Norm estimates for discrete singular integrals*

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May 13, 2023
Madison, WI

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†Partially supported by NSF

I. Motivation

1. A version of the sharp ℓ^p inequality for the discrete Hilbert transform in higher dimesnions.
2. A lot of interest in last $\sim 20/25$ years in studying "Discrete Analogues in Harmonic Analysis" With many works by many authors: J. Bourgain, E.M. Stein S. Wainger, A. Magyar, S. Wainger, Ionescu, Pierce, Mirek, and many others. Vast literature now.

L. Pierce, 2009 Ph.D. "Discrete analogues in harmonic analysis," Princeton University. Beautifully written!

II. Motivation for techniques used

Probabilistic ideas and tools have been quite successful in obtain (a) sharp inequalities or (b) dimension free inequalities for various singular integrals and Fourier multipliers on $\mathbb{R}^d$, and extensions to other geometric settings–Lie groups, manifolds, vector bundles, infinite dimensions, . . . Vast literature now!

RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

SOME RESULTS IN HARMONIC ANALYSIS IN $\mathbf{R}^n$, FOR $n \to \infty$

BY E. M. STEIN

1. Introduction. The purpose of this note is to bring to light some further results whose thrust is that certain fundamental estimates in harmonic analysis in $\mathbf{R}^n$ have formulations with bounds independent of n, as $n \to \infty$.

2. The theorem. In $\mathbf{R}^n$ we define the familiar Riesz transforms by $(R_j f)\hat{}\,(\xi) = i(\xi_j/|\xi|)\hat{f}(\xi)$, $j = 1, \ldots, n$, and write $R = (R_1, \ldots, R_n)$; also $|R(f)(x)|$ will stand for $(\sum_{j=1}^n |R_j(f)(x)|^2)^{1/2}$.

THEOREM.

$$\|R(f)\|_p \leq A_p \|f\|_p, \qquad 1 < p < \infty,$$

with A_p independent of n.

"Proof" (Littlewood-Paley)

$$\|Rf\|_p \leqslant a_p \|g_v(Rf)\|_p = a_p \|g_h(f)\|_p \leqslant a_p b_p \|f\|_p$$

Behaviour in p? $a_p \sim p$, $b_p \sim \sqrt{p} \Rightarrow A_p \sim p^{3/2}$, $p \to \infty$.

Open: What is the sharp constant A_p?

Stein concludes:

"The above results raise the following question. Can one find an appropriate infinite dimensional formulation of (that part of) harmonic analysis in $\mathbb{R}^n$, which displays, in a natural way the above uniformity in n? A related question is to study the "limit as $n \to \infty$" of the above results, insofar as such limits may have a meaning. "One might guess that a further understanding of these questions would involve, among other things, notions from probability theory: i.e. Brownian motion and possibly variants of the central limit theorem."

$$p_y(x) = \frac{c_d\ y}{(|x|^2 + y^2)^{\frac{d+1}{2}}}, \quad x \in R^d, y > 0, \quad u_f(x,y) = (p_y * f)(x)$$

$$R^{(k)} f(x) = \int_0^\infty \frac{\partial u_f(x,y)}{\partial x_k}\, dy = c_d \int_{\mathbb{R}^d} \frac{z_k}{|z|^{d+1}} f(x-z)\, dz$$

Gundy-Varopoulos (1978)

$$R^{(k)} f(x) = 2\mathrm{E}\left(\int_{-\infty}^{0} \frac{\partial u_f(W_s)}{\partial x_k}\, dY_s \middle| W_0 = x \right)$$

$W_t = (X_t^1, \ldots X_t^d, Y_t), -\infty < t \leqslant 0$ "background radiation" in $\mathbb{R}_+^{d+1}$,

$B_t = (X_t^1, \ldots X_t^d, Y_t)$, $(d+1)$-Brownian motion, τ exit time from $\mathbb{R}_+^{d+1}$

$$\begin{aligned} R^{(k)} f(x) &= 2 \lim_{y\to\infty} \mathrm{E}_{(0,y)} \left(\int_0^\tau \frac{\partial u_f(B_s)}{\partial x_k} dY_s | B_\tau = x \right) \\ &= \lim_{y\to\infty} \left(\mathrm{E}_{(0,y)} \int_0^\tau \left[\frac{\partial u_f(B_s)}{\partial x_k} dY_s - \frac{\partial u_f(B_s)}{\partial y} dX_s^k \right] \middle| B_\tau = x \right) \end{aligned}$$

$\mathcal{S}(x,y)=\{A_j(x,y)\}_{j=1}^m$, $A_j(x,y)\in\mathfrak{M}_{(d+1)}=(d+1)\times(d+1)$ matrices, $x\in\mathbb{R}^d, y\geqslant 0$

$$\|\mathcal{S}(x,y)\|=\left(\sup_{v\in R^{d+1},|v|\leqslant 1}\sum_{j=1}^m |A_j(x,y)v|^2\right)^{1/2}, \quad \|\mathcal{S}\|=\|\|\mathcal{S}(x,y)\|\|_{L^\infty(\mathbb{R}^d\times[0,\infty))}$$

$$\nabla u_f=\left(\frac{\partial u_f}{\partial x_1},\cdots,\frac{\partial u_f}{\partial x_d},\frac{\partial u_f}{\partial y}\right)$$

$$T_{A_j}f(x)=\lim_{y\to\infty}\mathrm{E}_{(0,y)}\left(\int_0^\tau [A_j(B_s)\nabla u_f(B_s)]\cdot dB_s\middle| B_\tau=x\right)$$

The $T_{A_j}{}'s$ are bounded on $L^p(\mathbb{R}^d)$, $1<p<\infty$: For a universal constant C ($C=7$ works)

$$\left\|\left(\sum_{j=1}^m |T_{A_j}f(x)|^2\right)^{1/2}\right\|_p\leqslant C(p^*-1)\|\mathcal{S}\|\|f\|_p, \quad \text{(R.B. 1984, 1986)}$$

$$p^*-1=\begin{cases}\frac{1}{p-1}, & 1<p\leqslant 2\\ p-1, & 2\leqslant p<\infty\end{cases}$$

Stein's inequality follows with $A_p\leqslant C(p^*-1)$

The $T_A's$ are bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. With $p^* = \max\{p, q\}$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left\| \left(\sum_{j=1}^{\infty} |T_{A_j} f(x)|^2 \right)^{1/2} \right\|_p \leqslant (p^* - 1)\|\mathcal{S}\|\|f\|_p = \begin{cases} (p-1)\|\mathcal{S}\|\|f\|_p, & 2 \leqslant p < \infty \\ \frac{1}{(p-1)}\|\mathcal{S}\|\|f\|_p, & 1 < p \leqslant 2 \end{cases}$$

(G. Wang & R.B. 1995)

If $A(x,y)v \cdot v = 0$, all $v \in \mathbb{R}^{d+1}$, all $(x,y) \in \mathbb{R}^{d+1}_+$ (orthogonality property)

$$\|T_A\|_{L^p \to L^p} \leqslant cot\left(\frac{\pi}{2p^*}\right) = \begin{cases} tan\left(\frac{\pi}{2p}\right), & 1 < p \leqslant 2 \\ cot\left(\frac{\pi}{2p}\right), & 2 \leqslant p < \infty \end{cases}$$ (G. Wang & R.B. 1995)

(M. Perlmutter 2015) The $T_A{}'s$ are C-Z $\left(|K(x,\tilde{x})| \leqslant \frac{\kappa}{|x-\tilde{x}|^d}, \quad |\nabla K(x,\tilde{x})| \leqslant \frac{\kappa}{|x-\tilde{x}|^{d+1}}\right)$

$$\begin{aligned} T_A f(x) &= p.v. \int K(x,\tilde{x}) f(\tilde{x}) d\tilde{x} \\ K_A(x,\tilde{x}) &= \int_{\mathbb{R}^{d+1}_+} 2y \left[A(z,y)\nabla p_y(z-\tilde{x})\right] \cdot \nabla p_y(z-x) dz\, dy \end{aligned}$$

For A constant (or function of y only) $K(x,\tilde{x}) = K(x-\tilde{x})$.

For $k = 1, 2, \ldots, d$,

$$\mathbb{H}^{(k)} = (a_{ij}^{(k)}) = \begin{cases} -1, & i = k, j = d+1 \\ 1, & i = d+1, j = k \\ 0, & \text{otherwise}, \end{cases}$$

$d = 2:$

$$\mathbb{H}^{(1)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathbb{H}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R^{(1)} = T_{\mathbb{H}^{(1)}}, \ R^{(2)} = T_{\mathbb{H}^{(2)}},$$

$$\mathbb{H}_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathbb{H}_0^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_{\mathbb{H}_0^{(1)}} = \frac{1}{2} R^{(1)}, \quad T_{\mathbb{H}_0^{(2)}} = \frac{1}{2} R^{(2)}$$

- $\|\mathbb{H}^{(k)}\| \leqslant 1$
- $\mathbb{H}^{(k)} v \perp v = 0$, all $v \in \mathbb{R}+^{d+1}$ (orthogonality property)
- The sequence $\mathcal{S} = \{\mathbb{H}_0^{(k)}\}_{k=1}^{d}$ has $\|\mathcal{S}\| \leqslant 1$

1. T. Iwaniec & G. Martin (1996) (method of rotations)/G. Wang & R.B. (1995) (martingales as above)

$$\|R^{(k)}\|_{L^p\to L^p} = \|H\|_{L^p\to L^p} = cot\left(\tfrac{\pi}{2p^*}\right) = \begin{cases} tan\left(\frac{\pi}{2p}\right), & 1 < p \leqslant 2 \\ cot\left(\frac{\pi}{2p}\right), & 2 \leqslant p < \infty. \end{cases}$$

$$Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x-y)}{y}\,dy$$

Second = S. Pichorides (1972) and B. Cole (Published in 1978 by T. Gamelin)

2. For the vector:

$$\|Rf\|_p \leqslant \sqrt{2}cot\left(\tfrac{\pi}{2p^*}\right),\ \ p \geqslant 2, \quad \text{Iwaniec–Martin (1996)}$$

$$\|Rf\|_p \leqslant 2(p^*-1), \quad 1 < p < \infty \quad \text{Wang–B. (1995)}$$

Where do the inequalities for the $T_A's$ come from?

From inequalities on "subordination" of martingales/stochastic integrals. They can be applied in a wide variety of settings where a "method of rotations" is not available.

Sharp martingale/stochastic inegrals inequalities

$$N_t = \int_0^t K_s \cdot dB_s, \quad M_t = \int_0^t H_s \cdot dB_s,$$

1. N is subordinate to M ($N \ll M$) if $|K_s| \leqslant |H_s|$ a.s. for all s, and
2. N is orthogonal to M ($N \perp M$) if $K_s \cdot H_s = 0$ a.s. for all s.

$$\|N_t\|_p \leqslant \begin{cases} (p^* - 1)\|M_t\|_p, & \text{if } N \ll M \text{ (Burkholder 1984)}, \\ cot\left(\frac{\pi}{2p^*}\right)\|M_t\|_p, & \text{if } N \ll M, N \perp M \text{ (G. Wang \& R.B. 1995)}. \end{cases}$$

These are are sharp

The discrete Hilbert transform

M. Riesz–Proved 1923, Published 1927: $1 < p < \infty$,

$$\|Hf\|_{L^p(\mathbb{R})} \leqslant C_p \|f\|_{L^p(\mathbb{R})}$$

Same paper: L^p boundedness of H implies l^p boundedness for H_{dis}.

$$\|H_{dis}f\|_{\ell^p(\mathbb{Z})} \leqslant C_p{}' \|f\|_{\ell^p(\mathbb{Z})}$$

$$H_{dis}f(n) = \frac{1}{\pi} \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{f(n-m)}{m} \quad \text{(Hilbert 1907)}$$

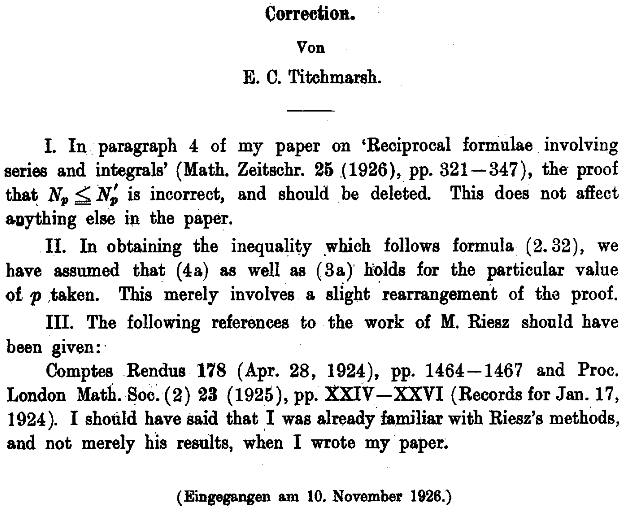
Further he showed: $\|H\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leqslant \|H_{dis}\|_{\ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})} \leqslant C\|H\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})}$

E. C. Titchmarsh (Proved 1924, Published 1926)

1. $\|H_{dis}f\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leqslant C_p \|f\|_{\ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})}$
2. $\|H_{dis}\|_{\ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})} = \|H\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})}$

Nice read of History: Mary Cartwright: "Manuscripts of Hardy, Littlewood, Marcel Riesz and Titchmarsh," Bull London. Math. Soc, 14 (1982), 472-532

Unfortunately there was an error in Titchmarsh's proof of $\|H_{dis}\|_p \leqslant \|H\|_p$.


Correction.

Von

E. C. Titchmarsh.

I. In paragraph 4 of my paper on 'Reciprocal formulae involving series and integrals' (Math. Zeitschr. 25 (1926), pp. 321—347), the proof that $N_p \leqq N'_p$ is incorrect, and should be deleted. This does not affect anything else in the paper.

II. In obtaining the inequality which follows formula (2.32), we have assumed that (4a) as well as (3a) holds for the particular value of p taken. This merely involves a slight rearrangement of the proof.

III. The following references to the work of M. Riesz should have been given:

Comptes Rendus 178 (Apr. 28, 1924), pp. 1464—1467 and Proc. London Math. Soc. (2) 23 (1925), pp. XXIV—XXVI (Records for Jan. 17, 1924). I should have said that I was already familiar with Riesz's methods, and not merely his results, when I wrote my paper.

(Eingegangen am 10. November 1926.)

The "tantalizing" problem: Prove the two operators have the same norm

Many proofs of the ℓ^p boundedness of H_{dis} exists

Hardy, Littlewood and Pólya (1934), S. Kak (1977), E. Laeng (2007), O. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea, and J. L. Varona (2017), Arcozzi, K. Domelevo and S. Petermichl (2022)

For $p = 2^m, m \in \mathbb{N}$ (or its conjugate)

Equality of norms: I. Gohberg y N.Y. Krupnik (1968). Using variant of M. Cotlar's identity in his: "A unified theory of Hilbert transforms and ergodic theorems" Revista matemática Cuyana (1955): $|Hf|^2 = 2H(f \cdot Hf) + |f|^2$

$$\|Hf\|_{2p}^2 = \|(Hf)^2\|_p \leqslant 2\|H(f \cdot Hf)\|_p + \|f^2\|_p \ldots$$

... leads to

$$\|H\|_{2p} \leqslant \|H\|_p + \sqrt{1 + \|H\|_p^2}$$

Use

$$cot(\alpha) + \sqrt{1 + cot^2(\alpha)} = cot(\tfrac{\alpha}{2}), \quad \alpha = \frac{\pi}{2^n}$$

$$|H_{dis}f|^2 = 2H_{dis}(f \cdot H_{dis}f) + J(f^2) + 2f \cdot Jf, \quad Jf = \frac{1}{\pi^2 k^2} * f$$

M. Kwaśnicki & R.B. (2019)

$$\|H_{dis}f\|_{\ell^p} \leqslant cot\left(\frac{\pi}{2p^*}\right)\|f\|_{\ell^p}, \quad 1 < p < \infty,$$

In particular,

$$\|H_{dis}\|_{\ell^p \to \ell^p} \leqslant \|H\|_{L^p \to L^p}.$$

Together with Riesz

$$\|H_{dis}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}$$

Another "tantalizing" question: Does this extend to Riesz transforms?

$$R^{(k)}_{dis}f(n) = c_d \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \frac{m_k}{|m|^{d+1}} f(n-m), \quad k = 1, 2, \ldots, d$$

That is,

$$\|R^{(k)}_{dis}\|_{\ell^p \to \ell^p} = \|R^{(k)}\|_{L^p \to L^p} = \|H_{dis}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}?$$

Calderón-Zygmund (1952)–Added in Proof, p. 138

$$Tf(x) = p.v. \int_{\mathbb{R}^d} K(y) f(x-y) dy, \quad T_{dis} f(n) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} K(m) f(n-m)$$

$$T : L^p \to L^p \quad \Longrightarrow \quad T_{dis} : \ell^p \to \ell^p$$

"For $n = 1$ this remark is due to M. Riesz, and the proof in the case of general n follows a similar pattern."

In fact, following Riesz one gets:

$$\|T_{dis}\|_{\ell^p \to \ell^p} \leqslant \|T_1\|_{L^p \to L^p} + C(d, \kappa)$$

$$\Rightarrow \quad \|R^{(k)}_{dis}\|_{\ell^p \to \ell^p} \leqslant \|R^{(k)}\|_{L^p \to p} + C_d = cot\left(\frac{\pi}{2p^*}\right) + C_d,$$

C_d depending only on d.

Following Riesz again: if $K(x) = \frac{\Omega(x)}{|x|^d}$, $\Omega(x) : S^{d-1} \to \mathbb{R}$, $\Omega(x) = \Omega(-x)$, mean zero,

$$\|T\|_{L^p \to L^p} \leqslant \|T_{dis}\|_{\ell^p \to \ell^p}$$

Above gives

$$\|R^{(k)}\|_{L^p\to L^p} \leqslant \|R^{(k)}_{dis}\|_{\ell^p\to\ell^p} \leqslant \|R^{(k)}\|_{L^p\to L^p} + C_d$$

New “Tantalizing” question. $d \geqslant 2$

$$\|R^{(k)}_{dis}\|_{p\to p} \leqslant \|R^{(k)}\|_{p\to p} = cot\left(\frac{\pi}{2p^*}\right)?$$

Weaker and also interesting:

$$\|R^{k}_{dis}\|_{\ell^p\to\ell^p} \leqslant A_p,$$

A_p is independent of d?

M. Kwaśnicki, D. Kim & R.B. (2023/2024): $A \in \mathfrak{M}_{(d+1)}$, $A \to T_A : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$

$$T_A(f)(n) = \sum_{m \in \mathbb{Z}^d, m \neq n} K_A(n,m) f(m)$$

$$K_A(n,m) = \int_{\mathbb{R}^d} \int_0^\infty 2y h(x,y) \left[A(x,y) \nabla \left(\frac{p_y(x-m)}{h(x,y)} \right) \right] \cdot \nabla \left(\frac{p_y(x-n)}{h(x,y)} \right) dy dx$$

$$h(x,y) = \sum_{n \in \mathbb{Z}^d} p_y(x-n) = \sum_{n \in \mathbb{Z}^d} e^{-2\pi|n|y} e^{2\pi i n \cdot x} \quad \text{(periodic Poisson kernel)}$$

$$= \left(\frac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi x)}, \quad d = 1 \right)$$

1. $$\|T_A f\|_{\ell^p} \leqslant (p^* - 1)\|A\| \|f\|_{\ell^p}$$
2. If A has the orthogonality property
$$\|T_A f\|_{\ell^p} \leqslant cot \left(\frac{\pi}{2p^*} \right) \|A\| \|f\|_{\ell^p}$$

With the matrices $\mathbb{H}^k$,

$$T_{\mathbb{H}^{(k)}} f(n) = \sum_{m \in \mathbb{Z}^d} K_{\mathbb{H}^{(k)}}(m) f(n-m) = K_{\mathbb{H}^{(k)}} * f(n),$$

$$K_{\mathbb{H}^{(k)}}(n) = \left(-4 \int_{\mathbb{R}^d} \int_0^\infty \frac{1}{h(x,y)} \frac{\partial p_y(x)}{\partial x_k} \frac{\partial}{\partial y} (y p_y(x-n))\, dy dx \right) \mathbf{1}_{\mathbb{Z}^d \setminus \{0\}}(n)$$

Kwasniki-B (2019) d=1 & Kim-Kwaśnicki-B (2023/2024), $d > 1$

$$\|T_{\mathbb{H}^{(k)}}\|_{\ell^p \to \ell^p} = cot\left(\frac{\pi}{2p^*}\right)$$

i.e.

$$\|T_{\mathbb{H}^{(k)}}\|_{\ell^p \to \ell^p)} = \|R^{(k)}\|_{L^p \to L^p}$$

The $T_{\mathbb{H}^{(k)}}$'s are not the same as the R^k_{dis}'s, unlike in the Gundy-Varopoulos classical case.

M.Kwaśnicki & R.B.for $d = 1$, there exist a probability kernel $\mathcal{K}$ such that

$$H_{dis} f(n) = \mathcal{K} * T_{\mathbb{H}}(f)(n), \quad n \in Z$$

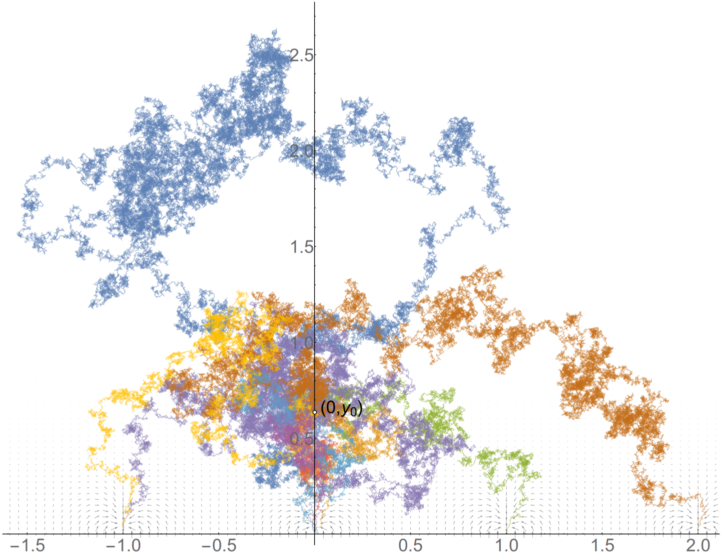
$$\implies \quad \|H_{dis}\|_{\ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})} \leqslant \|T_{\mathbb{H}}\|_{\ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})} \leqslant cot(\frac{\pi}{2p^*}) = \|H\|_{L^p \to L^p}$$

Question: for $d > 1$, does there exist a probability kernel $\mathcal{K}^{(k)}$ such that

$$R^{(k)}_{dis} f(n) = \mathcal{K}^{(k)} * T_{\mathbb{H}^{(k)}}(f)(n), \quad n \in \mathbb{Z}^d?$$

If so, this will resolve the "tantalizing" problem in several dimensions.

How to construct the $T'_A s$? Build them as conditional expectations of stochastic integrals on a BM in $\mathbb{R}_+^{d+1}$ that exit only on the lattice $\mathbb{Z}^d$. "Doob h-process"



$d = 1$ picture

For $f : \mathbb{Z}^d \to \mathbb{R}$ (can take with compact support, i.e., finite sequence)

$$u_f(x,y) = \sum_{n \in \mathbb{Z}^d} f(n) \frac{p_y(x-n)}{h(x,y)}, \quad \left(\sum_{n \in \mathbb{Z}^d} \left(\frac{p_y(x-n)}{h(x,y)} \right) = 1 \right)$$

u_f is h-harmonic in $\mathbb{R}^{d+1}_+$: $\Delta(h u_f) = 0$. Equivalent

$$\frac{1}{2} \Delta u_f(x,y) + \frac{\nabla h(x,y) \cdot \nabla u_f(x,y)}{h(x,y)} = 0,$$

Boundary values of u_f:

- $$\lim_{y \downarrow 0} u_f(x,y) = f_{\text{ext}}(x) = \sum_{m \in \mathbb{Z}^d} f(m) \Psi(x-m))$$
- $$f_{\text{ext}}(n) = f(n), \quad n \in \mathbb{Z}^d \quad \text{and} \quad \|f_{\text{ext}}\|_{L^p} \leqslant \|f\|_{\ell^p}$$

$$dZ_t = dB_t + \frac{\nabla h(Z_t)}{h(Z_t)} dt, \quad \text{(Doob h-process)}$$

τ exist time of Z_t from $\mathbb{R}^{d+1}_+$. $\tau < \infty$ a,s. & Z_τ takes values only on $\mathbb{Z}^d$:

$$\mathbb{P}_{(0,y)}\{Z_\tau = n\} = \frac{p_y(n)}{h(0,y)}$$

Two Martingales/stochastic integrals: $M_t = u_f(Z_{t\wedge\tau})$

$$\begin{aligned} M_t &= M_0 + \int_0^{t\wedge\tau} \nabla u_f(Z_s) \cdot dZ_s + \frac{1}{2}\int_0^{t\wedge\tau} \Delta u_f(Z_s) ds \\ &= M_0 + \int_0^{t\wedge\tau} \nabla u_f(Z_s) \cdot dZ_s - \int_0^{t\wedge\tau} \frac{\nabla h(Z_s) \cdot \nabla u_f(Z_s)}{h(Z_s)} ds \\ &= M_0 + \int_0^{t\wedge\tau} \nabla u_f(Z_s) \cdot dB_s. \end{aligned}$$

$$N_t = \int_0^{t\wedge\tau} A\nabla u_f(Z_s) \cdot dB_s, \quad A \in \mathfrak{M}_{(d+1)}$$

1. $\left(\mathbb{E}_{(0,y)}|A * f|^p\right)^{1/p} \leqslant (p^* - 1)\|A\| \left(\mathbb{E}_{(0,y)}|f(Z_\tau)|^p\right)^{1/p}$, (any A)
2. $\left(\mathbb{E}_{(0,y)}|A * f|^p\right)^{1/p} \leqslant cot\left(\frac{\pi}{2p^*}\right) \|A\| \left(\mathbb{E}_{(0,y)}|f(Z_\tau)|^p\right)^{1/p}$ (ortho A)

For $y > 0$, define

$$T_A^y(f)(n) = \mathbb{E}_{(0,y)}\left[N_\tau \,\middle|\, Z_\tau = n\right]$$

1. For any A: $\mathbb{E}_{(0,y)}|T_A^y f(Z_\tau)|^p \leqslant \mathbb{E}_{(0,y)}|N_\tau|^p \leqslant (p^* - 1)^p \|A\|^p \mathbb{E}_{(0,y)}|f(Z_\tau)|^p$.
2. A ortho: $\mathbb{E}_{(0,y)}|T_A^y f(Z_\tau)|^p \leqslant \mathbb{E}_{(0,y)}|N_\tau|^p \leqslant \left(cot\left(\frac{\pi}{2p^*}\right)\right)^p \|A\|^p \mathbb{E}_{(0,y)}|f(Z_\tau)|^p$.

1. $$\sum_{n \in \mathbb{Z}^d} |T_A^y(f)(n)|^p \frac{p_y(n)}{h(0,y)} \leqslant (p^* - 1)^p \|A\|^p \sum_{n \in \mathbb{Z}^d} |f(n)|^p \frac{p_y(n)}{h(0,y)}$$
2. $$\sum_{n \in \mathbb{Z}^d} |T_A^y(f)(n)|^p \frac{p_y(n)}{h(0,y)} \leqslant \left(cot\left(\frac{\pi}{2p^*}\right)\right)^p \|A\|^p \sum_{n \in \mathbb{Z}^d} |f(n)|^p \frac{p_y(n)}{h(0,y)}$$

Want to let $y \to \infty$ in T^y and $\frac{p_y(0)}{h(0,y)}$. Second is trivial.

$$\lim_{y \to \infty} \frac{1}{c_d} y^d p_y(n) = 1$$

For first:

Theorem (D. Kim, M.Kwaśnicki & R.B.)

$$\lim_{y\to\infty} T_A^y f(n) = T_A(f)(n) = \sum_{m\in\mathbb{Z}^d, m\neq n} K_A(n,m) f(m)$$

$$K_A(n,m) = \int_{\mathbb{R}^d}\int_0^\infty 2y h(x,y)\left[A(x,y)\nabla\left(\frac{p_y(x-m)}{h(x,y)}\right)\right]\cdot\nabla\left(\frac{p_y(x-n)}{h(x,y)}\right) dydx$$

$$K_{\mathbb{H}^{(k)}}(n) = \left(-4\int_{\mathbb{R}^d}\int_0^\infty \frac{1}{h(x,y)}\frac{\partial p_y(x)}{\partial x_k}\frac{\partial}{\partial y}(y p_y(x-n))\, dydx\right)\mathbf{1}_{\mathbb{Z}^d\setminus\{0\}}(n)$$

Remarks

Stein's vector version for the discrete operators

$$\left\| \left(\sum_{k=1}^{d} |T_{\mathbb{H}^{(k)}} f|^2 \right)^{1/2} \right\|_{\ell^p} \leqslant 2(p^* - 1)\|f\|_{\ell^p}.$$

Are the $T_{\mathbb{H}^{(k)}}$ discretization of C-Z-O operators? Yes

The following Kernels are C-Z kernels.

$$d = 1: \;\; K_{\mathbb{H}}(z)$$
$$= \frac{1}{\pi z}\left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 z^2)\sinh^2(y)}\, dy\right) \mathbf{1}_{\{|z|\geqslant 1\}}(z) + \frac{1}{\pi z}\mathbf{1}_{\{|z|<1\}}(z)$$

$$d \geqslant 2: \;\; K_{\mathbb{H}^{(k)}}(z)$$
$$= \left(-4\int_{\mathbb{R}^d}\int_0^\infty \frac{1}{h(x,y)}\frac{\partial p_0}{\partial x_k}\frac{\partial}{\partial y}(y p_y(x-z))\, dy dx\right)\mathbf{1}_{\{|z|\geqslant 1\}}(z) + c_d \frac{z_k}{|z|^{d+1}}\mathbf{1}_{\{|z|<1\}}(z)$$

What are their L^p norms? Open but...

Case $d = 1$: let $T =$ convolution w/kernel $K_{\mathbb{H}}$. Case $d > 1$: let T^k convolution w/kernel $K_{\mathbb{H}^{(k)}}$

$$cot(\pi/(2p^*)) \leqslant \|T\|_{L^p \to L^p} \leqslant 0.09956 + cot(\pi/(2p^*))$$

and

$$cot(\pi/(2p^*)) \leqslant \|T^k\|_{L^p \to L^p} \leqslant C_d + cot(\pi/(2p^*)),$$

where C_d depends on the dimension d.

References

- Bañuelos & Kwaśnicki: *The ℓ^p-norm of the discrete Hilbert transforms, 2019 Duke Math J,*
- Bañuelos &Kwaśnicki: *The ℓ^p-norm Riesz–Titchmarsh transform for even integer . J. London Math. Soc, 2024*
- Bañuelos, Kim, & Kwaśnicki: Sharp inequalities for discrete singular integrals: ArXiv https://arxiv.org/abs/2209.09737

- A similar constructions works for second order Riesz transforms (Work with Daesung Kim)

Thank You!