A TWO-PARAMETER EXTENSION OF BOURGAIN'S LOGARITHMIC LEMMA

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1. Abstract

The study of pointwise ergodic theorems goes back to 1931, when Birkhoff proved his classical pointwise ergodic theorem for Cesaro averages: for any non-atomic probability space, (X, μ), and $T : X \to X$ a measure-preserving transformation, the means

$$\left\{ \frac{1}{t} \sum_{n \leq t} T^n f(x) \right\}$$

converge pointwise μ-a.e. for each $f \in L^1(X)$. As is the case with many questions involving pointwise convergence, controlling the maximal function

$$\sup_{t \in \mathbb{N}} \left| \frac{1}{t} \sum_{n \leq t} T^n f \right|$$

played a key roll in the proof.

It took almost 60 years before the linear averages could be replaced by more complicated polynomial ones; in the late eighties, Bourgain managed to generalize Birkhoff's theorem as follows:

For any non-atomic probability space, (X, μ), and $T : X \to X$ a measure-preserving transformation, the means

$$\left\{ \frac{1}{t} \sum_{n \leq t} T^{P(n)} f(x) \right\}$$

converge pointwise μ-a.e. for each $f \in L^p(X)$, $p > 1$. Here, $P(n)$ is *any* polynomial with integer coefficients.

Although his theorem is very abstract, Bourgain achieved his result through (surprisingly) hard-analytic technique: he used the analytic number-theoretic circle method of Hardy and Littlewood to prove favorable estimates on the maximal function

$$\sup_{t \in \mathbb{N}} \left| \frac{1}{t} \sum_{n \leq t} T^{P(n)} f \right|.$$

The key ingredient in his analysis was the replacement of the polynomial multipliers on the torus

$$\mathbb{T} \ni \beta \mapsto \frac{1}{t} \sum_{n \leq t} e(-P(n)\beta)$$

with a more tractable family of "multi-frequency projections." To control the maximal function governing these projections, Bourgain developed a beautiful multi-frequency maximal function theory, generalizing the (single-frequency) theory of the Hardy-Littlewood maximal function.

In this talk we will first review Bourgain's theorem in the context of pointwise ergodic theory, before discussing two-parameter generalizations. This is joint work with Mariusz Mirek and Bartosz Trojan.

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