A LOWER BOUND FOR A VARIATION NORM OPERATOR ASSOCIATED WITH CIRCULAR MEANS

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Dedicated to Professor Hans Triebel on the occasion of his 90th birthday

Abstract. We prove that a local $L^p(V_2)$ variation norm estimate fails for circular means in two dimensions, and quantify this failure by proving lower bounds for functions of exponential type. This is related to lower bounds for Fourier multipliers supported on annuli, of the type considered by Córdoba.

1. Introduction

Consider the circular averages

$$A_t f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x_1 - t\cos\alpha, x_2 - t\sin\alpha)\, \mathrm{d}\alpha$$

for functions in $L^p(\mathbb{R}^2)$.

Let $I = [1,2]$ and let $V_r^I A$ be the operator obtained from taking the r-variation semi-norm with respect to the t variable of the map $t \mapsto A_t f$ over the interval I, that is

$$V_r^I Af(x) := \sup_{N\in\mathbb{N}} \sup_{\substack{t_1<\cdots<t_N \\ t_j\in I}} \Big(\sum_{j=1}^{N-1} |A_{t_{j+1}} f(x) - A_{t_j} f(x)|^r \Big)^{1/r}.$$

For the family of spherical means the variation operator $V_r^I A$ and its global analogue have been studied in [11] and then more recently in [3]. The L^p-boundedness for $V_r^I A$ for some r implies the L^p-boundedness of the local circular maximal operator [5] (with dilations restricted to I); therefore boundedness fails for $p \le 2$. It is known [11] that $V_r^I A$ maps L^p into itself for $2 < p \le 4$, $r > 2$ and for $p > 4$, $r > p/2$; moreover L^p-boundedness fails for $r < \max\{2, p/2\}$. A key question, namely the L^p-boundedness for the local variation operator $V_2^I A$ in the range $2 < p \le 4$, remained open. We remark that the usual restriction $r > 2$ in variation-norm bounds is related

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to Lépingle's theorem [16] which applies to the *global* variation norm operator (that is, when I is replaced by $(0,\infty)$). However, this does not suggest any necessary condition for the local variant $V_r^I A$.

It turns out that $V_2^I A$ is not bounded on any L^p space.

Theorem 1.1. *For all* $p \geq 1$,

$$\sup_{\substack{f\in\mathcal{S}(\mathbb{R}^2)\\ \|f\|_p\le 1}} \big\|V_2^I Af\|_{L^p(\mathbb{R}^2)} = \infty.$$

As pointed out above, this is well-known for $p \notin (2,4]$. In the interesting range $2 < p \leq 4$ we will obtain the result as an immediate consequence of a more quantitative version which we now present. Motivated somewhat by the presentation in [23, §I.1.], we test $V_2^I A$ on L^p-functions of exponential type, i.e. functions whose Fourier transform is supported in large balls.

For $\lambda > 1$ we let $E(\lambda)$ be the space of all tempered distributions whose Fourier transform is supported in $\{\xi : |\xi| \leq \lambda\}$. Define

$$B_p(\lambda) = \sup\big\{\|V_2^I Af\|_{L^p(\mathbb{R}^2)} : \|f\|_{L^p(\mathbb{R}^2)} \leq 1,\ f \in E(\lambda)\big\}. \tag{1.1}$$

Clearly $B_p(\lambda)$ is finite and increasing in λ; note that for all $1 \leq p \leq \infty$

$$\|V_2^I Af\|_p \lesssim \sup_{1\leq t\leq 2} \|\partial_t A_t f\|_p \lesssim \lambda\|f\|_p \quad \text{for } f \in E(\lambda), \tag{1.2}$$

which implies $B_p(\lambda) \leq C\lambda$. More refined arguments in [11] (related to a square function estimate in [6] essentially via [12]) yield

$$B_p(\lambda) \lesssim (\log\lambda)^C \quad \text{for } 2 \leq p \leq 4,$$

for a suitable positive exponent C. Our quantification of Theorem 1.1 is

Theorem 1.2. *Let* $p > 2$. *Then there are constants* $\lambda_0 > 1$, $c > 0$ *depending on* p *such that*

$$B_p(\lambda) \geq c(\log\lambda)^{\frac12-\frac1p}$$

for all $\lambda > \lambda_0$.

We are only interested in the range $2 < p \leq 4$ as larger lower bounds are known for $p > 4$ (see [11], [3]). In the proof of the theorem we reduce matters to lower bounds for multipliers of the form $\chi(|\xi| - \lambda)$ for large λ, with nonnegative $\chi \in C_c^\infty(\mathbb{R})$. Córdoba [8] showed that for $4/3 \leq p \leq 4$ the operator norm of the corresponding convolution operators is $O((\log\lambda)^{|1/p-1/2|})$, and this also matches the lower bound in Lemma 2.3 below.

Notation. For nonnegative quantities a, b, we write $a \lesssim b$ or $a \lesssim_L b$ to indicate $a \leq Cb$ for some constant C which may depend on some list L.

2. Proof of Theorem 1.2

In order to establish Theorem 1.2, we prove several auxiliary lemmas which link the problem to a class of radial Fourier multipliers. Our proof is inspired by a result of Kaneko and Sunouchi [12], who proved the pointwise equivalence of two global square functions first occurring in work by Stein: one associated with Bochner–Riesz means [19] (see also [6, 7, 18, 15]), and one associated with spherical means [21, 22].

For $\lambda_1 < \lambda_2$, let $E_{\mathrm{ann}}(\lambda_1, \lambda_2)$ be the space of tempered distributions whose Fourier transform is compactly supported in $\{\xi : \lambda_1 \le |\xi| \le \lambda_2\}$. Let σ denote the normalized surface measure on the unit circle S^1. Also, given a distribution μ, we define the dilate $\mu_t := t^{-2}\mu(t^{-1}\cdot)$, in the sense of distributions.

Lemma 2.1. *Let* $\lambda \ge 1$ *and* $p > 2$. *Then*

$$\Big\|\Big(\int_{5/4}^{7/4} |(\chi\sigma)_t * g|^2 \,\mathrm{d}t\Big)^{1/2}\Big\|_p \lesssim_p \lambda^{-1/2}(B_p(2\lambda)+1)\|g\|_p \tag{2.1}$$

for all $g \in E_{\mathrm{ann}}(\lambda/4, \lambda)$ *and all* $\chi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$.

Proof. Let $\upsilon \in C_c^\infty(\mathbb{R})$ be supported in $[1,2]$ and such that $\upsilon(t) = 1$ in a neighborhood of $[5/4, 7/4]$. Define

$$\mathcal{A}f(x,t) := \upsilon(t)A_t f(x) \tag{2.2}$$

and consider the associated variation-norm operator $V_2^I \mathcal{A}f(x)$. It is easy to see via the triangle inequality and the mean value theorem that

$$V_2^I \mathcal{A}f(x) \le \|\upsilon\|_\infty V_2^I Af(x) + \|\upsilon'\|_\infty \sup_{t\in[1,2]} |A_t f(x)|,$$

and by using Bourgain's circular maximal theorem we get for $2 < p \le \infty$

$$\|V_2^I \mathcal{A}f\|_p \lesssim (B_p(2\lambda) + C_p)\|f\|_p \quad \text{ if } f \in E(2\lambda).$$

Let $\{\Lambda_j\}_{j\in\mathbb{Z}}$ be a standard dyadic frequency decomposition $\{\Lambda_j\}_{j\in\mathbb{Z}}$ in the t variable (so that Λ_j localizes to frequencies τ with $2^{j-1} \le |\tau| \le 2^{j+1}$). Then the Besov space seminorm for $a \in \dot{B}^{1/2}_{2,\infty}$ is given by $\sup_{j\in\mathbb{Z}} 2^{j/2}\|\Lambda_j a\|_2$. By the continuous embedding $V_2 \hookrightarrow \dot{B}^{1/2}_{2,\infty}$ ([4]) we have

$$\begin{aligned} 2^{j/2}\|\Lambda_j \mathcal{A}f\|_{L^p(L^2)} &\le \|\sup_{j>0} 2^{j/2}|\Lambda_j \mathcal{A}f|\|_{L^p(L^2)} \le \|\mathcal{A}f\|_{L^p(\dot{B}^{1/2}_{2,\infty})} \\ &\lesssim \|V_2^I \mathcal{A}f\|_p \lesssim (B_p(2\lambda) + C_p)\|f\|_p \quad \text{ for } f \in E(2\lambda). \end{aligned}$$

We use this bound for $2^{-10} \le \lambda/2^j \le 2^{10}$. For $f \in E_{\mathrm{ann}}(\lambda/8, 2\lambda)$ we refer to an error estimate involving a negligible upper bound for other values of j to [3, Lemma 2.5] and we get

$$\sum_{\substack{j\ge 0:\\ 2^j \notin [2^{-10}\lambda, 2^{10}\lambda]}} \|\Lambda_j \mathcal{A}f\|_{L^p(L^2)} \lesssim_N \lambda^{-N}\|f\|_p \text{ if } f \in L^p \cap E_{\mathrm{ann}}(\lambda/8, 2\lambda).$$

Consequently we have

$$\|\mathcal{A}f\|_{L^p(L^2)} \lesssim \lambda^{-1/2}(B_p(2\lambda)+C_p+1)\|f\|_p, \quad \text{for } f\in L^p\cap E_{\mathrm{ann}}(\lambda/8,2\lambda).$$

Since $v(t)=1$ on $[5/4,7/4]$ we obtain, for $f\in L^p\cap E_{\mathrm{ann}}(\lambda/8,2\lambda)$,

$$\Big\|\Big(\int_{5/4}^{7/4}|A_tf|^2\,\mathrm{d}t\Big)^{1/2}\Big\|_p \lesssim \lambda^{-1/2}(B_p(2\lambda)+C_p+1)\|f\|_p. \tag{2.3}$$

We wish to replace A_t by the convolution operator with $(\chi\sigma)_t$ where χ has small compact support. To this end, for $g\in E_{\mathrm{ann}}(\lambda/4,\lambda)$ we write

$$(\chi\sigma)_t*g(x)=(2\pi)^{-2}\int t^2\widehat{\chi}(t\eta)e^{i\langle x,\eta\rangle}\sigma_t*(ge^{-i\langle\cdot,\eta\rangle})(x)\,\mathrm{d}\eta.$$

Observe that $t^2\widehat{\chi}(t\eta)\lesssim_N (1+|\eta|)^{-N}$ for $t\in[5/4,7/4]$. Hence, for $p>2$,

$$\begin{aligned}&\Big\|\Big(\int_{5/4}^{7/4}|(\chi\sigma)_t*g|^2\,\mathrm{d}t\Big)^{1/2}\Big\|_p\\ &\quad\lesssim \int_{|\eta|\le\lambda/8}(1+|\eta|)^{-N}\Big\|\Big(\int_{5/4}^{7/4}|A_t[ge^{-i\langle\cdot,\eta\rangle}]|^2\,\mathrm{d}t\Big)^{1/2}\Big\|_p\,\mathrm{d}\eta\\ &\qquad+\int_{|\eta|\ge\lambda/8}(1+|\eta|)^{-N}\Big(\int_{5/4}^{7/4}\|A_t[ge^{-i\langle\cdot,\eta\rangle}]\|_p^2\,\mathrm{d}t\Big)^{1/2}\,\mathrm{d}\eta.\end{aligned}$$

For the first integral, we observe that for $|\eta|\le\lambda/8$ and $g\in E_{\mathrm{ann}}(\lambda/4,\lambda)$ the modulated function $ge^{-i\langle\cdot,\eta\rangle}$ belongs to $E_{\mathrm{ann}}(\lambda/8,9\lambda/8)$, and thus one can apply (2.3) with $f=ge^{-i\langle\cdot,\eta\rangle}$. For the second integral we get a decay factor of $O(\lambda^{1-N})$ from the η-integration. This leads to the claimed inequality (2.1). □

In what follows we use the differential notation for convolution operators that are given by a Fourier multiplier m, i.e. $m(D)f=\mathcal{F}^{-1}[m\widehat{f}]$.

Lemma 2.2. *Let u_1, u_2 be $C_c^\infty(\mathbb{R})$ functions supported in $[-1,1]$ and let $u=u_1*u_2$. Then, for any $\lambda\ge1$, $p>2$ and any integer $N\ge0$,*

$$\|u(|D|-\lambda)\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)}\le C(u_1,u_2,p,N)(1+\sup_{\rho\ge1}\rho^{-N}B_p(\rho\lambda)).$$

Proof. Let χ_1, χ be C_c^∞ functions, supported in a narrow sector and a neighborhood of a unit vector, so that $\chi(x)=1$ on the support of χ_1. Let $Pg=\chi_1(\frac{D}{|D|})g$, which is a Fourier localization of g to a sector. By Lemma 2.1,

$$\Big\|\Big(\int_{5/4}^{7/4}|(\chi\sigma)_t*Pg|^2\,\mathrm{d}t\Big)^{1/2}\Big\|_p\lesssim\lambda^{-1/2}(B_p(2\lambda)+1)\|Pg\|_p \text{ for } g\in E_{\mathrm{ann}}(\tfrac{\lambda}{4},\lambda).$$

We use the method of stationary phase to get the usual asymptotics of the Fourier transform of $\chi\sigma$ in the conic support of $\chi_1(\xi/|\xi|)$. We obtain

$$(\chi\sigma)_t*Pg(x)=ct^{-1/2}\mathcal{F}^{-1}[\chi(\tfrac{\xi}{|\xi|})|\xi|^{-1/2}e^{-it|\xi|}\widehat{Pg}](x)+R_1g(x,t)$$

where the remainder term R_1 is a smoothing operator of order $-3/2$ satisfying for $g \in E_{\mathrm{ann}}(\frac{\lambda}{4}, \lambda)$ the (negligible) bound

$$\|R_1 g(\cdot, t)\|_{L^p} \lesssim \lambda^{-1}\|g\|_p, \quad 1 \le p \le \infty.$$

Since $\chi(\frac{D}{|D|})Pg = Pg$ (by the support properties of χ_1 and χ) we get

$$\Big\|\Big(\int_{5/4}^{7/4} |e^{-it|D|}Pg|^2\, \mathrm{d}t\Big)^{1/2}\Big\|_p \lesssim (B_p(2\lambda) + 1)\|Pg\|_p, \text{ for } g \in E_{\mathrm{ann}}(\lambda/4, \lambda). \tag{2.4}$$

It will be convenient to switch to an inequality which involves an integral over $\mathbb{R}$ instead of $[5/4, 7/4]$. We first look at constributions for $|t| \le 2^{10}\lambda^{-1}$. If $\zeta \in C^\infty$ is supported in $(2^{-5}, 2^5)$ then it is not hard to see that the multiplier $e^{it|\xi|}\zeta(\lambda^{-1}|\xi|)$ is the Fourier transform of an L^1 function with L^1 norm uniformly bounded in $|t| \le 2^{10}\lambda^{-1}$. Hence, for $p \ge 2$

$$\Big\|\Big(\int_{-2^{10}\lambda^{-1}}^{2^{10}\lambda^{-1}} |e^{-it|D|}Pg|^2\, \mathrm{d}t\Big)^{1/2}\Big\|_p \lesssim \Big(\int_{-2^{10}\lambda^{-1}}^{2^{10}\lambda^{-1}} \|e^{-it|D|}Pg\|_p^2\, \mathrm{d}t\Big)^{1/2} \lesssim \lambda^{-1/2}\|g\|_p$$

provided that $g \in E_{\mathrm{ann}}(\lambda/4, \lambda)$.

Next, we look at contributions for $|t| \ge \lambda^{-1}$. Let $a \in C^\infty$ supported in $\{\xi : 1/4 < |\xi| < 4\}$ and let $\widetilde{a}(\xi) = a(\xi)\chi_1(\xi/|\xi|)$. Then for $R \ge \lambda^{-1}$ we have

$$\Big(\int_{5R/4}^{7R/4} |e^{-it|D|}a(\lambda^{-1}D)Pg(x)|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2} = \Big(\int_{5/4}^{7/4} |e^{-is|D|}\widetilde{a}(R^{-1}\lambda^{-1}D)[g(R\cdot)](R^{-1}x)|^2 \frac{\mathrm{d}s}{s}\Big)^{1/2}.$$

Thus, by scaling, we get from (2.4) that

$$\Big\|\Big(\int_{5R/4}^{7R/4} |e^{-it|D|}a(\lambda^{-1}D)Pg|^2\, \mathrm{d}t\Big)^{1/2}\Big\|_p \lesssim R^{1/2}(B_p(2R\lambda) + 1)\|g\|_p$$

for all $R \ge \lambda^{-1}$. Changing t to $-t$ yields a similar inequality for the interval $[-7R/4, -5R/4]$. Combining these estimates, for any Schwartz function ϑ and $N_1 > 0$ we obtain

$$\Big\|\Big(\int |\vartheta(t)e^{it|D|}Pg|^2\, \mathrm{d}t\Big)^{1/2}\Big\|_p \lesssim C(\vartheta)\Big(\lambda^{-1/2} + \sum_{\lambda^{-1} \le 2^k \le 1} 2^{k/2}B_p(\lambda 2^{k+1}) + \sum_{k \ge 0} 2^{k(\frac{1}{2} - N_1)}B_p(\lambda 2^{k+1})\Big)\|g\|_p$$

provided that $g \in E_{\mathrm{ann}}(\lambda/4, \lambda)$. For the second sum we have used the rapid decay of ϑ for $|t| \geq 1$. This implies, for $g \in E_{\mathrm{ann}}(\lambda/4, \lambda)$ and $N \geq 0$ that

$$\Big\| \Big(\int |\vartheta(t) e^{it|D|} Pg|^2 \, \mathrm{d}t \Big)^{1/2} \Big\|_p \lesssim_N (1 + \sup_{\rho \geq 1} \rho^{-N} B_p(\rho\lambda)), \tag{2.5}$$

using that $B_p(\lambda 2^{k+1}) \leq B_p(2\lambda)$ for $\lambda^{-1} \leq 2^k \leq 1$. We now come to the inequality asserted in the lemma. Note that

$$u(|D| - \lambda) Pg(x) = \frac{1}{2\pi} \int \widehat{u}_2(\tau) \widehat{u}_1(\tau) e^{-i\lambda\tau} e^{i\tau|D|} Pg(x) \, \mathrm{d}\tau,$$

and by the Cauchy–Schwarz inequality

$$|u(|D| - \lambda) Pg(x)| \lesssim \Big(\int |\widehat{u}_1(\tau) e^{i\tau|D|} Pg(x)|^2 \, \mathrm{d}\tau \Big)^{1/2}.$$

If we apply (2.5) with $\vartheta(\tau) = \widehat{u}_1(\tau)$ we get the inequality asserted in the lemma, first for functions $g \in E_{\mathrm{ann}}(\lambda/4, \lambda)$ but by the support property of u it is implied for general $g \in L^p$. □

A last lemma deals with lower bounds for such multipliers.

Lemma 2.3. *There exist $\varepsilon > 0$, $\lambda_1 = \lambda_1(p) > 1$ and $c > 0$ such that for all $\lambda > \lambda_1$ the following holds for $p > 2$:*

For all nonnegative L^∞ functions u supported in $[-2\varepsilon^2, 2\varepsilon^2]$ that are bounded below by 1 in $[-\varepsilon^2, \varepsilon^2]$, and all integers $n \geq 10$,

$$\|u(|D| - 2^{2n})\|_{L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)} \geq c n^{\frac{1}{2} - \frac{1}{p}} . \tag{2.6}$$

This is proved by a variant of Fefferman's proof for the ball multiplier [10], using the Besicovitch construction, together with a standard randomization argument. It is known but not well-documented that Fefferman's proof also gives lower bounds for multipliers such as in Lemma 2.3; in fact the second-named author had presented a version of the lemma in a graduate course at the University of Chicago in 1985. Because of the lack of an appropriate reference in the literature in the precise form needed here, we give the proof for the convenience of the reader. Other applications of Fefferman's argument have been used, for example, in Fourier restriction theory [2], [9], for resolvent bounds for certain partial differential equations [14], [17] and recently in a local theory for cone multipliers with applications to Cauchy–Szegő projections [1]. We remark that the lower bound (2.6) matches for $2 \leq p \leq 4$ the upper bound

$$\|u(|D| - 2^{2n})\|_{L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)} \leq C n^{\frac{1}{2} - \frac{1}{p}}, \quad 2 \leq p \leq 4,$$

for which one requires an additional regularity assumption, say $u \in C^2$. It follows from Córdoba's work [8].

Proof of Lemma 2.3. We use the construction by Keich [13] which gives a slightly better upper bound in the construction of Besicovitch sets than the one used in, say, [10]. Consider line segments $\ell = \{(s, as + b), s \in [0, 1]\}$

where $a = a(\ell) \in [0,1]$ and $b = b(\ell) \in [-1,0]$. We write $\ell(s) := a(\ell)s + b(\ell)$. For given large $n \in \mathbb{N}$ let $T(\ell) \equiv T^n(\ell)$ be the triangle with vertices $(0, \ell(0))$, $(0, \ell(0) - 2^{-n})$, $(1, \ell(1))$. Let $\vec{T}(\ell)$ be the *reach of* $T(\ell)$, defined to be the triangle obtained by translating $T(\ell)$ by $2\sqrt{2}$ along the direction of ℓ.

Fix $n \in \mathbb{N}$, $n \geq 10$. It is shown in [13] that there exists a collection of line segments $\{\ell_\nu\}_{\nu=0}^{2^n-1}$ with $a_\nu \equiv a(\ell_\nu) = \nu 2^{-n}$ such that the triangles $T(\ell_\nu)$ satisfy

$$\operatorname{meas}\big(\bigcup_{\nu=0}^{2^n-1} T(\ell_\nu)\big) < n^{-1} \tag{2.7}$$

and the corresponding reaches $\vec{T}(\ell_\nu)$ are pairwise disjoint.

For each $\nu = 0, \dots, 2^n - 1$, let

$$e_\nu = \frac{(1, a_\nu)}{\sqrt{1 + a_\nu^2}}, \qquad e_\nu^\perp = \frac{(-a_\nu, 1)}{\sqrt{1 + a_\nu^2}}$$

and consider the function

$$f_\nu(y) = \mathbb{1}_{T(\ell_\nu)}(y) e^{i\langle y, 2^{2n} e_\nu\rangle}.$$

Let $\varepsilon > 0$ be sufficiently small, chosen to satisfy the requirements of the forthcoming argument. Let $\psi \in C_c^\infty$ be non-negative, supported in $(-1/2, 1/2)$ and bounded below by 1 in $(-1/4, 1/4)$. Define

$$h_\nu(\xi_2) = \psi\big(2^{-n}\varepsilon^{-1}(\xi_2 - 2^{2n}\tfrac{a_\nu}{\sqrt{1+a_\nu^2}})\big), \quad \kappa_\nu(x_2) = \frac{1}{2\pi}\int h_\nu(\xi_2) e^{ix_2\xi_2}\, \mathrm{d}\xi_2$$

and let L_ν denote the operator given by $L_\nu g := \kappa_\nu *_2 g$, where $*_2$ denotes the convolution in the second variable. We will first show the lower bound

$$|u(|D| - 2^{2n}) L_\nu f_\nu(x)| \geq c, \quad x \in \vec{T}(\ell_\nu), \tag{2.8}$$

for some $c > 0$. To this end, let K_ν denote the convolution kernel of the operator $u(|D| - 2^{2n})L_\nu$, given by

$$K_\nu(x) = (2\pi)^{-2} \int u(|\xi| - 2^{2n}) h_\nu(\xi_2) e^{i\langle x, \xi\rangle}\, \mathrm{d}\xi. \tag{2.9}$$

Then

$$K_\nu(x) e^{-i\langle x, 2^{2n} e_\nu\rangle} = \\ (2\pi)^{-2} \int u(|\xi| - 2^{2n}) h_\nu(\xi_2) e^{i\langle x, e_\nu\rangle\langle e_\nu, \xi - 2^{2n} e_\nu\rangle} e^{i\langle x, e_\nu^\perp\rangle\langle e_\nu^\perp, \xi\rangle}\, \mathrm{d}\xi$$

and a computation shows that on the support of integration we have

$$|\langle e_\nu, \xi - 2^{2n} e_\nu\rangle| \leq 2^6 \varepsilon^2 \quad \text{and} \quad |\langle \xi, e_\nu^\perp\rangle| \leq 2^{n+2}\varepsilon. \tag{2.10}$$

To see this, let $\xi = (\xi_1, \xi_2)$ satisfying $\big||\xi| - 2^{2n}\big| \leq 2\varepsilon^2$ and $|\xi_2 - 2^{2n}\frac{a_\nu}{(1+a_\nu^2)^{1/2}}| \leq 2^{n-1}\varepsilon$. Set $\eta = 2^{-2n}\xi$, so that $|\eta| = 1 + \varrho$ with $|\varrho| \leq \varepsilon^2 2^{-2n+1}$ and $\eta_2 =$

$\frac{a_\nu}{\sqrt{1+a_\nu^2}} + v$ with $|v| \le \varepsilon 2^{-n-1}$. We show that $|\eta_1 - \frac{1}{\sqrt{1+a_\nu^2}}| \le \varepsilon 2^{-n+1}$. Write

$$\begin{aligned} \eta_1 = \sqrt{|\eta|^2 - \eta_2^2} &= \big((1+\varrho)^2 - (\tfrac{a_\nu}{\sqrt{1+a_\nu^2}} + v)^2\big)^{1/2} \\ &= \big(1 - \tfrac{a_\nu^2}{1+a_\nu^2} + 2\varrho + \varrho^2 - 2\tfrac{a_\nu}{\sqrt{1+a_\nu^2}} v - v^2\big)^{1/2} = \big(\tfrac{1}{1+a_\nu^2} + \Delta\big)^{1/2} \end{aligned}$$

where $\Delta = 2\varrho + \varrho^2 - 2\frac{a_\nu}{\sqrt{1+a_\nu^2}} v - v^2$ and thus $|\Delta| \le \varepsilon^2 2^{2n+2} + \varepsilon^4 2^{-4n+2} + \varepsilon 2^{-n} + \varepsilon^2 2^{-2n-2} \le \varepsilon 2^{-n+1}$. Hence

$$|\eta_1 - \tfrac{1}{\sqrt{1+a_\nu^2}}| = \big|\big(\tfrac{1}{1+a_\nu^2} + \Delta\big)^{1/2} - \big(\tfrac{1}{1+a_\nu^2}\big)^{1/2}\big| \le |\Delta| \le \varepsilon 2^{-n+1}$$

and then also

$$|\eta - e_\nu| \le |\eta_1 - \tfrac{1}{\sqrt{1+a_\nu^2}}| + |\eta_2 - \tfrac{a_\nu}{\sqrt{1+a_\nu^2}}| \le \varepsilon 2^{-n+2}.$$

Next write $\eta = e_\nu + \omega_1 e_\nu + \omega_2 e_\nu^\perp$ and observe that $\sqrt{\omega_1^2 + \omega_2^2} = |\eta - e_\nu|$, so $|\omega|^2 \le (\varepsilon 2^{-n+2})^2$. We have $\langle e_\nu, \eta - e_\nu\rangle = \langle e_\nu, \omega_1 e_\nu + \omega_2 e_\nu^\perp\rangle = \omega_1$. Moreover

$$\begin{aligned} |\eta| - 1 &= |e_\nu + \omega_1 e_\nu + \omega_2 e_\nu^\perp| - 1 = \big((1+\omega_1)^2 + \omega_2^2\big)^{1/2} - 1 \\ &= (1 + 2\omega_1 + |\omega|^2)^{1/2} - 1 = \omega_1 + \tfrac{|\omega|^2}{2} + E(\omega) \end{aligned}$$

and using $|(1+s)^{1/2} - 1 - \frac{s}{2}| \le \frac14 (1-|s|)^{-3/2}\frac{s^2}{2}$, which follows from Taylor's expansion on both sides of the inequality, we estimate the error by $|E(\omega)| \le \frac18 \frac{(2|\omega_1|+|\omega|^2)^2}{(1-2|\omega_1|-|\omega|^2)^{3/2}} \le |\omega|^2$ (recall $|\omega|^2 \le (\varepsilon 2^{-n+2})^2$ and $n \ge 10$). Hence, since $|\eta| = 1 + \varrho$ with $|\varrho| \le \varepsilon^2 2^{-2n+1}$,

$$|\omega_1| \le \big||\eta| - 1\big| + 2|\omega|^2 = \big||\eta| - 1\big| + 2|\eta - e_\nu|^2 \le 2^{-2n+1}\varepsilon^2 + 2^{-2n+5}\varepsilon^2$$

and thus $|\omega_1| \le 2^{-2n+6}\varepsilon^2$. We get

$$\begin{aligned} &|\langle e_\nu, \eta - e_\nu\rangle| = |\omega_1| \le 2^{-2n+6}\varepsilon^2, \\ &|\langle \eta, e_\nu^\perp\rangle| = |\langle \eta - e_\nu, e_\nu^\perp\rangle| \le |\eta - e_\nu| \le 2^{-n+2}\varepsilon, \end{aligned}$$

and from this $|\langle e_\nu, \xi - 2^{2n} e_\nu\rangle| \le 2^6\varepsilon^2$ and $|\langle \xi, e_\nu^\perp\rangle| \le 2^{n+2}\varepsilon$, which correspond to the claimed bounds (2.10).

Hence, choosing ε sufficiently small,

$$\mathrm{Re}\,\big(e^{-i2^{2n}\langle x, e_\nu\rangle} K_\nu(x)\big) \ge c\varepsilon^3 2^n \text{ if } |\langle x, e_\nu\rangle| \le 2^4 \text{ and } |\langle x, e_\nu^\perp\rangle| \le 2^{-n+4}.$$

As a consequence we get the lower bound (2.8) on the reach of $T(\ell_\nu)$, namely

$$|K_\nu * f_\nu(x)| = \Big|\int K_\nu(x-y) e^{-i\langle x-y, 2^{2n} e_\nu\rangle} \mathbb{1}_{T(\ell_\nu)}(y)\, \mathrm{d}y\Big| \ge c, \quad x \in \vec{T}(\ell_\nu). \tag{2.11}$$

Now, define for $\omega \in [0,1]$

$$f^\omega(y) = \sum_{\nu=0}^{2^n-1} r_\nu(\omega) \kappa_\nu *_2 f_\nu$$

where $(r_\nu)_{\nu\in\mathbb{N}}$ is the sequence of Rademacher functions. If

$$\mathcal{C}_{p,n} = \big\| u(|D| - 2^{2n}) \big\|_{L^p\to L^p}$$

we have by duality

$$\|u(|D|-2^{2n})f^\omega\|_{p'} \le \mathcal{C}_{p,n}\|f^\omega\|_{p'}, \quad \omega\in[0,1].$$

Integrating in ω and using the above definitions we get

$$\Big(\int_0^1 \Big\|\sum_{\nu=0}^{2^n-1} r_\nu(\omega) K_\nu * f_\nu\Big\|_{p'}^{p'} \mathrm{d}\omega\Big)^{1/p'} \le \mathcal{C}_{p,n}\Big(\int_0^1 \Big\|\sum_{\nu=0}^{2^n-1} r_\nu(\omega)\kappa_\nu *_2 f_\nu\Big\|_{p'}^{p'} \mathrm{d}\omega\Big)^{1/p'}.$$

We interchange the x and ω integration on both sides and using both lower and upper bounds in Khinchine's inequality (see e.g. [20, Appendix D]) we get

$$\Big\|\Big(\sum_{\nu=0}^{2^n-1} |K_\nu * f_\nu|^2\Big)^{1/2}\Big\|_{p'} \le C(p)\mathcal{C}_{p,n}\Big\|\Big(\sum_{\nu=0}^{2^n-1} |\kappa_\nu *_2 f_\nu|^2\Big)^{1/2}\Big\|_{p'}. \tag{2.12}$$

We first give a lower bound for the left hand side of (2.12). By the disjointness of the $\vec{T}(\ell_\nu)$, and (2.11)

$$\begin{aligned} \Big\|\Big(\sum_{\nu=0}^{2^n-1} |K_\nu * f_\nu|^2\Big)^{1/2}\Big\|_{p'} &\ge \Big(\sum_{\nu'=0}^{2^n-1}\int_{\vec{T}(\ell_{\nu'})}\Big(\sum_{\nu=0}^{2^n-1}|K_\nu * f_\nu|^2\Big)^{p'/2}\mathrm{d}x\Big)^{1/p'} \\ &\ge \Big(\sum_{\nu'=0}^{2^n-1}\int_{\vec{T}(\ell_{\nu'})}|K_{\nu'} * f_{\nu'}|^{p'}\,\mathrm{d}x\Big)^{1/p'} \ge c\Big(\sum_{\nu=0}^{2^n-1}|\vec{T}(\ell_\nu)|\Big)^{1/p'} \ge 2^{-1/p'}c. \end{aligned} \tag{2.13}$$

We give an upper bound for the right hand side of (2.12). Use the uniform pointwise bound

$$|\kappa_\nu(x_2)| \le C2^n(1+2^n|x_2|)^{-2}$$

and the fact that all f_ν are supported in $\cup_{\nu=0}^{2^n-1}T(\ell_\nu)$ which by (2.7) is a set of measure $< 1/n$. It follows

$$\begin{aligned} \Big\|\Big(\sum_{\nu=0}^{2^n-1}|\kappa_\nu *_2 f_\nu|^2\Big)^{1/2}\Big\|_{p'} &\le C\Big\|\Big(\sum_{\nu=0}^{2^n-1}|f_\nu|^2\Big)^{1/2}\Big\|_{p'} \\ &\le C\,\mathrm{meas}\Big(\bigcup_{\nu=0}^{2^n-1}T(\ell_\nu)\Big)^{1/p'-1/2}\Big\|\Big(\sum_{\nu=0}^{2^n-1}|f_\nu|^2\Big)^{1/2}\Big\|_2 \\ &\le Cn^{-1/p'+1/2}\Big(\sum_{\nu=0}^{2^n-1}|T(\ell_\nu)|\Big)^{1/2} \le C2^{-1/2}n^{1/p-1/2}. \end{aligned} \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) we get

$$c2^{-1/p'} \le C2^{-1/2}C(p)\mathcal{C}_{p,n}n^{1/p-1/2}$$

and thus the assertion of the lemma. □

Proof of Theorem 1.2, conclusion. By a scaling argument we can replace 2^{2n} in Lemma 2.3 with $\lambda \in [2^{2n}, 2^{2n+2}]$. From Lemma 2.2 and Lemma 2.3 it follows that there is a $\mu_0 > 2$, $c_0 > 0$ such that for all $\mu > \mu_0$

$$\sup_{\rho \geq 1} B_p(\rho\mu)\rho^{-N} \geq c_0 (\log \mu)^{1/2-1/p}.$$

The trivial bound (1.2) implies $B_p(\lambda) \leq C\lambda$ and hence

$$\sup_{\rho \geq \mu} B_p(\rho\mu)\rho^{-N} \leq C_N \mu \sup_{\rho \geq \mu} \rho^{1-N} \leq C_N \mu^{2-N}.$$

Then

$$\begin{aligned} B_p(\mu^2) &\geq \sup_{1 \leq \rho \leq \mu} B_p(\rho\mu)\rho^{-N} \geq \sup_{\rho \geq 1} B_p(\rho\mu)\rho^{-N} - C_N \mu^{2-N} \\ &\geq c_0 (\log \mu)^{1/2-1/p} - C_N \mu^{2-N} \end{aligned}$$

and thus we get for $\mu > \mu_1 = \max\{\mu_0, \exp((2C_2/c_0)^{\frac{2p}{p-2}})\}$

$$B_p(\mu^2) \geq \tfrac{c_0}{2} (\tfrac{1}{2} \log \mu^2)^{\frac{1}{2}-\frac{1}{p}}$$

which implies the theorem. $\square$

Remark 2.4. Let $\mathfrak{A}f(x,t) = \chi(t)A_t f(x)$ where χ is a nontrivial bump function compactly supported in $(1,2)$. An examination of our proof (in particular the proof of Lemma 2.1) also shows that for $p > 2$

$$\sup_{f \in \mathcal{S}(\mathbb{R}^2)} \sup_{\|f\|_p \leq 1} \|\mathfrak{A}f\|_{L^p(\dot{B}^{1/2}_{2,\infty})} = \infty. \tag{2.15}$$

In view of the embedding $V_2 \hookrightarrow \dot{B}^{1/2}_{2,\infty}$ this gives a stronger lower bound than stated in Theorem 1.1. Note that V_2 embeds into L^∞ while $B^{1/2}_{2,\infty}$ does not. Moreover, we may strengthen this formulation of our result by replacing in (2.15) the $B^{1/2}_{2,\infty}$ norm or $\dot{B}^{1/2}_{2,\infty}$ semi-norm of $a(t) = \mathfrak{A}(x,t)$ with

$$\sup_{n>0} \|\Lambda_n a\|_2 2^{n/2} \omega_n,$$

where ω_n may be small for large n, such that $\omega_n = n^{\frac{1}{p}-\frac{1}{2}+\varepsilon}$; in fact we can choose any sequence satisfying $\limsup_{n\to\infty} n^{1/2-1/p}\omega_n = \infty$.

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