A new conjecture to unify Fourier restriction and Bochner-Riesz

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Contents

Partial integrals for the Fourier transform

- Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi \mathrm{i} x\cdot\xi} \mathrm{d}x$.
- Fourier inversion: $f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi \mathrm{i} x\cdot\xi} \mathrm{d}\xi$.
- Partial integrals: For each $R > 0$.
  $S_R f(x) = \int_{|\xi|\leq R} \hat{f}(\xi) e^{2\pi \mathrm{i} x\cdot\xi} \mathrm{d}\xi$.

Question

Does $S_R f$ always converge to f in L^p norm for arbitrary $f \in L^p$ when $R \to \infty$? ($1 < p < \infty$ fixed)

Fefferman's theorem

Question

Do we always have $S_R f \xrightarrow{L^p} f$ *when* $R \to \infty$*? (*$1 < p < \infty$ *fixed)*

- Reduces to the L^p-boundedness for S_1.
- True for $p = 2$ by Plancherel.
- True for $n = 1$ and arbitrary $1 < p < \infty$ (Hilbert transform, singular integral theory).

Theorem (Fefferman, 1971)

The question has a negative answer for all $p \neq 2$ *when* $n > 1$*.*

Some L^p-boundedness barely fails

- Fix $\delta > 0$. $S_1^\delta f(x) = \int_{|\xi|\leq 1} (1 - |\xi|^2)^\delta \hat{f}(\xi) e^{2\pi \mathrm{i} x\cdot\xi} \mathrm{d}\xi$.

Conjecture (Bochner-Riesz)

S_1^δ is bounded on L^p if $|\frac{1}{p} - \frac{1}{2}| < \frac{2\delta+1}{2n}$.

- Condition is necessary (Herz, 1962).
- Easier for larger $p > 2$.
- Can focus on: δ small and p very close to $\frac{2n}{n-1}$.
- Known in dimension $n = 2$ (Carleson-Sjölin, 1972).
- Widely open for $n \geq 3$.

Related oscillatory integral operators

$$S^\lambda f(x) = \int_{\mathbb{R}^n} e^{2\pi \mathrm{i} \lambda |x-y|} a(x-y) f(y) \mathrm{d}y.$$

[$\lambda \geq 1$. a: smooth cutoff away from 0.]

Conjecture

$\|S^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^{-\frac{n}{p}+\varepsilon}$, $p > \frac{2n}{n-1}$.

$$\tilde{S}^\lambda g(x', t) = \int_{\mathbb{R}^{n-1}} e^{2\pi \mathrm{i} \lambda t^{-1} \sqrt{\lambda^2 + |x' - t\xi|^2}} \tilde{a}(\frac{x' - t\xi}{\lambda}, \frac{t}{\lambda}, \xi) g(\xi) \mathrm{d}\xi.$$

[$\lambda \geq 1$. $\tilde{a}$: smooth cutoff away from 0 in the n-th variable.]

Conjecture

$\|\tilde{S}^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^{\varepsilon}$, $p > \frac{2n}{n-1}$.

Both conjectures imply Bochner-Riesz.

The Fourier restriction phenomenon

- For $f \in L^1(\mathbb{R}^n)$, $\hat{f}$ can be defined everywhere.
- Not true for general L^2 functions.
- Fourier restriction phenomenon (Stein (1967), ...): There exists $1 < p' < 2$ s.t. for every $f \in L^{p'}$, $\hat{f}$ can be meaningfully restricted to S^{n-1} as an integrable function (w.r.t. the hypersurface measure $\mathrm{d}\sigma$).
- Conjectural range: $p' < \frac{2n}{n+1}$. i.e. $p > \frac{2n}{n-1}$.
- Curvature is key.

$L^\infty \to L^p$ Fourier extension estimates

- By duality, Fourier restriction is equivalent to the following *Fourier extension estimate*:

  $$\|Ef\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(\mathrm{d}\sigma)}$$

  where f is a function on S^{n-1} and

  $$Ef(x) = \int_{S^{n-1}} e^{2\pi \mathrm{i} x \cdot \xi} f(\xi) \mathrm{d}\sigma(\xi).$$

- Conjectural range: $p > \frac{2n}{n-1}$.
- Known for $n = 2$ (Fefferman, Stein (1970)).
- Widely open for $n \geq 3$.

Hörmander type operators: setup

- We had two oscillatory integral operators mapping functions on $\mathbb{R}^{n-1}$ to functions on $\mathbb{R}^n$.
- For $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$, real $\phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ smooth in a neighborhood of supp$a$ and $\lambda > 1$, consider the operator

$$T^\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{2\pi \mathrm{i} \phi^\lambda(x;\xi)} a^\lambda(x;\xi) f(\xi) \mathrm{d}\xi$$

where $\phi^\lambda(x;\xi) = \lambda\phi(\frac{x}{\lambda};\xi)$ and $a^\lambda(x;\xi) = a(\frac{x}{\lambda};\xi)$.

Hörmander conditions

If we have

- (H1) The rank of $\nabla_x \nabla_\xi \phi$ is $n-1$ throughout supp$a$.
- (H2) For the *Gauss map* $G(x;\xi)$ with $G = \frac{G_0(x;\xi)}{|G_0(x;\xi)|}$ and

  $$G_0(x;\xi) = \wedge_{j=1}^{n-1} \partial_{\xi_j} \nabla_x \phi(x;\xi),$$

  we have

  $$\det(\nabla_\xi)^2 \langle \nabla_x \phi(x;\xi), G(x;\xi_0) \rangle|_{\xi=\xi_0} \neq 0$$

  ,

then T^λ is called a (family of) *Hörmander type operator*(s).

Hörmander's question

Question

(Hörmander, 1972) For a family of Hörmander type operators T^λ, is it true that $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$, $\forall p > \frac{2n}{n-1}$?

- Originally a constant in place of λ^ε. Standard to allow λ^ε-loss.
- True for $n = 2$ (Hörmander, also Carleson-Sjölin).
- Partial progress by Stein (1984): True in the Stein-Tomas range $p \geq \frac{2(n+1)}{n-1}$.
- Would imply Bochner-Riesz and Fourier restriction. Also other applications ...

Bad operators

Question

(Hörmander, 1972) For a family of Hörmander type operators T^λ, is it true that $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$, $\forall p > \frac{2n}{n-1}$?

- Fails in all dimensions > 2! (Bourgain (1991), Bourgain-Guth (2011), also Wisewell (2005))
- Example: $n = 3$.

  $$\phi(x;\xi) = x_1\xi_1 + x_2\xi_2 + x_3\xi_1\xi_2 + \frac{1}{2}x_3^2\xi_1^2$$

  makes the boundedness fail unless $p \geq 4$.
- Best p known for all dimensions (Bourgain, Bourgain-Guth).

The Positive curvature condition

Lee (2006) noticed if we also have

- (H2+) $(\nabla_\xi)^2 \langle \nabla_x \phi(x;\xi), G(x;\xi_0)\rangle|_{\xi=\xi_0}$ is always positive definite,

then the range of p may be improved.

(H2+) holds for the operators of interest in Bochner-Riesz and Fourier restriction (original ver.).

Question

For a family of Hörmander type operators T^λ satisfying (H2+), is it true that $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$, $\forall p > \frac{2n}{n-1}$?

Modified question

Question

For a family of Hörmander type operators T^λ satisfying (H2+), is it true that $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$, $\forall p > \frac{2n}{n-1}$?

- Again fails in all dimensions > 2! (Bourgain (1991), Guth-Hickman-Iliopoulou (2017))
- Best p known for all dimensions (Guth-Hickman-Iliopoulou, previous sharp examples in $3D$ by Wisewell (2005) and Minicozzi-Sogge (1997)).
- This was the main approach to Bochner-Riesz in high dimensions before the work of Wu (2020), Guo-Oh-Wang-Wu-Z. (2021) and Guo-Wang-Z. (2022).

A bad operator (positive curvature version)

Question

For a family of Hörmander type operators T^λ satisfying (H2+), is it true that $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$, $\forall p > \frac{2n}{n-1}$?

Answer: *Again, not necessarily*.

- A "worst example": $n = 3$.

$$\phi(x;\xi) = -x_1\xi_1 - x_2\xi_2 + \frac{1}{2}x_3\xi_1^2 + \frac{1}{2}x_3\xi_2^2 + x_3^2\xi_1\xi_2 + \frac{1}{2}x_3^3\xi_2^2$$

makes the boundedness fail unless $p \geq \frac{10}{3}$.

Generic failure

Diffeomorphisms in x and in ξ (separately) can change ϕ to a *normal form* around any point (taken to 0) in supp$a$:

$$\phi(x;\xi) = x_1\xi_1 + \cdots + x_{n-1}\xi_{n-1} + x_n\langle A\xi, \xi\rangle + O(|x_n||\xi|^3 + |x|^2|\xi|^2).$$

Theorem (Bourgain (1991))

Suppose $n = 3$. If ϕ is in a normal form and $a \neq 0$ at the origin, then $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$ fails for $p < \frac{118}{39}$ if $\partial_{x_3}^2 (\nabla_\xi)^2 \phi|_{(0;0)}$ is not a multiple of $\partial_{x_3} (\nabla_\xi)^2 \phi|_{(0;0)}$.

For the operators of interest in Bochner-Riesz and Fourier restriction, $\partial_{x_3}^2 (\nabla_\xi)^2 \phi|_{(0;0)}$ is always a multiple of $\partial_{x_3} (\nabla_\xi)^2 \phi|_{(0;0)}$ in the normal form expansion around every point.

An open question

What can we do if we move further along this direction?

Question

If ϕ satisfies the proportionality condition everywhere, can we use differeomorphisms in x and ξ (separately) to change ϕ to a good form?

We tried to compute cases of low degree polynomials in Mathematica and did not have much clue.

Other directions?

One can try to prove positive results if the proportionality condition is satisfied everywhere. We tried and succeeded in dimension 3.

Higher dimensions?

What would happen in high dimensions? To prove positive results do one need more derivatives?

Surprisingly, no!

Our discovery

We think a natural generalization of Bourgain's proportionality condition in all dimensions should be a good one to unify Bochner-Riesz and Fourier restriction.
We say ϕ satisfies *Bourgain's condition* at a point if there are two diffeomorphisms in x and ξ, resp., sending the point to $(0;0)$ and changing ϕ to a normal form with $\partial_{x_n}^2(\nabla_\xi)^2\phi|_{(0;0)}$ being a multiple of $\partial_{x_n}(\nabla_\xi)^2\phi|_{(0;0)}$.

- This is intrinsic.

Conjecture (Guo-Wang-Z. (2022))

For a family of Hörmander type operators T^λ satisfying (H2+), $\|T^\lambda\|_{L^p\to L^p}\lesssim_\varepsilon \lambda^\varepsilon$ holds for every $p>\frac{2n}{n-1}$ if and only if ϕ satisfies Bourgain's condition everywhere in supp$a$.

Generic failure in arbitrary dimension

Theorem (Guo-Wang-Z. (2022))

If Bourgain's condition fails at a point, then $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$ *fails for* $p < \frac{2(2n^2+n-1)}{2n^2-n-2}$.

- This number is $> \frac{2n}{n-1}$.

Positive result

Theorem (Guo-Wang-Z. (2022))

If Bourgain's condition is satisfied everywhere in $\mathrm{supp}\, a$*, then* $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$ *holds for* $p > p_{n,\mathsf{GWZ}}$.

- This gives asymptotic improvement on the previously best-known range of exponents in both Bochner-Riesz and Fourier restriction in all high dimensions.

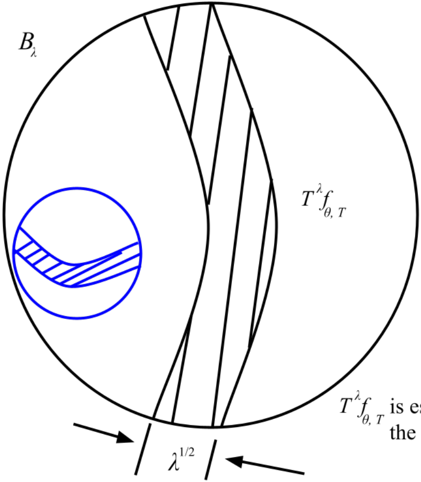
An interesting connection to geometry

Recent work on (reduced) Carleson-Sjölin operators for manifolds (Dai-Gong-Guo-Z., 2023):
Bourgain's condition $\Leftrightarrow$ constant curvature.

Wave packet decomposition

- One can cut the ξ-space into small caps of size $\lambda^{-\frac{1}{2}}$. The contribution from each cap is a superposition of *wave packets* that live in curved $\lambda^{\frac{1}{2}} \times \cdots \times \lambda^{\frac{1}{2}} \times \lambda$-tubes.
- Tubes are straight (Kakeya setting) in Fourier restriction.
- Generally they develop along polynomial curves.
- Similar structure at other scales; parabolic rescaling.

Wave packet in a ball



$T^\lambda f_{\theta,T}$ is essentially supported in the shaded region.

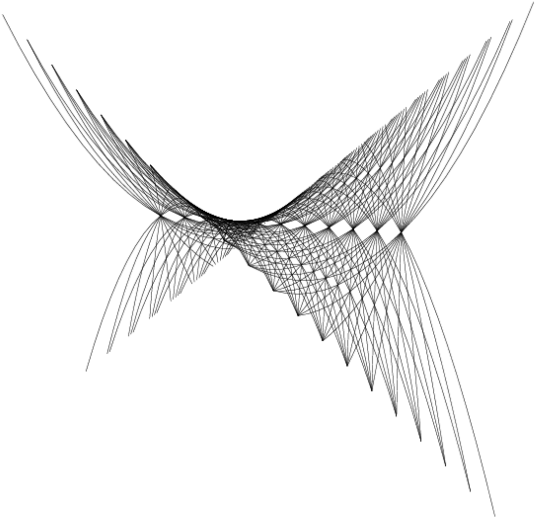
Generic failure: linear algebra of polynomials

Theorem (Guo-Wang-Z. (2022))

If Bourgain's condition fails at a point, then $\|T^\lambda\|_{L^p \to L^p} \lesssim_\varepsilon \lambda^\varepsilon$ *fails for* $p < \frac{2(2n^2+n-1)}{2n^2-n-2}$.

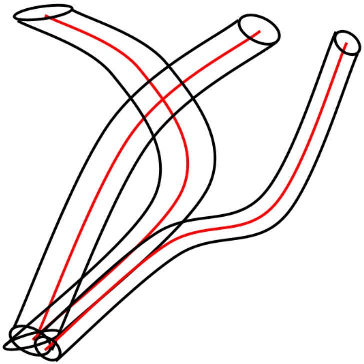
- Bourgain proved generic failure in $\mathbb{R}^3$ by *Kakeya compression*: One can compress part of tubes, one from each cap, locally into a neighborhood of a surface.
- We prove this phenomenon in every dimension > 2.
- Use calculus to compute the volume of the union of the central curves. A bit of semialgebraic geometry to control the surface area of the union.
- The choice of the "initial position" function has to make an $(n-1) \times (n-1)$-determinant of n-variate polynomials have order n. Done by very involved linear algebra.

Kakeya compression for curves



Picture taken from [GHI]

Bounding volume of the union of tubes



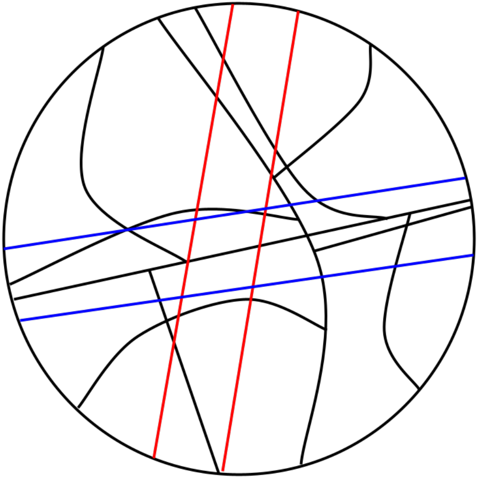
Look at the union of central curves (in red). We need to bound its volume and surface area.

Polynomial partitioning

Guth (2014, 2016) developed a framework of studying $T^\lambda f$ by cutting up the function repeatedly using zero sets of polynomials (originally in the Fourier restriction setting).

- Inspired by previous works of Dvir (2008), Guth (2008), Guth-Katz (2008, 2010), Solymosi-Tao (2011).
- The advantage of polynomials: Zero sets don't intersect lines a lot, but can cut the function into far more pieces "evenly".
- One then cares a lot about possible tubes near the zero set of the polynomial.

Cells cut out by zero sets of polynomials



Use zero sets of polynomials (black) repeatedly to cut B_R into cells. If a tube enters too many cells it has to be close to the zero set.

Polynomial Wolff axioms

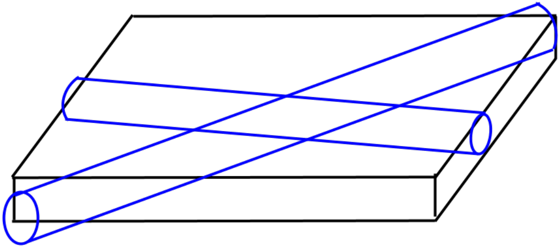
Theorem (Polynomial Wolff axioms, Guth ($n = 3$, 2014), Zahl ($n = 4$, 2018), Katz-Rogers (all n, 2018))

Let Z be the zero set of a polynomial of degree $O(1)$ in $B_1^n \subset \mathbb{R}^n$. Then the number of δ-separated directions such that there is a δ-tube of length ~ 1 in that direction lying in the $(C\delta)$-neighborhood of Z is bounded by

$$O_\varepsilon(\delta^{-(n-2+\varepsilon)}).$$

- Constant only depends on the dimension and the degree.
- A key ingredient: Tarski-Seidenberg theorem.
- Proof intuition: Kakeya compression for lines cannot happen if the "initial position" depends very "nicely" on the directions.
- We prove that natural generalizations of this theorem continue to hold if one has Bourgain's condition.

Understanding the Polynomial Wolff Axiom



Among all polynomial hypersurfaces of degree ≤ 100, the hyperplane essentially has the largest possible number of "almost tangential directions". For each of those directions, there is a unit line segment with all points on it close to the hypersurface.

The Variety Uncertainty Principle

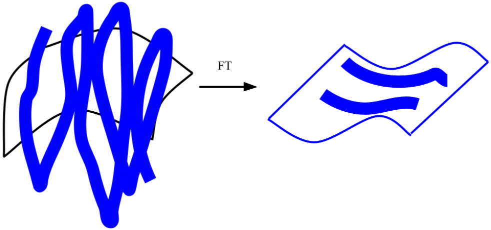
Theorem (Variety Uncertainty Principle, codim 1 case, essentially in Guo-Wang-Z. (2022))

For Z_1 and Z_2 being the zero sets of polynomials of degree $O(1)$ in $\mathbb{R}^n$, take the subset Y_1 (inside R-ball) and Y_2 (inside 1-ball) in Z_1 and Z_2, resp., with the tangent hyperplane everywhere on Y_1 and Y_2 having angle $\leq 100^{-n}$ against the $x_1 \cdots x_{n-1}$-hyperplane. Then for all $f \in L^\infty(Y_2)$,

$$\|(f \mathrm{d}\sigma_2)\check{}\|_{L^2(Y_1, \mathrm{d}\sigma_1)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^2(Y_2, \mathrm{d}\sigma_2)}.$$

- Constant only depends on the dimension and the degree.
- Has to do with the "broom" approach of Wang (2018). Easy version needed in her setting as one can take Z_2 to be a line.
- Proved by induction on scales and a geometric lemma of Guth.

Uncertainty for varieties



If $\hat{g}$ is on one "essentially horizontal" variety, then you can expect $\|g\|_{L^2}$ to be "smallest possible" on any "essentially horizontal" variety.

Thank you!