ANALYSIS ON SPARSE SETS

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Differentiation of integrals is a venerable theme in analysis, dealing with almost everywhere convergence for averages of functions. A fundamental example is the Lebesgue differentiation theorem, which says that the averages of any locally integrable function f over the family of balls $\{B(x;r) : r > 0\}$ converge to $f(x)$ as $r \to 0$, for almost every x. We say that the family $\{B(0;r) : r > 0\}$ differentiates L^1_{loc} and hence L^p, for all $1 \leq p < \infty$.

This has led to a natural question: which families of sets have the differentiation property with respect to Lebesgue spaces? For instance, does the family of parallelepipeds centered at the origin in $\mathbb{R}^d$ with arbitrary eccentricities differentiate $L^p(\mathbb{R}^d)$ for some $p \in [1, \infty)$? Does the family of spheres? Apart from their connection to differentiation, such questions have deep roots in Euclidean harmonic analysis and in integral geometry, tying in with the study of averaging and maximal operators over submanifolds.

Not surprisingly, many differentiation theorems in $\mathbb{R}^d$, $d \geq 2$, exploit the explicit geometric structure of the underlying sets (such as curves or surfaces) which have integer dimension $\leq d$. No such theory exists for the real line, which does not have any nontrivial lower-dimensional submanifolds. Nevertheless there are many sparse sets in $\mathbb{R}$, namely, sets that have Lebesgue measure zero or fractional Hausdorff dimension. Aversa and Preiss asked: does there exist a zero-measure set on $\mathbb{R}$ such that the family of all dilates of this set possesses L^2-differentiability? Phrased differently, can one obtain an L^p-differentiation theorem on $\mathbb{R}$ independently of the Lebesgue differentiation theorem?

The first part of the talk will be an overview of maximal and differentiation theorems in $\mathbb{R}^d$ for sparse sets such as hypersurfaces, and their connections with Fourier analysis and incidence theorems. I will also present joint work with Izabella Łaba that provides an affirmative answer to the question of Aversa and Preiss.