ANALYTIC CAPACITY AND SOME PROBLEMS IN APPROXIMATION THEORY

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I shall discuss a certain connection between uniform rational approximation, and approximation in the mean by polynomials on compact nowhere dense subsets of the complex plane $\mathbb{C}$. In particular, if X is compact and $R(X) \neq C(X)$ it can be shown that

(1) $H^p(X, dA) \neq L^p(X, dA)$ whenever $1 \leq p < \infty$, but nevertheless
(2) $R^p(X, dA) = L^p(X, dA)$ for all $p < \infty$ is still possible.

Item (1) depends on the fact that $H^p(X, dA)$ has a bounded point evaluation at every non-peak point for $R(X)$. The proof makes essential use of the semi-additivity of analytic capacity, and settles a question dating from 1973. Item (2) was first established by Sinanjan in 1966. His argument, however, depends on earlier work of Mergeljan and is computationally rather difficult. I shall present a proof that is conceptually much clearer, and depends only on the fundamentally different behavior of L^p-capacity and analytic capacity under a contraction. It follows from the work of Davie on bounded pointwise approximation by analytic functions that every non-peak point for $R(X)$ admits a representing measure absolutely continuous with respect to area, that is a representing measure of the form $k\,dA$ with $k \in L^1(dA)$. On the other hand, it is clear from (2) that k may not belong to $L^p(dA)$ for any $p > 1$. With this as background I shall also indicate how the existence of absolutely continuous representing measures for $R(X)$ can be obtained directly from Mel'nikov's peak point criterion, which is expressed solely in terms of analytic capacity.

My talk represents, in part, joint work with Erin Militzer.