VARIATION BOUNDS FOR SPHERICAL AVERAGES

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Abstract. We consider r-variation operators for the family of spherical means, with special emphasis on $L^p \to L^q$ estimates.

1. Introduction

Given a subset $E \subset \mathbb{R}$ and a family of complex valued functions $t \mapsto a_t$ defined on E, the r-variation of $a = \{a_t\}_{t\in E}$ is defined by

$$|a|_{V_r(E)} := \sup_{N\in\mathbb{N}} \sup_{\substack{t_1<\cdots<t_N\\ t_j\in E}} \Big(\sum_{j=1}^{N-1} |a_{t_{j+1}} - a_{t_j}|^r\Big)^{1/r}$$

for all $1 \le r < \infty$, and replacing the ℓ^r-sum by a sup in the case $r = \infty$. When $E = \mathbb{R}$ we simply use the notation V_r for $V_r(\mathbb{R})$. A norm on the space $V_r(E)$ is given by $\|a\|_{V_r(E)} := \|a\|_\infty + |a|_{V_r(E)}$. Variation norms have received considerable attention in analysis as they are used to strengthen pointwise convergence results for families of operators $\{A_t\}$. Of particular interest is Lépingle's inequality on the r-variation of martingales for $r > 2$ [29] (see also [33], [8], [21], [31]) and its consequences on families of operators in ergodic theory and harmonic analysis; see e.g. the papers [20], [21], [32], [30], [16] which contain many other references.

In this paper we focus on local and global r-variation estimates for the family of spherical averages $A = \{A_t\}_{t>0}$, given by

$$A_t f(x) = \int_{S^{d-1}} f(x - ty)\, \mathrm{d}\sigma(y)$$

where $\mathrm{d}\sigma$ denotes the normalized surface measure on the unit sphere S^{d-1}. By a classical result of Stein [43] ($d \ge 3$) and Bourgain [7] ($d = 2$) the spherical maximal function $Sf(x) := \sup_{t>0} |A_t f(x)|$ defines a bounded operator on $L^p(\mathbb{R}^d)$ if and only if $p > \frac{d}{d-1}$. Thus, for p in this range, we have $\lim_{t\to 0} A_t f(x) = f(x)$ a.e. for all $f \in L^p(\mathbb{R}^d)$. A strengthening of this result can be obtained by considering the variation norm operator $V_r A$ given by

$$V_r Af(x) \equiv V_r[Af](x) := |Af(x)|_{V_r((0,\infty))};$$

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note that $V_r[Af](x) \ge \sup_t |A_t f(x) - A_{t_0} f(x)|$ for all $x \in \mathbb{R}^d$, $t_0 \in \mathbb{R}$. In this context, Jones, Wright and one of the authors [21] obtained an almost optimal result, namely $V_r A$ is bounded on $L^p(\mathbb{R}^d)$ for all $r > 2$ if $\frac{d}{d-1} < p \le 2d$, and both the condition $r > 2$ and the p-range are sharp. In the range $p > 2d$, it was shown in [21] that $V_r A$ is L^p bounded if $r > p/d$, and fails to be bounded if $r < p/d$, but no information was known for the critical case $r = p/d$, $p > 2d$. Here we show an endpoint result for $V_{p/d}A$ in three and higher dimensions.

Theorem 1.1. *Let $d \ge 3$, $p > 2d$. Then the operator $V_{p/d}A$ is of restricted weak type (p,p), i.e. maps $L^{p,1}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$.*

We conjecture that a similar endpoint result holds true in two dimensions, but this remains open.

Our main focus will be on $L^p \to L^q$ results when $p < q$ for local r-variation operators, that is, when the variation is taken over a compact subinterval I of $(0,\infty)$; without loss of generality we take $I = [1,2]$. Scaling reasons quickly reveal that one needs to consider compact intervals for $L^p \to L^q$ bounds to hold if $p < q$. While this is an interesting problem in itself, it is also motivated by a question posed by Lacey [23] concerning sparse domination for the global $V_r A$ operator (see also [1, Problem 3.1]). See Theorem 1.7 below.

Results for the local variation operators are meant to improve on existing $L^p \to L^q$ results for the spherical local maximal function $S^I f(x) := \sup_{1\le t\le 2} A_t f(x)$, which we will now review. Schlag [37] (see also [38]) showed that if $d \ge 2$ there are $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds if $(1/p, 1/q)$ lies in the interior of $\mathfrak{Q}_d$, which denotes the quadrangle formed by the vertices

$$\begin{aligned} Q_1 &= (0,0), & Q_2 &= (\tfrac{d-1}{d}, \tfrac{d-1}{d}), \\ Q_3 &= (\tfrac{d-1}{d}, \tfrac{1}{d}), & Q_4 &= (\tfrac{d(d-1)}{d^2+1}, \tfrac{d-1}{d^2+1}). \end{aligned} \tag{1.1}$$

Moreover, S^I fails to be bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ outside the closure of $\mathfrak{Q}_d$. Note that Q_2 coincides with Q_3 when $d = 2$, so the quadrangle becomes a triangle in two dimensions.

The boundary segment $p = q$ amounts to the classical results of Stein and Bourgain for S. L^p-boundedness fails at the endpoint Q_2 but Bourgain showed in dimensions $d \ge 3$ that S is of restricted weak type at Q_2, i.e. bounded from $L^{\frac{d}{d-1},1}$ to $L^{\frac{d}{d-1},\infty}$ in dimensions $d \ge 3$ (and any better Lorentz estimate fails). The restricted weak type estimate at Q_2 fails in two dimensions [40] (even though it is true for radial functions [24]). For the remaining boundary cases Lee [25] showed that S^I is of restricted weak type at Q_4, and also at Q_3 in dimensions $d \ge 3$. The two-dimensional restricted weak type endpoint result at Q_4 was also shown in [25], and relied on the deep work by Tao [45] on endpoint bilinear Fourier extension bounds for the cone. The restricted weak type inequalities imply $L^p \to L^q$ boundedness on $[Q_1, Q_4)$ and on (Q_3, Q_4), however on (Q_2, Q_3) the operator is of restricted strong type and no better (the necessity follows from the standard counterexample; for the positive result one uses real interpolation on a vertical line, with a constant target exponent). Incidentally, for the local operator S^I this also implies restricted strong type at Q_2, which improves over the restricted weak type of S at Q_2.

Here we explore the existence of $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ inequalities for

$$V_r^I Af(x) := |Af(x)|_{V_r([1,2])}.$$

In two dimensions the values of r are restricted to $r > 2$ (see §3) but in higher dimensions all $r \in [1, \infty]$ may occur. For our sparse domination inequality for the global V_r, the version for $r > 2$ is most relevant because Lépingle's result requires the restriction $r > 2$ (see [35]); indeed this necessary condition can be shown to carry over to other results for the global V_r.

We start stating our results for $d \geq 3$. We first focus on the range $r > \frac{d^2+1}{d(d-1)}$ which is the reciprocal of the $1/p$ coordinate of the point Q_4 in (1.1). Note that this large range includes $r > 2$, so the following sharp $L^p \to L^q$ results for $V_r^I A$ will yield, in particular, satisfactory results for the sparse domination problem in dimension $d \geq 3$.

Theorem 1.2. *Suppose $d \geq 3$ and $r > \frac{d^2+1}{d(d-1)}$. Let $\mathfrak{P}_d(r)$ be the pentagon (Figure 1) with vertices*

$$P(r) = (\tfrac{1}{r}, \tfrac{1}{rd}), \quad Q_1(r) = (\tfrac{1}{rd}, \tfrac{1}{rd}), \quad Q_2 = (\tfrac{d-1}{d}, \tfrac{d-1}{d})$$
$$Q_3 = (\tfrac{d-1}{d}, \tfrac{1}{d}), \quad Q_4 = (\tfrac{d(d-1)}{d^2+1}, \tfrac{d-1}{d^2+1}).$$

Then

(i) $V_r^I A : L^p \to L^q$ is bounded for all $(\frac{1}{p}, \frac{1}{q})$ in the interior of $\mathfrak{P}_d(r)$ and unbounded for all $(\frac{1}{p}, \frac{1}{q}) \notin \mathfrak{P}_d(r)$.

(ii) $V_r^I A : L^p \to L^q$ is bounded for all $(\frac{1}{p}, \frac{1}{q})$ on the half open line segment $[Q_1(r), Q_2)$, on the closed line segment $[P(r), Q_1(r)]$, on the half open line segment $[P(r), Q_4)$, and on the open line segment (Q_4, Q_3).

(iii) $V_r^I A : L^{p,1} \to L^q$ is bounded (i.e. of restricted strong type (p, q)) if $(\frac{1}{p}, \frac{1}{q})$ belongs to the half open line segment $[Q_2, Q_3)$. $V_r^I A$ fails to be of strong type on $[Q_2, Q_3]$.

(iv) $V_r^I A : L^{p,1} \to L^{q,\infty}$ is bounded (i.e. of restricted weak type (p, q)) if $(\frac{1}{p}, \frac{1}{q}) \in \{Q_3, Q_4\}$.

For an explicit description of the various conditions at the boundary see §3.1.

We leave open what exactly happens at the points Q_3 and Q_4; it is not even known whether the local maximal function is of restricted strong type at Q_3 and whether it is any better than restricted weak type at Q_4. If we take $r = \infty$ we recover the known theorem for the local spherical maximal operator. Note that both $P(r)$ and $Q_1(r)$ tend to $Q_1 = (0, 0)$ as $r \to \infty$.

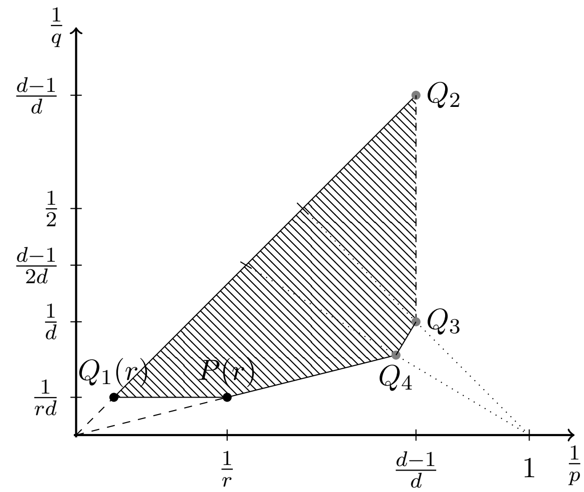

FIGURE 1. The pentagon $\mathfrak{P}_d(r)$ for $r > \frac{d^2+1}{d^2-d}$ and $d \geq 3$ (Theorem 1.2). The outer (dashed) quadrangle shows the region of boundedness as $r \to \infty$, i.e. for the maximal operator. Shown with $d = 4$ and $r = 3$.

Theorem 1.2 covers an interesting consequence for a sharp strong type estimate at the lower edge $q^{-1} = p^{-1}/d$ of the type set for the maximal function.

Corollary 1.3. *Let $d \geq 3$ and let $\frac{d^2+1}{d(d-1)} < p < \infty$. Then $V_r^I A : L^p \to L^{pd}$ is bounded if and only if $r \geq p$.*

When the value of r is between the reciprocal of the $1/p$ coordinate of Q_4 and Q_3, that is, $\frac{d}{d-1} < r \leq \frac{d^2+1}{d(d-1)}$, we obtain the following.

Theorem 1.4. *Suppose $d \geq 3$ and $\frac{d}{d-1} < r \leq \frac{d^2+1}{d(d-1)}$. Let $\mathfrak{P}_d(r)$ be the pentagon (Figure 2) with vertices*

$$Q_1(r) = \left(\tfrac{1}{rd}, \tfrac{1}{rd}\right), \quad Q_2 = \left(\tfrac{d-1}{d}, \tfrac{d-1}{d}\right), \quad Q_3 = \left(\tfrac{d-1}{d}, \tfrac{1}{d}\right)$$
$$P(r) = \left(\tfrac{1}{r}, \tfrac{d+1-r(d-1)}{r(d-1)}\right), \quad Q_4(r) = \left(1 - \tfrac{d+1}{rd(d-1)}, \tfrac{1}{rd}\right).$$

Then

(i) $V_r^I A : L^p \to L^q$ is bounded for $(\frac{1}{p}, \frac{1}{q})$ in the interior of $\mathfrak{P}_d(r)$ and unbounded for $(\frac{1}{p}, \frac{1}{q}) \notin \mathfrak{P}_d(r)$.

(ii) $V_r^I A : L^p \to L^q$ is bounded for $(\frac{1}{p}, \frac{1}{q})$ on the half open line segment $(Q_4(r), Q_1(r)]$ and on the half open line segment $[Q_1(r), Q_2)$.

(iii) $V_r^I A$ is of restricted strong type (p, q) if $(\frac{1}{p}, \frac{1}{q})$ belongs to the half open line segment $[Q_2, Q_3)$. $V_r^I A$ fails to be of strong type on $[Q_2, Q_3]$.

(iv) $V_r^I A$ is of restricted weak type (p, q) if $(\frac{1}{p}, \frac{1}{q}) = Q_3$.

Note that for $r = \frac{d^2+1}{d(d-1)}$ the pentagon $\mathfrak{P}_d(r)$ in Figure 2 degenerates to a quadrangle, as $P(r) = Q_4(r) = Q_4$. We leave open what happens at the closed boundary segment $[Q_4(r), P(r)]$ and the half-open boundary segment $[P(r), Q_3)$.

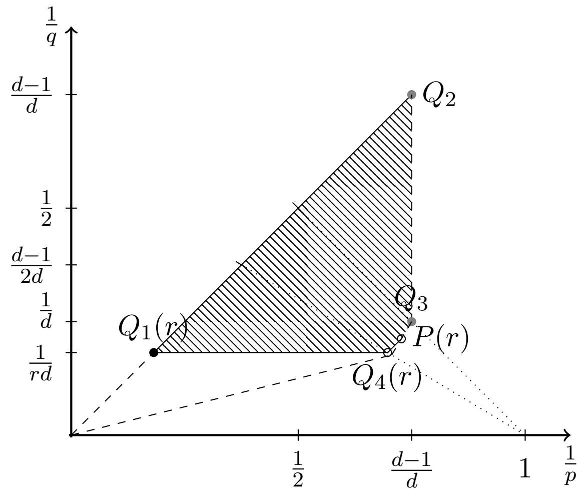

FIGURE 2. The pentagon $\mathfrak{P}_d(r)$ for $\frac{d}{d-1} < r \leq \frac{d^2+1}{d^2-d}$ and $d \geq 3$ (Theorem 1.4). The outer (dashed) quadrangle is the region of boundedness for the maximal operator. Shown with $d = 4$ and $r = \frac{11}{8}$.

Finally, we address small values of r.

Theorem 1.5. *Suppose that either $d \geq 4$ and $1 \leq r \leq \frac{d}{d-1}$ or $d = 3$ and $\frac{4}{3} < r \leq \frac{3}{2}$. Let $\mathfrak{Q}_d(r)$ be the quadrangle (Figure 3) with vertices*

$$Q_1(r) = \left(\tfrac{1}{rd}, \tfrac{1}{rd}\right), \quad Q_2(r) = \left(\tfrac{r(d-1)-1}{r(d-1)}, \tfrac{r(d-1)-1}{r(d-1)}\right),$$

$$Q_3(r) = \left(\tfrac{r(d-1)-1}{r(d-1)}, \tfrac{1}{r(d-1)}\right), \quad Q_4(r) = \left(1 - \tfrac{d+1}{rd(d-1)}, \tfrac{1}{rd}\right).$$

Then

(i) $V_r^I A : L^p \to L^q$ is bounded for $(\frac{1}{p}, \frac{1}{q})$ in the interior of $\mathfrak{Q}_d(r)$ and unbounded for $(\frac{1}{p}, \frac{1}{q}) \notin \mathfrak{Q}_d(r)$.

(ii) $V_r^I A : L^p \to L^q$ is bounded if $(\frac{1}{p}, \frac{1}{q})$ is in the half open line segment $(Q_4(r), Q_1(r)]$ and $[Q_1(r), Q_2(r))$.

(iii) For the case $r = 1$, $d \geq 4$, the operator $V_1^I A$ is of restricted weak type $(\frac{d-1}{d-2}, d-1)$ (that is, at $Q_3(1)$) and of restricted strong type $(\frac{d-1}{d-2}, q)$ for $\frac{d-1}{d-2} \leq q < d-1$ (that is, on $[Q_2(1), Q_3(1))$). In three dimensions, $V_1^I A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is bounded.

We leave open what happens at the closed boundary segments $[Q_2(r), Q_3(r)]$ for $1 < r \leq \frac{d}{d-1}$ and $[Q_3(r), Q_4(r)]$ for $1 \leq r \leq \frac{d}{d-1}$.

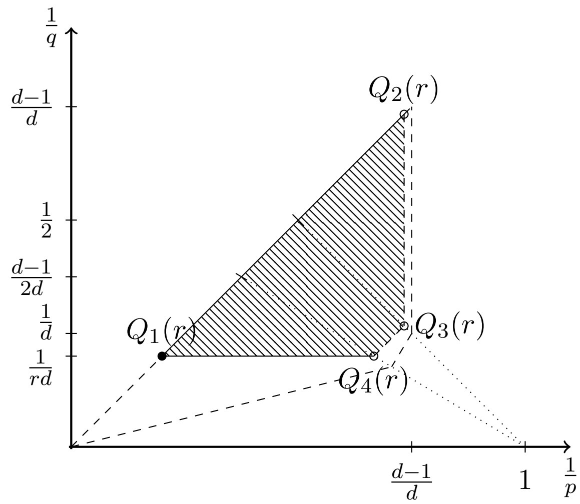

FIGURE 3. The quadrangle $\mathfrak{Q}_d(r)$ for $1 \le r \le \frac{d}{d-1}$ and $d \ge 4$ (Theorem 1.5). The outer (dashed) quadrangle is the boundedness region for the maximal function. Shown with $d = 4$ and $r = \frac{5}{4}$.

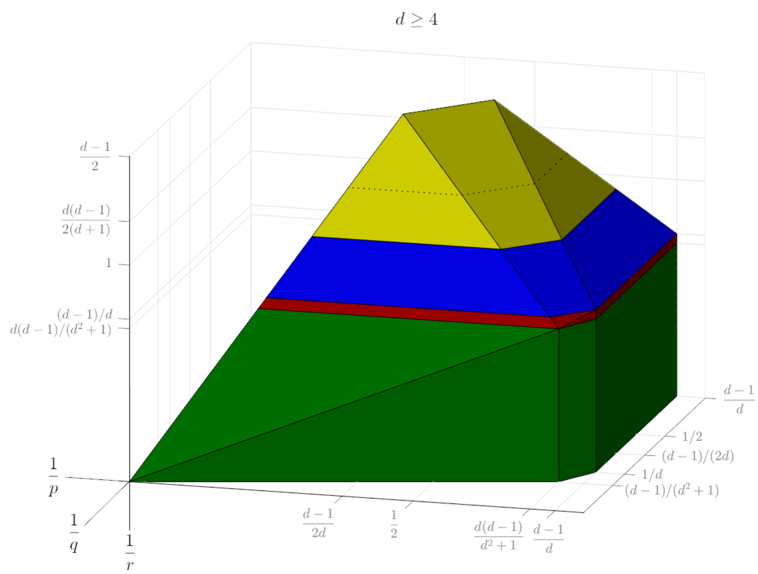

FIGURE 4. A diagram of the typeset of $V_r^I A$ in $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$-space for large values of d. The green region corresponds to Theorem 1.2 (Figure 1), the red region corresponds to Theorem 1.4 (Figure 2), and the blue region corresponds to Theorem 1.5 (Figure 3). The yellow region is conjectural.

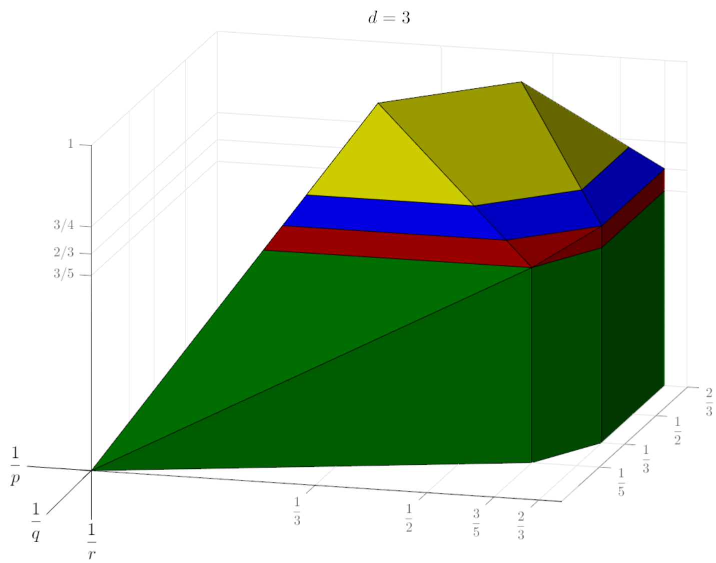

FIGURE 5. A diagram of the typeset of $V_r^I A$ in $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$-space for $d = 3$. The green region corresponds to Theorem 1.2 (Figure 1), the red region corresponds to Theorem 1.4 (Figure 2), and the blue region corresponds to Theorem 1.5 (Figure 3). The yellow region is conjectural.

Note that there is a discrepancy in our results between $d = 3$, for which we only obtain sharp results in the partial range $\frac{4}{3} < r \leq \frac{d}{d-1}$ and the case $d \geq 4$, where results are obtained for all $1 \leq r \leq \frac{d}{d-1}$. The reason is because we restrict ourselves to the traditional range $1 \leq r \leq \infty$ for the variation norm. The definition of V_r can be extended, with modifications, to the range $0 < r < 1$ (see for example [4]). In that context, one can formulate conjectural results for $V_r^I A$ for $\frac{2}{d-1} < r < 1$ (see Figure 4) for $d \geq 4$. We remark that a positive solution to Sogge's local smoothing conjecture [41] in $d + 1$ dimensions would imply a complete result up to endpoints. Partial results in the range $r > \frac{2(d+1)}{d(d-1)}$ can be proved using the techniques of this paper. We shall address issues for $r < 1$ in a follow up paper.

Similarly, in three dimensions, the range $1 \leq r \leq 4/3$ remains open as a conjecture (see Figure 5). Note that here we are in the traditional range for the V_r spaces.

In dimension 2, due to the recent full resolution of Sogge's problem in $2 + 1$ dimensions by Guth, Wang and Zhang [17], that is,

$$\partial_t^{1/2-\varepsilon} A : L^4 \to L^4(L^4),$$

it is possible to get an almost optimal result (up to endpoints) for the variation norm estimates.

Theorem 1.6. *Let $d = 2$.*

(i) If $r > 5/2$ then $V_r^I A : L^p \to L^q$ is bounded if $(\frac{1}{p}, \frac{1}{q})$ is either in the interior of the quadrangle $\mathfrak{Q}_2(r)$ (Figure 6) formed by the vertices

$$P(r) = (\tfrac{1}{r}, \tfrac{1}{2r}), \quad Q_1(r) = (\tfrac{1}{2r}, \tfrac{1}{2r}),$$
$$Q_2 = Q_3 = (\tfrac{1}{2}, \tfrac{1}{2}), \quad Q_4 = (\tfrac{2}{5}, \tfrac{1}{5})$$

or in the open line segment between $Q_2 = Q_3$ and $Q_1(r)$.

(ii) If $2 < r \le 5/2$ then $V_r^I A : L^p \to L^q$ is bounded if $(\frac{1}{p}, \frac{1}{q})$ is either in the interior of the quadrangle $\mathfrak{Q}_2(r)$ (Figure 7) formed by the vertices

$$Q_1(r) = \big(\tfrac{1}{2r}, \tfrac{1}{2r}\big), \quad Q_2 = Q_3 = \big(\tfrac{1}{2}, \tfrac{1}{2}\big),$$
$$P(r) = (\tfrac{1}{r}, \tfrac{3-r}{r}), \quad Q_4(r) = (1 - \tfrac{3}{2r}, \tfrac{1}{2r})$$

or in the open line segment between $Q_2 = Q_3$ and $Q_1(r)$.

(iii) If $r < 2$ then $V_r^I A$ does not map any $L^p(\mathbb{R}^2)$ to any $L^q(\mathbb{R}^2)$.

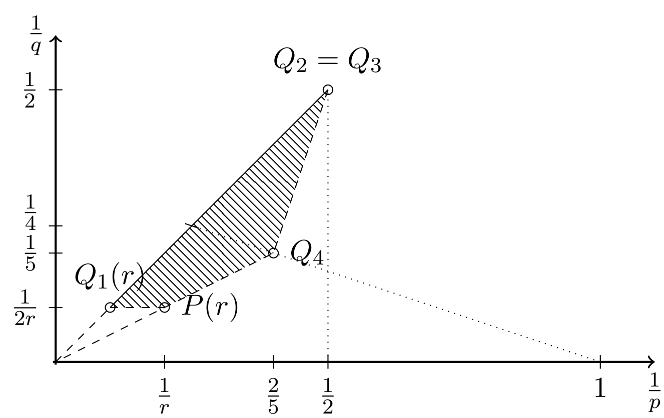

FIGURE 6. The region $\mathfrak{Q}_2(r)$ if $r > 5/2$ (Theorem 1.6 i). The outer (dashed) triangle is the region of boundedness for the maximal operator. Shown with $r = 5$.

Note that, as for the circular maximal function theorem, the points Q_2 and Q_3 coincide if $d = 2$; therefore the pentagon (Figures 1 and 2) in Theorems 1.2 and 1.4 becomes a quadrangle for $r > 2$. Moreover, $P(5/2) = Q_4(5/2) = Q_4$, so the quadrangle becomes a triangle for $r = 5/2$. The bounds are subsumed in Figure 8; note that in contrast with $d \ge 3$, the blue/yellow region disappears, as $\frac{d}{d-1} = \frac{2}{d-1}$ coincide for $d = 2$.

It is also possible to show unboundedness for $r = 2$ via an argument involving the Besicovitch set, which will be addressed in a forthcoming paper.

We note that an affirmative answer to *endpoint versions* of Sogge's problem as formulated and conjectured in [18] would also settle strong type bounds on the half-open boundary segment $(Q_4, Q_1(r)]$. Unfortunately such endpoint bounds in Sogge's problem are currently only available in dimensions four and higher.

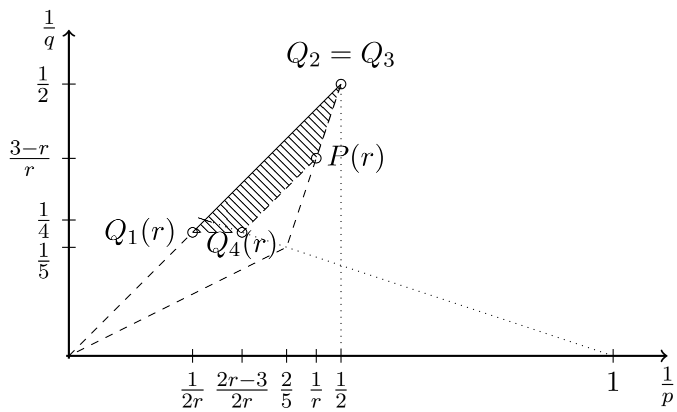

FIGURE 7. The region $\mathfrak{Q}_2(r)$ if $d = 2$ and $2 < r \leq 5/2$ (Theorem 1.6 ii). The outer (dashed) triangle is the region of boundedness for the maximal operator. Shown with $r = 2.2$.

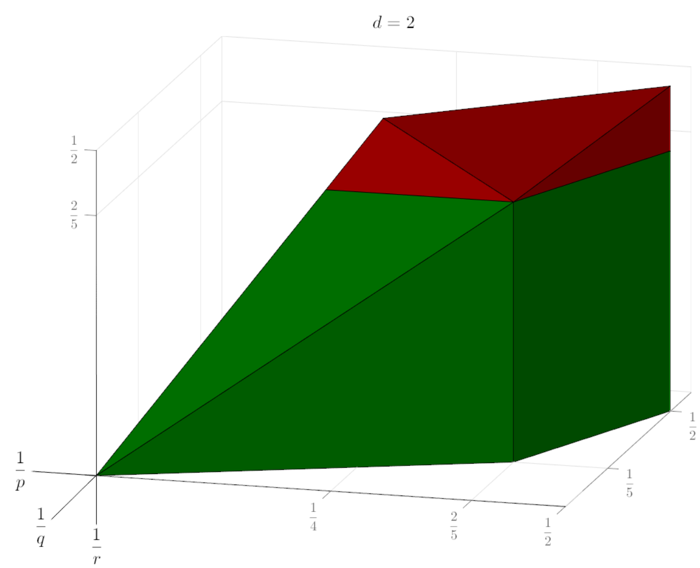

FIGURE 8. A diagram of the typeset of $V_r^I A$ in $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$-space for $d = 2$. The green region corresponds to Figure 6 and the red region corresponds to Figure 7.

Sparse domination. We now formulate a sparse domination result for the global operator V_rA, $r>2$. Recall that a family of cubes $\mathfrak{S}$ in $\mathbb{R}^d$ is called *sparse* if for every $Q\in\mathfrak{S}$ there is a measurable subset $E_Q\subset Q$ such that $|E_Q|\ge |Q|/2$ and such that the sets on the family $\{E_Q : Q\in\mathfrak{S}\}$ are pairwise disjoint. In what follows we abbreviate $\langle f\rangle_{Q,s}=(|Q|^{-1}\int_Q |f|^s)^{1/s}$.

Theorem 1.7. *Assume one of the following holds:*

(i) $d\ge 3$, $r>2$, *and* $(\frac1p,\frac1q)$ *in the interior of* $\mathfrak{P}_d(r)$.

(ii) $d=2$, $r>2$ *and* $(\frac1p,\frac1q)$ *in the interior of* $\mathfrak{Q}_2(r)$.

Then there is a constant $C=C(p,q)$ *such that for each pair of compactly supported bounded functions* f_1, f_2 *there is a sparse family of cubes* $\mathfrak{S}$ *such that*

$$\int_{\mathbb{R}^d} V_rAf_1(x)f_2(x)\,\mathrm{d}x\le C\sum_{Q\in\mathfrak{S}}|Q|\langle f_1\rangle_{Q,p}\langle f_2\rangle_{Q,q'}, \tag{1.2}$$

where $\frac1q+\frac1{q'}=1$. *Furthermore, the* $(1/p,1/q)$ *range is sharp up to endpoints in the sense that no such result can hold if* $(1/p,1/q)$ *does not lie in the closure of* $\mathfrak{P}_d(r)$, *or* $\mathfrak{Q}_2(r)$, *respectively.*

Theorem 1.7 can be obtained as an immediate consequence of a (more general) sparse domination result in [2], together with the L^p results in [21] and Theorems 1.2 and 1.6; see §3.9 and §8. Sparse domination is known to imply as a corollary a number of weighted inequalities in the context of Muckenhoupt and reverse Hölder classes. We refer the interested reader to [5] for the weighted consequences for V_rA of Theorem 1.7.

Structure of the paper. We start gathering some well known facts about spherical averages and function spaces in §2. In §3 we provide the examples showing the necessary conditions for our theorems. In §4 we exploit a single frequency analysis to deduce the claimed bounds in the interior of the regions, as well as some restricted weak and strong type endpoints, in Theorems 1.2, 1.4, 1.5 and 1.6. The proof of the harder off-diagonal strong type boundary results in those theorems, and therefore Corollary 1.3, is provided in §§5-6. In §7 we prove the restricted weak type inequality for the global operator in Theorem 1.1. Finally, the sparse domination result is discussed in §8.

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2. Preliminaries

It will be convenient to consider the t-parameter as a variable. To this end, let $\chi \in C_c^\infty(\mathbb{R})$ so that $\chi(t) = 1$ for t in a neighborhood of $[1, 2]$ and supported in $[1/2, 4]$, and define

$$\mathcal{A}f(x,t) := \chi(t)A_t f(x). \tag{2.1}$$

In view of future frequency decompositions, let $\beta_0 \in C_c^\infty(\mathbb{R})$ so that $\beta_0(s) = 1$ for $|s| < 1/2$ and $\beta_0(s) = 0$ for $|s| > 1$. For every integer $j \geq 1$, set

$$\beta_j(s) = \beta_0(2^{-j}s) - \beta_0(2^{1-j}s).$$

For functions g on $\mathbb{R}$, and $l \in \mathbb{N}_0$, define the operators Λ_l by

$$\widehat{\Lambda_l g}(\tau) = \beta_l(\tau)\widehat{g}(\tau). \tag{2.2}$$

For functions f on $\mathbb{R}^d$, and $j \in \mathbb{N}_0$, define the operators L_j by

$$\widehat{L_j f}(\xi) = \beta_j(|\xi|)\widehat{f}(\xi), \tag{2.3}$$

and let $\widetilde{L}_j$ be a modification of L_j satisfying $\widetilde{L}_j L_j = L_j$.

2.1. *V_r and related function spaces.* It will be convenient to work with the Besov space $B^{1/r}_{r,1}$. The Besov spaces $B^s_{p,q}(\mathbb{R})$ can be defined using the dyadic frequency decompositions $\{\Lambda_l\}_{l=0}^\infty$ on the real line and we have $\|u\|_{B^s_{p,q}} = (\sum_{l=0}^\infty [2^{ls}\|\Lambda_l u\|_p]^q)^{1/q}$. From the Plancherel–Polya inequality we know the embedding

$$B^{1/r}_{r,1} \hookrightarrow V_r \hookrightarrow B^{1/r}_{r,\infty}, \tag{2.4}$$

see [46, Ch.1]. One can also consult the paper by Bergh and Peetre [4] (who however work with a different type of variation space when $r = 1$) or refer to [16, Proposition 2.2]. Thus an inequality for the variation operator $V_r^I \mathcal{A}$ follows if we can control the $B^{1/r}_{r,1}$ norm of $t \mapsto \mathcal{A}f(x,t)$.

Note that, by our definition, $V_1(\mathbb{R})$ coincides with the space of bounded functions of bounded variations. The fundamental theorem of calculus implies

$$\|V_1^E A\|_{L^p \to L^q} \leq \|\partial_t \mathcal{A}\|_{L^p \to L^q(L^1(E))}, \tag{2.5}$$

so we shall focus on obtaining bounds for the right-hand side when studying $V_1^E A$.

2.2. *Frequency decomposition in space.* Given $j \geq 0$, write

$$A_t L_j f = K_{j,t} * f, \tag{2.6}$$

where L_j is as in (2.3), so that $\widehat{K_{j,t}}(\xi) = \widehat{\sigma}(t\xi)\beta_j(|\xi|)$. Note that $K_{j,t}$ is a Schwartz convolution kernel and therefore we restrict our attention to the case $j \geq 1$.

An immediate computation yields the following pointwise estimates for the convolution kernel.

Lemma 2.1. *For all $N \in \mathbb{N}_0$, there exists a constant $C_N > 0$ such that*

$$|\partial_t^{\varsigma} K_{j,t}(x)| \lesssim_{\varsigma} C_N 2^{j\varsigma} \frac{2^j}{(1 + 2^j\big||x| - t\big|)^N} \tag{2.7}$$

holds for all $x \in \mathbb{R}^d$, all $t > 0$ and all $\varsigma \in \mathbb{N}_0$. Consequently,

$$|K_{j,t}(x)| \lesssim_N (2^j|x|)^{-N} \qquad \text{if} \quad |x| \geq 10, \quad t \in [1/2, 4]. \tag{2.8}$$

In analogy to the definition of $\mathcal{A}$ in (2.1), define

$$\mathcal{A}_j f(x,t) := \chi(t) A_t L_j f(x) = \chi(t) K_{j,t} * f(x).$$

We gather some estimates for $\mathcal{A}_j$ when the inequalities involve L^1 or L^∞ spaces.

First, from the trivial fact that $\|A_t f\|_\infty \lesssim \|f\|_\infty$ uniformly in $t \in \mathbb{R}$, one immediately has

$$\|\mathcal{A}_j f\|_{L^\infty(L^\infty)} \lesssim \|f\|_\infty. \tag{2.9}$$

Moreover, one has the following estimates for L^1 functions.

Lemma 2.2. *For $1 \leq q \leq \infty$,*

$$\|\mathcal{A}_j f\|_{L^q(L^1)} + 2^{-j}\|\partial_t \mathcal{A}_j f\|_{L^q(L^1)} \lesssim \|f\|_1.$$

Proof. By (2.7) one has

$$\big|\mathcal{A}_j f(x,t)\big| + 2^{-j}\big|\partial_t \mathcal{A}_j f(x,t)\big| \lesssim \int_{\mathbb{R}^d} |f(y)| \frac{2^j}{(1 + 2^j||x-y| - t|)^N} \, \mathrm{d}y \tag{2.10}$$

for all $N \in \mathbb{N}_0$. Integrating in t over the support of χ one sees that, for fixed x,

$$\begin{aligned} &\int_{1/2}^4 \big|\mathcal{A}_j f(x,t)\big| \, \mathrm{d}t + 2^{-j} \int_{1/2}^4 \big|\partial_t \mathcal{A}_j f(x,t)\big| \, \mathrm{d}t \\ &\lesssim \int_{\mathbb{R}^d} |f(y)| \int_{1/2}^4 \frac{2^j}{(1 + 2^j||x-y| - t|)^N} \, \mathrm{d}t \, \mathrm{d}y \lesssim \|f\|_1. \end{aligned}$$

This gives the assertion for $q = \infty$.

For $q = 1$, the result follows from integrating in x instead, using the decay in (2.10) and taking into account that the integration in t is over $[1/2, 4]$.

The remaining cases $1 < q < \infty$ follow from combining the above through Young's convolution inequality. $\square$

Corollary 2.3. *For* $1 \le r \le \infty$,

$$\|\mathcal{A}_j f\|_{L^\infty(L^r)} \lesssim 2^{j(1-\frac{1}{r})} \|f\|_1.$$

Proof. Interpolate between

$$\|\mathcal{A}_j\|_{L^\infty(L^\infty)} \lesssim 2^j \|f\|_1,$$

which follows from (2.7), and Lemma 2.2 with $q = \infty$. □

2.3. *Oscillatory integral representation.* Given $m \in \mathbb{R}$, let $S^m(\mathbb{R}^d)$ denote the class of all functions $a \in C^\infty(\mathbb{R}^d)$ satisfying

$$|\partial^\alpha a(\xi)| \lesssim_\alpha (1 + |\xi|)^{m-|\alpha|}$$

for all multiindex $\alpha \in \mathbb{N}_0^d$ and all $\xi \in \mathbb{R}^d$. Given $a \in S^m(\mathbb{R})$, define

$$T_j^{\pm}[a, f](x, t) = \int_{\mathbb{R}^d} \beta_j(|\xi|) a(t|\xi|) e^{i\langle x, \xi\rangle \pm it|\xi|} \widehat{f}(\xi)\, \mathrm{d}\xi. \tag{2.11}$$

It is well known that the Fourier transform of the spherical measure is

$$\widehat{\sigma}(\xi) = (2\pi)^{d/2} |\xi|^{-(d-2)/2} J_{\frac{d-2}{2}}(|\xi|) = b_0(|\xi|) + \sum_{\pm} b_{\pm}(|\xi|) e^{\pm i|\xi|},$$

where $b_0 \in C_c^\infty(\mathbb{R})$ is supported in $\{|\xi| \le 1\}$ and $b_\pm \in S^{-(d-1)/2}(\mathbb{R})$ are supported in $\{|\xi| \ge 1/2\}$ (*c.f.* [44, Chapter VIII]). Thus one can write

$$\mathcal{A}_j f(x, t) = 2^{-j(d-1)/2} (2\pi)^{-d} \sum_{\pm} T_j^{\pm}[a_\pm, f](x, t) \chi(t) \tag{2.12}$$

where $a_\pm \in S^0(\mathbb{R})$. We note that the kernel estimate (2.7) could also be obtained through integration by parts in (2.11) using the above representation. It is clear from the expression of $T_j^\pm$ that

$$\partial_t\big(T_j^{\pm}[a, f](x, t)\chi(t)\big) = T_j^{\pm}[a, f](x, t)\chi'(t) + T_j^{\pm}[\widetilde{a}, f](x, t)\chi(t)$$

where $\widetilde{a}(\xi) = a'(t|\xi|)|\xi| \pm i|\xi| a(\xi)$. This and Plancherel's theorem yield

$$\|\mathcal{A}_j f\|_{L^2(L^2)} \lesssim 2^{-j(d-1)/2} \|f\|_2, \qquad \|\partial_t \mathcal{A}_j f\|_{L^2(L^2)} \lesssim 2^{-j(d-3)/2} \|f\|_2. \tag{2.13}$$

2.4. *A Stein–Tomas estimate.* In [21], in order to obtain L^p bounds for the global $V_r A$, the estimate

$$\Big\| \Big(\int_1^2 |e^{it\sqrt{-\Delta}} L_j f|^2 \, \mathrm{d}t \Big)^{1/2} \Big\|_p \lesssim 2^{j(d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2} + \varepsilon)} \|f\|_p \tag{2.14}$$

with $\varepsilon > 0$ is used for $\frac{2(d+1)}{d-1} \le p < \infty$ if $d \ge 3$; it holds for $4 < p < \infty$ if $d = 2$. This statement is closely related to estimates for Stein's square-function generated by Bochner–Riesz multipliers in [11], [13] and [39], and the connection is given by the theorem of Kaneko and Sunouchi [22]. See also [28] for endpoint bounds and historical remarks, and [27], [26] for recent work on Stein's square function. The Stein–Tomas L^2 Fourier restriction theorem together with a localization result (*cf.* Lemma 4.1 below) yields an analogue of (2.14) with $\varepsilon = 0$ for $p \geq \frac{2(d+1)}{d-1}$. The method is well known [14] but we include the statement with a proof for completeness.

Lemma 2.4. *Let* $\frac{2(d+1)}{d-1} \leq q \leq \infty$. *Then for all* $j \geq 0$,

$$\|\mathcal{A}_j f\|_{L^q(L^2)} \lesssim 2^{-jd/q} \|f\|_{L^2}.$$

Proof. We use the oscillatory integral representation in (2.12) and (2.11). We only discuss the estimate for $T_j^+[a,f](x,t)\chi(t)$ and abbreviate it with $T_j f(x,t)$ (the corresponding estimate for T_j^- is analogous). It then suffices to show

$$2^{-j(d-1)/2} \|T_j f\|_{L^q(L^2)} \lesssim 2^{-jd/q} \|f\|_2, \qquad \tfrac{2(d+1)}{d-1} \leq q \leq \infty.$$

Let

$$\widetilde{T}_j g(x,t) = \chi(t) \int_{\mathbb{R}^d} \beta_j(|\xi|) a(t|\xi|) e^{-it|\xi|} \widehat{g}(\xi,t) e^{i\langle x,\xi\rangle} \, \mathrm{d}\xi$$

and observe that in view of the support of χ we have $\widetilde{T}_j g(\cdot,t) = 0$ for $t \notin [1/2,4]$. By a duality argument, it suffices to show that for $g \in L^p(L^2)$ the inequality

$$\Big\| \int \widetilde{T}_j g(\cdot,t) \, \mathrm{d}t \Big\|_2 \lesssim 2^{j(\frac{d}{p} - \frac{d}{2} - \frac{1}{2})} \|g\|_{L^p(L^2)}, \qquad 1 \leq p \leq \tfrac{2(d+1)}{d+3} \tag{2.15}$$

holds. By Plancherel's theorem the square of the left-hand side is equal to

$$\begin{aligned} &\int_{\mathbb{R}^d} \Big| \int \chi(t) \beta_j(|\xi|) a(t|\xi|) e^{-it|\xi|} \widehat{g}(\xi,t) \, \mathrm{d}t \Big|^2 \mathrm{d}\xi \\ &= \int_0^\infty \int_{S^{d-1}} \Big| \int \chi(t) \beta_j(r) a(tr) e^{-itr} \widehat{g}(r\theta,t) \, \mathrm{d}t \Big|^2 \mathrm{d}\theta \, r^{d-1} \, \mathrm{d}r. \end{aligned}$$

We now apply the Stein–Tomas inequality for the Fourier restriction operator for the sphere (valid for $1 \leq p \leq 2(d+1)/(d+3)$), and see that the last expression is dominated by a constant times

$$\begin{aligned} &\int_0^\infty \Big\| \int \chi(t) \beta_j(r) a(tr) e^{-itr} r^{-d} g(r^{-1}\cdot,t) \, \mathrm{d}t \Big\|_p^2 r^{d-1} \, \mathrm{d}r \\ &\lesssim \int_0^\infty \Big\| \int \chi(t) \beta_j(r) a(tr) e^{-itr} g(\cdot,t) \, \mathrm{d}t \Big\|_p^2 r^{\frac{2d}{p} - d - 1} \, \mathrm{d}r \\ &\lesssim 2^{2j(\frac{d}{p} - \frac{d}{2} - \frac{1}{2})} \Big\| \Big(\int_0^\infty \Big| \int \chi(t) \beta_j(r) a(tr) e^{-itr} g(\cdot,t) \, \mathrm{d}t \Big|^2 \mathrm{d}r \Big)^{1/2} \Big\|_p^2 \end{aligned} \tag{2.16}$$

where in the last inequality we have used Minkowski's integral inequality. Next, observe that

$$\begin{aligned} &\int_0^\infty \Big| \int \chi(t) \beta_j(r) a(tr) e^{-itr} g(x,t) \, \mathrm{d}t \Big|^2 \mathrm{d}r \\ &= \iint \int_0^\infty \chi(t) \chi(t') |\beta_j(r)|^2 a(tr) \overline{a(t'r)} e^{i(t'-t)r} \, \mathrm{d}r \, g(x,t) \overline{g}(x,t') \, \mathrm{d}t \, \mathrm{d}t'. \end{aligned}$$

We integrate by parts in r and then estimate the absolute value of the displayed expression by a constant times

$$\begin{aligned}&\iint \frac{2^j}{(1+2^j|t-t'|)^2}|g(x,t)g(x,t')|\,\mathrm{d}t\,\mathrm{d}t'\\&=\int_{-\infty}^{\infty}\frac{2^j}{(1+2^j|h|)^2}\int|g(x,t)g(x,t+h)|\,\mathrm{d}t\,\mathrm{d}h\lesssim\int|g(x,t)|^2\,\mathrm{d}t.\end{aligned}$$

Using this in (2.16) yields (2.15) and hence the assertion. $\square$

2.5. *Frequency decompositions in time.* In order to deduce Besov space estimates for $t\mapsto \mathcal{A}_j f(x,t)$, we also work with a frequency decomposition in the t-variable. We extend the definition of Λ_l in (2.2) to functions of x and t and apply that decomposition to the operators $\mathcal{A}_j$ in the t-variable.

It is useful to observe that dyadic frequency decompositions in the variable dual to t essentially correspond in our situation to dyadic frequency decompositions in the variables dual to x. To see this, we show that the terms $\Lambda_l\mathcal{A}_j$ are mostly negligible when $|j-l|\geq 10$. We write

$$\Lambda_l\mathcal{A}_j f(x,t)=2^{-j(d-1)/2}(2\pi)^{-(d+1)}\sum_{\pm}\int_{\mathbb{R}^d}\kappa_{j,l}^{\pm}(y,t)f(x-y)\,\mathrm{d}y$$

where, in view of (2.11), one has

$$\kappa_{j,l}^{\pm}(y,t)=\int_{\mathbb{R}}\int_{\mathbb{R}^d}e^{i\langle y,\xi\rangle+it\tau}\beta_l(\tau)\beta_j(|\xi|)\int\chi(s)\,a_{\pm}(s\xi)e^{is(\pm|\xi|-\tau)}\,\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{2.17}$$

Lemma 2.5. *(i) For every $N\in\mathbb{N}_0$, there exists a finite $C_N>0$ such that*

$$|\kappa_{j,l}^{\pm}(y,t)|\leq C_N(1+|y|+|t|)^{-N}\min\{2^{-jN},2^{-lN}\},\quad |j-l|\geq 10. \tag{2.18}$$

(ii) Suppose $1\leq p,r\leq q\leq\infty$. Then, there exists a finite $C_N(p,q,r)>0$ such that

$$\|\Lambda_l\mathcal{A}_j f\|_{L^q(L^r)}\leq C_N(p,q,r)\min\{2^{-jN},2^{-lN}\}\|f\|_p,\quad |j-l|\geq 10.$$

Proof. Part (i) follows from (2.17) after multiple integration by parts in s and subsequent integration by parts in ξ,τ. Part (ii) is an immediate consequence of (i) using Minkowski's and Young's convolution inequality. $\square$

The above lemma allows one to only focus on the spatial frequency decomposition when looking for estimates of the type $L^p\to L^q(B^{1/r}_{r,1})$ for the operator $\mathcal{A}$ in most cases of interest. In particular, we get the following.

Corollary 2.6. *Let $s\in\mathbb{R}$, $1\leq p,q,r\leq\infty$. Then for all $j\in\mathbb{N}_0$,*

$$\|\mathcal{A}_j\|_{L^p\to L^q(B^s_{r,1})}\lesssim 2^{js}\|\mathcal{A}_j\|_{L^p\to L^q(L^r)}+C_N2^{-jN}$$

Proof. We write $\|\mathcal{A}_j f\|_{L^q(B^s_{r,1})} \le I + II$ where

$$I = \Big\| \sum_{\substack{l\ge 0\\ |j-l|\le 10}} 2^{ls} \Big\|\Lambda_l \mathcal{A}_j f\Big\|_{L^r(\mathbb{R})}\Big\|_{L^q(\mathbb{R}^d)},$$

$$II = \Big\| \sum_{\substack{l\ge 0\\ |j-l|> 10}} 2^{ls} \Big\|\Lambda_l \mathcal{A}_j f\Big\|_{L^r(\mathbb{R})}\Big\|_{L^q(\mathbb{R}^d)}.$$

Clearly

$$I \lesssim 2^{js}\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{js}\|\mathcal{A}_j\|_{L^p\to L^q(L^r)}\|f\|_p$$

and by (ii) in Lemma 2.5

$$II \lesssim \sum_{l\ge 0} \min\{2^{-jN}, 2^{-lN}\}\|f\|_p \lesssim 2^{-jN}\|f\|_p.$$

Combining both estimates, the assertion follows. □

In certain endpoint estimates in §6, we use an upgraded version of Corollary 2.6 in conjunction with Littlewood–Paley theory, as presented in the next lemma.

Lemma 2.7. *Let $1\le r<\infty$, $2\le q<\infty$, $1<p<\infty$ such that $r,p\le q$. Let $s\in\mathbb{R}$. Assume that for all $\{f_j\}_{j\ge 0}$ with $f_j\in L^p$,*

$$\Big\|\sum_{j\ge 0}\|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})}\Big\|_{L^q(\mathbb{R}^d)} \lesssim \Big(\sum_{j\ge 0} 2^{-jsq}\|f_j\|_p^q\Big)^{1/q} \tag{2.19}$$

holds. Then

$$\big\|\mathcal{A}f\big\|_{L^q(B^s_{r,1})} \lesssim \|f\|_{L^p}. \tag{2.20}$$

Proof. Write $\|\mathcal{A}f\|_{L^q(B^s_{r,1})}\le I + II$, where I and II are as in the proof of Corollary 2.6 but with an additional sum in the j-parameter. Recall that $\mathcal{A}_j f = \mathcal{A}_j(\widetilde{L}_j f)$. Applying the assumption (2.19) in I, one obtains

$$\begin{aligned} I &\lesssim \Big\|\sum_{j=0}^{\infty} 2^{js}\|\mathcal{A}_j(\widetilde{L}_j f)\|_{L^r(\mathbb{R})}\Big\|_{L^q(\mathbb{R}^d)} \lesssim \Big(\sum_{j=0}^{\infty}\|\widetilde{L}_j f\|_p^q\Big)^{\frac1q} \\ &\lesssim \Big\|\Big(\sum_{j=0}^{\infty}|\widetilde{L}_j f|^q\Big)^{\frac1q}\Big\|_p \lesssim \Big\|\Big(\sum_{j=0}^{\infty}|\widetilde{L}_j f|^2\Big)^{\frac12}\Big\|_p \lesssim \|f\|_p \end{aligned}$$

since $q\ge 2$ and $1<p\le q<\infty$; note that the second line follows from Minkowski's inequality, the embedding $\ell^2\hookrightarrow \ell^q$ and the Littlewood–Paley inequality. For the error term II, one applies (ii) in Lemma 2.5 to obtain

$$II \lesssim_N \sum_{l\ge 0}\sum_{j\ge 0} 2^{ls}\min\{2^{-lN}, 2^{-jN}\}\|f\|_p \lesssim \|f\|_p$$

for $N>s$. Combining both estimates, (2.20) follows. □

Remark. The previous lemma also extends to $q=\infty$ with the obvious notational modifications.

2.6. *Bourgain's interpolation lemma.* For the proof of restricted weak type inequalities we will repeatedly apply a result of Bourgain [6] that leads to restricted weak type inequalities in certain endpoint situations. We cite the abstract version of this lemma given in [12, §6.2] for the Lions–Peetre real interpolation spaces (see [3]).

Let $\overline{A}=(A_0,A_1)$, $\overline{B}=(B_0,B_1)$ be compatible Banach spaces in the sense of interpolation theory. Let $T_j:\overline{A}\to\overline{B}$ be sublinear operators satisfying for all $j\in\mathbb{Z}$

$$\|T_j\|_{A_0\to B_0}\le C_0 2^{j\gamma_0},\quad \|T_j\|_{A_1\to B_1}\le C_1 2^{-j\gamma_1},\quad \gamma_0,\gamma_1>0. \tag{2.21}$$

This assumption and real interpolation immediately gives $\|T_j\|_{\overline{A}_{\theta,\rho}\to\overline{B}_{\theta,\rho}}=O(1)$ for all $0<\rho\le\infty$ and all $\theta=\gamma_0/(\gamma_0+\gamma_1)$, but one also gets a weaker conclusion for the sum of the operators.

Lemma 2.8. *Suppose* (2.21) *holds for all* $j\in\mathbb{Z}$. *Then*

$$\Big\|\sum_j T_j\Big\|_{\overline{A}_{\theta,1}\to\overline{B}_{\theta,\infty}}\le C(\gamma_0,\gamma_1)C_0^{\frac{\gamma_1}{\gamma_0+\gamma_1}}C_1^{\frac{\gamma_0}{\gamma_0+\gamma_1}}.$$

3. Necessary conditions

In this section we modify known examples for the spherical maximal operators to give some necessary conditions for $L^p\to L^q$ boundedness of the local variation operator $V_r^I A$. For $r>\frac{d}{d-1}$ these conditions show that $L^p\to L^q$ boundedness does not hold in the complement of the region $\mathfrak{P}_d(r)$ in Theorems 1.2 and 1.4 and the complement of $\mathfrak{Q}_2(r)$ in Theorem 1.6. For $1\le r\le\frac{d}{d-1}$ they show that $L^p\to L^q$ boundedness does not hold in the complement of $\mathfrak{Q}_d(r)$ defined in Theorem 1.5. They also show that V_r^I is unbounded from any $L^p(\mathbb{R}^2)$ to any $L^q(\mathbb{R}^2)$ if $r<2$, that is, part (iii) in Theorem 1.6. Finally, we also prove sharpness of the sparse bounds in Theorem 1.7 up to the endpoints.

3.1. *Description of the edges.* It will be helpful to make explicit the equations for the edges of the boundedness regions in the above theorems.

(i) Consider the case $r>\frac{d^2+1}{d(d-1)}$ and the region $\mathfrak{P}_d(r)$ in Theorem 1.2. In this case the point $P(r)$ is on the line through $(0,0)$ and Q_4, which is given by $\{\frac1q=\frac{1}{dp}\}$. The boundary lines describing $\mathfrak{P}_d(r)$ are

$$\overline{P(r)Q_1(r)}=\{\tfrac1q=\tfrac{1}{dr}\},\qquad \overline{Q_1(r)Q_2}=\{\tfrac1q=\tfrac1p\},\qquad \overline{Q_2Q_3}=\{\tfrac1p=\tfrac{d-1}{d}\},$$
$$\overline{Q_3Q_4}=\{\tfrac1q=\tfrac{d+1}{d-1}\tfrac1p-1\},\qquad \overline{Q_4P(r)}=\{\tfrac1q=\tfrac{1}{dp}\}.$$

If $d=2$, the points Q_2 and Q_3 coincide, and the lines $\overline{Q_1(r)Q_2}$, $\overline{Q_3Q_4}$, $\overline{Q_4P(r)}$ and $\overline{P(r)Q_1(r)}$ describe the quadrangle $\mathfrak{Q}_2(r)$ in Theorem 1.6, (i).

(ii) For the case $\frac{d}{d-1}<r\le\frac{d^2+1}{d(d-1)}$ the point $P(r)$ moves to the line connecting Q_3 and Q_4 and only the part between $P(r)$ and Q_3 will be part of the boundary. Note that for $r=\frac{d^2+1}{d(d-1)}$ the points $P(r)$ and Q_4 coincide

so that the pentagon degenerates to a quadrangle. As $r \to \frac{d}{d-1}$ the point $P(r)$ moves to Q_3. The boundary lines of $\mathfrak{P}_d(r)$ in Theorem 1.4 are given in this case by

$$\overline{Q_1(r)Q_2} = \{\tfrac{1}{q} = \tfrac{1}{p}\}, \qquad \overline{Q_2Q_3} = \{\tfrac{1}{p} = \tfrac{d-1}{d}\},$$
$$\overline{Q_3P(r)} = \{\tfrac{1}{q} = \tfrac{d+1}{d-1}\cdot\tfrac{1}{p} - 1\}, \qquad \overline{P(r)Q_4(r)} = \{\tfrac{1}{q} = \tfrac{1}{p} + \tfrac{2}{r(d-1)} - 1\},$$
$$\overline{Q_4(r)Q_1(r)} = \{\tfrac{1}{q} = \tfrac{1}{dr}\}.$$

It is convenient to note, in view of §4, that the equation $\frac{1}{q} = \frac{1}{p} + \frac{2}{r(d-1)} - 1$ is equivalent to $\frac{1}{r} = \frac{d-1}{2}\big(\frac{1}{q} + \frac{1}{p'}\big)$.

Again, if $d = 2$, the points Q_2 and Q_3 coincide, and the lines $\overline{Q_1(r)Q_2}$, $\overline{Q_3P(r)}$, $\overline{P(r)Q_4(r)}$ and $\overline{Q_4(r)(r)Q_1(r)}$ describe the quadrangle $\mathfrak{Q}_2(r)$ in Theorem 1.6, (ii).

(iii) In the case $1 \le r < \frac{d}{d-1}$ we now have a quadrangle $\mathfrak{Q}_d(r)$ in Theorem 1.5, whose boundary lines are

$$\overline{Q_1(r)Q_2(r)} = \{\tfrac{1}{p} = \tfrac{1}{q}\}, \qquad \overline{Q_2(r)Q_3(r)} = \{\tfrac{1}{p} = 1 - \tfrac{1}{r(d-1)}\},$$
$$\overline{Q_3(r)Q_4(r)} = \{\tfrac{1}{q} = \tfrac{1}{p} + \tfrac{2}{r(d-1)} - 1\}, \qquad \overline{Q_4(r)Q_1(r)} = \{\tfrac{1}{q} = \tfrac{1}{dr}\}.$$

We next list our necessary conditions for bounds on $V_r^I A$. We remark that the sharpness in the conditions §§3.2 – 3.5 corresponds to the necessary conditions for the spherical maximal function S^I.

3.2. *The condition* $p \le q$. This is the standard necessary condition for translation operators mapping $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, see [19].

3.3. *The condition* $p > \frac{d}{d-1}$. This is (a variant of) Stein's example for spherical maximal functions [43]. Let B be the ball of radius $1/10$ centered at the origin and let $f(y) = \mathbb{1}_B(y)|y|^{1-d}(\log|y|)^{-1}(\log\log|y|)^{-1}$. Then $f \in L^{\frac{d}{d-1},q}$ for all $q > 1$, but for $1 < |x| < 2$ and $t(x) = |x|$ we have $A_{t(x)}f(x) = \infty$.

3.4. *The condition* $d/q \ge 1/p$. For the condition $d/q \ge 1/p$ we just take the standard example for the spherical averages [38], namely consider a fixed shell $S_{j,0}$ (as in (3.3) below) and $g_j = \mathbb{1}_{S_{j,0}}$ so that $\|g_j\|_p \le 2^{-j/p}$. For $|x| \le 2^{-j-2}$ we have $A_1 g_j(x) \ge c > 0$ and evaluating the L^q norm over $\{x : |x| \le 2^{-j-2}\}$ we get $\|V_r^I A g_j\|_q \ge 2^{-jd/q}$ and obtain the necessity of $d/q \ge 1/p$.

3.5. *The condition* $\frac{1}{q} \ge \frac{d+1}{(d-1)p} - 1$. This is the standard Knapp example in [38]. Given $0 < \delta \ll 1$, one tests the maximal operator on f_δ being the characteristic function of $\{y : |y'| \le \delta,\ |y_d| \le \delta^2\}$ and evaluates $A_{x_d} f_\delta(x)$ for $|x'| \le \delta$ and $1 < x_d < 2$.

3.6. *The condition* $\frac{1}{p} \le 1 - \frac{1}{r(d-1)}$. In view of §3.3 this example is only relevant for $r < \frac{d}{d-1}$. For large j define

$$c_{j,n} = -n2^{-j}, \quad n = 1, \dots, N \tag{3.1}$$

where $N = 2^{j-2}$. Let $B_{j,n}$ be the ball of radius 2^{-j-4} centered at $c_{j,n}e_d$.

Let $f_j(x) = \sum_{n=1}^{N} (-1)^n \mathbb{1}_{B_{j,n}}(x)$, so that

$$\|f_j\|_p \lesssim N^{1/p} 2^{-jd/p}.$$

Consider

$$R = \{(x', x_d) : |x'| \le (4d)^{-1},\ 1 \le x_d \le 3/2\}. \tag{3.2}$$

Note that for $x \in R$ we have $|x - c_{j,n}e_d| \in [1, 2]$; indeed, $|x - c_{j,n}e_d| \ge |x_d - c_{j,n}| \ge 1$ and $|x - c_{j,n}e_d| \le (|x_d - c_{j,n}|^2 + (4d)^{-2})^{1/2} \le 2$.

For $x \in R$ pick $t_n(x) = |x - c_{j,n}e_d|$ and observe that there is a constant $a > 0$ such that $A_{t_{2\nu}(x)} f_j(x) \ge a2^{-j(d-1)}$ and $A_{t_{2\nu-1}(x)} f_j(x) \le -a2^{-j(d-1)}$, and thus

$$|A_{t_{2\nu}(x)} f_j(x) - A_{t_{2\nu-1}(x)} f_j(x)| \ge 2a2^{-j(d-1)}.$$

Hence, for any r we get $V_r^I Af(x) \gtrsim N^{1/r} 2^{-j(d-1)}$ for $x \in R$ and thus for any $q > 0$

$$\frac{\|V_r^I Af_j\|_q}{\|f_j\|_p} \gtrsim N^{\frac{1}{r} - \frac{1}{p}} 2^{-j(d-1-\frac{d}{p})}$$

Since $N = 2^{j-2}$ the assumption of $L^p \to L^q$ boundedness of $V_r^I A$ implies $\frac{1}{r} \le \frac{d-1}{p'}$ or equivalently $\frac{1}{p} \le 1 - \frac{1}{r(d-1)}$.

3.7. *The condition* $\frac{1}{q} \ge \frac{1}{p} + \frac{2}{(d-1)r} - 1$, *i.e.* $\frac{d-1}{2}(\frac{1}{q} + \frac{1}{p'}) \ge \frac{1}{r}$. This is a variant of the example in §3.5. We let $c_{j,n}$ be as in (3.1) and $P_{j,n} = \{y : |y'| \le 2^{-j/2-2}, |y_d - c_{j,n}| \le 2^{-j-4}\}$. Let $N \le 2^{j-2}$. Let $f_j = \sum_{n=1}^{N} (-1)^n \mathbb{1}_{P_{j,n}}(x)$. Then $\|f_j\|_p \lesssim N^{1/p} 2^{-j\frac{d+1}{2p}}$. Let $\Omega = \{x : |x'| \le 2^{-j/2-2},\ 1 \le x_d \le 3/2\}$ so that $|\Omega| \approx 2^{-j(d-1)/2}$. Let $t_n(x) = |x_d - c_{j,n}| \in [1, 2]$. Then for $x \in \Omega$, $A_{t_{2\nu}(x)} f_j(x) \ge a2^{-j(d-1)/2}$ and $A_{t_{2\nu-1}(x)} f_j(x) \le -a2^{-j(d-1)/2}$ for some constant $a > 0$. Hence $V_r^I Af_j(x) \gtrsim N^{1/r} 2^{-j\frac{d-1}{2}}$ and thus $\|V_r^I Af_j\|_q \gtrsim N^{1/r} 2^{-j\frac{d-1}{2}(1+\frac{1}{q})}$. Consequently with $N = 2^{j-2}$

$$\frac{\|V_r^I Af_j\|_q}{\|f_j\|_p} \gtrsim N^{\frac{1}{r} - \frac{1}{p}} 2^{-j\frac{d-1}{2}(1+\frac{1}{q}) + j\frac{d+1}{2p}} \gtrsim 2^{j(\frac{1}{r} - \frac{d-1}{2}(\frac{1}{q} + \frac{1}{p'}))}.$$

Hence the condition $\frac{d-1}{2}(\frac{1}{q} + \frac{1}{p'}) \ge \frac{1}{r}$ is necessary for $V_r^I A : L^p \to L^q$ to be bounded. Moreover, as $p \le q$ by §3.2, this also implies that no $L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2)$ bounds hold for $r < 2$.

3.8. *The condition $d/q \geq 1/r$.* Consider the shells

$$S_{j,n} = \{y : \big||y| - 1 - n2^j\big| \leq 2^{-j-2}\}. \tag{3.3}$$

We set $f_j = \sum_{n=1}^{N}(-1)^n \mathbb{1}_{S_{j,n}}$, with $N = 2^{j-2}$. Then clearly $\|f_j\|_p \lesssim 1$ uniformly in j.

For $|x| \leq 2^{-j-5}$ let $t_n(x) = 1 + n2^{-j} \in [1,2]$. Then $A_{t_{2\nu}(x)} f_j(x) \geq a$ and $A_{t_{2\nu-1}(x)} f_j(x) \leq -a$ for some a independent of j. Hence $V_r^I f(x) \gtrsim N^{1/r} \approx 2^{j/r}$ for $|x| \leq 2^{-j}$ and thus $\|V_r^I f_j\|_q \gtrsim 2^{j(\frac{1}{r}-\frac{d}{q})}$. This implies the necessity of the condition $1/r \leq d/q$.

Remark. An alternative (more complicated) example for the condition $d/q \geq 1/r$ is in [21, §8].

3.9. *Sharpness of the sparse bounds.* The sparse domination result in Theorem 1.7 is sharp, and this is immediate from the examples just described in this section. The argument, shown by Lacey in [23, Section 5] for the spherical maximal function, can be extended in our context and even more general ones [2, Proposition 7.2].

We exemplify this considering the example in §3.6, with the choice $N = 2^{j-2}$. With f_j as in this example we have $|f_j| = \mathbb{1}_U$ where U is the union of the balls $B_{j,n}$ which is essentially a 2^{-j}-neighborhood of the x_d-axis segment $[-1/4, 0]$. $V_r A f_j$ is evaluated at R as in (3.2). Then for large j we have

$$\langle V_r A f_j, \mathbb{1}_R\rangle = \int_{\mathbb{R}^d} V_r A f(x) \mathbb{1}_R(x)\, \mathrm{d}x \gtrsim 2^{j(\frac{1}{r}-d+1)}.$$

On the other hand, suppose that $p < q$ and the sparse bound

$$\int_{\mathbb{R}^d} V_r A f_j(x) \mathbb{1}_R(x)\, \mathrm{d}x \leq C_0 \sup_{\mathfrak{S}:\text{sparse}} \Lambda_{p,q'}^{\mathfrak{S}}(f, \mathbb{1}_R)$$

holds for some positive C_0, with $\Lambda_{p,q'}^{\mathfrak{S}}(f,g) = \sum_{Q\in\mathfrak{S}} |Q| \langle f_j\rangle_{Q,p} \langle \mathbb{1}_R\rangle_{Q,q'}$. By the definition of supremum there is a sparse collection $\mathfrak{S}_0$ such that

$$\int_{\mathbb{R}^d} V_r A f_j(x) \mathbb{1}_R(x)\, \mathrm{d}x \leq 2C_0 \sum_{Q\in\mathfrak{S}_0} |Q| \langle f_j\rangle_{Q,p} \langle \mathbb{1}_R\rangle_{Q,q'}.$$

It is crucial in the example that

$$\operatorname{dist}(\operatorname{supp}(f_j), R) \geq 1 \tag{3.4}$$

which implies that all cubes contributing to the sum have side length at least 1. Moreover, for each $l \geq 0$ there are only $O(1)$ cubes of sidelength 2^l contributing. For each such term we can estimate

$$|Q| \langle f_j\rangle_{Q,p} \langle \mathbb{1}_R\rangle_{Q,q'} \lesssim |Q|^{\frac{1}{q}-\frac{1}{p}} 2^{-j\frac{d-1}{p}}$$

and by summing over all terms (taking advantage of $p < q$) we obtain

$$2^{j(\frac{1}{r}-d+1)} \lesssim \langle V_r A f_j, \mathbb{1}_R\rangle = \int_{\mathbb{R}^d} V_r A f_j(x) \mathbb{1}_R(x)\, \mathrm{d}x \lesssim C_0 2^{-j\frac{d-1}{p}}$$

and letting $j \to \infty$ we obtain the same necessary condition as in §3.6, i.e. $\frac{1}{p} \le 1 - \frac{1}{r(d-1)}$.

The remaining examples in §§3.3–3.8 yield similar necessary conditions for sparse bounds, and this is proved by essentially the same idea, always taking advantage of a support-separation property analogous to (3.4). We leave the details to the reader.

4. $L^p \to L^q(L^r)$ ESTIMATES FOR $\mathcal{A}_j$

In this section we prove $L^p \to L^q(L^r)$ bounds for the dyadic frequency localized operators $\mathcal{A}_j$ in the closure of the regions $\mathfrak{P}_d(r)$ and $\mathfrak{Q}_d(r)$ featuring in Theorems 1.2, 1.4, 1.5 and 1.6. This will lead to the proofs for $L^p \to L^q$ bounds for $V_r^I A$ if $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of $\mathfrak{P}_d(r)$ and $\mathfrak{Q}_d(r)$ respectively, as well as several restricted weak-type results through Bourgain's interpolation trick.

4.1. *Localization.* The following observation relies on the localization property (2.6) of the kernel $K_{j,t}$.

Lemma 4.1. *(i) For $p_0 \le p_1 \le q_1 \le q_0$, $1 \le r \le \infty$, and every $N \in \mathbb{N}$,*

$$\|\mathcal{A}_j\|_{L^{p_1} \to L^{q_1}(L^r)} \lesssim \|\mathcal{A}_j\|_{L^{p_0} \to L^{q_0}(L^r)} + C_N 2^{-jN}.$$

(ii) For $r_0 \le r_1$, $1 \le p \le q \le \infty$,

$$\|\mathcal{A}_j\|_{L^p \to L^q(L^{r_0})} \lesssim \|\mathcal{A}_j\|_{L^p \to L^q(L^{r_1})}.$$

Proof. Assume that $\|\mathcal{A}_j\|_{L^{p_0} \to L^{q_0}(L^r)} < \infty$. Let $f \in L^{p_1}$. For $\mathfrak{z} \in \mathbb{Z}^d$ let $Q_\mathfrak{z} = \prod_{i=1}^d [\mathfrak{z}_i, \mathfrak{z}_i + 1)$. Let $Q_\mathfrak{z}^*$ be a cube centered at $\mathfrak{z}$ with side-length $20d$. Write $f = \sum_\mathfrak{z} f_\mathfrak{z}$ with $f_\mathfrak{z} = f \mathbb{1}_{Q_\mathfrak{z}}$ and estimate

$$\|\mathcal{A}_j f\|_{L^{q_1}(L^r)} \le \Big\| \sum_\mathfrak{z} \mathbb{1}_{Q_\mathfrak{z}^*} \mathcal{A}_j f_\mathfrak{z} \Big\|_{L^{q_1}(L^r)} + \Big\| \sum_\mathfrak{z} \mathbb{1}_{\mathbb{R}^d \setminus Q_\mathfrak{z}^*} \mathcal{A}_j f_\mathfrak{z} \Big\|_{L^{q_1}(L^r)} = I + II.$$

Since the $Q_\mathfrak{z}^*$ have bounded overlap, by Hölder's inequality for $q_1 \le q_0$,

$$I \lesssim \Big(\sum_\mathfrak{z} \|\mathbb{1}_{Q_\mathfrak{z}^*} \mathcal{A}_j f_\mathfrak{z}\|_{L^{q_1}(L^r)}^{q_1} \Big)^{1/q_1} \lesssim \Big(\sum_\mathfrak{z} \|\mathcal{A}_j f_\mathfrak{z}\|_{L^{q_0}(L^r)}^{q_1} \Big)^{1/q_1}.$$

Applying the bound for the operator $\mathcal{A}_j$,

$$\Big(\sum_\mathfrak{z} \|\mathcal{A}_j f_\mathfrak{z}\|_{L^{q_0}(L^r)}^{q_1} \Big)^{1/q_1} \lesssim \|\mathcal{A}_j\|_{L^{p_0} \to L^{q_0}(L^r)} \Big(\sum_\mathfrak{z} \|f_\mathfrak{z}\|_{L^{p_0}}^{q_1} \Big)^{1/q_1}$$

and, since $p_0 \le p_1 \le q_1$, we also have

$$\Big(\sum_\mathfrak{z} \|f_\mathfrak{z}\|_{L^{p_0}}^{q_1} \Big)^{1/q_1} \lesssim \Big(\sum_\mathfrak{z} \|f_\mathfrak{z}\|_{L^{p_1}}^{q_1} \Big)^{1/q_1} \lesssim \Big(\sum_\mathfrak{z} \|f_\mathfrak{z}\|_{L^{p_1}}^{p_1} \Big)^{1/p_1} \lesssim \|f\|_{p_1}.$$

Moreover, by (2.8) with $N > d$,

$$II \le \Big(\int \Big[\int_{|y-x| \ge 1} (2^j |x-y|)^{-N} |f(y)| \, \mathrm{d}y \Big]^{q_1} \mathrm{d}x \Big)^{1/q_1} \lesssim_N 2^{-jN} \|f\|_{p_1}.$$

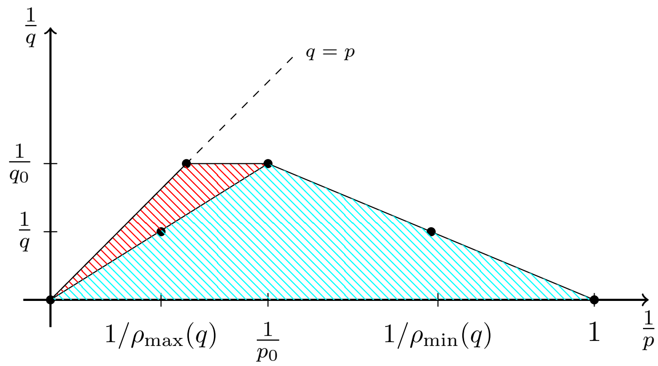

FIGURE 9. Interpolation and localization lemmas. If $\|\mathcal{A}_j\|_{L^{p_0}\to L^{q_0}(L^{p_0})} \lesssim 2^{-jd/q_0}$, then $\|\mathcal{A}_j\|_{L^p\to L^q(L^p)} \lesssim 2^{-jd/q}$ in the blue triangle and $\|\mathcal{A}_j\|_{L^p\to L^q(L^{\rho_{\max}(q)})} \lesssim 2^{-jd/q}$ in the red triangle.

Combining the two estimates we obtain

$$\|\mathcal{A}_j f\|_{L^{q_1}(L^r)} \lesssim \big(\|\mathcal{A}_j\|_{L^{p_0}\to L^{q_0}(L^r)} + C_N 2^{-jN}\big)\|f\|_{p_1},$$

which is the assertion in part (i).

Part (ii) is immediate and simply follows from Hölder's inequality in the t-variable. ∎

4.2. *Interpolation.* Lemma 2.4 can be extended to a larger range of exponents by interpolation with (2.9) and Lemma 2.2 and by the localization property in Lemma 4.1. We state this in more generality; see Figure 9.

Lemma 4.2. *Let p_0 and q_0 such that $1 \le p_0 \le q_0 \le \infty$. Assume that*

$$\sup_{j\ge 0} 2^{jd/q_0}\|\mathcal{A}_j\|_{L^{p_0}\to L^{q_0}(L^{p_0})} \le C < \infty. \tag{4.1}$$

Let $q_0 \le q \le \infty$ and define $\rho_{\min}(q)$ and $\rho_{\max}(q)$ by

$$1 - \frac{1}{\rho_{\min}(q)} = \frac{q_0}{q}\Big(1 - \frac{1}{p_0}\Big), \qquad \frac{1}{\rho_{\max}(q)} = \frac{q_0}{q}\frac{1}{p_0}. \tag{4.2}$$

Assume that $\rho_{\min}(q) \le p \le q$ and $0 < r \le \min\{p, \rho_{\max}(q)\}$. Then

$$\sup_{j\ge 0} 2^{jd/q}\|\mathcal{A}_j\|_{L^p\to L^q(L^r)} < \infty.$$

Proof. Note that $\rho_{\min}(q) \le \rho_{\max}(q)$ when $q \ge q_0$, with strict inequality when $q > q_0$, and $\rho_{\min}(q_0) = \rho_{\max}(q_0) = p_0$. Assume $q > q_0$ and let $\vartheta = 1 - q_0/q$. Note that $(1-\vartheta)/p_0 = 1/\rho_{\max}(q)$ and $(1-\vartheta)/p_0 + \vartheta = 1/\rho_{\min}(q)$. We interpolate (4.1) with the inequality

$$\sup_{j\ge 0}\|\mathcal{A}_j\|_{L^{p_1}\to L^\infty(L^{p_1})} < \infty, \quad 1 \le p_1 \le \infty$$

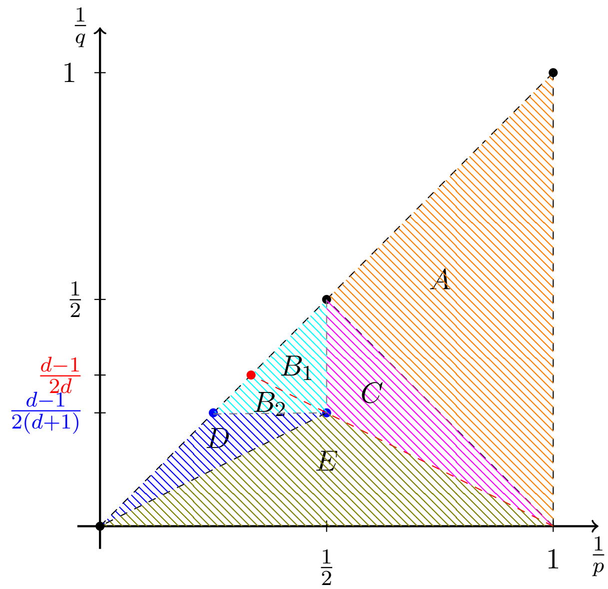

FIGURE 10. Regions for $L^p \to L^q(L^r)$ bounds for the single scale $\mathcal{A}_j$ for $0 < r \le 1$. As r increases the regions shrink due to the constraints $r \le p$ or $r \le \frac{q(d-1)}{d+1}$.

for the choices $p_1 = 1$ and $p_1 = \infty$ (by Lemma 2.2 and (2.9)) and obtain the $L^p \to L^q(L^p)$ inequality for $p = \rho_{\min}(q)$ and $p = \rho_{\max}(q)$. A further interpolation gives

$$\sup_{j\ge0} \|\mathcal{A}_j\|_{L^p\to L^q(L^p)} \lesssim \big(1 + \sup_{j\ge0} \|\mathcal{A}_j\|_{L^{p_0}\to L^{q_0}(L^{p_0})}\big), \quad \rho_{\min}(q) \le p \le \rho_{\max}(q).$$

We now combine this with Lemma 4.1 and see that the $L^p \to L^q(L^r)$ estimates hold when $\rho_{\min}(q) \le p \le \rho_{\max}(q)$ and $r \le p$ and moreover when $\rho_{\min}(q) \le r \le \rho_{\max(q)}$ and $r \le p \le q$. $\square$

4.3. *Bounds for $\mathcal{A}_j$.* The previous lemma and the estimates in §2 yield the following bounds; see Figure 10 for the regions.

Proposition 4.3. *Let $d \ge 2$.*

(A) Let $1 \le p \le 2$, $p \le q \le p'$ and $0 < r \le p$. Then

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-j(d-1)/p'} \|f\|_{L^p}.$$

(B) Let $2 \le p \le q \le \frac{2(d+1)}{d-1}$. Let $0 < r \le 2$. Then

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-j\frac{d-1}{2}(\frac{1}{q}+\frac{1}{2})} \|f\|_{L^p}.$$

(C) Let $1 \le p \le 2$, $\frac{d-1}{d+1}\frac{1}{p'} \le \frac{1}{q} \le \frac{1}{p'}$ and $0 < r \le p$. Then

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-j\frac{d-1}{2}(\frac{1}{q}+\frac{1}{p'})} \|f\|_{L^p}.$$

(D) Let $\frac{2(d+1)}{d-1} \le q \le \infty$, $\frac{d-1}{d+1}\frac{1}{p} \le \frac{1}{q} \le \frac{1}{p}$ *and* $0 < r \le \frac{q(d-1)}{d+1}$. *Then*

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-jd/q}\|f\|_{L^p}.$$

(E) Let $\frac{2(d+1)}{d-1} \le q \le \infty$, $\frac{1}{q} \le \frac{d-1}{d+1}\frac{1}{p}$, $\frac{1}{q} \le \frac{1}{p} \le 1 - \frac{d+1}{d-1}\frac{1}{q}$, *and* $0 < r \le p$. *Then*

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-jd/q}\|f\|_{L^p}.$$

Proof. The bounds in (A) for $r = p$ follow from interpolation of Lemma 2.2 and the L^2-estimate (2.13), whilst the remaining values of $0 < r < p$ follow from (ii) in Lemma 4.1.

The bounds in (D) and (E) are an application of Lemma 4.2 with $p_0 = 2$, $q_0 = \frac{2(d+1)}{d-1}$, which is the estimate in Lemma 2.4.

The bounds in (C) follow from interpolation of those in (A) if $q = p'$ and those in (E) if $\frac{1}{q} = \frac{1}{p'}\frac{d-1}{d+1}$, $1 \le p \le 2$.

Finally, the bounds in (B) follow from interpolation of the L^2 estimate (2.13) with the $L^p \to L^p(L^2)$ estimate in (D) for $p = \frac{2(d+1)}{d-1}$, and a further interpolation of those with the estimates in (C) for $p = 2$. □

The above bounds on (A), (C) and (E) are sharp. However, the bounds in (B) and the r-range in (D) can be improved; for example, if information on the *local smoothing* phenomenon for the wave equation is known. Recall that these estimates, first noted by Sogge in [41], are of the type

$$\Big\|\Big(\int_1^2 |e^{it\sqrt{-\Delta}}L_j f|^p \,\mathrm{d}t\Big)^{1/p}\Big\|_{L^p} \lesssim 2^{j(\bar{s}_p-\sigma)}\|f\|_{L^p} \tag{4.3}$$

for some $\sigma > 0$ if $2 < p < \infty$, where $\bar{s}_p := (d-1)\big(\frac{1}{2} - \frac{1}{p}\big)$. It is conjectured that (4.3) holds for all $\sigma < \sigma_p$, where

$$\sigma_p := \begin{cases} 1/p & \text{if } \quad \frac{2d}{d-1} \le p < \infty, \\ \bar{s}_p & \text{if } \quad 2 \le p \le \frac{2d}{d-1}. \end{cases}$$

This conjecture is strongest at $p = \frac{2d}{d-1}$. After contributions by many, it has recently been solved by Guth, Wang and Zhang [17] for $d = 2$, and is known to hold for all $p \ge \frac{2(d+1)}{d-1}$ if $d \ge 3$ by the sharp decoupling inequalities of Bourgain and Demeter [9]. It is also expected that endpoint regularity results with $\sigma = 1/p$ should hold if $p > 2d/(d-1)$; see [18] for results in this direction if $d \ge 4$. The validity of the local smoothing conjecture would imply the following bounds on spherical averages on the region (B). We remark that these improved bounds are only relevant for our variational bounds if $d = 2, 3$; for $d \ge 4$ the bounds in Proposition 4.3 will suffice (see the discussion after Theorem 1.5 in the Introduction).

Proposition 4.4. *Let* $d \ge 2$. *Assume that the local smoothing conjecture holds, that is,* (4.3) *holds at* $p = \frac{2d}{d-1}$ *for all* $\sigma < 1/p$.

(B_1) If $\frac{d-1}{d+1}\frac{1}{p'} \le \frac{1}{q} \le \frac{1}{p}$ and $2 < q \le \frac{2d}{d-1}$, $2 < p \le \frac{2d}{d-1}$ and $0 < r \le p$, then

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-j\frac{d-1}{2}(\frac{1}{q}+\frac{1}{p'})+j\varepsilon}\|f\|_{L^p}$$

for all $\varepsilon > 0$.

(B_2) If $\frac{1}{q} \le \min\{\frac{d-1}{d+1}\frac{1}{p'}, \frac{1}{p}\}$ and $\frac{2d}{d-1} \le q \le \frac{2(d+1)}{d-1}$ and $0 < r \le p$, then

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-jd/q+j\varepsilon}\|f\|_{L^p}$$

for all $\varepsilon > 0$.

In particular, the above estimates hold for $d = 2$.

Proof. By the oscillatory integral representation in (2.12) and (2.11), the estimate (4.3) implies

$$\|\mathcal{A}_j f\|_{L^p(L^p)} \lesssim 2^{-j\frac{d-1}{2}+j\varepsilon}\|f\|_{L^p} \tag{4.4}$$

for $p = \frac{2d}{d-1}$. Interpolation of (4.4) and Lemma 2.4 yields

$$\|\mathcal{A}_j f\|_{L^q(L^p)} \lesssim 2^{-jd/q+j\varepsilon}\|f\|_{L^p} \tag{4.5}$$

for $\frac{1}{q} = \frac{d-1}{d+1}\frac{1}{p'}$ and $2 < p \le \frac{2d}{d-1} \le q < \frac{2(d+1)}{d-1}$. Moreover, interpolation of (4.4) and the L^2-estimate (2.13) yields

$$\|\mathcal{A}_j\|_{L^p(L^p)} \lesssim 2^{-j\frac{d-1}{2}+j\varepsilon}\|f\|_{L^p} \tag{4.6}$$

for $2 < p \le \frac{2d}{d-1}$. The region (B_1) then follows from interpolating (4.5) and (4.6).

For the region (B_2), interpolate (4.4) and (2.9) to obtain

$$\|\mathcal{A}_j f\|_{L^q(L^q)} \lesssim 2^{-jd/q+j\varepsilon}\|f\|_{L^q} \tag{4.7}$$

for all $\frac{2d}{d-1} \le q \le \infty$. A further interpolation of (4.7) with (4.5) for $\frac{2d}{d-1} \le q \le \frac{2(d+1)}{d-1}$ yields the estimates in (B_2).

The assertion for $d = 2$ follows since the local smoothing assumption was established in [17]. □

The range of r in the estimates in (D) can also be improved to $0 < r \le p$ using the known local smoothing estimates at $p = \frac{2(d+1)}{d-1}$ for all $\sigma < 1/p$. For our variational problem, this only becomes relevant if $d = 2$, as otherwise the results in Proposition 4.3 will suffice. We note that the use of such local smoothing estimates induces an ε-loss with respect to (D) in Proposition 4.3, although this will have no consequences on our proof in $d = 2$. The ε-loss in the forthcoming proposition can be removed if $p > \frac{2(d-1)}{d-3}$ when $d \ge 4$ by the currently known sharp regularity estimates in [18].

Proposition 4.5 (Improved bounds in (D))**.** *Let $d \ge 2$. Let $\frac{2(d+1)}{d-1} \le q \le \infty$, $\frac{d-1}{d+1}\frac{1}{p} \le \frac{1}{q} \le \frac{1}{p}$ and $r \le p$. Then*

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-j(d/q-\varepsilon)}\|f\|_p$$

for all $\varepsilon > 0$.

Proof. By (2.12), the estimates (4.3) for $p \geq \frac{2(d+1)}{d-1}$ imply that, given any $\varepsilon > 0$,

$$\|\mathcal{A}_j f\|_{L^p(L^p)} \lesssim 2^{-j(d/q-\varepsilon)}\|f\|_{L^q}$$

holds for all $\frac{2(d+1)}{d-1} \leq q \leq \infty$. It then suffices to interpolate this with the estimates in Proposition 4.3, (D), when $q = \frac{p(d+1)}{d-1}$ and $r = p$. $\square$

4.4. *Bounds for* $V_r^I \mathcal{A}_j$. Let $1 \leq r \leq \infty$. By the embedding (2.4) and Corollary 2.6, $V_r A$ maps $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if there exists an $\varepsilon > 0$ such that

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-j(\frac{1}{r}+\varepsilon)}\|f\|_{L^p}, \tag{4.8}$$

for all $f \in L^p$. This will suffice to show all the bounds in the interiors of $\mathfrak{P}_d(r), \mathfrak{Q}_d(r)$ claimed in Theorems 1.2, 1.4, 1.5 and 1.6.

We start with the case $d \geq 3$. We will only have to identify in each region $A - E$ of Proposition 4.3 the conditions under which (4.8) holds and to relate this to the corresponding statements in the theorems in the introduction.

Proposition 4.6. *Let* $d \geq 3$. *The inequality* (4.8) *holds for some* $\varepsilon > 0$ *under the following conditions on* $1 \leq p, q \leq \infty$, $0 < r \leq \infty$:

- *(A')* $1 \leq p \leq 2$, $p \leq q \leq p'$, *and*
  - $\frac{d}{d-1} < r \leq p$; *or*
  - $\frac{2}{d-1} < r \leq \frac{d}{d-1}$ *and* $\frac{1}{p} < 1 - \frac{1}{(d-1)r}$.
- *(B')* $2 \leq p \leq q \leq \frac{2(d+1)}{d-1}$ *and*
  - $\frac{2(d+1)}{d(d-1)} < r \leq 2$; *or*
  - $\frac{2}{d-1} < r \leq \frac{2(d+1)}{d(d-1)}$ *and* $\frac{1}{q} > \frac{2}{(d-1)r} - \frac{1}{2}$.
- *(C')* $1 \leq p \leq 2$, $\frac{d-1}{d+1}\frac{1}{p'} \leq \frac{1}{q} \leq \frac{1}{p'}$, *and*
  - $\frac{d^2+1}{d(d-1)} < r \leq p$ *and* $\frac{1}{q} > \frac{d+1}{d-1}\frac{1}{p} - 1$, $\frac{1}{p} < \frac{d-1}{d}$; *or*
  - $\frac{d}{d-1} < r \leq \min\{\frac{d^2+1}{d(d-1)}, p\}$ *and* $\frac{1}{q} > \frac{d+1}{d-1}\frac{1}{p} - 1$, $\frac{1}{q} > \frac{1}{p} + \frac{2}{r(d-1)} - 1$; *or*
  - $\frac{2}{d-1} < r \leq \frac{d}{d-1}$ *and* $\frac{1}{q} > \frac{1}{p} + \frac{2}{r(d-1)} - 1$.
- *(D')* $\frac{2(d+1)}{d-1} \leq q \leq \infty$, $\frac{d-1}{d+1}\frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{p}$ *and* $\frac{1}{q} > \frac{1}{dr}$ *for* $\frac{2(d+1)}{d(d-1)} < r \leq \frac{q(d-1)}{d+1}$.
- *(E')* $\frac{2(d+1)}{d-1} \leq q \leq \infty$, $\frac{1}{q} \leq \frac{d-1}{d+1}\frac{1}{p}$, $\frac{1}{q} \leq \frac{1}{p} \leq 1 - \frac{d+1}{d-1}\frac{1}{q}$ *and* $\frac{1}{q} > \frac{1}{dr}$ *for* $\frac{2(d+1)}{d(d-1)} < r \leq p$.

Proof. It suffices to check that the exponents appearing in the inequalities $A - E$ in Proposition 4.3 are strictly greater than $1/r$ under the claimed conditions.

(A') The exponent in (A), Proposition 4.3, is $\frac{d-1}{p'}$. Note that $\frac{d-1}{p'} > \frac{1}{r}$ is satisfied if $\frac{d}{d-1} < r \leq p$. Moreover, it also holds if $\frac{1}{p} < \frac{(d-1)r-1}{(d-1)r}$ and

$r \le \frac{d}{d-1}$. The additional constraint $r > \frac{2}{d-1}$ follows since $p \ge 2$ in (A). Note this requires $d \ge 3$.

(B') The exponent in (B), Proposition 4.3, is $\frac{d-1}{2}(\frac{1}{q}+\frac{1}{2})$. Note that $\frac{d-1}{2}(\frac{1}{q}+\frac{1}{2}) > \frac{1}{r}$ is satisfied if $\frac{2(d+1)}{d(d-1)} < r \le 2$, as $q \le \frac{2(d+1)}{d-1}$. Moreover, it also holds if $\frac{1}{q} > \frac{2}{(d-1)r} - \frac{1}{2}$ and $r \le \frac{2(d+1)}{d(d-1)}$. The additional constraint $r > \frac{2}{d-1}$ follows since $q \ge 2$ in (B). Note this requires $d \ge 3$.

(C') The exponent in (C), Proposition 4.3, is $\frac{d-1}{2}(\frac{1}{q}+\frac{1}{p'})$. Note that $\frac{d-1}{2}(\frac{1}{q}+\frac{1}{p'}) > \frac{1}{r}$ is satisfied if $\frac{d^2+1}{d(d-1)} < r \le p$, as $\frac{1}{q} \ge \frac{d-1}{d+1}\frac{1}{p'}$. The additional constraint $\frac{1}{q} > \frac{d+1}{d-1}\frac{1}{p} - 1$ follows from $r \le p$. Note that this and $q \ge p'$, also yield the additional constraint $\frac{1}{p} < \frac{d-1}{d}$.

For the remaining values $r \le \frac{d^2+1}{d(d-1)}$, it simply holds by the assumption $\frac{1}{q} > \frac{1}{p} + \frac{2}{r(d-1)} - 1$. Note that $r \le p$ is automatically satisfied if $r \le \frac{d}{d-1}$. The lower bound $r > \frac{2}{d-1}$ follows from the assumption $\frac{1}{q} > \frac{1}{p} + \frac{2}{r(d-1)} - 1$ with $q \ge p'$ and $p \le 2$. This yields $\frac{2}{d-1} < r \le p \le 2$, which requires $d \ge 3$.

(D') The exponent in (D), Proposition 4.3, is $\frac{d}{q}$. Note that $\frac{d}{q} > \frac{1}{r}$ is trivially satisfied if $\frac{1}{q} > \frac{1}{dr}$. The lower bound $r > \frac{2(d+1)}{d(d-1)}$, follows from $q \le \frac{2(d+1)}{d-1}$. Note that when combined with $r \le \frac{q(d-1)}{d+1}$ requires $d \ge 3$.

(E') The exponent in (E), Proposition 4.3, is $\frac{d}{q}$. Note that $\frac{d}{q} > \frac{1}{r}$ is trivially satisfied if $\frac{1}{q} > \frac{1}{dr}$. The constraint $r > \frac{2(d+1)}{d(d-1)}$ follows from $q \ge \frac{2(d+1)}{d-1}$. Note the above constraints combined yield the additional condition $\frac{1}{d} \le \frac{r}{q} \le \frac{d-1}{d+1}$, which requires $d \ge 3$. □

We next turn to the case $d = 2$. As observed in the proof of the previous proposition, the bounds in Proposition 4.3 do not yield any bound of the type (4.8) for $d = 2$. We use instead the upgraded bounds from Propositions 4.4 and 4.5.

Proposition 4.7. *Let $d = 2$. The inequality (4.8) holds for some $\varepsilon > 0$ under the following conditions on $1 \le p, q \le \infty$, $0 < r \le \infty$:*

(B_1') $\frac{1}{3p'} \le \frac{1}{q} \le \frac{1}{p}$ and $2 < q \le 4$, $2 < p \le 4$, and
- *$5/2 < r \le p$; or*
- *$2 < r \le \min\{5/2, p\}$ and $\frac{1}{q} > \frac{1}{p} + \frac{2}{r} - 1$.*

(B_2') $\frac{1}{q} \le \min\{\frac{1}{3p'}, \frac{1}{p}\}$, $4 \le q \le 6$ and $\frac{1}{q} > \frac{1}{2r}$ for $2 < r \le p$.

(D') $6 \le q \le \infty$, $\frac{1}{3p} \le \frac{1}{q} \le \frac{1}{p}$ and $\frac{1}{q} > \frac{1}{2r}$ for $3 < r \le p$.

Proof. As in Proposition 4.6, it suffices to check that the exponents appearing in the inequalities B_1, B_2 in Proposition 4.4 and in Proposition 4.5 are strictly greater than $1/r$ under the claimed conditions.

(B_1') The exponent in (B_1'), Proposition 4.4 is $\frac{1}{2}(\frac{1}{q}+\frac{1}{p'})-\varepsilon$. Choosing $\varepsilon>0$ small enough, $\frac{1}{2}(\frac{1}{q}+\frac{1}{p'})-\varepsilon>\frac{1}{r}$ is satisfied using $q\le 3p'$ and $5/2<r\le p$. If $r\le\min\{5/2,p\}$, the required condition follows simply by assumption choosing $\varepsilon>0$ to be small enough. Note that $r>2$ follows from the assumptions $\frac{1}{2}(\frac{1}{q}+\frac{1}{p'})>\frac{1}{r}$ and $p\le q$.

(B_2') The exponent in (B_2'), Proposition 4.4 is $2/q-\varepsilon$. Choosing $\varepsilon>0$ small enough, $2/q-\varepsilon>\frac{1}{r}$ is trivially satisfied by the assumption $\frac{1}{q}>\frac{1}{2r}$. The lower bound $r>2$ follows from the assumptions $\frac{1}{q}>\frac{1}{2r}$ and $q\ge 4$.

(D') The exponent in (D'), Proposition 4.5 is $2/q-\varepsilon$. Choosing $\varepsilon>0$ small enough, $2/q-\varepsilon>\frac{1}{r}$ is trivially satisfied by the assumption $\frac{1}{q}>\frac{1}{2r}$. Note that the lower bound $r>3$ follows combining the assumptions $\frac{1}{q}>\frac{1}{2r}$ and $q\ge 6$. □

Combining Propositions 4.6 and 4.7 with the observations in §3.1 we get the following estimates for $V_r^I A$ for all $r\ge 1$. We use the trivial fact that $L^q(V_{r_0})$ is embedded in $L^q(V_{r_1})$ for $r_0<r_1$, which allows to overcome the $r\le p$ or $r\le\frac{q(d-1)}{d+1}$ constraints in the above Propositions.

Corollary 4.8. *Let $d\ge 3$. $V_r^I A : L^p\to L^q$ is bounded if one of the following conditions is satisfied:*

(i) $(\frac{1}{p},\frac{1}{q})$ belongs to the open line segment $(Q_1(r),Q_2)$ or the interior of the domain $\mathfrak{P}_d(r)$ in Theorem 1.2 ($r>\frac{d^2+1}{d(d-1)}$).

(ii) $(\frac{1}{p},\frac{1}{q})$ belongs to the open line segment $(Q_1(r),Q_2)$ or the interior of the domain $\mathfrak{P}_d(r)$ in Theorem 1.4 ($\frac{d}{d-1}<r\le\frac{d^2+1}{d(d-1)}$).

(iii) $(\frac{1}{p},\frac{1}{q})$ belongs to the open line segment $(Q_1(r),Q_2(r))$ or the interior of the domain $\mathfrak{Q}_d(r)$ in Theorem 1.5 ($1\le r\le\frac{d}{d-1}$ for $d\ge 4$ or $\frac{4}{3}<r\le\frac{3}{2}$ for $d=3$).

Corollary 4.9. *Let $d=2$. $V_r^I A : L^p\to L^q$ is bounded if one of the following conditions is satisfied:*

(i) $(\frac{1}{p},\frac{1}{q})$ belongs to the open line segment $(Q_1(r),Q_2)$ or the interior of the domain $\mathfrak{Q}_2(r)$ in Theorem 1.6, (i) ($r>\frac{5}{2}$).

(ii) $(\frac{1}{p},\frac{1}{q})$ belongs to the open line segment $(Q_1(r),Q_2)$ or the interior of the domain $\mathfrak{Q}_2(r)$ in Theorem 1.6, (ii) ($2<r\le\frac{5}{2}$).

4.5. *Various endpoint bounds.* We shall discuss various endpoint bounds that can be obtained by interpolation (in particular Bourgain's interpolation lemma as formulated in §2.6). This will settle all endpoint results claimed in our theorems except for a more sophisticated strong type bound at the lower edges which will be discussed in the two subsequent sections.

We start by looking at the point Q_3.

Lemma 4.10. *Let $d\ge 3$, $r>\frac{d}{d-1}$. Let $p_3=\frac{d}{d-1}$, $q_3=d$.*

Then $\mathcal{A} : L^{p_3,1} \to L^{q_3,\infty}(B^{1/r}_{r,1})$ is bounded. Consequently, $V_r^I A$ is of restricted weak type at Q_3 in Theorems 1.2 and 1.4.

Proof. By standard embedding theorems, we can assume $r \le 2$. For $r > \frac{d}{d-1}$ we have $\frac{d-1}{r'} - \frac{1}{r} > 0$. We have from Corollary 2.3 and Proposition 4.3, (A),

$$\|\mathcal{A}_j f\|_{L^\infty(L^r)} \lesssim 2^{j(1-1/r)}\|f\|_1,$$
$$\|\mathcal{A}_j f\|_{L^{r'}(L^r)} \lesssim 2^{-j\frac{d-1}{r'}}\|f\|_r,$$

and by Corollary 2.6

$$\|\mathcal{A}_j f\|_{L^\infty(B^{1/r}_{r,1})} \lesssim 2^j\|f\|_1,$$
$$\|\mathcal{A}_j f\|_{L^{r'}(B^{1/r}_{r,1})} \lesssim 2^{-j(\frac{d-1}{r'}-\frac{1}{r})}\|f\|_r.$$

The lemma then follows by applying §2.6 to the last two inequalities. The bound for $V_r A$ is a simple corollary in view of (2.4). □

A similar argument yields a restricted weak type bound at Q_4.

Lemma 4.11. *Let $d \ge 3$, $r > \frac{d^2+1}{d(d-1)}$ and $p_4 = \frac{d^2+1}{d(d-1)}$, $q_4 = \frac{d^2+1}{d-1}$.*

Then $\mathcal{A} : L^{p_4,1} \to L^{q_4,\infty}(B^{1/r}_{r,1})$ is bounded. Consequently, $V_r^I A$ is of restricted weak type at Q_4 in Theorem 1.2.

Proof. By standard embedding theorems, we can assume $r \le 2$. By assumption on r we have $\frac{d(d-1)}{(d+1)r'} - \frac{1}{r} > 0$. It then suffices to interpolate using §2.6 the inequalities

$$\|\mathcal{A}_j f\|_{L^\infty(B^{1/r}_{r,1})} \lesssim 2^j\|f\|_{L^1}$$
$$\|\mathcal{A}_j f\|_{L^{q_\circ}(B^{1/r}_{r,1})} \lesssim 2^{-j(\frac{d}{q_\circ}-\frac{1}{r})} \lesssim \|f\|_{L^{p_\circ}} \quad \text{with } p_\circ = r, \quad q_\circ = \tfrac{d+1}{d-1}r';$$

the last inequality follows from Proposition 4.3, (E). □

Corollary 4.12. *Let $d \ge 3$. Then the following hold:*

(i) $V_r^I A : L^p \to L^q$ is bounded if $(1/p, 1/q)$ belongs to the open segment (Q_3, Q_4) in Theorem 1.2 ($r > \frac{d^2+1}{d(d-1)}$).

(ii) $V_r^I A : L^{p,1} \to L^q$ is bounded if $(1/p, 1/q)$ belongs to the half-open segment $[Q_2, Q_3)$ in Theorems 1.2 and 1.4 ($r > \frac{d}{d-1}$).

Proof. Part (i) just follows from interpolation between Lemma 4.10 and 4.11.

For part (ii), let $p = \frac{d}{d-1}$ and fix $q_2 = \frac{d}{d-1}$ and $q_3 = d$. For $\mathfrak{z} \in \mathbb{Z}^d$, let $Q_\mathfrak{z} = \prod_{i=1}^d [\mathfrak{z}_i, \mathfrak{z}_i + 1)$ and let $Q^*_\mathfrak{z}$ be a cube centered at $\mathfrak{z}$ with sidelength $20d$. Write $f = \sum_\mathfrak{z} f_\mathfrak{z}$ with $f_\mathfrak{z} = f\mathbb{1}_{Q_\mathfrak{z}}$. As $V_r^I A$ is local and the $Q^*_\mathfrak{z}$ have bounded overlap, by Hölder's inequality

$$\|V_r^I Af\|_{L^{q_2,\infty}} \le \Big(\sum_\mathfrak{z} \|\mathbb{1}_{Q^*_\mathfrak{z}} V_r^I A f_\mathfrak{z}\|^{q_2}_{L^{q_2,\infty}}\Big)^{1/q_2} \le \Big(\sum_\mathfrak{z} \|\mathbb{1}_{Q^*_\mathfrak{z}} V_r^I A f_\mathfrak{z}\|^{q_2}_{L^{q_3,\infty}}\Big)^{1/q_2}.$$

By Lemma 4.10, the right-hand side is further bounded by

$$\Big(\sum_{\mathfrak{z}} \|f_{\mathfrak{z}}\|_{L^{p,1}}^{q_2}\Big)^{1/q_2} \lesssim \|f\|_{p,1},$$

as $p = q_2 = p_3 = \frac{d}{d-1}$. This implies that $V_r^I A$ is of restricted weak type at Q_2 if $r > \frac{d}{d-1}$. By interpolation between Q_2 and Q_3, one has that $V_r^I A$ is of restricted strong type on the open line segment (Q_2, Q_3). Finally, the restricted strong type at Q_2 follows from the above localization argument, but using any of the just obtained $L^{p,1} \to L^q$ inequalities for $q_2 < q < q_3$ instead of the $L^{p,1} \to L^{q_3,\infty}$. □

Remark. One can obtain that $V_r^I A$ is of restricted weak type at Q_2 in Theorems 1.2 and 1.4 ($r > d/(d-1)$) by an application of §2.6 with the inequalities

$$\|\mathcal{A}_j f\|_{L^1(B^{1/r}_{r,1})} \lesssim 2^j \|f\|_1$$
$$\|\mathcal{A}_j f\|_{L^2(B^{1/r}_{r,1})} \lesssim 2^{-j(d-2)/2+j/r} \|f\|_2.$$

Interpolation with the restricted weak type bound at Q_3 yields the restricted strong type bounds on the open line segment (Q_2, Q_3). However, in order to deduce the restricted strong type at Q_2 we need to argue with the localization argument presented in the proof of Corollary 4.12 above.

We next address the claimed bounds for $V_1^I A$ in Theorem 1.5.

Lemma 4.13. *Let $d \geq 4$. The operator $\partial_t \mathcal{A}$ maps $L^{\frac{d-1}{d-2},1}$ boundedly to $L^{d-1,\infty}(L^1)$. Consequently, $V_r^I A$ is of restricted weak type at $Q_3(1)$ in Theorem 1.5.*

Proof. We have $\|\partial_t \mathcal{A}_j f\|_{L^2(L^1)} \lesssim \|\partial_t \mathcal{A}_j f\|_{L^2(L^2)} \lesssim 2^{-j\frac{d-3}{2}} \|f\|_2$. We interpolate the estimates (obtained from Corollary 2.3 and Proposition 4.3 together with Corollary 2.6)

$$\|\partial_t \mathcal{A}_j f\|_{L^\infty(L^1)} \lesssim 2^j \|f\|_1$$
$$\|\partial_t \mathcal{A}_j f\|_{L^2(L^1)} \lesssim 2^{-j\frac{d-3}{2}} \|f\|_2$$

and obtain the conclusion by application of §2.6. □

Corollary 4.14. *Let $d \geq 4$. The operator $V_r^I A : L^{p,1} \to L^q$ is bounded if $(1/p, 1/q)$ belongs to the half-open line segment $[Q_2(1), Q_3(1))$ in Theorem 1.5.*

Proof. The restricted strong type bounds on $[Q_2(1), Q_3(1))$ can be obtained as in Corollary 4.12. □

Lemma 4.15. *Let $d = 3$. The operator $V_1^I A$ is bounded on $L^2(\mathbb{R}^3)$.*

Proof. By (2.5) we have

$$\|V_1^I A\|_2 \leq \Big\| \int_I |\partial_t \mathcal{A} f(\cdot, t)| \, \mathrm{d}t \Big\|_2 \lesssim \Big(\iint |\partial_t \mathcal{A}_t f|^2 \, \mathrm{d}x \, \mathrm{d}t \Big)^{1/2} \lesssim \|f\|_2,$$

by (2.13) and orthogonality. □

5. A MAXIMAL OPERATOR

We first introduce an auxiliary maximal function which will be crucial in the proof of the endpoint bounds in §6.

For $L \in \mathbb{Z}$ let $\mathcal{Q}_L$ be the family of all cubes in $\mathbb{R}^d$ with side length in $(2^{L-1}, 2^L]$. Given Q we write

$$L(Q) = L \quad \text{ if } Q \in \mathcal{Q}_L. \tag{5.1}$$

We use the slashed integral to denote an average, i.e.

$$⨍_Q g(y)\,\mathrm{d}y = \frac{1}{|Q|}\int_Q g(y)\,\mathrm{d}y.$$

For $x \in \mathbb{R}^d$ we let $\mathcal{Q}_L(x)$ be the collection of all $Q \in \mathcal{Q}_L$ containing x. Given $n = 0, 1, 2, \dots$ and a sequence of functions $F = \{f_j\}_{j\geq 0}$, define the maximal function

$$\mathfrak{M}_{r,n}F(x) = \sup_{j\geq n}\sup_{Q\in\mathcal{Q}_{n-j}(x)} ⨍_Q \Big(\int |\mathcal{A}_j f_j(y,t)|^r\,\mathrm{d}t\Big)^{1/r}\,\mathrm{d}y. \tag{5.2}$$

The following result should be compared with Lemma 4.2. Away from the right boundary of the region in that lemma, we gain a crucial factor of $2^{-n\varepsilon}$. Related statements can be found in [36], [34] (see also [18] for dual versions).

Proposition 5.1. *Let p_0 and q_0 such that $1 < p_0 \leq q_0 < \infty$. Assume that*

$$\sup_{j\geq 0} 2^{jd/q_0}\|\mathcal{A}_j\|_{L^{p_0}\to L^{q_0}(L^{p_0})} < \infty. \tag{5.3}$$

Let $q_0 < q \leq \infty$ and $\frac{1}{\rho_{\max}(q)} = \frac{q_0}{q}\frac{1}{p_0}$ and $\frac{1}{\rho_{\min}(q)} = 1 - \frac{q_0}{q}(1-\frac{1}{p_0})$. Assume that

$$\rho_{\min}(q) < p \leq q \text{ and } \begin{cases} r \leq p & \text{if } \rho_{\min}(q) < p < \rho_{\max}(q), \\ r < \rho_{\max}(q) & \text{if } \rho_{\max}(q) \leq p \leq q. \end{cases} \tag{5.4}$$

Then there exists $\varepsilon(p,q,r) > 0$ such that

$$\|\mathfrak{M}_{r,n}F\|_q \leq C_{p,q,r}2^{-n\varepsilon(p,q,r)}\Big(\sum_{j\geq n} 2^{-jd}\|f_j\|_p^q\Big)^{1/q}. \tag{5.5}$$

For the proof we first observe a uniform estimate in n.

Lemma 5.2. *Let $p_0 \leq q_0$ and assume (5.3) holds. Let $q > 1$ and $q_0 \leq q \leq \infty$. Let $\rho_{\min}(q)$, $\rho_{\max}(q)$ be as in (4.2), and let $\rho_{\min}(q) \leq p \leq q$ and $0 < r \leq \min\{p, \rho_{\max}(q)\}$. Then*

$$\|\mathfrak{M}_{r,n}F\|_q \lesssim \Big(\sum_{j\geq n} 2^{-jd}\|f_j\|_p^q\Big)^{1/q}. \tag{5.6}$$

Proof. Let M_{HL} denote the Hardy–Littlewood maximal function. Then

$$|\mathfrak{M}_{r,n}F(x)| \lesssim \sup_{x\in Q} \fint_Q \sup_{j\ge n} \Big(\int |\mathcal{A}_j f_j(y,t)|^r \,\mathrm{d}t\Big)^{1/r} \,\mathrm{d}y \\ \le M_{HL}\big[\sup_{j\ge n} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})}\big](x)$$

and therefore, since $r \le q$ and $q > 1$

$$\|\mathfrak{M}_{r,n}F\|_q \lesssim \big\| \sup_{j\ge n} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})}\big\|_q \lesssim \Big\| \Big(\sum_{j\ge n} \|\mathcal{A}_j f_j\|^q_{L^r(\mathbb{R})}\Big)^{1/q}\Big\|_q \\ = \Big(\sum_{j\ge n} \|\mathcal{A}_j f_j\|^q_{L^q(L^r)}\Big)^{1/q} \lesssim \Big(\sum_{j\ge n} 2^{-jd} \|f_j\|_p^q\Big)^{1/q};$$

here in the last step we have used Lemma 4.2. □

We now show how to gain over this inequality in the special case $r = p_0$.

Lemma 5.3. *Let $p_0 \le q_0$ and assume* (5.3) *holds. Then for $q_0 \le q \le \infty$, $p_0 \le p \le q$*

$$\|\mathfrak{M}_{p_0,n}F\|_q \lesssim 2^{-nd(\frac{1}{q_0}-\frac{1}{q})} \Big(\sum_{j\ge n} 2^{-jd} \|f_j\|_p^q\Big)^{1/q}. \tag{5.7}$$

Proof. We use real interpolation for the sublinear operator $\mathfrak{M}_{p_0,n}$. Then (5.7) follows from

$$\|\mathfrak{M}_{p_0,n}F\|_{q_0} \lesssim \Big(\sum_{j\ge n} 2^{-jd} \|f_j\|_p^{q_0}\Big)^{1/q_0}, \quad p_0 \le p \le q_0, \tag{5.8a}$$

and

$$\|\mathfrak{M}_{p_0,n}F\|_\infty \lesssim 2^{-nd/q_0} \sup_{j\ge n} \|f_j\|_p, \quad p_0 \le p \le \infty. \tag{5.8b}$$

Note that (5.8a) immediately follows by Lemma 5.2.

We now show (5.8b). Fix $x \in \mathbb{R}^d$, $j \ge n$ and $Q \in \mathcal{Q}_{n-j}(x)$. Let R_x be a cube of diameter $20d$ centered at x. Then split

$$\fint_Q \Big(\int |\mathcal{A}_j f_j(y,t)|^{p_0} \,\mathrm{d}t\Big)^{1/p_0} \,\mathrm{d}y \le I(x) + II(x)$$

where

$$I(x) = \fint_Q \Big(\int |\mathcal{A}_j[\mathbb{1}_{R_x} f_j](y,t)|^{p_0} \,\mathrm{d}t\Big)^{1/p_0} \,\mathrm{d}y, \\ II(x) = \fint_Q \Big(\int |\mathcal{A}_j[\mathbb{1}_{R_x^\complement} f_j](y,t)|^{p_0} \,\mathrm{d}t\Big)^{1/p_0} \,\mathrm{d}y.$$

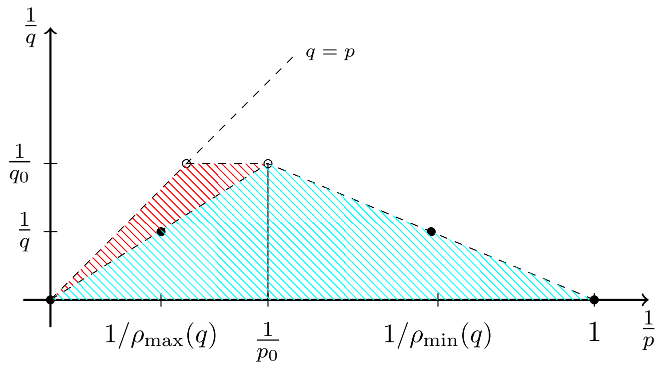

FIGURE 11. Bounds for $\mathfrak{M}_{r,n}$. At $r = p = p_0$, $q_0 < q \leq \infty$, we have a gain in n given by Lemma 5.3. Interpolation with the uniform estimates from Lemma 5.2 for $r = p = \rho_{\min}(p)$ and $r = p = \rho_{\max}(q)$ yields the estimates in the interior of the blue triangle. The bounds in the red triangle follow by the localization argument.

Using Hölder's inequality, then the assumption (5.3) and then again Hölder's inequality we get

$$\begin{aligned} I(x) &\leq \Big(\frac{1}{|Q|}\int \|\mathcal{A}_j[\mathbb{1}_{R_x} f_j](y,\cdot)\|_{p_0}^{q_0}\,\mathrm{d}y\Big)^{1/q_0} \\ &\lesssim |Q|^{-1/q_0} 2^{-jd/q_0}\|\mathbb{1}_{R_x} f_j\|_{p_0} \lesssim 2^{-nd/q_0}\|\mathbb{1}_{R_x} f_j\|_{p_0} \lesssim 2^{-nd/q_0}\|f_j\|_p, \end{aligned}$$

since $Q \in \mathcal{Q}_{n-j}$ and $p \geq p_0$.

Next we use estimate (2.8) (with $M > d$)

$$\begin{aligned} II(x) &\lesssim \fint_Q \int_{|y-w|\geq 1} C_M(2^j|y-w|)^{-M}|f_j(w)|\,\mathrm{d}w\,\mathrm{d}y \\ &\lesssim 2^{-jM}\|f_j\|_p \lesssim 2^{-nM}\|f_j\|_p. \end{aligned}$$

We combine the estimates for $I(x)$, $II(x)$ and then, after taking suprema in x, in $Q \in \mathcal{Q}_{n-j}(x)$ and in j, (5.8b) follows. $\square$

The proof of Proposition 5.1 follows from (carefully) interpolating the two previous lemmas and a localization argument, as indicated in Figure 11.

Conclusion of the proof of Proposition 5.1. We fix $q > q_0$. Observe that $\frac{1}{\rho_{\min}(q)} - \frac{1}{p_0} = (1-\frac{q_0}{q})(1-\frac{1}{p_0}) > 0$ and $\frac{1}{p_0} - \frac{1}{\rho_{\max}(q)} = (1-\frac{q_0}{q})\frac{1}{p_0} > 0$ so that $\rho_{\min}(q) < p_0 < \rho_{\max}(q)$. We first focus on the case $\rho_{\min}(q) < p < \rho_{\max}(q)$, for which it suffices to consider $r = p$; the corresponding inequality for smaller r follows by Hölder's inequality on $[1/2, 4]$. The remaining case $\rho_{\max}(q) \leq p \leq q$ will follow as a consequence of the previous range via a localization argument.

Let p be as in (5.4). In what follows set $w(j) = 2^{-jd}$ and let $\ell^q_w(L^p)$ be the space of L^p-valued sequences with

$$\|F\|_{\ell^q_w(L^p)} = \Big(\sum_j 2^{-jd}\|f_j\|_p^q\Big)^{1/q}.$$

By linearization it suffices to consider, for any measurable choices of positive integers $x \mapsto j(x) \in \mathbb{N}$, with $j(x) \geq n$, cubes $Q(x) \in \mathcal{Q}_{n-j}(x)$, and measurable $L^{r'}(\mathbb{R})$ valued functions $(x,y) \mapsto v(x,y,\cdot)$ in $L^\infty(\mathbb{R}^{2d})$, the bilinear operator

$$\mathcal{M}_n[F,v](x) = \fint_{Q(x)} \int v(x,y,s)\mathcal{A}_{j(x)} f_{j(x)}(y,s)\, \mathrm{d}s\, \mathrm{d}y$$

and show that

$$\|\mathcal{M}_n[F,v]\|_{L^q} \lesssim 2^{-n\varepsilon(p,q,r)}\|F\|_{\ell^q_w(L^p)}\|v\|_{L^\infty(L^{r'})}, \tag{5.9}$$

The conclusion for $r \leq p = p_0$ is immediate from Lemma 5.3, and in the study of the range $\rho_{\min}(q) < p < \rho_{\max}(q)$ we shall distinguish in what follows between the $\rho_{\min}(q) < p < p_0$ and $p_0 < p < \rho_{\max}(q)$.

The case $\rho_{\min}(q) < p \leq p_0$. It suffices to prove (5.9) for $r = p$. We have from (5.6)

$$\|\mathcal{M}_n[F,v]\|_{L^q} \lesssim \|F\|_{\ell^q_w(L^p)}\|v\|_{L^\infty(L^{p'})} \quad \text{for } p = \rho_{\min}(q) \tag{5.10}$$

and from (5.7)

$$\|\mathcal{M}_n[F,v]\|_{L^q} \lesssim 2^{-nd(\frac{1}{q_0}-\frac{1}{q})}\|F\|_{\ell^q_w(L^{p_0})}\|v\|_{L^\infty(L^{p_0'})}. \tag{5.11}$$

One can interpolate (5.10) and (5.11), noting that for $0<\theta<1$ and $\frac{1}{\rho_{\min}(q)} = 1 - \frac{q_0}{q}\frac{1}{p_0'}$,

$$\left.\begin{aligned} &\rho_{\min}(q) < p < p_0 \\ &\tfrac{1-\theta}{p_0} + \tfrac{\theta}{\rho_{\min}(q)} = \tfrac{1}{p} \end{aligned}\right\} \quad \Longrightarrow \quad (1-\theta)d(\tfrac{1}{q_0}-\tfrac{1}{q}) = \tfrac{dp_0'}{q_0}\big(\tfrac{1}{\rho_{\min}(q)} - \tfrac{1}{p}\big),$$

and deduce

$$\begin{aligned} &\|\mathcal{M}_n[F,v]\|_{L^q} \lesssim 2^{-n\varepsilon(p,q,r)}\|F\|_{\ell^q_w(L^p)}\|v\|_{L^\infty(L^{p'})}, \\ &\qquad\qquad \text{for } \rho_{\min}(q) \leq p \leq p_0, \quad \varepsilon(p,q,p) = \tfrac{dp_0'}{q_0}\big(\tfrac{1}{\rho_{\min}(q)} - \tfrac{1}{p}\big) > 0 \end{aligned} \tag{5.12}$$

with the implicit constants independent of the choices $j(x)$, $Q(x)$. Thus we also get (5.5) for $r \leq p$, $\rho_{\min} < p \leq p_0$ and $\varepsilon(p,q,r) = \varepsilon(p,q,p)$ as in (5.12).

In order to carry out the interpolation argument one uses Stein's interpolation theorem on analytic families of operators, with an obvious analytic family suggested by the proof of the Riesz–Thorin theorem; we omit the details. Alternatively one can use Calderón's second method $[\cdot,\cdot]^\theta$, combining a result on multilinear maps with a result on Banach lattices such as $L^\infty(X)$, see [10, §11.1], [10, §13.6].

The case $p_0 < p < \rho_{\max}(q)$. Again, it suffices to consider the case $r = p$. Note that for $0 < \theta < 1$ and $\frac{1}{\rho_{\max}(q)} = \frac{q_0}{q}\frac{1}{p_0}$

$$\left.\begin{array}{l} p_0 < p < \rho_{\max}(q) \\ \frac{1-\vartheta}{p_0} + \frac{\vartheta}{\rho_{\max}(q)} = \frac{1}{p} \end{array}\right\} \quad \Longrightarrow \quad (1-\vartheta)d(\tfrac{1}{q_0} - \tfrac{1}{q}) = \tfrac{dp_0}{q_0}\big(\tfrac{1}{p} - \tfrac{1}{\rho_{\max}(q)}\big) \tag{5.13}$$

We claim

$$\begin{aligned} &\|\mathcal{M}_n[F,v]\|_{L^q} \lesssim 2^{-n\varepsilon(p,q,p)} \|F\|_{\ell^q_w(L^p)} \|v\|_{L^\infty(L^{p'})}, \\ &\qquad\qquad \text{for } p_0 \le p \le \rho_{\max}(q), \quad \varepsilon(p,q) = \tfrac{dp_0}{q_0}\big(\tfrac{1}{p} - \tfrac{1}{\rho_{\max}(q)}\big) > 0, \end{aligned} \tag{5.14}$$

Given (5.13), the inequalities (5.14) can then be deduced by the above indicated interpolation arguments from

$$\|\mathcal{M}_n[F,v]\|_{L^q} \lesssim \|F\|_{\ell^q_w(L^p)} \|v\|_{L^\infty(L^{p'})} \quad \text{for } p = \rho_{\max}(q) \tag{5.15}$$

and

$$\|\mathcal{M}_n[F,v]\|_{L^q} \lesssim 2^{-nd(\frac{1}{q_0} - \frac{1}{q})} \|F\|_{\ell^q_w(L^{p_0})} \|v\|_{L^\infty(L^{p_0'})}. \tag{5.16}$$

Note that (5.15), (5.16) are immediate consequences of (5.6) and (5.7), respectively.

The case $\rho_{\max}(q) \le p \le q$, $0 < r < \rho_{\max}(q)$. This case just follows by the localization argument used in the proof of Lemma 4.1 via the kernel estimates (2.8), which allows to show that if

$$\|\mathfrak{M}_{r,n} F\|_q \lesssim 2^{-n\varepsilon} \Big(\sum_{j \ge n} 2^{-jd} \|f_j\|_{p_*}^q\Big)^{1/q}$$

holds for all $0 < r \le r_*$ and some $1 \le p_* \le q$, then

$$\|\mathfrak{M}_{r,n} F\|_q \lesssim 2^{-n\varepsilon} \Big(\sum_{j \ge n} 2^{-jd} \|f_j\|_{p}^q\Big)^{1/q}$$

also holds for all $p_* \le p \le q$ and all $0 < r \le r_*$. For fixed $q \in (q_0, \infty]$, the desired estimates for $\rho_{\max}(q) \le p \le q$ then follow from the above with input inequalities $r_* = p_* = \rho_{\max}(q) - \epsilon$ for ϵ arbitrarily small. Note that this argument has already been used in the context of $\mathfrak{M}_{r,n}$ in the proof of (5.8b) in Lemma 5.3. We omit the details. □

6. The sharp $L^p \to L^{pd}(L^p)$ bound

In this section we will give bounds for the sums of the operators $\mathcal{A}_j$ which, in particular, will cover the crucial endpoint bound at $P(r) = (\frac{1}{r}, \frac{1}{rd})$ in Theorem 1.2, as well as the remaining endpoints bounds stated in Theorems 1.2, 1.4 and 1.5.

Proposition 6.1. *Let $1 < p_0 \le q_0 < \infty$. Assume that*

$$\sup_{j \ge 0} 2^{jd/q_0} \|\mathcal{A}_j\|_{L^{p_0} \to L^{q_0}(L^{p_0})} \le C_0 \le \infty. \tag{6.1}$$

Let $q_0 < q < \infty$ *and define* $\frac{1}{\rho_{\max}(q)} = \frac{q_0}{q}\frac{1}{p_0}$ *and* $\frac{1}{\rho_{\min}(q)} = 1 - \frac{q_0}{q}(1-\frac{1}{p_0})$. *Assume that* p, r *are as in* (5.4), *i.e.*

$$\rho_{\min}(q) < p \le q \text{ and } \begin{cases} r \le p & \text{if } \rho_{\min}(q) < p < \rho_{\max}(q), \\ r < \rho_{\max}(q) & \text{if } \rho_{\max}(q) \le p \le q. \end{cases}$$

Then for all $\{f_j\}_{j\ge 0}$,

$$\Big\| \sum_{j\ge 0} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})} \Big\|_{L^q(\mathbb{R}^d)} \le C(p,q)(1+C_0)\Big(\sum_{j\ge 0} 2^{-jd} \|f_j\|_p^q \Big)^{1/q}. \tag{6.2}$$

Proof. We first observe that by the monotone convergence theorem it suffices to show (6.2) for any *finite* collection of functions $\{f_j\}_{j=0}^{\mathfrak{n}-1}$, with uniform bounds in $\mathfrak{n} \in \mathbb{N}$; moreover, all f_j can be assumed to be in the Schwartz class. From Lemma 4.2 we get

$$\Big\| \sum_{j=0}^{\mathfrak{n}-1} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})} \Big\|_{L^q(\mathbb{R}^d)} \lesssim \mathfrak{n}^{1-1/q} \Big(\sum_j 2^{-jd} \|f_j\|_p^q \Big)^{1/q}, \quad q_0 \le q \le \infty \tag{6.3}$$

and it is our task to remove the $\mathfrak{n}$-dependence in this estimate for p, q, r as in the statement of the Proposition.

For a function $G \in L^{q_0}(\mathbb{R}^d)$ we recall the definition of the Fefferman–Stein sharp maximal function

$$G^{\#}(x) := \sup_{x\in Q} \fint_Q \Big| G(y) - \fint_Q G(w)\,\mathrm{d}w \Big| \,\mathrm{d}y$$

which satisfies the bound $\|G\|_q \le c(q)\|G^{\#}\|_q$ for every q with $q_0 < q < \infty$, and the implicit constant in this inequality is independent of the L^{q_0}-norm of G. This was proved in [15]. We may apply this inequality to

$$G(x) = \sum_{j\ge 0} \Big(\int |\mathcal{A}_j f_j(x,t)|^r \,\mathrm{d}t \Big)^{1/r},$$

as its L^{q_0}-norm is finite by (6.3); recall the sum is assumed to be finite. Let $\mathcal{Q}(x) = \cup_{L\in\mathbb{Z}} \mathcal{Q}_L(x)$, i.e. the family of cubes containing x. We estimate

$$G^{\sharp}(x) \lesssim \mathcal{G}_I(x) + \mathcal{G}_{II}(x) + \mathcal{G}_{III}(x)$$

where, with $L(Q)$ as in (5.1),

$$\mathcal{G}_I(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\le 0}} \fint_Q \Big| \sum_{0\le j\le -L(Q)} \Big(\|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} - \fint_Q \|\mathcal{A}_j f_j(w,\cdot)\|_{L^r} \,\mathrm{d}w \Big) \Big| \,\mathrm{d}y,$$

$$\mathcal{G}_{II}(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\le 0}} \fint_Q \sum_{j\ge -L(Q)} \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} \,\mathrm{d}y,$$

$$\mathcal{G}_{III}(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)> 0}} \fint_Q \sum_{j\ge 0} \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} \,\mathrm{d}y.$$

The estimate for $\mathcal{G}_{III}$ follows from the estimate for $\mathcal{G}_{II}$. To see this let

$$U(y) = \sum_{j\geq 0} \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r}, \qquad U_*(w) = \sup_{\substack{Q\in\mathcal{Q}(w)\\ L(Q)=0}} \fint_Q U(y)\,\mathrm{d}y.$$

Given a cube $\widetilde{Q} \in \mathcal{Q}(x)$ with $L(\widetilde{Q}) > 0$ we may tile $\widetilde{Q}$ into cubes of side length 1 and get

$$\fint_{\widetilde{Q}} U(y)\,\mathrm{d}y \leq \fint_{\widetilde{Q}} U_*(w)\,\mathrm{d}w \leq M_{HL}[U_*](x)$$

where M_{HL} denotes the Hardy–Littlewood maximal operator. By a very crude estimate we can replace U_* by $\mathcal{G}_{II}$ and get

$$\mathcal{G}_{III}(x) \leq M_{HL}[\mathcal{G}_{II}](x). \tag{6.4}$$

The term $\mathcal{G}_{II}$ is the most interesting but it has been already taken care of in §5. We can write, with $U_j(y) := \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r}$

$$\begin{aligned}\mathcal{G}_{II}(x) &= \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leq 0}} \fint_Q \sum_{n=0}^{\infty} U_{-L(Q)+n}(y)\,\mathrm{d}y \\ &\leq \sum_{n=0}^{\infty} \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leq 0}} \fint_Q U_{-L(Q)+n}(y)\,\mathrm{d}y = \sum_{n=0}^{\infty} \sup_{j\geq n} \sup_{Q\in\mathcal{Q}_{n-j}(x)} \fint_Q U_j(y)\,\mathrm{d}y.\end{aligned}$$

Hence, with $\mathfrak{M}_{r,n}$ defined in (5.2) and $F = \{f_j\}_{j\geq 0}$, we get

$$\mathcal{G}_{II}(x) \leq \sum_{n\geq 0} \mathfrak{M}_{r,n} F(x).$$

From Proposition 5.1, (6.4) and the L^q-boundedness of the Hardy-Littlewood maximal operator we obtain

$$\|\mathcal{G}_{II}\|_q + \|\mathcal{G}_{III}\|_q \lesssim \Big(\sum_{j\geq 0} 2^{-jd} \|f_j\|_p^q\Big)^{1/q} \tag{6.5}$$

for the range of (p,q,r) assumed in the proposition.

It remains to consider the term $\mathcal{G}_I$, where we can use the cancellative properties of the #-function. We will show that

$$\|\mathcal{G}_I\|_q \lesssim \Big(\sum_{j\geq 0} 2^{-jd} \|f_j\|_p^q\Big)^{1/q} \quad \text{for } \rho_{\min}(q) \leq p \leq q. \tag{6.6}$$

For $n \geq 0$ define

$$\mathcal{G}_{I,n}(x) := \sup_{j\geq 0} \sup_{Q\in\mathcal{Q}_{-n-j}(x)} \fint_Q \Big| \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} - \fint_Q \|\mathcal{A}_j f_j(w,\cdot)\|_{L^r}\,\mathrm{d}w \Big|\,\mathrm{d}y$$

and, arguing as for $\mathcal{G}_{II}$, we observe that $\mathcal{G}_I(x) \le \sum_{n\ge 0} \mathcal{G}_{I,n}(x)$. Our claim (6.6) will follow from the estimate

$$\|\mathcal{G}_{I,n}\|_q \lesssim 2^{-n}\Big(\sum_{j\ge 0} 2^{-jd}\|f_j\|_p^q\Big)^{1/q} \quad \text{for } \rho_{\min}(q) \le p \le q \tag{6.7}$$

uniformly in n. In order to show this, note that by the triangle inequality

$$\mathcal{G}_{I,n}(x) \le \sup_{j\ge 0} \sup_{Q\in\mathcal{Q}_{-n-j}(x)} \fint_Q \fint_Q \big\|\mathcal{A}_j f_j(y,\cdot) - \mathcal{A}_j f_j(w,\cdot)\big\|_{L^r} \, \mathrm{d}w \, \mathrm{d}y.$$

Write $\mathcal{A}_j f_j = \widetilde{L}_j \mathcal{A}_j f_j$ and let θ_j be the convolution kernel of $\widetilde{L}_j$. Then for $j \le n$, $x, y, w \in Q$, $Q \in \mathcal{Q}_{-n-j}$

$$\begin{aligned}
&\Big(\int \big|\mathcal{A}_j f_j(y,t) - \mathcal{A}_j f_j(w,t)\big|^r \, \mathrm{d}t\Big)^{1/r} \\
&\le \Big(\int \Big| \int (\theta_j(y-z) - \theta_j(w-z))\mathcal{A}_j f_j(z,t) \, \mathrm{d}z\Big|^r \, \mathrm{d}t\Big)^{1/r} \\
&\le \int \int_0^1 |\langle y-w, \nabla\theta_j(w+\tau(y-w)-z)\rangle| \, \mathrm{d}\tau \Big(\int \big|\mathcal{A}_j f_j(z,t)\big|^r \, \mathrm{d}t\Big)^{1/r} \mathrm{d}z.
\end{aligned}$$

Since $1+2^j|w+\tau(y-w)-z| \approx 1+2^j|x-z|$ for $x,y,w \in Q$, $Q \in \mathcal{Q}_{-n-j}(x)$, $\tau \in [0,1]$, we can estimate the last expression by

$$C_N 2^{-n-j} \int \frac{2^{j(d+1)}}{(1+2^j|x-z|)^N} \Big(\int |\mathcal{A}_j f_j(z,t)|^r \, \mathrm{d}t\Big)^{1/r} \mathrm{d}z$$

and hence we get for $n \ge 0$ and $N > d$

$$|\mathcal{G}_{I,n}(x)| \lesssim 2^{-n} M_{HL}\big[\sup_{j\ge 0} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})}\big](x).$$

This implies, using the L^q boundedness of the Hardy–Littlewood maximal operator M_{HL} and Lemma 4.2,

$$\begin{aligned}
\|\mathcal{G}_{I,n}\|_{L^q} &\lesssim 2^{-n}\big\|\sup_{j\ge 0}\|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})}\big\|_{L^q} \lesssim 2^{-n}\Big\|\Big(\sum_{j\ge 0}\|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})}^q\Big)^{1/q}\Big\|_{L^q} \\
&= 2^{-n}\Big(\sum_{j\ge 0}\|\mathcal{A}_j f_j\|_{L^q(L^r)}^q\Big)^{1/q} \lesssim 2^{-n}\Big(\sum_{j\ge 0} 2^{-jd}\|f_j\|_p^q\Big)^{1/q},
\end{aligned}$$

which is (6.7) and thus (6.6) is proved. The proof is complete after combining (6.6) and (6.5). □

As (6.1) holds with $p_0 = 2$, $q_0 = \frac{2(d+1)}{d-1}$ by Lemma 2.4, Proposition 6.1 yields the following.

Corollary 6.2. *Assume that*

$$\tfrac{2(d+1)}{d-1} < q < \infty, \quad \tfrac{d+1}{d-1}\tfrac{1}{q} < \tfrac{1}{p} < 1 - \tfrac{d+1}{d-1}\tfrac{1}{q}, \quad r \le p \tag{6.8}$$

or

$$\tfrac{2(d+1)}{d-1} < q < \infty, \quad \tfrac{1}{q} \le \tfrac{1}{p} \le \tfrac{d+1}{d-1}\tfrac{1}{q}, \quad r < \tfrac{q(d-1)}{d+1}. \tag{6.9}$$

Then for all $\{f_j\}_{j\geq 0}$,

$$\Big\| \sum_{j\geq 0} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})} \Big\|_{L^q(\mathbb{R}^d)} \lesssim \Big(\sum_{j\geq 0} 2^{-jd} \|f_j\|_p^q \Big)^{1/q}.$$

Combining this with Lemma 2.7 one obtains the strong type bound at $P(r)$ in Theorem 1.2 (that is, Corollary 1.3).

Proposition 6.3. *Let* $d \geq 3$, $\frac{d^2+1}{d(d-1)} < p < \infty$. *Then*

$$\|\mathcal{A}f\|_{L^{pd}(B^{1/p}_{p,1})} \leq \|f\|_{L^p}. \tag{6.10}$$

Moreover,

$$\|V_p^I Af\|_{L^{pd}} \leq \|f\|_{L^p}. \tag{6.11}$$

Proof. The bound (6.11) follows from (6.10) via (2.4).

To prove (6.10) we apply Corollary 6.2 and Lemma 2.7 with $s = d/q = 1/p$. We verify the assumption (6.8) for $q = pd$, $r = p$. The condition $\frac{d+1}{d-1}\frac{1}{q} < \frac{1}{p}$ is satisfied (for $q = pd$) when $d^2 - 2d - 1 > 0$, which holds when $d \geq 3$. The condition $\frac{1}{p} < 1 - \frac{d+1}{d-1}\frac{1}{q}$ is satisfied (for $q = pd$) if $p > \frac{d^2+1}{d(d-1)}$. The condition $q > \frac{2(d+1)}{d-1}$ is also satisfied if $q = pd$, $p > \frac{d^2+1}{d(d-1)}$. In particular, the latter implies that $q = pd > 2$ in this range, so the hypothesis of Lemma 2.7 are also satisfied and thus (6.10) follows. □

Arguing in a similar way, we obtain the remaining claimed endpoint bounds in Theorems 1.2, 1.4 and 1.5.

Proposition 6.4. *Let* $d \geq 3$ *and* $r > \frac{d^2+1}{d(d-1)}$.

(i) Let $r \leq p \leq rd$. *Then the operators* $\mathcal{A} : L^p \to L^{rd}(B^{1/r}_{r,1})$ *and* $V_r^I A : L^p \to L^{rd}$ *are bounded. That is,* $V_r^I : L^p \to L^q$ *is bounded if* $(1/p, 1/q)$ *belongs to the closed segment* $[Q_1(r), P(r)]$ *in Theorem 1.2.*

(ii) Let $\frac{d^2+1}{d(d-1)} < p \leq r$. *Then the operators* $\mathcal{A} : L^p \to L^{pd}(B^{1/r}_{r,1})$ *and* $V_r^I A : L^p \to L^{pd}$ *are bounded. That is,* $V_r^I : L^p \to L^q$ *is bounded if* $(1/p, 1/q)$ *belongs to the half-open segment* $[P(r), Q_4)$ *in Theorem 1.2.*

Proof. For part (i) we use again Corollary 6.2 and Lemma 2.7 with $s = d/q = 1/r$. The condition (6.8) yields the ranges $\frac{rd(d-1)}{rd(d-1)-(d+1)} < p < \frac{rd(d-1)}{d+1}$ and $r \leq p$. Note that $r > \frac{rd(d-1)}{rd(d-1)-(d+1)}$ if and only if $r > \frac{d^2+1}{d(d-1)}$; moreover, the condition $q > \frac{2(d+1)}{d-1}$ (for $q = rd$) is satisfied in the range $r > \frac{d^2+1}{d(d-1)}$ whenever $d^2 - 2d - 1 > 0$, which holds for $d \geq 3$. This settles the range $r \leq p < \frac{rd(d-1)}{d+1}$. The range $\frac{rd(d-1)}{d+1} \leq p \leq rd$ corresponds to (6.9). Note that the condition $r < \frac{q(d-1)}{d+1}$ requires (for $q = rd$) $d^2 - 2d - 1 > 0$, which holds when $d \geq 3$. The condition $q > \frac{2(d+1)}{d-1}$ was already verified for (6.8). This concludes the bounds in (i).

Part (ii) follows from standard embedding theorems from Proposition 6.3. □

Proposition 6.5. *Let $d \ge 4$ and $1 \le r \le \frac{d^2+1}{d(d-1)}$ or $d = 3$ and $4/3 < r \le 5/3$. Let $\frac{rd(d-1)}{rd(d-1)-d-1} < p \le rd$. Then the operators $\mathcal{A} : L^p \to L^{rd}(B^{1/r}_{r,1})$ and $V_r^I A : L^p \to L^{rd}$ are bounded. That is, $V_r^I : L^p \to L^q$ is bounded if $(1/p, 1/q)$ belongs to the half-open segment $[Q_1(r), Q_4(r))$ in Theorems 1.4 and 1.5.*

Proof. This follows arguing as in the proof of Proposition 6.4. The only difference is that in (6.8) the relevant range for p (for $q = rd$) if $r \le \frac{d^2+1}{d(d-1)}$ is $\frac{rd(d-1)}{rd(d-1)-d-1} < p < \frac{rd(d-1)}{d+1}$. Moreover, the condition $q > \frac{2(d+1)}{d-1}$ requires (for $q = rd$) that $r > \frac{2(d+1)}{d(d-1)}$. As $r \ge 1$, this condition is satisfied if $d^2 - 3d - 2 > 0$, which holds for $d \ge 4$. If $d = 3$, we require the restriction $r > 4/3$. □

7. An endpoint bound for the global variation

The purpose of this section is to prove Theorem 1.1.

It will be useful to work with the standard homogeneous Littlewood–Paley decomposition. We define $P_j f$, $j \in \mathbb{Z}$ by

$$\widehat{P_j f}(\xi) = \big(\beta_0(2^{-j}|\xi|) - \beta_0(2^{1-j}|\xi|)\big)\widehat{f}(\xi)$$

so that P_j localizes to frequencies of size $\approx 2^j$. We have $P_j = L_j$ for $j \ge 1$ and $L_0 f = \sum_{j\le 0} P_j f$ for $f \in L^p$, $p \in (1,\infty)$ with convergence in L^p. It will also be convenient to use reproducing operators $\widetilde{P}_j$ localizing to slightly larger frequency annuli, with $\widetilde{P}_j P_j = P_j$ for $j \in \mathbb{Z}$.

Let $j \ge 0$. We recall the definition

$$\mathcal{A}_j f(x,t) = \chi(t) A_t L_j f(x) = \chi(t) K_{j,t} * f(x) \text{ where } \widehat{K_{j,t}}(\xi) = \widehat{\sigma}(t\xi)\beta_j(|\xi|).$$

We combine this with dyadic dilations, and define for $k \in \mathbb{Z}$, $t \in [1/2, 4]$,

$$\mathcal{A}_j^k f(x,t) = \chi(t) A_{2^k t} P_{j-k} f(x) = \chi(t) K^k_{j,t} * f(x) \tag{7.1}$$

where $K^k_{j,t} = 2^{-kd} K_{j,t}(2^{-k}\cdot)$. Below we shall often use a scaled version of (2.8), namely

$$|K^k_{j,t}(x)| \lesssim_N 2^{-kd}(2^{j-k}|x|)^{-N}, \qquad |x| \ge 10 \cdot 2^k, \quad t \in [1/2, 4]. \tag{7.2}$$

We start recording the following special case of Proposition 4.3, which will be relevant for the proof of Theorem 1.1 (when $p = q$).

Corollary 7.1. *For $2 \le r \le \infty$, $\frac{r(d+1)}{d-1} \le q \le \infty$, $r \le p \le q$ we have*

$$\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-jd/q} \|f\|_p.$$

By Corollary 7.1 and rescaling we have

$$\|\mathcal{A}_j^k\|_{L^p \to L^p(L^r)} \lesssim 2^{-jd/p}, \quad \frac{r(d+1)}{d-1} \le p \le \infty. \tag{7.3}$$

One can improve over this result and extend it to sums in k whenever $\frac{r(d+1)}{d-1} < p < \infty$.

Proposition 7.2. *For $2 \le r < \infty$, $\frac{r(d+1)}{d-1} < p < \infty$ we have, for all $j \ge 1$,*

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} \|\mathcal{A}_j^k f\|_{L^r(\mathbb{R})}^r \Big)^{1/r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-jd/p} \|f\|_p. \tag{7.4}$$

Proof. As in the proof of Proposition 6.1, by the monotone convergence theorem, it suffices to show (7.4) for any finite collection $\{\mathcal{A}_j^k\}_{k\in K}$, with uniform bounds on the cardinality of the finite set $K \subset \mathbb{Z}$.

We use again the sharp maximal function of Fefferman–Stein. Let

$$G(x) = \big(\sum_{k \in K} \|\mathcal{A}_j^k f(x, \cdot)\|_{L^r}^r \big)^{1/r},$$

which has finite L^{p_0} norm whenever $p_0 = \frac{r(d+1)}{d-1}$; note that (7.3) and Minkowski's inequality imply that

$$\|G\|_{p_0} \lesssim |K|^{1/r} 2^{-jd/p} \|f\|_{p_0}.$$

By the bound $\|G\|_p \lesssim_p \|G^{\#}\|_p$, it suffices to show $\|G^{\#}\|_p \lesssim 2^{-jd/p}\|f\|_p$, uniformly on the finite set K for $p_0 < p < \infty$. By the triangle inequality, we dominate

$$\begin{aligned} G^{\#}(x) &\le \sup_{Q \in \mathcal{Q}(x)} \fint_Q \fint_Q \Big(\sum_{k \in \mathbb{Z}} \|\mathcal{A}_j^k f(y, \cdot) - \mathcal{A}_j^k f(w, \cdot)\|_{L^r}^r \Big)^{1/r} \mathrm{d}w \, \mathrm{d}y \\ &\le 2\, \mathcal{G} f(x) + \sum_{n=1}^{\infty} \mathcal{U}_n f(x) \end{aligned}$$

where

$$\mathcal{G} f(x) := \sup_{L \in \mathbb{Z}} \sup_{Q \in \mathcal{Q}_L(x)} \fint_Q \Big(\sum_{k \le L} \|\mathcal{A}_j^k f(y, \cdot)\|_{L^r}^r \Big)^{1/r} \mathrm{d}y$$

and

$$\mathcal{U}_n f(x) := \sup_{L \in \mathbb{Z}} \sup_{Q \in \mathcal{Q}_L(x)} \fint_Q \fint_Q \big\| \mathcal{A}_j^{L+n} f(y, \cdot) - \mathcal{A}_j^{L+n} f(w, \cdot) \big\|_{L^r} \mathrm{d}w \, \mathrm{d}y.$$

We claim that for $\frac{r(d+1)}{d-1} \le p \le \infty$, $2 \le r < \infty$ we have

$$\|\mathcal{G} f\|_p \lesssim 2^{-j\alpha(r)} \|f\|_p, \quad \text{for some } \alpha(r) > d/p \tag{7.5}$$

and

$$\|\mathcal{U}_n f\|_p \lesssim \begin{cases} 2^{(n-j)d(\frac{d-1}{r(d+1)} - \frac{1}{p})} 2^{-jd/p} \|f\|_p, & \text{if } 1 \le n \le j, \\ 2^{j-n} 2^{-jd/p} \|f\|_p, & \text{if } n > j. \end{cases} \tag{7.6}$$

The desired bound $\|G^{\#}\|_p \lesssim 2^{-jd/p}\|f\|_p$ follows summing in n if $p > \frac{r(d+1)}{d-1}$.

Proof of (7.5). We prove inequalities for $\mathcal{G}$ on L^r and L^∞ which will yield (7.5) by interpolation.

Let $p_0 = \frac{r(d+1)}{d-1}$. We first observe the inequalities

$$\|\mathcal{A}_j^k f\|_{L^2(L^r)} \lesssim 2^{-j(\frac{d-2}{2}+\frac{1}{r})}\|f\|_{L^2} \tag{7.7}$$

$$\|\mathcal{A}_j^k f\|_{L^{p_0}(L^r)} \lesssim 2^{-jd/p_0}\|f\|_{L^{p_0}} \tag{7.8}$$

uniformly in k. The estimate (7.7) holds from the bounds $\|\mathcal{A}_j^k\|_{L^2\to L^2(L^2)} \lesssim 2^{-j(d-1)/2}$ and $\|\partial_t \mathcal{A}_j^k\|_{L^2\to L^2(L^2)} \lesssim 2^{-j(d-3)/2}$ (which follow from (2.13)) and the Sobolev embedding theorem, and (7.8) is (7.3) with $p = p_0$. Since $2 \le r < p_0$ and $\frac{d-2}{2} + \frac{1}{r} > \frac{d}{p_0}$ one has by interpolation that

$$\|\mathcal{A}_j^k f\|_{L^r(L^r)} \lesssim 2^{-j\alpha(r)}\|f\|_{L^r}, \quad \text{for some } \alpha(r) > d/p_0. \tag{7.9}$$

This implies

$$\begin{aligned}\|\mathcal{G}f\|_{L^r} &\lesssim \Big\| M_{HL}\big[\big(\sum_{k\in\mathbb{Z}} \|\mathcal{A}_j^k \widetilde{P}_{j-k} f\|_{L^r(\mathbb{R})}^r\big)^{1/r}\big]\Big\|_r \\ &\lesssim \Big(\sum_{k\in\mathbb{Z}} \|\mathcal{A}_j^k \widetilde{P}_{j-k} f\|_{L^r(L^r)}^r\Big)^{1/r} \\ &\lesssim 2^{-j\alpha(r)}\Big(\sum_{k\in\mathbb{Z}} \|\widetilde{P}_{j-k} f\|_r^r\Big)^{1/r} \lesssim 2^{-j\alpha(r)}\|f\|_r \end{aligned} \tag{7.10}$$

by Littlewood–Paley theory, since $r \ge 2$.

We now prove an L^∞ bound. Fix x, L, $Q \in \mathcal{Q}_L(x)$ and let B_x^L be the ball centered at x with radius $d2^{L+10}$. Using Hölder's inequality and the embedding $\ell^1 \subseteq \ell^r$ for $r \ge 1$ we estimate

$$⨍_Q \Big(\sum_{k\le L} \big\|\mathcal{A}_j^k f(y,\cdot)\big\|_{L^r}^r\Big)^{1/r} \mathrm{d}y \le I(x) + II(x)$$

where

$$I(x) = \Big(⨍_Q \sum_{k\le L} \big\|\mathcal{A}_j^k[\mathbb{1}_{B_x^L} f](y,\cdot)\big\|_{L^r}^r \,\mathrm{d}y\Big)^{1/r}$$

$$II(x) = ⨍_Q \sum_{k\le L} \big\|\mathcal{A}_j^k[\mathbb{1}_{\mathbb{R}^d\setminus B_x^L} f](y,\cdot)\big\|_{L^r} \,\mathrm{d}y.$$

We have, in view of $\mathcal{A}_j^k = \mathcal{A}_j^k \widetilde{P}_{j-k}$ and using (7.9),

$$\begin{aligned} I(x) &\lesssim 2^{-Ld/r}\Big(\sum_{k\le L} \big\|\mathcal{A}_j^k \widetilde{P}_{j-k}[\mathbb{1}_{B_x^L} f]\big\|_{L^r(L^r)}^r\Big)^{1/r} \\ &\lesssim 2^{-j\alpha(r)} 2^{-Ld/r}\Big(\sum_{k\le L} \big\|\widetilde{P}_{j-k}[\mathbb{1}_{B_x^L} f]\big\|_{L^r}^r\Big)^{1/r} \\ &\lesssim 2^{-j\alpha(r)} 2^{-Ld/r}\|\mathbb{1}_{B_x^L} f\|_{L^r} \lesssim 2^{-j\alpha(r)}\|f\|_{L^\infty}, \end{aligned}$$

using $r \geq 2$ and Littlewood–Paley theory in the third inequality. For the term $II(x)$ we use (7.2) and estimate

$$\begin{aligned} II(x) &\lesssim_N \sum_{k \leq L} \fint_Q \int_{\mathbb{R}^d \setminus B_x^L} \frac{2^{-kd}}{(2^{j-k}|y-w|)^N} |f(w)| \, \mathrm{d}w \, \mathrm{d}y \\ &\lesssim 2^{-jN} \sum_{k \leq L} 2^{(L-k)(d-N)} \|f\|_\infty \lesssim 2^{-jN} \|f\|_\infty, \end{aligned}$$

where $N > d$. We combine the estimates for $I(x)$ and $II(x)$ to obtain

$$\|\mathcal{G} f\|_\infty \lesssim 2^{-j\alpha(r)} \|f\|_\infty. \tag{7.11}$$

Interpolating (7.10) and (7.11) and noting that that $\alpha(r) > d/p_0 \geq d/p$ for $p \geq p_0$ we obtain (7.5).

Proof of (7.6) *for* $1 \leq n \leq j$. This case is similar to that of $\mathcal{G}$. Let $p_0 = \frac{r(d+1)}{d-1}$. We get the asserted estimate by interpolating between the inequalities

$$\|\mathcal{U}_n f\|_{p_0} \lesssim 2^{-jd/p_0} \|f\|_{p_0} \tag{7.12}$$

$$\|\mathcal{U}_n f\|_\infty \lesssim 2^{(n-j)d/p_0} \|f\|_\infty. \tag{7.13}$$

To see (7.12), we use $\mathcal{A}_j^k = \mathcal{A}_j^k \widetilde{P}_{j-k}$ and (7.8) to estimate

$$\begin{aligned} \|\mathcal{U}_n f\|_{p_0} &\lesssim \Big\| M_{HL} \big[\big(\sum_{k \in \mathbb{Z}} \|\mathcal{A}_j^k \widetilde{P}_{j-k} f\|_{L^r(\mathbb{R})}^{p_0} \big)^{1/p_0} \big] \Big\|_{p_0} \\ &\lesssim \Big(\sum_{k \in \mathbb{Z}} \|\mathcal{A}_j^k \widetilde{P}_{j-k} f\|_{L^{p_0}(L^r)}^{p_0} \Big)^{1/p_0} \\ &\lesssim 2^{-jd/p_0} \big(\sum_{k \in \mathbb{Z}} \|\widetilde{P}_{j-k} f\|_{p_0}^{p_0} \big)^{1/p_0} \lesssim 2^{-jd/p_0} \|f\|_{p_0}, \end{aligned}$$

using that $p_0 \geq 2$ and Littlewood–Paley theory in the last inequality.

To see (7.13) we fix x, L, $Q \in \mathcal{Q}_L(x)$, let B_x^{L+n} be the ball centered at x with radius $d2^{L+n+10}$ and estimate

$$\begin{aligned} &\fint_Q \fint_Q \big\| \mathcal{A}_j^{L+n} f(y, \cdot) - \mathcal{A}_j^{L+n} f(w, \cdot) \big\|_{L^r} \mathrm{d}w \, \mathrm{d}y \\ &\lesssim \fint_Q \big\| \mathcal{A}_j^{L+n} \widetilde{P}_{j-L-n} f(y, \cdot) \big\|_{L^r} \mathrm{d}y \leq III(x) + IV(x) \end{aligned}$$

where

$$\begin{aligned} III(x) &= \Big(\fint_Q \big\| \mathcal{A}_j^{L+n} [\mathbb{1}_{B_x^{L+n}} f](y, \cdot) \big\|_{L^r}^{p_0} \mathrm{d}y \Big)^{1/p_0}, \\ IV(x) &= \fint_Q \big\| \mathcal{A}_j^{L+n} [\mathbb{1}_{\mathbb{R}^d \setminus B_x^{L+n}} f](y, \cdot) \big\|_{L^r} \mathrm{d}y. \end{aligned}$$

We get by (7.3)

$$III(x) \lesssim 2^{-Ld/p_0} 2^{-jd/p_0} \|\mathbb{1}_{B_x^{L+n}} f\|_{p_0} \lesssim 2^{(n-j)d/p_0} \|f\|_\infty.$$

Moreover, by (7.2),

$$IV(x) \lesssim \fint_Q \int_{\mathbb{R}^d \setminus B_x^{L+n}} \frac{2^{-(L+n)d}}{(2^{j-L-n}|y-w|)^N} |f(w)|\, \mathrm{d}w\, \mathrm{d}y \lesssim 2^{-jN} \|f\|_\infty$$

for any $N > d$. The estimates for $III(x)$ and $IV(x)$ yield (7.13) for $1 \le n \le j$.

Proof of (7.6) *for* $n > j$. Here we use cancellation and note that for $x \in Q$

$$\fint_Q \fint_Q \big\| \widetilde{P}_{j-L(Q)-n} g(y,\cdot) - \widetilde{P}_{j-L(Q)-n} g(w,\cdot) \big\|_{L^r} \,\mathrm{d}w\, \mathrm{d}y \\ \lesssim 2^{j-n} M_{HL}[\|g\|_{L^r(\mathbb{R})}](x).$$

Using this with $g = \mathcal{A}_j^{L+n} f = \widetilde{P}_{j-L-n} \mathcal{A}_j^{L+n}$ and the Fefferman–Stein inequality for sequences of Hardy–Littlewood maximal functions, we may then estimate

$$\begin{aligned} &\Big\| \sup_{L\in\mathbb{Z}} \sup_{Q \in \mathcal{Q}_L(x)} \fint_Q \fint_Q \big\| \mathcal{A}_j^{L+n} f(y,\cdot) - \mathcal{A}_j^{L+n} f(w,\cdot) \big\|_{L^r} \,\mathrm{d}w\, \mathrm{d}y \Big\|_{L^p(dx)} \\ &\lesssim 2^{j-n} \Big\| \sup_{k\in\mathbb{Z}} M_{HL}\big[\|\mathcal{A}_j^k f\|_{L^r(\mathbb{R})}\big] \Big\|_p \lesssim 2^{j-n} \Big(\sum_{k\in\mathbb{Z}} \|\mathcal{A}_j^k \widetilde{P}_{j-k} f\|_{L^p(L^r)}^p \Big)^{1/p} \\ &\lesssim 2^{j-n} 2^{-jd/p} \Big(\sum_{k\in\mathbb{Z}} \|\widetilde{P}_{j-k} f\|_p^p \Big)^{1/p} \lesssim 2^{j-n} 2^{-jd/p} \|f\|_p \end{aligned}$$

for $\frac{r(d+1)}{d-1} \le p \le \infty$, using (7.3) in the third inequality and $p \ge 2$ and Littlewood–Paley theory in the last inequality. Thus (7.6) is established for $n > j$.

This finishes the proof of the proposition. □

Remark. The difficulty for putting the pieces together comes because it is assumed $\frac{r(d+1)}{d-1} < p$. If one had $r \ge p$, one can simply put pieces together by standard Littlewood–Paley theory as, for instance, in (7.10).

A consequence of Proposition 7.2 is the following restricted weak type bound.

Proposition 7.3. *For* $d \ge 3$, $r \ge 2$,

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \Big\| \sum_{j\ge1} \mathcal{A}_j^k f \Big\|_{B^{1/r}_{r,1}(\mathbb{R})}^r \Big)^{1/r} \Big\|_{L^{rd,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^{rd,1}(\mathbb{R}^d)}. \tag{7.14}$$

Proof. Write

$$\Lambda_l \mathcal{A}_j^k f(x,t) = 2^{-j(d-1)/2} (2\pi)^{-(d+1)} \sum_{\pm} \int 2^{-kd} \kappa_{j,l}^{\pm}(2^{-k}y, t) f(x-y) dy$$

where $\kappa_{j,l}^{\pm}$ is as in (2.17).

We first show that for all $j \ge 1$, $2 \le r < \infty$, $\frac{r(d+1)}{d-1} < p < \infty$,

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \|\mathcal{A}_j^k f\|^r_{B^{1/r}_{r,1}(\mathbb{R})} \Big)^{1/r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-j(d/p-1/r)} \|f\|_p. \tag{7.15}$$

Indeed, by Proposition 7.2, for $|j-l| \le 10$

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{l/r} \|\Lambda_l \mathcal{A}_j^k f\|^r_{L^r(\mathbb{R})} \Big)^{1/r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-j(d/p-1/r)} \|f\|_p. \tag{7.16}$$

Moreover for $|j-l| \ge 10$, we get from (2.18)

$$\begin{aligned} &\Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{l/r} \|\Lambda_l \mathcal{A}_j^k f\|^r_{L^r(\mathbb{R})} \Big)^{1/r} \Big\|_{L^p(\mathbb{R}^d)} \\ &\qquad \lesssim_N \min\{2^{-j(N-\frac1r)}, 2^{-l(N-\frac1r)}\} \Big\| \Big(\sum_{k\in\mathbb{Z}} \big| M_{HL}[\widetilde{P}_{j-k} f] \big|^r \Big)^{1/r} \Big\|_p \\ &\qquad \lesssim_N \min\{2^{-j(N-\frac1r)}, 2^{-l(N-\frac1r)}\} \|f\|_p, \end{aligned} \tag{7.17}$$

using that the Fefferman–Stein and Littlewood–Paley inequalities together imply

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \big| M_{HL}[\widetilde{P}_{j-k} f] \big|^r \Big)^{1/r} \Big\|_p \lesssim_p \|f\|_p, \quad 1 < p < \infty,\ r \ge 2.$$

Then (7.15) follows summing over $\ell \ge 0$ in (7.16) and (7.17).

We finish the proof using Bourgain's interpolation trick (see §2.6). Consider

$$\mathfrak{A}_j f(x) := \Big(\sum_{k\in\mathbb{Z}} \|\mathcal{A}_j^k f(x,\cdot)\|^r_{B^{1/r}_{r,1}(\mathbb{R})} \Big)^{1/r}.$$

Note that $\frac{r(d+1)}{d-1} < rd$ when $d^2 - 2d - 1 > 0$, that is, $d \ge 3$. Let p_0, p_1 be such that $\frac{r(d+1)}{d-1} < p_0 < rd < p_1$. By (7.15) we have that $\|\mathfrak{A}_j f\|_{p_i} \lesssim 2^{-j(d/p_i-1/r)} \|f\|_{p_i}$, $i = 0, 1$ and then a restricted weak type (rd, rd) inequality for $\sum_{j\ge0} \mathfrak{A}_j$ follows from Lemma 2.8. This implies the assertion. □

Conclusion of the proof of Theorem 1.1. Following [21], we write

$$V_r Af(x) \le V_r^{\mathrm{dyad}} Af(x) + V_r^{\mathrm{sh}} Af(x)$$

where

$$V_r^{\mathrm{dyad}} Af(x) := \sup_{N\in\mathbb{N}} \sup_{k_1<\dots<k_N} \Big(\sum_{i=1}^{N-1} |A_{2^{k_{i+1}}} f(x) - A_{2^{k_i}} f(x)|^r \Big)^{1/r}$$

is the *dyadic or long variation operator* and

$$V_r^{\mathrm{sh}} Af(x) := \Big(\sum_{k\in\mathbb{Z}} |V_r^{I_k} Af(x)|^r \Big)^{1/r}$$

is the *short variation operator*, using only variation within the dyadic intervals $I_k = [2^k, 2^{k+1}]$; recall that $V_r^{I_k} Af(x)$ denotes the r-variation of

$t \mapsto A_t f(x)$ over the interval I_k. It then suffices to establish the claimed bound in Theorem 1.1 for the operators V_r^{dyad} and V_r^{sh}.

Regarding $V_r^{\mathrm{dyad}} A$, the inequality

$$\|V_r^{\mathrm{dyad}} A f\|_p \lesssim_{p,r} \|f\|_p, \quad r > 2,\ 1 < p < \infty$$

was proved in [21]. This of course implies a L^p bound for $V_{p/d}$ if $p > 2d$ and, in particular, the claimed restricted weak type bound follows by the embedding $L^{p,1} \hookrightarrow L^p \hookrightarrow L^{p,\infty}$.

We next proceed with $V_r^{\mathrm{sh}} A$. Since $\chi(t) = 1$ on $I = [1,2]$ we get

$$V_r^{I_k} A f(x) \le \Big| \sum_{j=0}^{\infty} \mathcal{A}_j^k f(x,\cdot) \Big|_{V_r(I)}$$

by the definition of $\mathcal{A}_j^k$ in (7.1). The term corresponding to $j = 0$ is easily estimated by a square function

$$\Big(\sum_{k\in\mathbb{Z}} |\mathcal{A}_0^k f(x,\cdot)|_{V_r(I)}^r \Big)^{1/r} \lesssim \Big(\sum_{k\in\mathbb{Z}} \int_1^2 \big| \tfrac{d}{dt} \mathcal{A}_0^k f(x,t) \big|^2 \, \mathrm{d}t \Big)^{1/2}.$$

We claim for $1 < p < \infty$

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \int_1^2 \big| \tfrac{d}{dt} \mathcal{A}_0^k f(x,t) \big|^2 \, \mathrm{d}t \Big)^{1/2} \Big\|_p \le C_p \|f\|_p. \tag{7.18}$$

Since $\chi'(t) = 0$ for $1 \le t \le 2$ we have

$$\tfrac{d}{dt} \widehat{\mathcal{A}_0^k f}(\xi,t) = \chi(t) \langle 2^k \xi, \nabla \widehat{\sigma}(2^k t \xi) \rangle \beta_0(2^k |\xi|) \widehat{f}(\xi)$$

Using Plancherel's theorem and interchanging sums and integrals one gets (7.18) for $p = 2$. We then invoke standard Calderón–Zygmund theory in the Hilbert-space setting (see [42, ch. II.5]) to see that (7.18) holds in the full range $1 < p < \infty$. It follows that for $r \ge 2$

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mathcal{A}_0^k f(x,\cdot)|_{V_r(I)}^r \Big)^{1/r} \Big\|_p \lesssim_p \|f\|_p$$

which is stronger than the required $L^{p,1} \to L^{p,\infty}$ bound.

It remains to consider the cases $j \ge 1$. By the embedding (2.4) we have

$$\Big(\sum_{k\in\mathbb{Z}} \Big| \sum_{j\ge 1} \mathcal{A}_j^k f(x,\cdot) \Big|_{V_r}^r \Big)^{1/r} \lesssim \Big(\sum_{k\in\mathbb{Z}} \Big\| \sum_{j\ge 1} \mathcal{A}_j^k f(x,\cdot) \Big\|_{B^{1/r}_{r,1}}^r \Big)^{1/r}.$$

We apply the restricted weak type inequality of Proposition 7.3 to the expression on the right-hand side to conclude the desired bound for V_r^{sh}. This finishes the proof. □

Remark. If in two dimensions one has the conjectured local smoothing *endpoint* results for $p > 4$ then one can also show the restricted weak type $(2r, 2r)$ estimate (7.14) for $r > 2$. The conjectured endpoint estimate in the assumptions seems currently out of reach.

8. A sparse domination result

We conclude the paper with a discussion of the sparse domination result for the global $V_r A$ in Theorem 1.7. It is indeed an immediate consequence of a special case of a result on convolution operators with compactly supported distributions which can be found in [2, Prop.7.2].

We let u be a compactly supported distribution, define the dilate in the sense of distributions by $\langle u_t, f\rangle = \langle u, f(t\cdot)\rangle$ and let $Tf(x,t) = f * u_t$. For fixed x let $V_r Tf$ denote the r-variation norm of $t \mapsto Tf(x,t)$. As before let $I = [1,2]$ and $V_r^I f(x)$ the corresponding variation norm over I.

Theorem 8.1. [2]. *Let $1 < p \le q < \infty$, and $u \in \mathcal{S}'(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d \setminus \{0\}$.*

(i) Suppose that

$$\|V_r T\|_{L^p \to L^{p,\infty}} + \|V_r T\|_{L^{q,1} \to L^q} < \infty, \tag{8.1}$$

$$\|V_r^I T\|_{L^p \to L^q} < \infty, \tag{8.2}$$

and that there is an $\varepsilon > 0$ so that for all $\lambda \ge 2$, and all Schwartz function f with $\operatorname{supp} \widehat{f} \subset \{\xi : \lambda/2 < |\xi| < 2\lambda\}$,

$$\|V_r^I Tf\|_q \le C\lambda^{-\varepsilon}\|f\|_p. \tag{8.3}$$

Then there is a constant $C = C(p,q)$ such that for each pair of compactly supported bounded functions f_1, f_2 there is a sparse family of cubes $\mathfrak{S}(f_1, f_2)$ such that

$$\int V_r T f_1(x)\, f_2(x) dx \le C \sum_{Q \in \mathfrak{S}(f_1,f_2)} |Q| \langle f_1\rangle_{Q,p} \langle f_2\rangle_{Q,q'}. \tag{8.4}$$

(ii) Suppose that in addition $p < q$, and suppose that (8.4) *holds with a constant independently of f_1, f_2. Then conditions* (8.1), (8.2) *hold.*

Proof of Theorem 1.7. We let u be surface measure on the unit sphere. As discussed in the introduction the inequalities in (8.1) were already proved in the relevant ranges of Theorem 1.7 in [21]. The inequalities (8.2) and (8.3) in the asserted ranges follow from the single-scale frequency bounds in Propositions 4.6 and 4.7. Thus the sparse bounds in Theorem 1.7 are a consequence of part (i) of Theorem 8.1. The sharpness of the sparse bounds follows from part (ii); see also §3.9 for a direct argument. □

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