Faber expansions and sampling in mixed order Sobolev spaces

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We consider the mixed order Sobolev space

$$S^r_pW(\mathbb{T}^d) := \Big\{ f \in L_p(\mathbb{T}^d) : \Big\| \Big(\sum_{j\in\mathbb{N}_0^d} 2^{2r|j|_1} |\delta_j[f](\cdot)|^2 \Big)^{1/2} \Big\|_p < \infty \Big\},$$

where $1 < p < \infty$ and $\frac{1}{p} < r < \infty$. Here $\delta_j[f]$ is that part of the Fourier series of f with frequencies in a dyadic anisotropic rectangle. We study a replacement of $\delta_j[f]$ by building blocks that use only discrete information of f (function evaluations). Such a replacement (in the sense of equivalent norms) can be achieved with the help of tensorized Faber basis where a continuous function f is decomposed into tensor products of dilated and translated hat functions. Here we need the condition $r > \frac{1}{p}$. The obtained discrete characterization is well suited for studying sampling issues in $S^r_pW(\mathbb{T}^d)$. We construct a sampling algorithm taking values on a sparse grid that allows for proving asymptotically optimal error bounds for the linear sampling numbers

$$g_n(S^r_pW(\mathbb{T}^d), Y) = \inf_{\substack{(\xi_i)_{i=1}^n \subset \mathbb{T}^d \\ (\psi_i)_{i=1}^n \subset Y}} \sup_{\|f\|_{S^r_pW} \le 1} \Big\| f(\cdot) - \sum_{i=1}^n f(\xi_i)\psi_i(\cdot) \Big\|_Y,$$

where the error is measured in a Lebesgue space $L_q(\mathbb{T}^d)$, $1 < q \le \infty$ as well as in isotropic Sobolev spaces $W^s_q(\mathbb{T}^d)$, $s > r$, $1 < p \le q < \infty$. The talk is based on a joint work with Tino Ullrich.