Quantitative norm convergence of triple ergodic averages for commuting transformations

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A purpose of this talk is to show how estimates in multilinear harmonic analysis, of the form

$$|\Lambda(f_1, \dots, f_n)| \leq C\|f_1\|_{L^{p_1}} \cdots \|f_n\|_{L^{p_n}},$$

where Λ is a multilinear singular integral form, can be used to show *quantitative* results on convergence of various ergodic averages.

$(X, \mathcal{F}, \mu)$ probability space, $T : X \to X$ measure preserving, i.e. $\mu(T^{-1}E) = \mu(E)$, $f \in L^\infty(X)$, $x \in X$

$$M_n(f)(x) := \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

- L^2 convergence as $n \to \infty$: von Neumann '30
- Pontwise a.e. convergence as $n \to \infty$: Birkhoff '31
- Norm-variation estimates: Jones, Ostrovskii, Rosenblatt '96

$$\sup_{n_0 < \ldots < n_m} \sum_{j=1}^{m} \|M_{n_j}(f) - M_{n_{j-1}}(f)\|_{\mathrm{L}^2(X)}^2 \lesssim \|f\|_{\mathrm{L}^2(X)}^2$$

The variation exponent 2 is sharp.
Gives an upper bound on the # of jumps of $(M_n(f))_{n\in\mathbb{N}}$ of fixed size.

$(X, \mathcal{F}, \mu)$ probability space, $T_1, \dots T_k : X \to X$ mutually commuting and measure-preserving, $f_1, \dots, f_k \in L^\infty(X)$

$$M_n(f_1, \dots, f_k)(x) := \frac{1}{n} \sum_{i=0}^{n-1} f_1(T_1^i x) \cdots f_k(T_k^i x)$$

- L^2 convergence as $n \to \infty$: $k = 2$ Conze and Lesigne '84; Tao '08
- Pontwise a.e. convergence as $n \to \infty$: open problem
- Norm-variation estimates: For some $1 < p_i < \infty$, $2 \le r < \infty$,

$$\sup_{n_0 < \ldots < n_m} \sum_{j=1}^{m} \|M_{n_j}(f_1, \dots, f_k) - M_{n_{j-1}}(f_1, \dots, f_k)\|_{\mathrm{L}^2(X)}^r \lesssim \prod_{i=1}^{k} \|f_i\|_{L^{p_1}(X)}^r$$

- $k = 2$, $r = 2$, $(p_1, p_2) = (4, 4)$: D., Kovač, Škreb, Thiele '16
- $k = 3$, $r > 4$, $(p_1, p_2, p_3) = (4, 8, 8)$: D., Thiele, Slavíková '23
- Other L^p exponents by interpolation, monotonicity, etc.
- Open problem: $k \ge 4$; $k = 3$ and $r \le 4$

$$M_n(f,g)(x) = \frac{1}{n}\sum_{i=0}^{n-1} f(T_1^i x) g(T_2^i x)$$

- Show

$$\sup_{n_0 < \ldots < n_m} \sum_{j=1}^{m} \|M_{n_{j-1}}(f,g) - M_{n_j}(f,g)\|_{\mathrm{L}^2(X)}^2 \lesssim \|f\|_{\mathrm{L}^4(X)}^2 \|g\|_{\mathrm{L}^4(X)}^2$$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T_1^i x) g(T_2^i x)$$

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- Show

$$\sup_{n_0 < \ldots < n_m} \sum_{j=1}^{m} \|M_{n_{j-1}}(f,g) - M_{n_j}(f,g)\|_{L^2(X)}^2 \lesssim \|f\|_{L^4(X)}^2 \|g\|_{L^4(X)}^2$$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(k+i,l)g(k,l+i)$$

- Show

$$\sup_{n_0<\ldots<n_m}\sum_{j=1}^{m}\|M_{n_{j-1}}(f,g)-M_{n_j}(f,g)\|_{\mathrm{L}^2(X)}^2 \lesssim \|f\|_{\mathrm{L}^4(X)}^2\|g\|_{\mathrm{L}^4(X)}^2$$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(k+i,l)g(k,l+i)$$

- Show

$$\sup_{n_0<\ldots<n_m} \sum_{j=1}^{m} \|M_{n_{j-1}}(f,g) - M_{n_j}(f,g)\|^2_{\mathrm{L}^2(X)} \lesssim \|f\|^2_{\mathrm{L}^4(X)}\|g\|^2_{\mathrm{L}^4(X)}$$

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- Show

$$\sup_{n_0<\ldots<n_m} \sum_{j=1}^{m} \|M_{n_{j-1}}(f,g) - M_{n_j}(f,g)\|^2_{L^2(X)} \lesssim \|f\|^2_{L^4(X)}\|g\|^2_{L^4(X)}$$

$$\frac{1}{n} \int_{[0,n)} f(x+s, y) g(x, y+s) ds$$

- Show

$$\sup_{n_0 < \ldots < n_m} \sum_{j=1}^{m} \| M_{n_{j-1}}(f, g) - M_{n_j}(f, g) \|_{\mathrm{L}^2(X)}^2 \lesssim \|f\|_{\mathrm{L}^4(X)}^2 \|g\|_{\mathrm{L}^4(X)}^2$$

$$\int_{\mathbb{R}} f(x+s,y)g(x,y+s)\frac{1}{n}1_{[0,n)}(s)ds$$

- Show

$$\sup_{n_0<\ldots<n_m} \sum_{j=1}^{m} \|M_{n_{j-1}}(f,g) - M_{n_j}(f,g)\|^2_{L^2(X)} \lesssim \|f\|^2_{L^4(X)}\|g\|^2_{L^4(X)}$$

$$\int_{\mathbb{R}} f(x+s, y) g(x, y+s) \frac{1}{n} 1_{[0,1)}\left(\frac{s}{n}\right) ds$$

- Show

$$\sup_{n_0 < \ldots < n_m} \sum_{j=1}^{m} \| M_{n_{j-1}}(f, g) - M_{n_j}(f, g) \|_{L^2(X)}^2 \lesssim \|f\|_{L^4(X)}^2 \|g\|_{L^4(X)}^2$$

$$\int_{\mathbb{R}} f(x+s, y) g(x, y+s) \frac{1}{n} 1_{[0,1)}(\frac{s}{n}) ds$$

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$$\int_{\mathbb{R}} f(x+s,y)g(x,y+s)\frac{1}{n}\varphi(\frac{s}{n})ds$$

- Show

$$\sup_{n_0<\ldots<n_m} \sum_{j=1}^{m} \|M_{n_{j-1}}(f,g) - M_{n_j}(f,g)\|^2_{\mathrm{L}^2(X)} \lesssim \|f\|^2_{\mathrm{L}^4(X)} \|g\|^2_{\mathrm{L}^4(X)}$$

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$$\sup_{n_0 < \ldots < n_m} \sum_{j=1}^{m} \| M_{n_{j-1}}(f, g) - M_{n_j}(f, g) \|^2_{\mathrm{L}^2(X)} \lesssim \| f \|^2_{\mathrm{L}^4(X)} \| g \|^2_{\mathrm{L}^4(X)}$$

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$$\int_{\mathbb{R}} f(x+s, y)g(x, y+s)\varphi_n(s)ds$$

- Show

$$\sup_{n_0 < \ldots < n_m} \sum_{j=1}^{m} \| M_{n_{j-1}}(f,g) - M_{n_j}(f,g) \|_{\mathrm{L}^2(X)}^2 \lesssim \|f\|_{\mathrm{L}^4(X)}^2 \|g\|_{\mathrm{L}^4(X)}^2$$

$$A_n(f,g)(x,y) = \int_{\mathbb{R}} f(x+s,y)g(x,y+s)\varphi_n(s)ds$$

- Show

$$\sup_{0<n_0<\ldots<n_m} \sum_{j=1}^{m} \|A_{n_{j-1}}(f,g) - A_{n_j}(f,g)\|_{\mathrm{L}^2(\mathbb{R}^2)}^2 \lesssim_{\varphi} \|f\|_{\mathrm{L}^4(\mathbb{R}^2)}^2 \|g\|_{\mathrm{L}^4(\mathbb{R}^2)}^2$$

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- Long variation: $n_j = 2^{k_j}$; Short variation: $n_j \in [2^k, 2^{k+1})$, sum in k (Jones, Seeger, Wright '08)
- Fixing n_j, expanding the norm on LHS gives

$$\sum_{j=1}^{m} \int_{\mathbb{R}^4} f(x+s,y)g(x,y+s)f(x+t,y)g(x,y+t)$$
$$(\varphi_{n_{j-1}} - \varphi_{n_j})(s)(\varphi_{n_{j-1}} - \varphi_{n_j})(t)\, dxdydsdt$$

$$A_n(f,g)(x,y) = \int_{\mathbb{R}} f(x+s,y)g(x,y+s)\varphi_n(s)ds$$

- Show

$$\sup_{0<n_0<\ldots<n_m} \sum_{j=1}^{m} \|A_{n_{j-1}}(f,g) - A_{n_j}(f,g)\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \lesssim_\varphi \|f\|^2_{\mathrm{L}^4(\mathbb{R}^2)}\|g\|^2_{\mathrm{L}^4(\mathbb{R}^2)}$$

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- Show

$$\sup_{0<n_0<\ldots<n_m} \sum_{j=1}^{m} \|A_{n_{j-1}}(f,g) - A_{n_j}(f,g)\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \lesssim_\varphi \|f\|^2_{\mathrm{L}^4(\mathbb{R}^2)}\|g\|^2_{\mathrm{L}^4(\mathbb{R}^2)}$$

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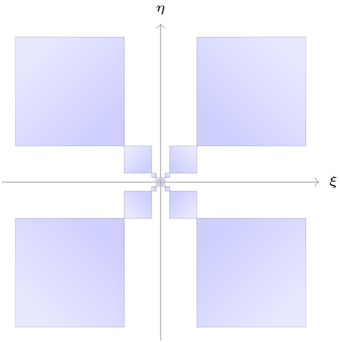
$$\sum_{j=1}^{m} \int_{\mathbb{R}^4} f(x+s,y)g(x,y+s)f(x+t,y)g(x,y+t)$$
$$(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(s)(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(t)\, dxdydsdt$$

Let $\widehat{\varphi}$ be supported in $[-1, 1]$ and constant on $[-1/2, 1/2]$.
The Fourier transform of the kernel is

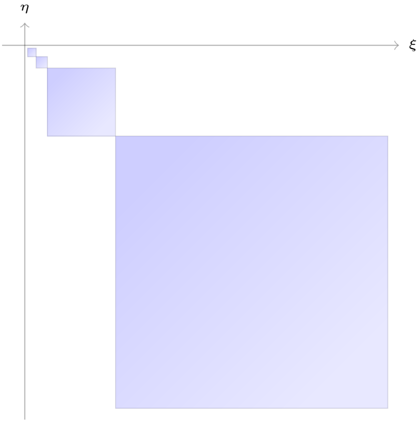
$$\sum_j \left(\widehat{\varphi}(2^{k_{j-1}}\xi) - \widehat{\varphi}(2^{k_j}\xi)\right)\left(\widehat{\varphi}(2^{k_{j-1}}\eta) - \widehat{\varphi}(2^{k_j}\eta)\right)$$



Not 2D Calderón-Zygmund kernel (which would be the case $k_j = j$).

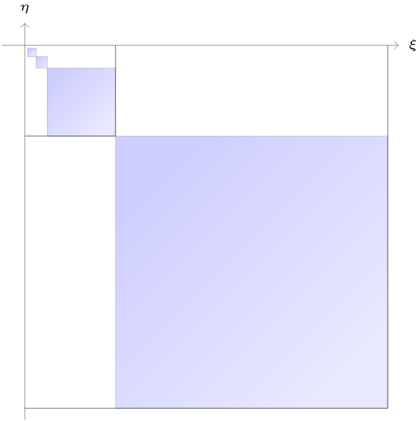
The kernel is a sum of terms

$$(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(s)(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(t)$$



The kernel is a sum of terms

$$(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(s)(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(t) \\ = \varphi_{2^{k_{j-1}}}(s)\varphi_{2^{k_{j-1}}}(t) - \varphi_{2^{k_j}}(s)\varphi_{2^{k_j}}(t)$$



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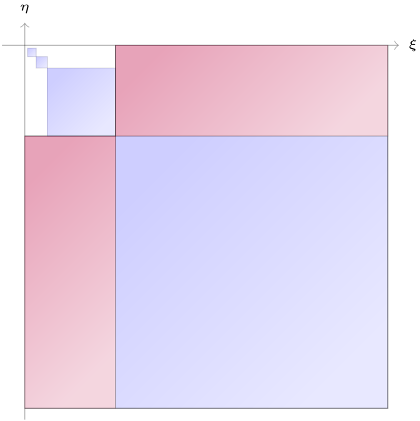
$$
\begin{aligned}
(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(s)(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(t) \\
&= \varphi_{2^{k_{j-1}}}(s)\varphi_{2^{k_{j-1}}}(t) - \varphi_{2^{k_j}}(s)\varphi_{2^{k_j}}(t) \\
&- \varphi_{2^{k_{j-1}}}(s)\left[\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}}\right](t) - \dots
\end{aligned}
$$



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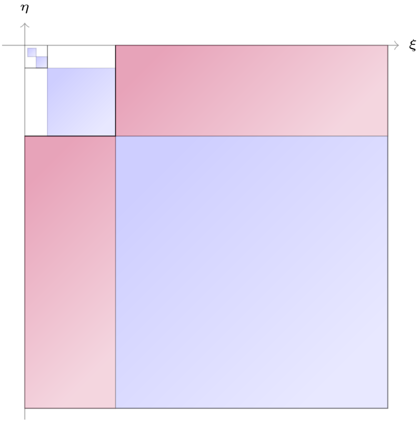

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\begin{aligned}
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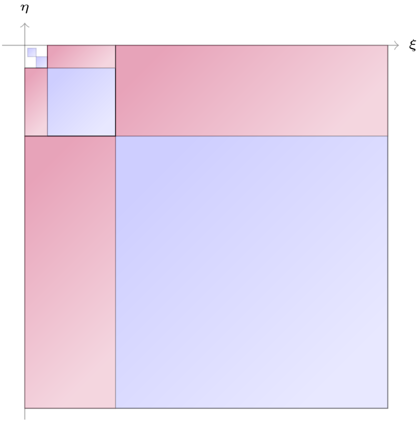

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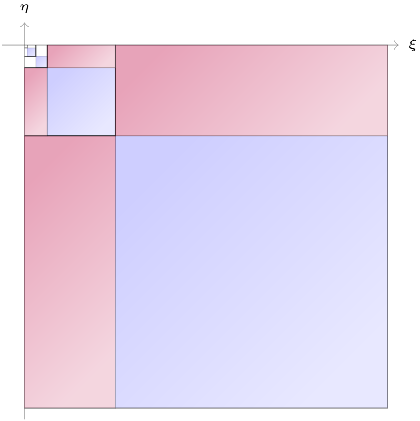
$$



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\begin{aligned}
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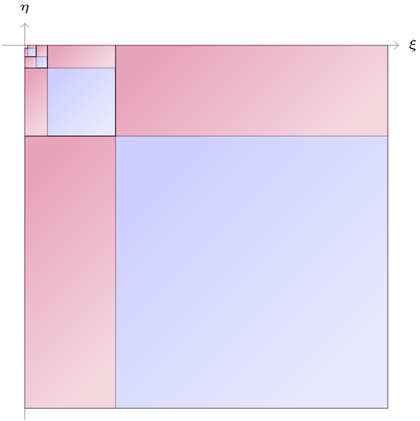

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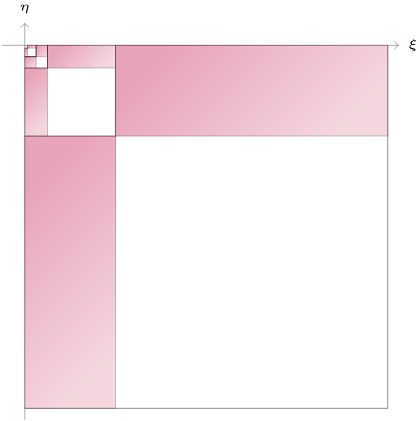

- Sum in j for the trivial piece

$$\sum_{j=1}^{m} \varphi_{2^{k_{j-1}}}(s)\varphi_{2^{k_{j-1}}}(t) - \varphi_{2^{k_j}}(s)\varphi_{2^{k_j}}(t)$$
$$= \varphi_{2^{k_0}}(s)\varphi_{2^{k_0}}(t) - \varphi_{2^{k_m}}(s)\varphi_{2^{k_m}}(t)$$

Hölder's inequality in (x, y) for the exponents $(4, 4, 4, 4)$

$$\Big| \int_{\mathbb{R}^4} f(x+s, y)g(x, y+s)f(x+t, y)g(x, y+t)$$
$$\varphi_{k_0}(s)\varphi_{k_0}(t)\,dxdydsdt\Big| \lesssim_{\varphi} \|f\|_{L^4}^2 \|g\|_{L^4}^2$$

- Remains to estimate

$$\sum_{j=1}^{m} \int_{\mathbb{R}^4} f(x+s, y)g(x, y+s)f(x+t, y)g(x, y+t)$$
$$\varphi_{2^{k_{j-1}}}(s)\left[\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}}\right](t)\,dxdydsdt$$

- Sum in j for the trivial piece

$$\sum_{j=1}^{m} \varphi_{2^{k_{j-1}}}(s)\varphi_{2^{k_{j-1}}}(t) - \varphi_{2^{k_j}}(s)\varphi_{2^{k_j}}(t)$$
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$$\varphi_{k_0}(s)\varphi_{k_0}(t)dxdydsdt\Big| \lesssim_\varphi \|f\|_{L^4}^2 \|g\|_{L^4}^2$$

- Remains to estimate $((x', y') = (s + x + y,\ t + x + y))$,

$$\sum_{j=1}^{m} \int_{\mathbb{R}^4} f(y, x')g(x, x')f(y, y')g(x, y')$$
$$\varphi_{2^{k_{j-1}}}(x' - x - y)\big[\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}}\big](y' - x - y)\, dxdydx'dy'$$

- Sum in j for the trivial piece

$$\sum_{j=1}^{m} \varphi_{2^{k_{j-1}}}(s)\varphi_{2^{k_{j-1}}}(t) - \varphi_{2^{k_j}}(s)\varphi_{2^{k_j}}(t)$$
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$$\varphi_{k_0}(s)\varphi_{k_0}(t)dxdydsdt\Big| \lesssim_{\varphi} \|f\|_{L^4}^2 \|g\|_{L^4}^2$$

- Remains to estimate $((x', y') = (s+x+y,\ t+x+y))$, $(\psi_t = -\partial_t \varphi_t)$

$$\sum_{j=1}^{m} \int_{\mathbb{R}^4} f(y, x')g(x, x')f(y, y')g(x, y')$$
$$\varphi_{2^{k_{j-1}}}(x'-x-y)\big[\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}}\big](y'-x-y)\,dxdydx'dy'$$

- Sum in j for the trivial piece

$$\sum_{j=1}^{m} \varphi_{2^{k_{j-1}}}(s)\varphi_{2^{k_{j-1}}}(t) - \varphi_{2^{k_j}}(s)\varphi_{2^{k_j}}(t)$$
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Hölder's inequality in (x, y) for the exponents $(4, 4, 4, 4)$

$$\Big| \int_{\mathbb{R}^4} f(x+s, y)g(x, y+s)f(x+t, y)g(x, y+t)$$
$$\varphi_{k_0}(s)\varphi_{k_0}(t)dxdydsdt\Big| \lesssim_{\varphi} \|f\|_{\mathrm{L}^4}^2 \|g\|_{\mathrm{L}^4}^2$$

- Remains to estimate $((x', y') = (s + x + y,\ t + x + y))$, $(\psi_t = -\partial_t \varphi_t)$

$$\sum_{j=1}^{m} \int_{2^{k_{j-1}}}^{2^{k_j}} \int_{\mathbb{R}^4} f(y, x')g(x, x')f(y, y')g(x, y')$$
$$\varphi_{2^{k_{j-1}}}(x' - x - y)\psi_t(y' - x - y)\, dxdydx'dy' \frac{dt}{t}$$

Decomposition into elementary tensors

$$\varphi_{2^{k_j}}(x' - x - y)\psi_t(y' - x - y)$$

$$= \int_{\mathbb{R}^2} \varphi_{2^{k_j}}(x' - p - q)\psi_t(y' - p - q)\omega_t(x - p)\omega_t(y - q)dpdq$$

$$\varphi_{2^{k_j}}(x' - x - y)\psi_t(y' - x - y)$$

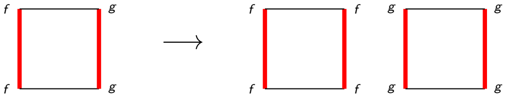
$$= \int_{\mathbb{R}^2} \varphi_{2^{k_j}}(x' - p - q)\psi_t(y' - p - q)\omega_t(x - p)\omega_t(y - q)dpdq$$

We would like to achieve this with $\int \omega = 0$.

$$\sum_{j=1}^{m} \int_{2^{k_{j-1}}}^{2^{k_j}} \int_{\mathbb{R}^6} f(y,x')g(x,x')f(y,y')g(x,y')$$
$$\varphi_{2^{k_j}}(x'-p-q)\psi_t(y'-p-q)\omega_t(x-p)\omega_t(y-q)\, dxdydx'dy'dpdq\frac{dt}{t}$$

$$= \sum_{j=1}^{m} \int_{2^{k_{j-1}}}^{2^{k_j}} \int_{\mathbb{R}^4} \Big(\int_{\mathbb{R}} f(y,x')f(y,y')\omega_t(y-q)dy \Big)$$
$$\Big(\int_{\mathbb{R}} g(x,x')g(x,y')\omega_t(x-p)dx \Big)$$
$$\varphi_{2^{k_j}}(x'-p-q)\psi_t(y'-p-q)dx'dy'dpdq\frac{dt}{t}$$

- Cauchy-Schwarz to separate f and g



$$\sum_{j=1}^{m} \int_{2^{k_{j-1}}}^{2^{k_j}} \int_{\mathbb{R}^6} f(y,x')g(x,x')f(y,y')g(x,y')$$
$$\varphi_{2^{k_j}}(x'-p-q)\psi_t(y'-p-q)\omega_t(x-p)\omega_t(y-q)\,dxdydx'dy'dpdq\frac{dt}{t}$$

$$\le \sum_{j=1}^{m} \int_{2^{k_{j-1}}}^{2^{k_j}} \int_{\mathbb{R}^4} \left| \int_{\mathbb{R}} f(y,x')f(y,y')\omega_t(y-q)dy \right|$$
$$\left| \int_{\mathbb{R}} g(x,x')g(x,y')\omega_t(x-p)dx \right|$$
$$|\varphi_{2^{k_j}}|(x'-p-q)|\psi_t|(y'-p-q)dx'dy'dpdq\frac{dt}{t}$$

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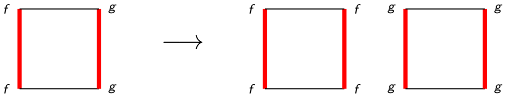
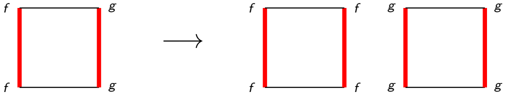


$$\sum_{j=1}^{m} \int_{2^{k_{j-1}}}^{2^{k_j}} \int_{\mathbb{R}^6} f(y,x')g(x,x')f(y,y')g(x,y')$$
$$\varphi_{2^{k_j}}(x'-p-q)\psi_t(y'-p-q)\omega_t(x-p)\omega_t(y-q)\,dxdydx'dy'dpdq\frac{dt}{t}$$

$$\leq \Big(\sum_{j=1}^{m} \int_{2^{k_{j-1}}}^{2^{k_j}} \int_{\mathbb{R}^4} \Big| \int_{\mathbb{R}} f(y,x')f(y,y')\omega_t(y-q)dy\Big|^2$$
$$|\varphi_{2^{k_j}}|(x'-p-q)|\psi_t|(y'-p-q)dx'dy'dpdq\frac{dt}{t}\Big)^{1/2}\cdot \ldots$$

- Cauchy-Schwarz to separate f and g



- Cauchy-Schwarz lowers complexity of the form by reducing to forms with occurrences of one function only
- Now dominate $|\varphi|$, $|\psi|$ by Gaussians to obtain a highly symmetric non-negative form whose proof can be completed with partial integration (heat equation) and monotonicity arguments

$$\Lambda_1(f) \leq \|f\|_4^4 - \Lambda_2(f) \leq \|f\|_4^4$$

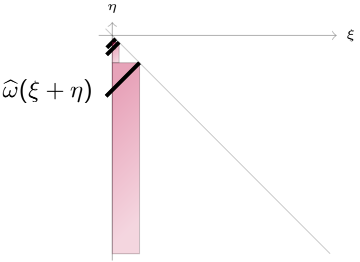
- Singular Brascamp-Lieb forms with cubical structure, do not fall under classical CZ theory (Even with one-parameter CZ kernel)
- Kovac '12, ..., D., Thiele, Slavikova '21, etc.

$$\varphi_{2^{k_j}}(x' - x - y)\psi_t(y' - x - y)$$

$$= \int_{\mathbb{R}^2} \varphi_{2^{k_j}}(x' - p - q)\psi_t(y' - p - q)\omega_t(x - p)\omega_t(y - q)dpdq$$

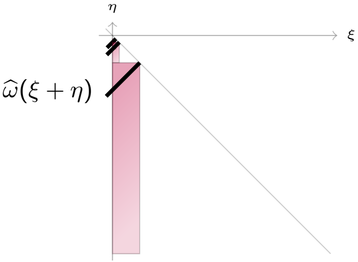
Decomposition into elementary tensors

$$\varphi_{2^{k_j}}(x' - x - y)\psi_t(y' - x - y)$$
$$= \int_{\mathbb{R}^2} \widehat{\varphi_{2^{k_j}}}(\xi) e^{2\pi i \xi(x'-x-y)} \widehat{\psi_t}(\eta) e^{2\pi i \eta(y'-x-y)} d\xi d\eta$$
$$= \int_{\mathbb{R}^2} \widehat{\varphi_{2^{k_j}}}(\xi) e^{2\pi i x'\xi} \widehat{\psi_t}(\eta) e^{2\pi i y'\eta} \widehat{\omega}_t(\xi+\eta) e^{-2\pi i x(\eta+\xi)} \widehat{\omega}_t(\xi+\eta) e^{-2\pi i y(\eta+\xi)} d\xi d\eta$$



$$= \int_{\mathbb{R}^2} \varphi_{2^{k_j}}(x' - p - q)\psi_t(y' - p - q)\omega_t(x - p)\omega_t(y - q) dp dq$$

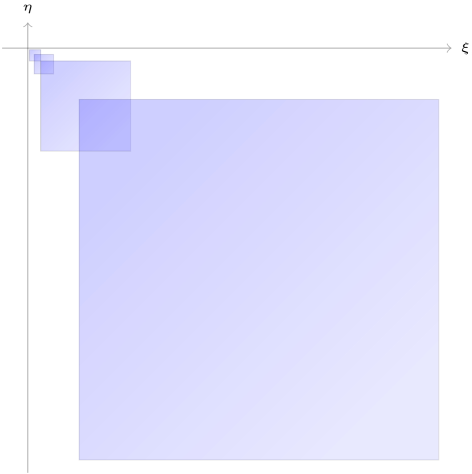
$$\varphi_{2^{k_j}}(x'-x-y)\psi_t(y'-x-y)$$
$$= \int_{\mathbb{R}^2} \widehat{\varphi_{2^{k_j}}}(\xi)e^{2\pi i\xi(x'-x-y)}\widehat{\psi_t}(\eta)e^{2\pi i\eta(y'-x-y)}d\xi d\eta$$
$$= \int_{\mathbb{R}^2} \widehat{\varphi_{2^{k_j}}}(\xi)e^{2\pi ix'\xi}\widehat{\psi_t}(\eta)e^{2\pi iy'\eta}\widehat{\omega}_t(\xi+\eta)e^{-2\pi ix(\eta+\xi)}\widehat{\omega}_t(\xi+\eta)e^{-2\pi iy(\eta+\xi)}d\xi d\eta$$



$$= \int_{\mathbb{R}^2} \varphi_{2^{k_j}}(x'-p-q)\psi_t(y'-p-q)\omega_t(x-p)\omega_t(y-q)dpdq$$

However: in reality the support of ω intersects the diagonal

Multiplier and telescoping (realistic)



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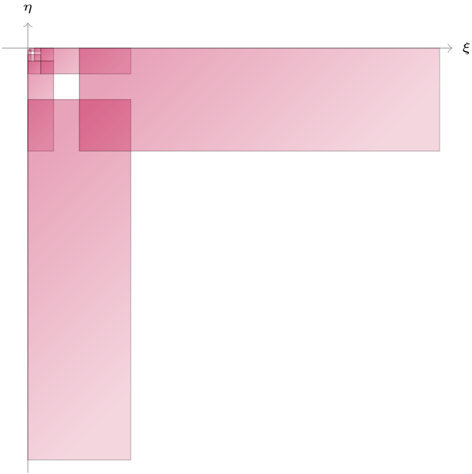

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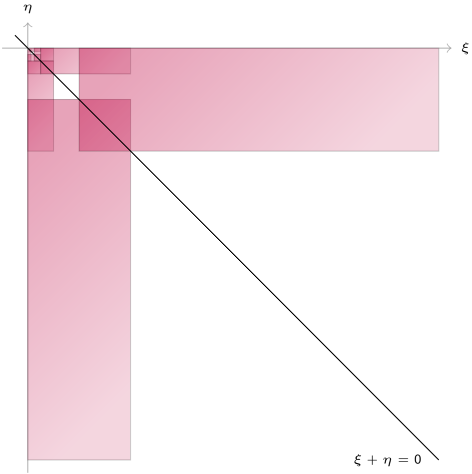
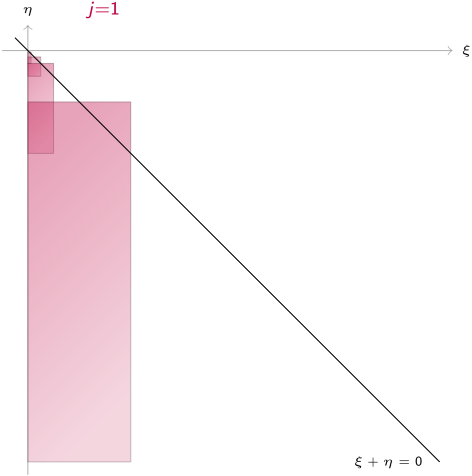
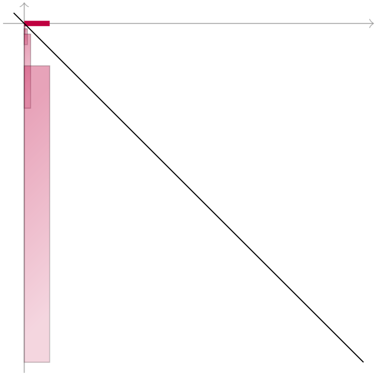


$$\sum_{j=1}^{m} \varphi_{2^{k_j}}(s)(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(t)$$



$$\sum_{j=1}^{m} \phi_{2^{k_j}}(s)(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(t)$$

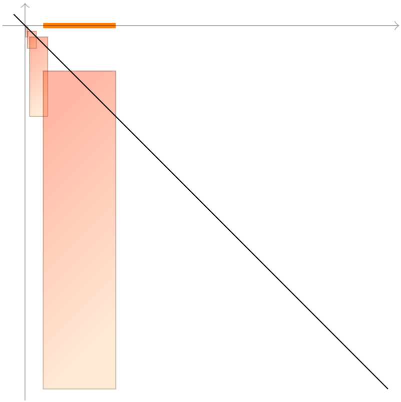
$$\phi = \varphi\chi_{(-2^{-4},2^{-4})}$$



stick

$$\sum_{j=1}^{m} \psi_{2^{k_j}}(s)(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(t)$$

$$\psi = \varphi\chi_{(-2,-2^{-5})\cup(2^{-5},2)}$$



bad, but CZ near $\xi + \eta = 0$

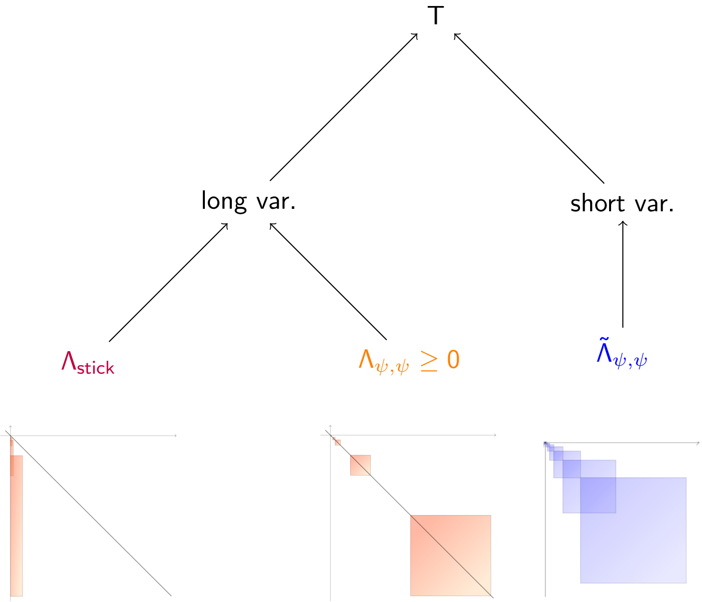
$$\Lambda_{\text{bad}} = \sum_{j=1}^{m} \int_{\mathbb{R}^4} f(x+s,y)g(x,y+s)f(x+t,y)g(x,y+t)$$
$$\psi_{2^{k_j}}(s)(\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}})(t)dxdydsdt$$
$$= \sum_{j=1}^{m} \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} f(x+s,y)g(x,y+s)\psi_{2^{k_j}}(s)ds \Big)$$
$$\Big(\int_{\mathbb{R}} f(x+t,y)g(x,y+t)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(t)dt \Big)\, dxdy$$
$$\leq \Lambda_{\psi,\psi}^{1/2} \Lambda^{1/2}$$

Therefore, long variation:

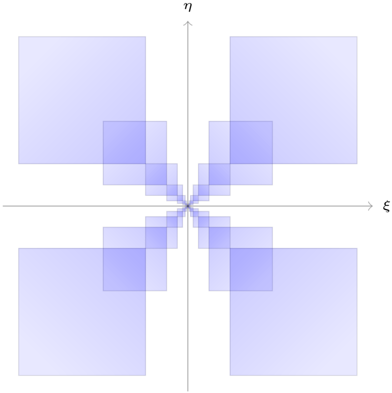
$$\begin{aligned} \Lambda &\lesssim \text{single-scale} + \Lambda_{\text{stick}} + \Lambda_{\text{bad}} \\ &\lesssim \text{single-scale} + \Lambda_{\text{stick}} + \Lambda_{\psi,\psi}^{1/2} \Lambda^{1/2} \end{aligned}$$

So it suffices to bound Λ_{stick} and $\Lambda_{\psi,\psi}$.

Summarizing

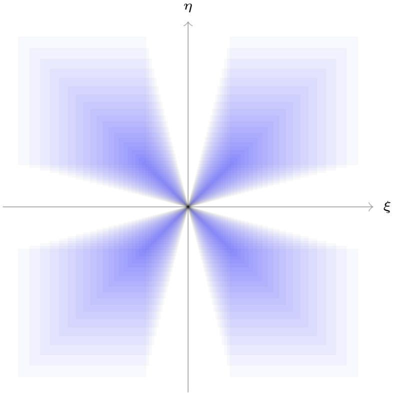


- $\Lambda_{\psi,\psi}$ and $\tilde{\Lambda}_{\psi,\psi}$: Calderón-Zygmund kernel with FT



- How to deal with the diagonal?

- Pass to a multiplier constant on lines through the origin with a C-S and domination
- Subtract the constant on $\xi + \eta = 0$ as form with constant multiplier is trivially bounded



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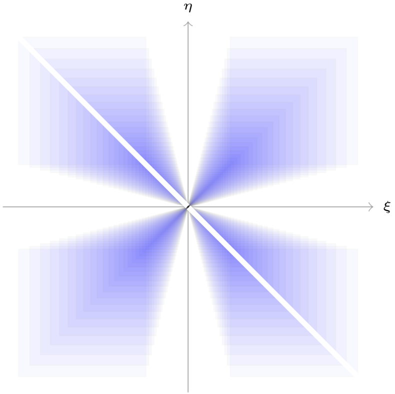
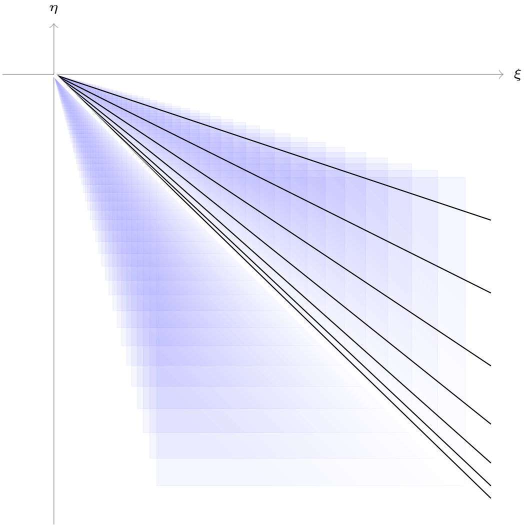
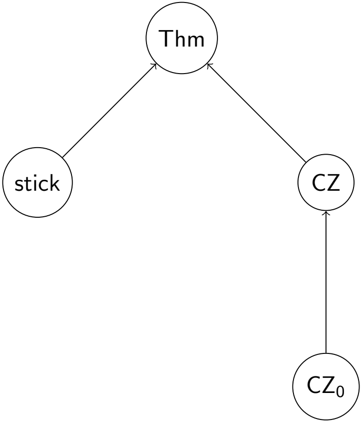


- Estimate each cone separately
- Estimates summable due to vanishing on $\xi + \eta = 0$

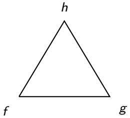




- In the proof it was important to "fill in the gaps" i.e. add missing scales; crucial positivity of the form.
- Used after the reduction in the long variation as well as to pass to a homogeneous multiplier.

Triangular Hilbert transform

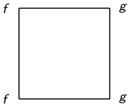
$$(x, y) \mapsto \text{p.v.} \int_{\mathbb{R}^2} f(x+t, y) g(x, y+t) \frac{1}{t} dt$$



- No L^p estimates known.
- As a byproduct we prove a bound for the associated square function,

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} \Big| \int_{\mathbb{R}} f(x+t, y) g(x, y+t) 2^{-j} \psi(2^{-j} t) dt \Big| \Big)^{1/2} \Big\|_{L^2_{x,y}(\mathbb{R}^2)}$$

$$\lesssim \|f\|_{L^4(\mathbb{R}^2)} \|g\|_{L^4(\mathbb{R}^2)}$$



Transference leads to averages on $\mathbb{R}^3$

$$A_n(f,g,h)(x,y,z) = \int_{\mathbb{R}} f(x+s,y,z)g(x,y+s,z)g(x,y,z+s)\varphi_n(s)ds$$

- We show, for $r > 4$ and $0 < n_0 < \cdots < n_m$,

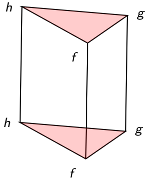
$$\sum_{j=1}^{m} \|A_{n_j}(f,g,h) - A_{n_{j-1}}(f,g,h)\|_{\mathrm{L}^2}^r \lesssim_\varphi \|f\|_{\mathrm{L}^8}^r \|g\|_{\mathrm{L}^8}^r \|h\|_{\mathrm{L}^4}^r$$

  via a "weak-type endpoint" estimate

$$\sum_{j=1}^{m} \|A_{n_j}(f,g,h) - A_{n_{j-1}}(f,g,h)\|_{\mathrm{L}^2}^2 \lesssim_\varphi m^{1/2} \|f\|_{\mathrm{L}^8}^2 \|g\|_{\mathrm{L}^8}^2 \|h\|_{\mathrm{L}^4}^2$$

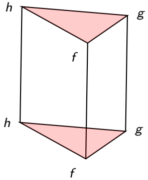
- Limitation of the techniques: not able to remove the loss in m

$$\sum_{j=1}^{m} \int_{\mathbb{R}^5} \quad f(x+s,y,z)g(x,y+s,z)h(x,y,z+s)$$
$$f(x+t,y,z)g(x,y+t,z)h(x,y,z+t)$$
$$(\varphi_{n_j} - \varphi_{n_{j-1}})(s)$$
$$(\varphi_{n_j} - \varphi_{n_{j-1}})(t) \quad d(x,y,z,s,t)$$



- No L^p estimates known, even if $n_j = 2^j$ (CZ kernel).
- Prove estimate with loss in $m^{1/2}$: shares some characteristics with cancellation estimates for the triangular HT, (D., Kovač, Thiele '16) but with arbitrary scales and 2D kernel

$$\sum_{j=1}^{m} \int_{\mathbb{R}^5} \begin{aligned} & f(u,y,z)g(x,u,z)h(x,y,u) \\ & f(u',y,z)g(x,u',z)h(x,y,u') \\ & (\varphi_{n_j} - \varphi_{n_{j-1}})(u-x-y-z) \\ & (\varphi_{n_j} - \varphi_{n_{j-1}})(u'-x-y-z) \quad d(x,y,z,u,u') \end{aligned}$$



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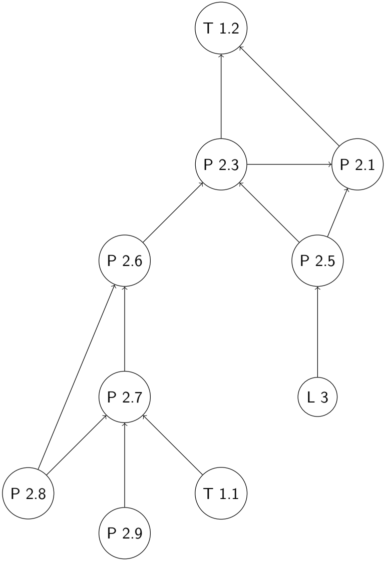
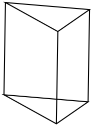
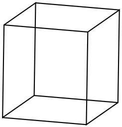
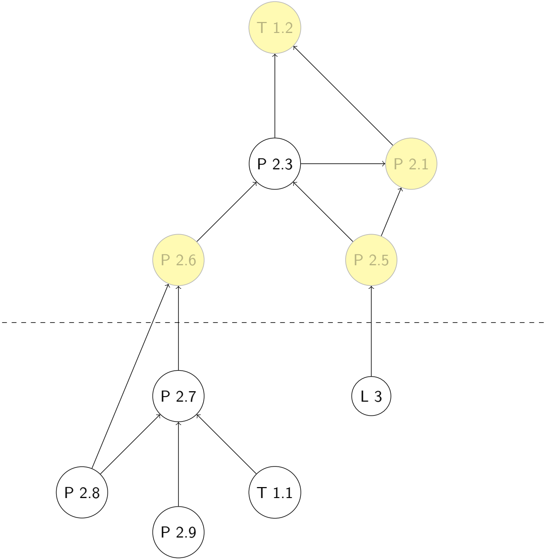
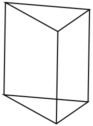

T 1.2
P 2.3
P 2.1
P 2.6
P 2.5
P 2.7
L 3
P 2.8
T 1.1
P 2.9

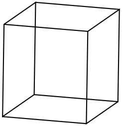




Structure of the proof







- **First:** cannot add additional scales to positive forms due to loss $m^{1/2}$, which is produced by the first Cauchy-Schwarz
- Workaround: Up to CZ error terms, pass to particular bump functions which allow to create multipliers constant on $\xi + \eta = 0$
- CZ error terms still need to preserve complexity m

- **Second:** new stick terms, associated with higher-dim kernels
- They are multi-parameter, but no need to keep track of m.
- Higher dimensional CZ pieces also arise, but this has been done previously (D., Thiele '18 and D., Thiele, Slavikova '21).

$$M_n(f_1, f_2, f_3, f_4)(x) := \frac{1}{n}\sum_{i=0}^{n-1} f_1(T_1^i x) f_2(T_2^i x) f_3(T_1^i x) f_4(T_1^i x)$$

- Do not aim for a sharp variation exponent.
- More "triangles", more iterations needed to reduce complexity to a higher dimensional cube.
- Not yet clear how to iterate the steps on higher dimensional multipliers that arise after multiple applications of Cauchy-Schwarz.
- In particular, need to keep the number of jumps m at multiple iterations
- Multiple transformations: likely induction with 4 transformations being the induction base.

Thank you!