New Improvement to Falconer's Distance Set Problem

Yumeng Ou
University of Pennsylvania

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INTRODUCTION

- **Phenomena**: the larger a set is, the richer geometric structure it should have.

  Size of a set: cardinality. measure. dimension.

  Geometric configurations: distances. graphs. areas. directions

- **Our focus**: fractal sets in $\mathbb{R}^d$. distances.

  Compact $E \subset \mathbb{R}^d$. Distance set $\Delta(E) = \{|x-y| : x, y \in E\}$.

  (Pinned distance set $\Delta_x(E) = \{|x-y| : y \in E\}$. x fixed.)

Falconer Distance Set Conjecture.

Cpt $E \subset \mathbb{R}^d$. $d \geq 2$. If $\dim_H(E) > \frac{d}{2}$. then its distance set satisfies $|\Delta(E)| > 0$. (i.e. positive 1D Lebesgue measure)

- "$\frac{d}{2}$" is sharp. (lattice like construction)
- "$d \geq 2$" is necessary. (positive result impossible in 1D.)

BRIEF HISTORY

- Partial progress has been made over the decades:

  " If $\dim(E) > \frac{d}{2} + \beta$, then $|\Delta(E)| > 0$ ". (Conj : $\beta = 0$)

→ Falconer (1985): $\dim(E) > \frac{d}{2} + \frac{1}{2}$ (The gap is merely $\frac{1}{2}$!)

→ Bourgain (1994): $d=2$. $\dim(E) > 1 + \frac{4}{9}$.

→ Wolff (1999): $d=2$. $\dim(E) > 1 + \frac{1}{3}$

→ Erdoğan (2005): $d \geq 3$. $\dim(E) > \frac{d}{2} + \frac{1}{3}$.

→ Since 2018. huge wave of activities.

(Du, Guth, Iosevich, O, Wang, Wilson, Zhang.)

$$\dim(E) > \begin{cases} \frac{d}{2} + \frac{1}{4} & \text{(GIOW 2D ; DIOWZ 4D+ . even)} \\ \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4} & \text{(DGOWWZ 3D ; DZ 5D+ . odd.)} \end{cases}$$

These are results of modern tools in analysis (e.g. decoupling, broad-narrow analysis), GMT (e.g. projection theory). and Cominatorics (e.g. Polynomial method. incidence estimates.)

BRIEF HISTORY

- The recent distance results actually hold in a strong form:

  If $\dim E > \frac{d}{2} + \beta$, then $\exists\, x \in E$ s.t. $|\Delta_x(E)| > 0$.

  (Recall: $\Delta_x(E) := \{ |x-y| : y \in E \}$ is the pinned distance set.)

→ The pinned problem was long considered significantly harder.

  Peres–Schlag (2000): $\dim(E) > \frac{d}{2} + \frac{1}{2}$.

→ Liu (2018): an L^2 identity that implies that the key intermediate estimate (weighted Fourier restriction) leads to not only the full, but also pinned distance result.

→ Following Liu, all the previous records hold for pinned distance.

- Also of interest:
  - How does $\dim \Delta(E)$ or $\dim \Delta_x(E)$ depend on $\dim E$?
  - structure of $\Delta(E)$ or $\Delta_x(E)$?

MAIN RESULT

Theorem (Du-O-Ren-Zhang. 2023)

$E \subset \mathbb{R}^d$, cpt, $d \geq 3$. If $\dim_H(E) > \frac{d}{2} + \frac{1}{4} - \frac{1}{8d+4}$.
Then $\exists\, x \in E$ s.t. $|\Delta_x(E)| > 0$. where
$\Delta_x(E) = \{|x-y| : y \in E\}$ denotes the pinned distance set.

- Finally more than half way there! (The gap is now $< \frac{1}{4}$.)
- 3D+ got ahead of 2D. (unusual even in discrete case).
- Our strategy also leads to improved lower bound of $\dim_H(\Delta_x(E))$. For example. a special case $\dim E = \frac{d}{2}$:

Theorem $\exists\, x \in E$ s.t. $\dim_H(\Delta_x(E)) \geq \frac{d+2}{2(d+1)}$.

$\rightarrow$ Recovers previously best known in 3D. (Shmerkin-Wang)

$\rightarrow$ New in 4D+. (Previous best: $\frac{1}{2}$. Falconer)

- A key new ingredient: Ren's Radial projection theorem.
- We found 2 proofs: GMT v.s. Fractal decoupling.

WHY IS THE RESULT IN $d \geq 3$ "BETTER"?

- Current record: $\begin{cases} d=2. & \dim E > \frac{d}{2} + \frac{1}{4} = \frac{5}{4}. \\ d \geq 3. & \dim E > \frac{d}{2} + \frac{1}{4} - \frac{1}{8d+4}. \end{cases}$

- Advantage (also challenge) in higher dim (e.g. 3D):

  $E \subset \mathbb{R}^3$. $\frac{3}{2} < \dim(E) < \frac{3}{2} + \frac{1}{3} < 2$.

  <u>Extreme Case 1</u>: E is truly 3D. "Broad".

  Multilinear arguments in Fourier analysis or Radial Projection techniques in GMT usually work well.

  <u>Extreme Case 2</u>: E is in some hyperplane. "Narrow".

  Use 2D Falconer result to solve the problem.

  ($\dim(E) > \frac{3}{2} > \frac{5}{4}$. 2D result applies to E.)

  A key difference between 2D and higher dim

- There are other technical reasons why our methods cannot improve 2D. ("good threshold" has no room for improvement)

GETTING (A BIT MORE) TECHNICAL.

- Convert to L^2 estimate:

Take $E_1, E_2 \subset E$ s.t. $\mathrm{dist}(E_1, E_2) \gtrsim 1$. $\mathcal{H}^\alpha(E_i) > 0$. ($\alpha < \dim E$)

Build Frostman measure μ_i on E_i, i.e. $\mathrm{supp}(\mu_i) \subset E_i$. μ_i is probability and $\mu_i(B(x,r)) \lesssim r^\alpha$. $\forall x$. $\forall r < 1$

Goal: $\exists\, x \in E_2$ s.t. $|\Delta_x(E_1)| > 0$.

Classical reduction: Fix x. consider pinned distance map

$$\begin{aligned} E_1 &\longrightarrow \Delta_x(E_1) \\ y &\longmapsto |x-y| \\ \mu_1 &\longrightarrow d^x_*(\mu_1) \end{aligned}$$

Defn: $\displaystyle\int_{\Delta_x(E_1)} f(t)\, d^x_*(\mu_1)(t) = \int_{E_1} f(|x-y|)\, d\mu_1(y)$

New Goal: $\displaystyle\int |d^x_*(\mu_1)(t)|^2 dt < \infty$.

(Indeed. $\displaystyle 1 = \mu_1(E_1) = \int_{\Delta_x(E_1)} d^x_*(\mu_1) \leq |\Delta_x(E_1)|^{1/2} \, \| d^x_*(\mu_1) \|_{L^2}$)

THE GOOD-BAD DECOMPOSITION

- **New Goal**: $\| d_*^x(\mu_1)(x) \|_{L^2} < \infty$ for some $x \in E_2$.
- **An idea**: Remove "heavy" part of μ_1 that generates redundancy.

  Decompose $\mu_1 = \mu_{1,g} + \mu_{1,b}$.

  Newer Goals: ① $\| d_*^x(\mu_{1,b}) \|_{L^1(t)} < \frac{1}{1000}$ for most $x \in E_2$.

  ② $\int_{E_2} \| d_*^x(\mu_{1,g}) \|^2_{L^2(t)} \, d\mu_2(x) < \infty$

- Such a decomposition is necessary if one wants to get below $\dim = \frac{4}{3}$ in 2D, and similarly in 3D+.
- Various ways to define $\mu_{1,g}$, $\mu_{1,b}$ have been used.

  $\mu_{1,g}$ = Sum of wave packets whose μ_2 measure is less than a threshold (Scale dependent)

① is usually shown using radial projections in GMT.

② is usually converted to a Fourier restriction estimate

THE TWO APPROACHES IN DU-O-REN-ZHANG

Pick your poison:

$\boxed{\text{Control a wild } \mu_{1,b}}$

Pro: $\mu_{1,g}$ satisfies better threshold. classical method via refined decoupling can handle it.

Con: Need to remove a lot more bad wave packets. Complicated new GMT argument needed. (capture interaction between tubes and plates)

or $\boxed{\text{Handle a problematic } \mu_{1,g}}$

Pro: Define $\mu_{1,g}$ with geometric info already incorporated. (Relatively) easy to remove bad wave packets.

Con: $\mu_{1,g}$ now looks very different. We need to use the geometric info to refine the Fourier restriction argument. A new fractal decoupling.

SOME PREP WORK

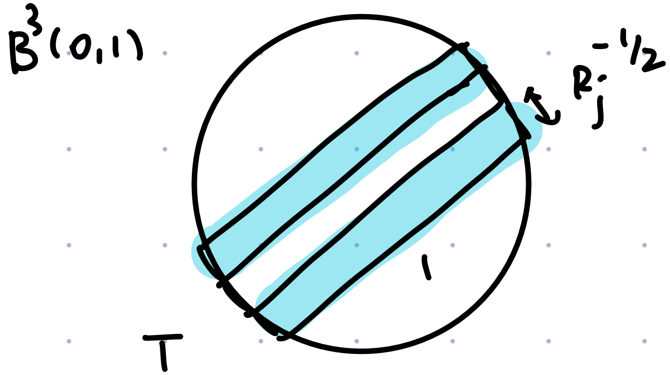
Example: $d = 3$. $\frac{3}{2} < \alpha < 2$.

Wave packet decomposition of $\mu_1 \sim \sum_{j=0}^{\infty} \sum_{T \in \mathbb{T}_j} M_T \mu_1$.

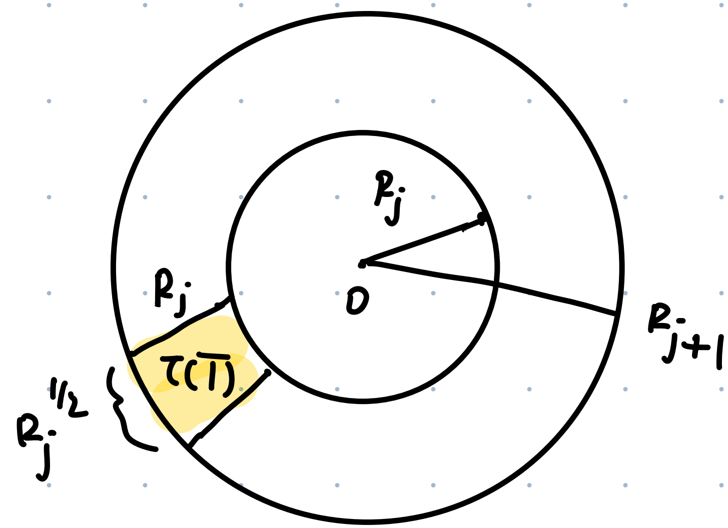
Fix R_0 large. $R_j = 2^j R_0$.

$M_T \mu_1$ concentrates on the $R_j^{-1/2} \times R_j^{-1/2} \times 1$ tube $T \subset B^3(0,1)$

$\widehat{M_T \mu_1}$ concentrates on $\tau(T)$



Physical side



Fourier side.

- $\mu_{1,b} := \sum_{T \text{ bad}} M_T \mu_1$. $\quad T$ is bad if $\begin{cases} \mu_2(T) \text{ very heavy, or} \\ T \subset \text{heavy plate.} \end{cases}$

Small thickening of a hyperplane

BAD TUBE & HEAVY PLATE.

- $\forall r > 0$. $\mathcal{E}_r \doteq$ essentially distinct collection of r-plate in $\mathbb{R}^3$.

  Property 1: $\forall \frac{r}{2}$ - plate $\cap B(0,1)$ lies in some r-plate in $\mathcal{E}_r$.

  Property 2: $\forall$ s-plate $(s \geq r)$ contains $\lesssim \left(\frac{s}{r}\right)^3$ r-plates in $\mathcal{E}_r$.

- "Heavy Plate": $\forall j \geq 1$. $\mathcal{H}_j = \{ H \in \mathcal{E}_{R_j^{-\kappa}} : \mu_1 + \mu_2(H) > R_j^{-\eta} \}$.

  Property: $\sum_{i=1}^{j} \# \mathcal{H}_i \lesssim R_j^{N\eta}$. ($N$ depends on dim of μ_i)

  The bad part of measure $\mu_{1,b} := \sum_{T \text{ bad}} M_T \mu_1$

- Defn. $R_j^{-1/2}$ - tube T is bad: T is contained in some $H \in \bigcup_{i \leq j} \mathcal{H}_i$.

  OR $\mu_2(4T) \geq R_j^{-\frac{\alpha}{2} + \varepsilon}$.

$\Rightarrow$ T Good: $\mu_2(4T) < R_j^{-\frac{\alpha}{2} + \varepsilon}$ AND T is not contained in any $H \in \bigcup_{i \leq j} \mathcal{H}_i$.

BAD TUBE & HEAVY PLATE

- After $\mu_{1,b}$ is removed, in weighted restriction step, only $\mu_2(4T) < R_j^{-\frac{\alpha}{2}+\varepsilon}$ is used.

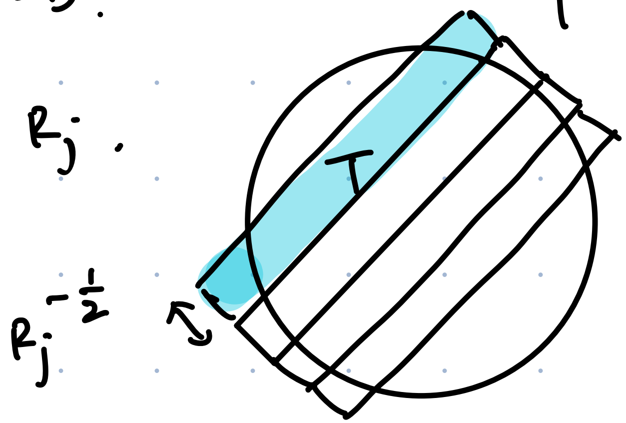
$\rightarrow$ Compare to good threshold in earlier works:

$$\mu_2(4T) < \begin{cases} R_j^{-\frac{d}{4}+\varepsilon}, & d \text{ even} \\ R_j^{-(\frac{d}{4}-\frac{1}{4})+\varepsilon}, & d \text{ odd.} \end{cases}$$

Since $\alpha > \frac{d}{2}$, is a better (lower) threshold.

$\rightarrow$ In 2D, such an improved threshold is impossible.

Fix R_j.



$\#$ parallel $T \sim R_j^{\frac{1}{2}}$.

If μ_2 evenly distributed,

then each $\mu_2(T) \sim R_j^{-\frac{1}{2}}$.

$\rightarrow$ is in some sense sharp:

If E_2 = union of $R_j^{\frac{\alpha}{2}}$ many $R_j^{-\frac{1}{2}}$-balls, each measuring $R_j^{-\frac{\alpha}{2}}$, then nonempty T needs to contain at least one ball.

REMOVAL OF BAD TUBES

- **Goal:** $\| d_*^x(\mu_{1,b}) \|_{L^1(t)} < \frac{1}{1000}$. for most $x \in E_2$.

  $\to d_*^x(\mu_{1,b})(t) \sim \mu_{1,b} * \sigma_t(x)$.

  (pushforward of $\mu_{1,b}$ under the pinned distance map.)

  $\to \mu_{1,b} = \sum_{j=0}^{\infty} \sum_{T \in \mathbb{T}_{j,bad}} M_T \mu_1$.

  $T \subset$ heavy plate thicker than T.
  or $\mu_2(4T) \geq R_j^{-\frac{\alpha}{2}+\varepsilon}$.

- **Simplified scenario: Single scale R.**

  Tubes: $R^{-1/2} \times R^{-1/2} \times 1$. Plates: R^{-K}-nbhd of hyperplane.

① Step 1. If $\mu_1(P) > 0$ for some hyperplane P. then we are trivially done. (directly apply 2D Falconer result to set E, where $\dim E > \frac{3}{2} > \frac{5}{4}$.)

Otherwise. $\mu_1(P) = 0$. $\forall P$. Hence. for sufficiently large R. $\mu_1(R^{-\beta}\text{-plate}) < \frac{1}{1000}$.

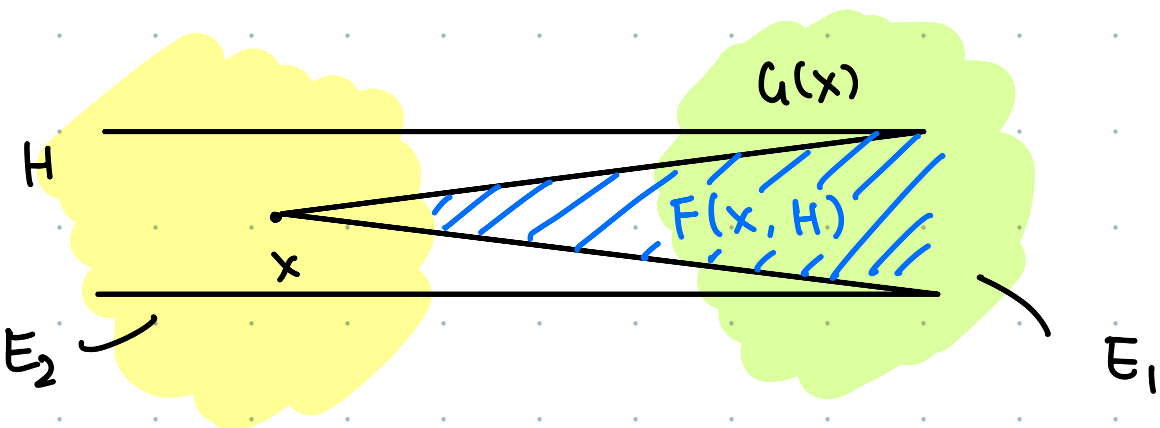
REMOVAL OF BAD TUBES

② **Step 2.** Fix $x \in E_2$ (TBD). remove T that are contained in any heavy plate through x. by removing such heavy plates directly. (adapted from a result of Shmerkin)

i.e., find large $E_2' \subset E_2$. s.t. $\forall x \in E_2'$

(1) Keep $G(x) \subset E_1$, where $G(x) = \{y : y \notin F(x,H),\ \forall \text{ heavy } H\}$.

(2). $E_1 \setminus G(x) \subset$ some $R^{-\beta}$-plate



$y \in E_1 \cap F(x.H)$

$\Rightarrow T(x.y) \subset H$.

Obs. $\|d_*^x(\mu_1) - d_*^x(\mu_1|_{G(x)})\|_{L^1(t)} \leq \mu_1(G(x)^c) < \frac{1}{1000}$, for R large enough. So we can replace μ_1 by $\mu_1|_{G(x)}$.

REMOVAL OF BAD TUBES

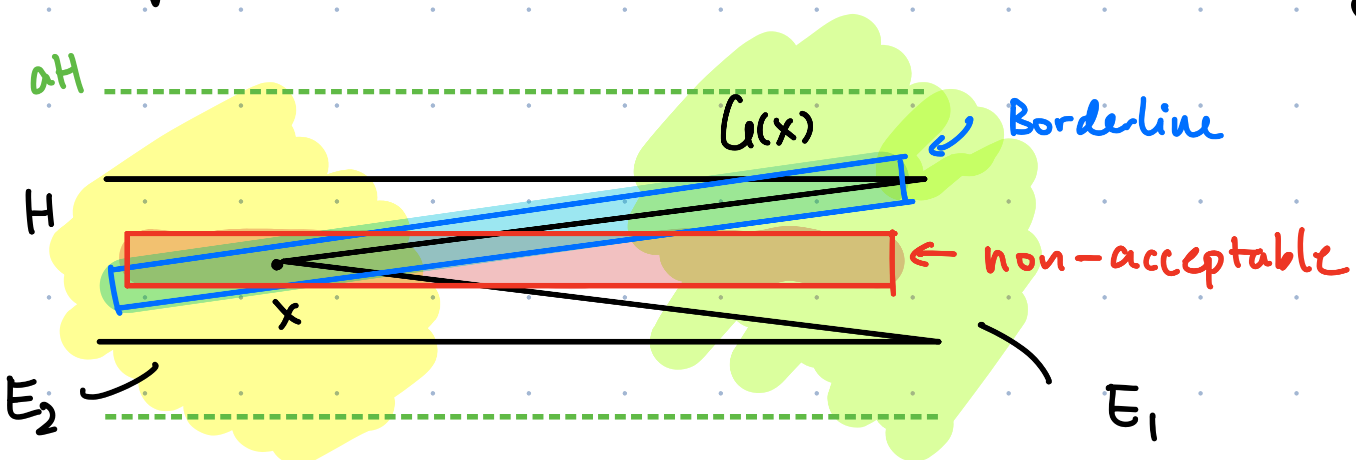
③ **Step 3.** $\mu_1|_{G(x)} \sim M_0(\mu_1|_{G(x)}) + \sum_T M_T(\mu_1|_{G(x)})$

Obs. $\mu_1|_{G(x)}$ doesn't see non-acceptable tubes :

$\|M_T(\mu_1|_{G(x)})\|_{L^1} \le \mathrm{RapDec}(R)$. $\quad$ ($M_T(\cdots)$ concentrates on $2T$).

$\Rightarrow$ Reduce to acceptable tubes T.

④ **Step 4.** Further remove borderline tubes. by introducing a random parameter a.



⑤ **Step 5.** We are left with tubes T that are away from any heavy plate. and $\mu_2(4T) \gtrsim R^{-\frac{\alpha}{2}+\varepsilon}$.

Apply Ren's Radial projection thm in $\mathbb{R}^d$.

REMOVAL OF BAD TUBES

- End product: reduced to $M_0(\mu_1|_{G(x)}) + \sum_{T \text{ good}} M_T(\mu_1|_{G(x)})$.

$\rightarrow$ The dependence on x is very scary. could be detrimental in the weighted restriction step.

$\rightarrow$ But we can replace $M_T(\mu_1|_{G(x)})$ by $M_T(\mu_1)$!

$$\| M_T(\mu_1|_{G(x)}) - M_T(\mu_1) \|_{L^1} = \| M_T(\mu_1|_{G(x)^c}) \|_{L^1} \leq \mathrm{RapDec}(R).$$

(T good & not borderline $\Rightarrow T \cap E_1 \subset G(x)$.)

$\rightarrow$ $\mu_1 . g = M_0(\mu_1|_{G(x)}) + \sum_{j=1}^{\infty} \sum_{T \in \mathbb{T}_j, \text{good}} M_T \mu_1.$

(Dependence on x in the M_0 term is ok. only need L^1 bd of $\mu_1|_{G(x)}$. no restriction esti. involved.)

REMOVAL OF BAD TUBES

- In reality, we need to consider a multi-scale problem. $R_j = 2^j R_0$, $j \geq 0$. This is a main technical difficulty.
  - construct $G_0(x) \supseteq G_1(x) \supseteq \cdots \supseteq G_j(x) \supseteq G_{j+1}(x) \supseteq \cdots$
  - reduce to $G_j(x)$ on j^{th} Littlewood-Paley piece of μ_1.
  - Show $\mu_1(G_j(x) \setminus G_{j+1}(x)) \lesssim R_j^{-1}$.
- Back to Step 5: remove T s.t. T is away from all heavy plates, and $\mu_2(4T) \geq R^{-\frac{\alpha}{2}+\varepsilon}$.

Ren's Radial Projection Theorem:

$E, F \subseteq \mathbb{R}^d$ with $\dim \leq k$. If E is NOT contained in a k-plane then $\sup_{x \in E} \dim(P_x(F \setminus \{x\})) \geq \min\{\dim E, \dim F\}$. $P_x(y) := \frac{x-y}{|x-y|}$

($d=2$, $k=1$ case was proved recently by Orponen-Shmerkin-Wang)

Cor: μ_i, E_i as before. α-dim. Fix $\eta, \kappa > 0$. $\exists\, \gamma > 0$ s.t.

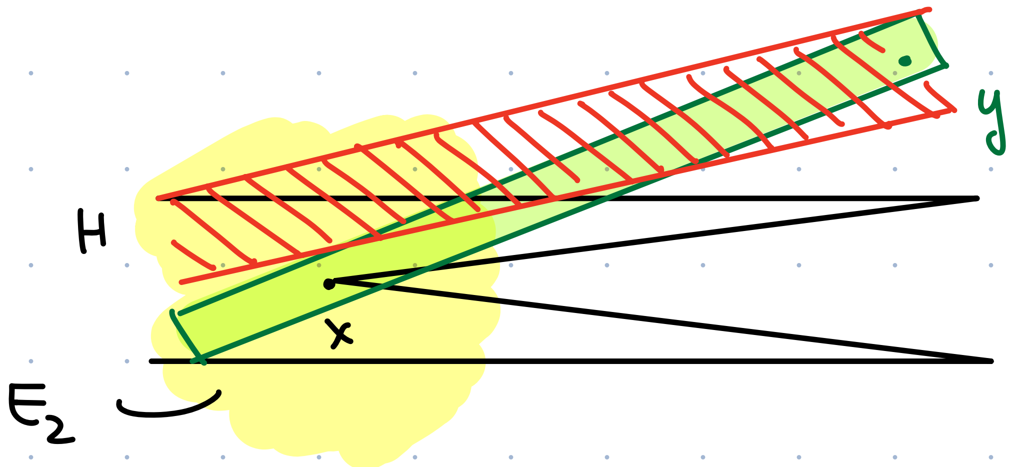
$\forall j \geq 0$, $\exists\, B \subset E_1 \times E_2$ s.t. ① $\mu_1 \times \mu_2(B) < R_j^{-\gamma}$, and

② $\forall\, y \in E_1$ and $R_j^{-1/2}$-tube $T \ni y$, $\mu_1(T \setminus (A|_y \cup B|_y)) \leq R_j^{-\frac{\alpha}{2} + O(\kappa) + \frac{\varepsilon}{2}}$

($A|_y := \{x \in E_2 : x, y$ in some $R_j^{-\eta}$-heavy $\frac{R_j^{-\kappa}}{C}$-plate.)

→ If T is away from heavy plates, A can be ignored.

($x \in T$, $T \subset G(x) \Rightarrow \forall\, y \in T \cap E_1$, $T \cap A|_y = \phi$.)



- $x, y \in T$.
- T away from all heavy plates through x.

→ B helps us find a large subset of good pin points x.

→ If $\mu_2(T)$ is large, fixing $y \in E_1$, $\#\{$such T thru $y\}$ is small. Combined with , one gets a decay for μ_1 of union of such T.

Thank you for listening!