v(L^B)} \le C \prod_{i=1}^{d} \|f_i\|_{L^{p_i,1}} \tag{3.14}$$

holds for all $(p_1^{-1}, \dots, p_d^{-1}) \in \Sigma(A,B)$.

We apply this inequality to the right hand side of (3.13) to obtain

$$\begin{aligned} &\Big\| \int S_h[F] dh \Big\|_{L^{q/d}(B_R)} \\ &\quad \le C(d,p)\, \sigma^{-\vartheta(p)/2} \Big(\mathfrak{c}\,\sigma^{-1}\, \mathcal{A}(Rd^3, M, \mathfrak{c}/d) \Big)^{1-\vartheta(p)} \prod_{j=1}^{d} \|f_j w^{1/p}\|_{L^{p_j,1}} \end{aligned}$$

whenever $(p_1^{-1},\dots,p_d^{-1})\in\Sigma(A_p,B_p)$. Summing over the permutations $\pi\in\mathfrak{S}_d$ then yields

$$\Big\|\prod_{i=1}^d Tf_i\Big\|_{L^{q/d}(B_R)}\le C(d,p)\,\sigma^{-\vartheta(p)/2}\Big(\mathfrak{c}\,\sigma^{-1}\,\mathcal{A}(Rd^3,M,\mathfrak{c}/d)\Big)^{1-\vartheta(p)}\prod_{j=1}^d\|f_jw^{1/p}\|_{L^{p_j,1}}. \tag{3.15}$$

We now use applications of Hölder's inequality and Christ's multilinear trick for the q_d-linear expression $\prod_{i=1}^{q_d}Tf_i$, exactly as in §6 of [2]. This yields

$$\Big\|\prod_{i=1}^{q_d}Tf_i\Big\|_{L^{q/q_d}(B_R)}\lesssim\sigma^{-q_d\vartheta(p)/2d}(\mathfrak{c}\,\sigma^{-1}\,\mathcal{A}(Rd^3,M,\sigma/d))^{(1-\vartheta(p))q_d/d}\prod_{i=1}^{q_d}\big\|f_i|\phi^{(d)}|^{\frac{2}{(d^2+d)p}}\big\|_{L^{p,q_d}}.$$

Since $p<q_d<q$ this implies (for $f_i\equiv f$)

$$\|Tf\|_{L^q(B_R)}\le C(d,p,q)\sigma^{-\vartheta(p)/2d}(\mathfrak{c}\,\sigma^{-1}\,\mathcal{A}(Rd^3,M,\sigma/d))^{(1-\vartheta(p))/d}\big\|f|\phi^{(d)}|^{\frac{2}{(d^2+d)p}}\big\|_p \tag{3.16}$$

provided that $\phi\in\mathcal{K}_{a,b,M}$ for some $M<\infty$. Observe that from the definition of $\mathcal{A}$ we get

$$\mathcal{A}(Rd^3,M,\mathfrak{c}/d))\le C_{d,p}\mathcal{A}(R,M,\mathfrak{c})$$

and thus (3.16) implies

$$\mathcal{A}(R,M,\mathfrak{c})\le C(d,p)\mathcal{A}(R,M,\mathfrak{c})^{(1-\vartheta(p))/d}\sigma^{-\vartheta(p)/2d}$$

which by (3.12) yields (3.3). □

4. Examples of curves covered by Theorem 1.1

4.1. Condition (1.6) (and *a fortiori* condition (1.4)) holds for $\phi(t)=t^\beta$ and the required monotonicity of the first d derivatives holds if $\beta>d-1$.

4.2. Consider the function $\phi(t)=\exp(-t^{-\beta})$ for $t>0$. Then induction shows that $\phi^{(d)}(t)=\beta^de^{-t^{-\beta}}t^{-d(\beta+1)}\big(1+\sum_{j=1}^d a_{j,d}t^{j\beta}\big)$ and the coefficients satisfies the recursive relation $a_{k,d+1}=\beta^{-1}a_{k,d}-a_{k-1,d}(d+1-k+d/\beta)$ if $k\le d-1$ and $a_{d,d+1}=-a_{d-1,d}(1+d/\beta)$ if $k=d$. It is obvious that if $A>1$, then condition (1.6) is satisfied on a (small) interval $(0,c(A))$.

4.3. Suppose that $\big(\prod_{j=1}^d g(s_j)\big)^{1/d} \le g\big(\sqrt[d]{s_1\cdots s_d}\big)$ for $0 < s_1 \le s_2 \le \cdots \le s_d < \infty$, and g is nonnegative and increasing. Set $f_g(s) = \exp(-1/g(s))$. Then we also have for $\overline{s} = (\prod_{i=1}^d s_j)^{1/d}$

$$f_g(\bar{s}) = \exp\big(-1/g(\bar{s})\big) \ge \exp\big(-(\prod_{j=1}^d 1/g(s_j))^{1/d}\big)$$
$$\ge \exp\Big(-\frac{1}{d}\sum_{j=1}^d \frac{1}{g(s_j)}\Big) = \Big(\prod_{j=1}^d f_g(s_j)\Big)^{1/d}.$$

Thus if the first d derivatives of a function ϕ are nonnegative and increasing on $(0,\infty)$ and if ϕ satisfies (1.6) with $A = 1$ then the same conditions are satisfied by $\psi(t) = \int_0^t (t-u)^{d-1}\exp(-1/\phi^{(d)}(u))du$. As mentioned in the introduction this leads to a sequence of progressively flatter functions mentioned following the statement of Theorem 1.1.

4.4. Similarly, suppose that $\big(\prod_{j=1}^d g(s_j)\big)^{1/d} \le g\big(\sqrt[d]{s_1\cdots s_d}\big)$ for $0 \le a < s_1 \le s_2 \le \cdots \le s_d < b$. Assume also that $g(s) > e$ if $s \in (a,b)$. Then

$$\Big(\prod_{j=1}^d \log(g(s_j))\Big)^{1/d} \le \frac{1}{d}\sum_{j=1}^d \log(g(s_j))$$
$$= \log\Big(\prod_{j=1}^d g(s_j)\Big)^{1/d} \le \log(g(\sqrt[d]{s_1\cdots s_d})).$$

Again if $\psi(t) = \int_a^t (t-u)^{d-1}\log(\phi^{(d)}(u))du$, if $\phi^{(d)}(s) > e$ and $\phi^{(d)}$ is nondecreasing on (a,b) then condition (1.6) with $A = 1$ for ϕ implies (1.6) with $A = 1$ for ψ.

5. Proof of Theorem 1.2

First assume that (1.7) holds. We will establish (1.8). For $\lambda > 1$ define

$$T_\lambda f(x) = \chi(x)\int_a^b f(t)e^{-i\lambda\langle x,\gamma(t)\rangle}dt,$$

where χ is the characteristic function of a set of diameter 1.

Definition. *For $-\infty < a < b < \infty$ and $\sigma > 0$, let $\mathcal{C}_{a,b}(\sigma)$ be the class of all real-valued functions ϕ defined on (a,b) for which*

(i) $\phi \in C^d((a,b))$ and the derivatives $\phi', \dots, \phi^{(d)}$ are nonnegative and nondecreasing on (a,b), and

(ii) the inequality

$$\phi^{(d-1)}(s) - \phi^{(d-1)}(t) \ge \sigma^{-\frac{1}{\alpha}}(s-t)^{\frac{1}{\alpha}+1-\frac{d(d+1)}{2}} \tag{5.1}$$

holds for all s and t such that $a < t < s < b$.

With $q = 1 + 1/\alpha$ and for $\lambda > 1$, $\sigma > 0$ and large r, define

$$\mathcal{B} \equiv \mathcal{B}(\lambda, \sigma, r) := \lambda^{d/q} \sup_{\substack{\phi \in \mathcal{C}_{a,b}(\sigma) \\ -r \le a < b \le r}} \sup_{\|f\|_{L^q((a,b))} \le 1} \|T_\lambda f\|_{L^{q,\infty}(\mathbb{R}^d)}.$$

By duality and Lemma 5.1 below, (1.8) is a consequence of the following estimate

$$\mathcal{B}(\lambda, \sigma, r) \le C(d, \alpha)\, \sigma^{\frac{1}{1+\alpha}}. \tag{5.2}$$

Lemma 5.1. *If (1.7) holds for all parallelepipeds E in $\mathbb{R}^d$ then the inequality*

$$B^{-\frac{1}{\alpha}}\, (s-t)^{\frac{1}{\alpha}+1-\frac{d(d+1)}{2}} \le \phi^{(d-1)}(s) - \phi^{(d-1)}(t)$$

holds whenever $a < t < s < b$.

We shall give the proof in §6.

To begin the proof of (5.2), fix a, b, σ, and $\phi \in \mathcal{C}_{a,b}(\sigma)$ and then define

$$\begin{aligned} M_\lambda(f_1, \cdots, f_d)(x) &= \prod_{j=1}^d T_\lambda f_j(x) \\ &= \chi(x) \int_{\mathbb{R}^{d-1}} \int_{\mathcal{I}_h} e^{-i\lambda \langle x, \sum_{j=1}^d \gamma(s+h_j)\rangle} \prod_{j=1}^d f_j(s+h_j)\, ds\, dh_1 \cdots dh_{d-1}, \end{aligned}$$

where our convention now is that $h_d = 0$ and $\mathcal{I}_h$ is the (possibly empty) intersection of the d intervals $(a - h_j, b - h_j)$. In what follows we will further simplify the notation by writing $h = (h_1, \ldots, h_{d-1})$ and $\Gamma(s, h) = \sum_{j=1}^d \gamma(s + h_j)$. With an eye to decomposing the multilinear operator M_λ we define

$$u(h) = \prod_{1 \le i < j \le d} |h_i - h_j| = h_1 \cdots h_{d-1} \prod_{1 \le i < j \le d-1} |h_i - h_j|$$

and

$$K(h) = u(h) \Big(\sup_{1 \le i < j \le d} |h_i - h_j| \Big)^{\frac{1}{\alpha} - \frac{d(d+1)}{2}}.$$

Note that K is homogeneous of degree $\alpha^{-1} - d$. Now, for $m \in \mathbb{Z}$, let

$$S_m = \{h \in \mathbb{R}^{d-1} : 2^{-m-1} < K(h) \le 2^{-m}\}$$

and, following [1], define

$$M_{\lambda,m}(f_1, \cdots, f_d)(x) = \chi(x) \int_{S_m} \int_{\mathcal{I}_h} e^{-i\lambda\langle x, \Gamma(s,h)\rangle} \prod_{j=1}^d f_j(s + h_j)\, ds\, dh.$$

We will need to observe that

$$m_{d-1}(S_m) \le C(d, \alpha)\, 2^{-m(d-1)\alpha/(1-d\alpha)}. \tag{5.3}$$

By homogeneity, it is enough to check that $m_{d-1}(\{h : K(h) \le 1\}) \le C(d)$. Since $\alpha \le 2/(d^2+d)$,

$$\{h : K(h) \le 1\} \subset (\{h : u(h) \le 1\} \cup \{h : \sup |h_i| \le 1\},$$

and so it is enough to check that

$$m_{d-1}(\{h : u(h) \le 1\}) \le C(d). \tag{5.4}$$

But it follows from [8] (see (i) of Proposition 2.4 in [2]) that

$$m_{d-1}(\{h : 0 \le h_1 \le \cdots \le h_{d-1}\,;\ u(h) \le 1\}) \le C(d)$$

and so $m_{d-1}(\{h : 0 \le h_j\,;\ u(h) \le 1\}) \le C'(d)$. Since

$$\prod_{1\le i\le j\le d} \big||h_i| - |h_j|\big| \le \prod_{1\le i\le j\le d} |h_i - h_j| = u(h),$$

(5.4) follows.

Now considerations similar to those which lead to (3.7) show that

$$\Gamma(s,h) = \mathfrak{v}(h) + d\mathfrak{A}(h)\widetilde{\gamma}(s+\overline{h},h)$$

where $\mathfrak{v}(h)$ is a vector, where $\mathfrak{A}(h)$ is a matrix with entries 1 on the diagonal and 0 below, where $\overline{h} = \sum_{j=1}^{d} h_j/d$, and where

$$\widetilde{\gamma}(s,h) = \Big(s, \frac{s^2}{2}, \ldots, \frac{s^{d-1}}{(d-1)!}, \widetilde{\phi}(s,h)\Big)$$

with

$$\widetilde{\phi}(s,h) = \frac{1}{d}\sum_{i=1}^{d} \phi(s - \overline{h} + h_i).$$

Since (5.1) holds for ϕ, it holds as well for each $\widetilde{\phi}(\cdot,h)$. Therefore we have the estimate

$$\Big\|\chi \int_{\mathcal{I}_h} e^{-i\lambda\langle\cdot,\Gamma(s,h)\rangle} f(s)\,ds\Big\|_{L^{q,\infty}(\mathbb{R}^d)} \le \lambda^{-d/q}\mathcal{B}(\lambda,\sigma,r)\,\|f\|_{L^q(\mathcal{I}_h)}.$$

Taking (5.3) into consideration, an application of Minkowski's inequality thus yields

$$\begin{aligned} &\|M_{\lambda,m}(f_1,\cdots,f_d)\|_{L^{q,\infty}(\mathbb{R}^d)} \\ &\qquad \le C(d,\alpha)\lambda^{-d/q}\mathcal{B}(\lambda,\sigma,r)\,2^{-m[(d-1)\alpha/(1-d\alpha)]}\|f_d\|_q \prod_{j=1}^{d-1}\|f_j\|_\infty, \end{aligned} \tag{5.5}$$

where $\|\cdot\|_q$ stands for the norm in $L^q(a,b)$.

Let $J(s,h)$ stand for the absolute value of the Jacobi-determinant of the transformation $(s,h) \mapsto \Gamma(s,h)$ (defined on $\{(s,h) : s \in \mathcal{I}_h\}$). To obtain an L^2 estimate for $M_{\lambda,m}$ we will need the following inequality:

$$J(s,h) \ge c(d)\,\sigma^{-1/\alpha}K(h). \tag{5.6}$$

This inequality follows from (5.1) and the next lemma whose proof is given in §6.

Lemma 5.2. *Suppose the inequality*

$$c\,(s-t)^\rho \le \phi^{(d-1)}(s) - \phi^{(d-1)}(t) \tag{5.7}$$

for some $\rho > 0$ and for $a < t < s < b$. Then there is also the inequality

$$c(d)\,c\,u(h)\Big(\sup_{1\le i<j\le d} |h_i - h_j|\Big)^{\rho-1} \le J(s,h)$$

whenever $s \in \mathcal{I}_h$.

Now the transformation $(s,h) \mapsto \Gamma(s,h)$ is at most $d!$ to one a.e., so

$$\|M_{\lambda,m}(f_1,\dots,f_d)\|^2_{L^2(\mathbb{R}^d)} \le d! \int_{S_m}\int_{\mathcal{I}_h} \Big|\prod_{j=1}^d f_j(s+h_j)\Big|^2 \frac{1}{J(s,h)}\,ds\,dh.$$

Applying (5.6) and recalling (5.3), we obtain

$$\begin{aligned} &\|M_{\lambda,m}(f_1,\cdots,f_d)\|_{L^2(\mathbb{R}^d)} \\ &\qquad \le C(d,\alpha)\,\lambda^{-d/2}\,\sigma^{1/2\alpha}\,2^{m[1-(2d-1)\alpha]/[2(1-d\alpha)]}\|f_d\|_2 \prod_{j=1}^{d-1}\|f_j\|_\infty. \end{aligned} \tag{5.8}$$

Interpolating the estimates (5.5) and (5.8) yields that

$$\begin{aligned} &\|M_{\lambda,m}(f_1,\cdots,f_d)\|_{L^{q/d,\infty}(\mathbb{R}^d)} \\ &\qquad \le C(d,\alpha)\,\lambda^{-d^2/q}\,\sigma^{(d-1)/(1-\alpha)}\,\mathcal{B}(\lambda,\sigma,r)^{\delta(\alpha)}\|f_d\|_{q/d} \prod_{j=1}^{d-1}\|f_j\|_\infty, \end{aligned} \tag{5.9}$$

with

$$\delta(\alpha) = \frac{1-(2d-1)\alpha}{1-\alpha} \in (0,1).$$

If one uses Bourgain's interpolation argument in [3] (see also the appendix of [4]) then one actually obtains an estimate for the sum $M_\lambda = \sum_m M_{\lambda,m}$, namely,

$$\begin{aligned} &\|M_{\lambda}(f_1,\cdots,f_d)\|_{L^{q/d,\infty}(\mathbb{R}^d)} \\ &\qquad \le C(d,\alpha)\,\lambda^{-d^2/q}\,\sigma^{(d-1)/(1-\alpha)}\,\mathcal{B}(\lambda,\sigma,r)^{\delta(\alpha)}\|f_d\|_{q/d,1} \prod_{j=1}^{d-1}\|f_j\|_\infty. \end{aligned} \tag{5.10}$$

To arrive at (5.10) it suffices to prove this bound for $f_d = \chi_U$, the characteristic function of a measurable set U. One then uses (5.8) to estimate the size of the set where $|\sum_{2^m\le\beta} M_{\lambda,m}(f_1,\dots,f_{d-1},\chi_U)| \ge s$, and one uses (5.5) to estimate the size of the set where $|\sum_{2^m>\beta} M_{\lambda,m}(f_1,\dots,f_{d-1},\chi_U)| \ge s$; here

$\beta > 0$ will be suitably chosen. This leads to

$$\begin{aligned} & m_d\big(\{x : \big|\sum_m M_{\lambda,m} f(x)\big| > 2s\}\big) \\ & \le\ \lambda^{-d}|U|\Big[C(d,\alpha)^q s^{-q}\mathcal{B}(\lambda,\sigma,r)^q\beta^{-\frac{(d-1)\alpha q}{1-d\alpha}}\prod_{i=1}^{d-1}\|f_i\|_\infty^q \\ & \qquad + C(d,\alpha)^2 s^{-2}\sigma^{1/\alpha}\beta^{-\frac{(1-(2d-1)\alpha}{1-d\alpha}}\prod_{i=1}^{d-1}\|f_i\|_\infty^2\Big], \end{aligned}$$

and the estimate (5.10) follows by choosing the optimal β. (5.10) gives

$$\begin{aligned} & \Big\|\prod_{j=1}^{d} T_\lambda f_j\Big\|_{L^{\frac{q}{d},\infty}(\mathbb{R}^d)} \\ & \qquad \le C(d,\alpha)\,\lambda^{-d^2/q}\,\sigma^{(d-1)/(1-\alpha)}\,\mathcal{B}(\lambda,\sigma,r)^{\delta(\alpha)}\,\|f_1\|_{\frac{q}{d},1}\prod_{j=2}^{d}\|f_j\|_\infty, \end{aligned} \tag{5.11}$$

and if we take for all f_j the same characteristic function of a set we also get

$$\|T_\lambda f\|_{q,\infty} \le C(d,\alpha)\,\lambda^{-d/q}\,\sigma^{(d-1)/(d-d\alpha)}\,\mathcal{B}(\lambda,\sigma,r)^{\delta(\alpha)/d}\|f\|_{q,1}. \tag{5.12}$$

Now fix an integer $N > q$. Applying a version of Hölder's inequality (see (2.1) in [2]) and permuting the functions, (5.11) and (5.12) yield

$$\begin{aligned} & \Big\|\prod_{j=1}^{N} T_\lambda f_j\Big\|_{L^{q/N,\infty}(\mathbb{R}^d)} \\ & \qquad \le C(d,\alpha)\,\lambda^{-Nd/q}\,\sigma^{N(d-1)/(d-d\alpha)}\,\mathcal{B}(\lambda,\sigma,r)^{N\delta(\alpha)/d}\prod_{j=1}^{N}\|f_j\|_{L^{q_j,1}} \end{aligned}$$

when $(q_1^{-1},\cdots,q_N^{-1})$ is one of the N points Q_j in $\mathbb{R}^N$ defined as follows: Q_1 is the point with the first component d/q, the next $d-1$ components 0, and the remaining $N-d$ components equal to $1/q$; Q_2 is obtained by shifting the components of Q_1 to the right by one and moving the last component to the front; etc. Here $L^{\infty,1}$ should be interpreted as L^∞. Applying Christ's multilinear trick (for multilinear operators with values in the quasi-normed q/N-convex space $L^{q/N,\infty}$, see Proposition 2.3 in [2] and also [11]), these estimates yield

$$\begin{aligned} & \Big\|\prod_{j=1}^{N} T_\lambda f_j\Big\|_{L^{q/N,\infty}(\mathbb{R}^d)} \\ & \qquad \le C(d,\alpha)\,\lambda^{-Nd/q}\,\sigma^{N(d-1)/(d-d\alpha)}\,\mathcal{B}(\lambda,\sigma,r)^{N\delta(\alpha)/d}\prod_{j=1}^{N}\|f_j\|_{L^{q_j,r_j}} \end{aligned}$$

when $(q_1^{-1}, \cdots, q_N^{-1})$ is in the interior of the convex hull Σ of $Q_1, \cdots, Q_N$ and when the $r_j \in [1, \infty]$ satisfy $\sum_{j=1}^N 1/r_j = N/q$. Note that the point $(1/q, \cdots, 1/q)$ is the center of Σ. Hence, taking $f_j = f$ and $q_j = r_j = q$, we obtain

$$\|T_\lambda f\|_{L^{q,\infty}(\mathbb{R}^d)} \leq C(d, \alpha)\, \lambda^{-d/q}\, \sigma^{(d-1)/(d-d\alpha)}\, \mathcal{B}(\lambda, \sigma, r)^{\delta(\alpha)/d} \|f\|_{L^q}.$$

Therefore, by the definition of $\mathcal{B}(\lambda, \sigma, r)$, we have

$$\mathcal{B}(\lambda, \sigma, r) \leq C(d, \alpha)\, \sigma^{(d-1)/(d-d\alpha)}\, \mathcal{B}(\lambda, \sigma, r)^{\delta(\alpha)/d}.$$

Recalling the definition of δ, some algebra yields (5.2). Thus (1.8) is established.

Now for the converse, we assume that (1.9) holds with $1/P' = \alpha/Q$ and will show that (1.7) holds with B replaced by $C(d, p)\, B$. Fix an $f \in C_c^\infty(\mathbb{R}^d)$ with f nonnegative and equal to 1 on $[0, 1]^d$. Consider a parallelepiped

$$E = x_0 + \big\{ \sum_{j=1}^d t_j x_j : 0 \leq t_j \leq 1 \big\}$$

and fix a linear isomorphism T of $\mathbb{R}^d$ which satisfies

$$T([0, 1]^d) = \big\{ \sum_{j=1}^d t_j x_j : 0 \leq t_j \leq 1 \big\}.$$

Let g be defined by $\widehat{g}(x) = f(T^{-1}(x - x_0))$ so that $\widehat{g}$ is nonnegative and equal to 1 on E . Then a computation shows $\|g\|_{L^P(\mathbb{R}^d)} = m_d(E)^{1/P'}\ \|\widehat{f}\|_{L^P(\mathbb{R}^d)}$. If (1.9) holds then it follows that

$$\lambda(E)^{1/Q} \leq \Big(\int_a^b |\widehat{g}(\gamma(t))|^Q dt \Big)^{1/Q} \leq B^{\frac{1}{Q}} m_d(E)^{1/P'}\ \|\widehat{f}\|_{L^P(\mathbb{R}^d)}.$$

Since $1/P' = \alpha/Q$ this yields (1.7) with B replaced by $\|\widehat{f}\|_{L^P(\mathbb{R}^d)}^Q B$ and therefore completes the proof of Theorem 1.2.

6. Proofs of Lemma 5.1 and Lemma 5.2

Proof of Lemma 5.1. Write $s = t + h$ and let E_{d-2} be the parallelogram in $\mathbb{R}^2$ with vertices

$$P_1 = (t, \phi^{(d-2)}(t)), \qquad P_2 = P_1 - \rho e_2$$
$$P_3 = (t + h, \phi^{(d-2)}(t + h)), \qquad P_4 = P_3 + \rho e_2$$

where $\rho = h\phi^{(d-1)}(t+h) + \phi^{(d-2)}(t) - \phi^{(d-2)}(t+h) \geq 0$, so that $\phi^{(d-1)}(t+h)$ is the slope of the line segments $\overline{P_2 P_3}$ and $\overline{P_1 P_4}$. Then (as a sketch will show)

$$
\begin{aligned}
m_2(E_{d-2}) &\le 2\int_t^{t+h}\big(\phi^{(d-2)}(t)+\phi^{(d-1)}(t+h)(s-t)-\phi^{(d-2)}(s)\big)\,ds \\
&= 2\int_t^{t+h}\int_t^s\big(\phi^{(d-1)}(t+h)-\phi^{(d-1)}(u)\big)\,du\,ds \\
&\le 2\int_t^{t+h}\int_t^s\big(\phi^{(d-1)}(t+h)-\phi^{(d-1)}(t)\big)\,du\,ds \\
&= h^2\big(\phi^{(d-1)}(t+h)-\phi^{(d-1)}(t)\big).
\end{aligned}
\tag{6.1}
$$

We now prove the following

Claim: For $2 \le k \le d$,

$$
\{\gamma^{(d-k)}(s) : t \le s \le t+h\} \subset \begin{cases} \{e_{d-k}\}\times E_{d-k}, & 2\le k\le d-1, \\ E_0, & k=d, \end{cases}
\tag{6.2}
$$

where $\{e_1,\dots,e_d\}$ is the standard basis in $\mathbb{R}^d$ and E_{d-k} is a parallelepiped in $\mathbb{R}^k$ with

$$
m_k(E_{d-k}) \le h^{\frac{k^2+k-2}{2}}\big(\phi^{(d-1)}(t+h)-\phi^{(d-1)}(t)\big).
\tag{6.3}
$$

The above calculation (6.1) verifies this claim for $k=2$, and all $d\ge 2$. We argue by induction on k and assume $3\le k\le d$ and that the induction hypothesis is true for $k-1$.

Now suppose $s\in[t,t+h]$. Then

$$
\gamma^{(d-k)}(s)-\gamma^{(d-k)}(t)=\int_t^s \gamma^{(d-k+1)}(u)\,du
$$

belongs to

$$
\begin{aligned}
&O_{d-k}\times(s-t)\big(\{e_{d-k+1}\}\times E_{d-k+1}\big) \\
&\subset O_{d-k}\times\{u(1,x)\in\mathbb{R}\times\mathbb{R}^{k-1} : 0\le u\le h, x\in E_{d-k+1}\}
\end{aligned}
$$

where O_{d-k} denotes the origin in $\mathbb{R}^{d-k}$ and where O_{d-k} is omitted if $k=d$.

Let x_0 be any point of E_{d-k+1} in $\mathbb{R}^{k-1}$. The set

$$
\widetilde{E}_{d-k} := \{(1,x)-v(1,x_0) : x\in E_{d-k+1},\ 0\le v\le 1\}
$$

is a parallelepiped in $\mathbb{R}^k$ which satisfies $m_k(\widetilde{E}_{d-k}) = m_{k-1}(E_{d-k+1})$, which contains O_k and $\{(1,x) : x\in E_{d-k+1}\}$, and which therefore (by convexity) contains

$$
\{u(1,x) : 0\le u\le 1,\ x\in E_{d-k+1}\}.
$$

Thus, with

$$
E_{d-k} := \big\{(t,\dots,\tfrac{t^{k-1}}{(k-1)!},\phi^{(d-k)}(t)) + uy : 0\le u\le h,\ y\in \widetilde{E}_{d-k}\big\},
$$

we have

$$
\{\gamma^{(d-k)}(s) : t\le s\le t+h\} \subset \begin{cases} \{e_{d-k}\}\times E_{d-k}, & 3\le k<d, \\ E_0, & k=d, \end{cases}
$$

where E_{d-k} is a parallelepiped in $\mathbb{R}^k$ and

$$m_k(E_{d-k}) = h^k m_{k-1}(E_{d-k+1}).$$

Since $m_{k-1}(E_{d-k+1}) \le h^{\frac{(k-1)^2+(k-1)-2}{2}}\big(\phi^{(d-1)}(t+h) - \phi^{(d-1)}(t)\big)$ we also obtain

$$m_k(E_{d-k}) \le h^{\frac{k^2+k-2}{2}}\big(\phi^{(d-1)}(t+h) - \phi^{(d-1)}(t)\big)$$

and the claim is proved.

Finally, if we apply the claim for $k = d$ and note that $\lambda(E_0) \ge h$, (1.7) yields the conclusion of the lemma. $\square$

Proof of Lemma 5.2. We begin by noting an inequality for the Vandermonde determinant (2.1), namely, if $\delta > 0$ and $t_1 < \cdots < t_n$ then (with $u = (u_1, \dots, u_{n-1})$)

$$\int_{t_1}^{t_2}\int_{t_2}^{t_3}\cdots\int_{t_{n-1}}^{t_n} V_{n-1}(u)(u_{n-1}-u_1)^\delta du_{n-1}\cdots du_1 \\ \ge C(n)\, V_n(t_1,\dots,t_n)\,(t_n-t_1)^\delta. \tag{6.4}$$

To see (6.4), we observe that the left hand side is bounded below by

$$\int_{t_1}^{(t_1+t_2)/2}\int_{t_2}^{t_3}\cdots\int_{t_{n-2}}^{t_{n-1}}\int_{(t_{n-1}+t_n)/2}^{t_n} V_{n-1}(u)(u_{n-1}-u_1)^\delta du \\ \ge \Big(\frac{t_n-t_1}{2}\Big)^\delta \int_{t_1}^{(t_1+t_2)/2}\int_{t_2}^{t_3}\cdots\int_{t_{n-2}}^{t_{n-1}}\int_{(t_{n-1}+t_n)/2}^{t_n} V_{n-1}(u)\,du.$$

Now we also use (2.14), and together with the estimate

$$\int_{t_1}^{(t_1+t_2)/2}\int_{(t_{n-1}+t_n)/2}^{t_n}(u_{n-1}-u_1)\prod_{j=2}^{n-2}[(u_j-u_1)(u_{n-1}-u_j)]\,du_{n-1}\,du_1 \\ \ge \frac{1}{4}\int_{t_1}^{t_2}\int_{t_{n-1}}^{t_n}(u_{n-1}-u_1)\prod_{j=2}^{n-2}[(u_j-u_1)(u_{n-1}-u_j)]\,du_{n-1}\,du_1,$$

this yields (6.4).

Now assume that the inequality (5.7) holds if $a < s < t < b$ and let $\mathcal{J}_k(t_1,\dots,t_k;\phi^{(d-k)})$ be defined as in Lemma (2.3), *i.e.*, as the determinant of the $k\times k$ matrix with columns $(1, t_j, \dots, \frac{t_j^{k-2}}{(k-2)!}, \phi^{(d-k+1}(t_j))^T$. We will show that if $2 \le k \le d$ and $a < t_1 < \cdots < t_k < b$, then

$$\mathcal{J}_k(t_1\dots,t_k;\phi^{(d-k)}) \ge c(k)\,c\,V_k(t_1,\dots,t_k)\,(t_k-t_1)^{\rho-1}. \tag{6.5}$$

By choosing $\{t_j\}$ to be a nondecreasing rearrangement of $\{s+h_j\}$, the case $k = d$ of (6.5) will imply Lemma 5.2. If $k = 2$ then (6.5) follows immediately from (5.7). So, proceeding by induction, assume that (6.5) holds for $k-1$. By (2.11)

$$\mathcal{J}_k((t_1,\dots,t_k;\phi^{(d-k)}) = \int_{t_1}^{t_2}\cdots\int_{t_{k-1}}^{t_k} \mathcal{J}_{k-1}(\sigma_1,\dots,\sigma_{k-1};\phi^{(d-k+1)})\,d\sigma_{k-1}\cdots d\sigma_1.$$

By our inductive assumption this exceeds

$$c(k-1)\,c\int_{t_1}^{t_2}\cdots\int_{t_{k-1}}^{t_k} V_{k-1}(\sigma_1,\dots,\sigma_{k-1})(\sigma_{k-1}-\sigma_1)^{\rho-1}d\sigma_{k-1}\cdots d\sigma_1$$

and so (6.4) gives (6.5), completing the proof of Lemma 5.2. □

7. Further results

In this section we gather some results about Fourier restriction with respect to Euclidean arclength measure on curves, mainly focusing on degenerate homogeneous curves. For related arguments see [12], [13], [8].

7.1. Homogeneous curves. The following result follows by rescaling techniques from the result in [2] on nondegenerate curves (analogous to (1.1)). Let

$$\gamma(t) = (t^{a_1}, t^{a_2}, \dots, t^{a_d}) \tag{7.1}$$

where $d \ge 3$, and $-\infty < a_1 < a_2 < \cdots < a_d < \infty$, and $a_i \ne 0$, $i = 1,\dots,d$. We let $\mathcal{R}$ be the Fourier restriction operator, setting $\mathcal{R}f(t) = \widehat{f}(\gamma(t))$. Let

$$D = a_1 + a_2 + \cdots + a_d$$

be the "homogeneous" dimension and assume $D > d(d+1)/2$.

Proposition 7.1. *Let $p_d = \frac{d^2+d+2}{d^2+d}$ and γ as in (7.1). Then $\mathcal{R}$ is of restricted weak type $(p_d, p_d'/D)$,*

$$\big\|\mathcal{R}f\big\|_{L^{p_d'/D,\infty}(dt)} \le C(a_1,\dots,a_n)\|f\|_{L^{p_d,1}}. \tag{7.2}$$

Proof. Define

$$(\mathcal{R}_k f)(t) = \widehat{f}(\gamma(t))\chi_{I_k}(t)$$

where $I_k = [2^{-k-1}, 2^{-k}]$. We may use the nonisotropic dilations adapted to the curve to rescale the result in the nondegenerate case (Theorem 1.1 in [2]); we obtain

$$\|\mathcal{R}_k f\|_{L^{p_d}(dt)} \le C 2^{k[(D+1)(1-\frac{1}{p_d})-1]}\|f\|_{L^{p_d,1}(\mathbb{R}^d)}. \tag{7.3}$$

Let $D_0 = d(d+1)/2$ and fix $0 < q_0 < p_d'/D$. Since $1/q_0 > D/p_d' > D_0/p_d' = 1/p_d$, by Hölder's inequality the last estimate implies

$$\|\mathcal{R}_k f\|_{L^{q_0}(dt)} \le C 2^{-k[\frac{1}{q_0}-D(1-\frac{1}{p_d})]}\|f\|_{L^{p_d,1}(\mathbb{R}^d)}. \tag{7.4}$$

Since $(D+1)/(p_d')-1 > (D_0+1)/(p_d')-1 = 0$, an application of Bourgain's interpolation lemma to (7.3) and (7.4) gives the assertion. □

7.2. An improvement. For some very specific classes we can improve the second Lorentz exponent on the left hand side of (7.2).

We now suppose the stronger restricted strong type estimate

$$\|\mathcal{R}f\|_{L^{p_d}(wdt)} \le C\|f\|_{p_d,1} \tag{7.5}$$

where wdt is affine arclength measure. *Assume* that

$$1/w \in L^{s,\infty}(dt) \tag{7.6}$$

for some $s \in (0,\infty)$. Define q by

$$\frac{1}{q} = \frac{1}{p_d} + \frac{1}{sp_d}. \tag{7.7}$$

Then as in [8] one can use the Lorentz space multiplication theorem (Theorem 4.5 in [10]), and it follows that

$$\|\mathcal{R}f\|_{L^{q,p_d}(dt)} = \|(\mathcal{R}f)w^{1/p_d}\cdot w^{-1/p_d}\|_{L^{q,p_d}(dt)}$$
$$\le C\|(\mathcal{R}f)w^{1/p_d}\|_{L^{p_d}(dt)}\,\|w^{-1/p_d}\|_{L^{sp_d,\infty}(dt)} = C\|w^{-1}\|^{1/p_d}_{L^{s,\infty}(dt)}\,\|\mathcal{R}f\|_{L^{p_d}(wdt)}.$$

Hence (7.5) and (7.6) imply that for q as in (7.7)

$$\|\mathcal{R}f\|_{L^{q,p_d}(dt)} \le C\|f\|_{p_d,1}. \tag{7.8}$$

Corollary 7.2. *Let $\gamma(t) = (t, t^\alpha, t^{5\alpha-1})$ with $\alpha > 1$. Then $\mathcal{R}$ maps $L^{7/6,1}$ boundedly to $L^{7/(6\alpha),7/6}$.*

Proof. Note that $D = 6\alpha > 6 = D_0$. Also one computes $w(t) = c(\alpha)t^{\alpha-1}$ with $c(\alpha) \ne 0$ so that $w^{-1} \in L^{s,\infty}$ for $s = 1/(\alpha-1)$. By Theorem 1.4 in [2] it follows that (7.5) holds with $p_3 = 7/6$, so that the assertion follows. □

7.3. $L^p \to L^q$ bounds. Finally, let us suppose that, instead of (7.5), the estimate

$$\|\mathcal{R}f\|_{L^Q(wdt)} \le C\|f\|_p \tag{7.9}$$

holds for $1/p + 1/(D_0Q) = 1$, and $1/w \in L^{s,\infty}(dt)$ with $1 < p < p_d$ and some $s \in (0,\infty)$. Then an argument similar to the one given above together with an interpolation show that

$$\|\mathcal{R}f\|_{L^{q,p}(dt)} \le C\|f\|_p$$

for $1 < p < p_d$ and

$$\frac{1}{p} + \frac{s}{(s+1)D_0q} = 1.$$

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