Generalized Sublevel Set Inequalities for Differential Forms

Philip T. Gressman

Madison Lectures in Harmonic Analysis

16 May 2024

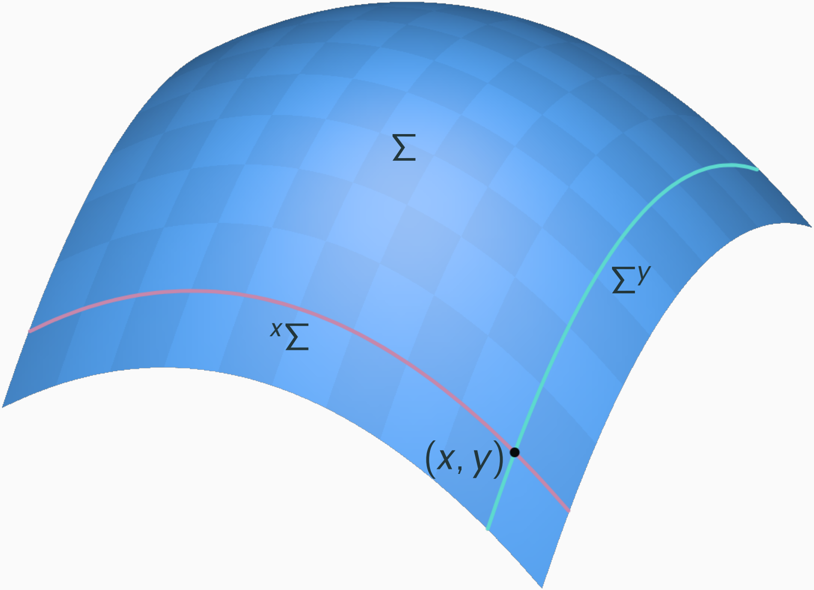
Penn
Department of Mathematics

Context: L^p–Improving Inequalities

- Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ be an open set and let Σ be a submanifold (algebraic variety) in Ω.
- Let $\pi(x, y)$ be a defining function associated to Σ: $(x, y) \in \Sigma$ if and only if $\pi(x, y) = 0$.
- For each x, let ${}^x\Sigma \subset \mathbb{R}^n$ consist of points y for which $\pi(x, y) = 0$.
- Consider the operator

$$Tf(x) := \int_{{}^x\Sigma} f(y) w(x, y) d\sigma(y).$$

What are the L^p–L^q mapping properties of T?


Σ
Σ^y
${}^x\Sigma$
(x, y)

Example: Spherical Averages

- $\Sigma \subset \mathbb{R}^n \times \mathbb{R}^n := \left\{(x, y) \;\middle|\; |x-y|^2 = 1\right\}$
- $\pi(x, y) := |x-y|^2 - 1$
- $Tf(x) := \int_{x+\mathbb{S}^{n-1}} f(y) d\mathcal{H}^{n-1}(y)$
- T maps $L^{(n+1)/n}$ to L^{n+1} (Strichartz 1970, Littman 1973)
- Proof is roughly analytic interpolation between $L^2 \to H^{(n-1)/2}$ and an $L^1 \to L^\infty$ bound for fractional integral of order 1 applied to T.

Local Geometry of Σ

- Let $\pi(x, y) := (\pi_1(x, y), \ldots, \pi_k(x, y))$.
- $d_x\pi_1, \ldots, d_x\pi_k$ are 1-forms which annihilate vectors tangent to $\Sigma^y := \{x \mid \pi(x, y) = 0\}$. They depend on the choice of π.
- The wedge product $d_x\pi := d_x\pi_1 \wedge \cdots \wedge d_x\pi_k$ is determined by Σ up to a nonvanishing scalar factor.
- There is no canonical norm for $d_x\pi$, but we can say

$$d_x\pi(x, y) = \sum_{i_1<\cdots<i_k} c_{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$

and take norm of coeff. vector for given $e_1, \ldots, e_n$.

Enemies of L^p-Improving

- For a given x, there are many y's such that $x \in \Sigma^y$ (namely, $y \in {}^x\Sigma$). The quantity $d_x\pi(x, y)$ indirectly encodes tangent space of Σ^y at x.
- For L^p-improving to happen, one needs the tangent space of Σ^y at x to vary robustly as y varies in ${}^x\Sigma$.
- No L^p-improving occurs when $d_x\pi(x, y)$ is constant (up to scalar factor) for each x as y varies over ${}^x\Sigma$.
- In this bad case, there is a choice of basis $e := \{e_1, \ldots, e_n\}$ with fixed volume such that $||d\pi_x||_e$ is uniformly as small as desired.

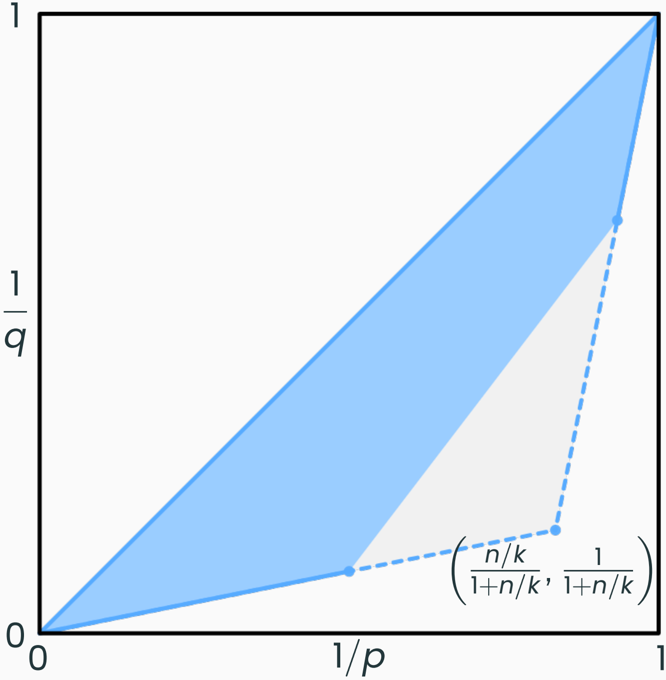
Theorem: Testing Condition (G 2022)

If π is polynomial and $p \in (1, \infty)$, the operator T maps L^p to $L^{np/k}$ iff

$$\int_{x\Sigma} \frac{|w(x,y)|^{p'} d\sigma(y)}{||d_x \pi(x,y)||_e^{p'-1}}$$

is uniformly bounded for all x and all $e := \{e_1, \ldots, e_n\}$ of fixed volume.

NB: In bad case, some choice of e makes denominator uniformly small for any given $p > 1$.



1
1/q
0
0
1/p
1
(n/k / 1+n/k, 1 / 1+n/k)

The Abstract "Sublevel" Problem

Suppose that $u^1(t), \ldots, u^k(t)$ are smooth $\mathbb{R}^n$-valued functions on some domain $\Omega \subset \mathbb{R}^\ell$. Let $u := u^1 \wedge \cdots \wedge u^k$. For what (nontrivial) weights w and exponents τ is it the case that

$$\int_\Omega \frac{w(t)dt}{[||u(t)||_e]^\tau}$$

is uniformly bounded for all e of normalized volume?

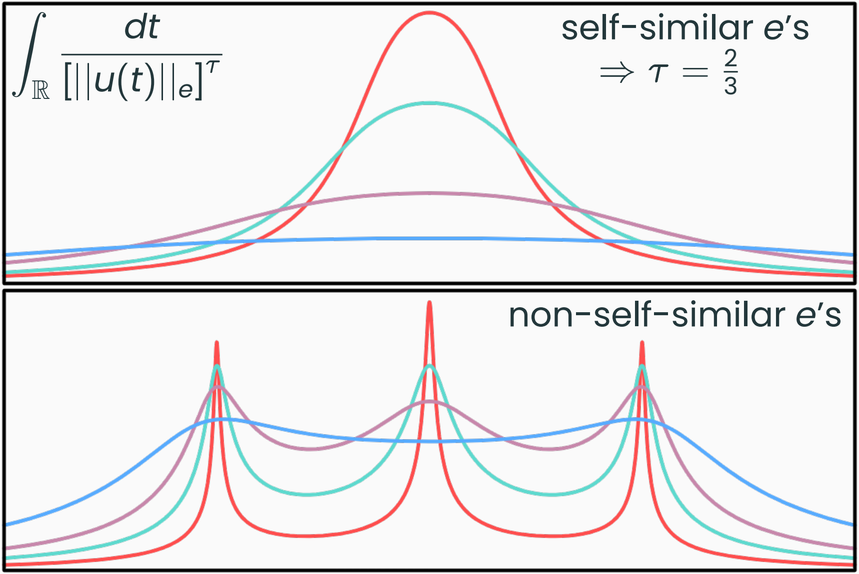
We will assume that the u^i are algebraic or Nash.

- Uniform estimation is challenging because $k < n$: This means that at any point $t \in \Omega$, there is always *some* basis e at which $||u(t)||_e$ is as small as desired.

- Take $\epsilon^{-1}u^1(t), \ldots, \epsilon^{-1}u^k(t)$ to be the first k elements of the basis, then attach additional elements rescaled to make volume 1.

- In this basis, $u^1(t) \wedge \cdots \wedge u^k(t)$ looks small.

Example: $u(t) = dx_0 + tdx_1 + \cdots + t^3dx_3$



Smooth Row and Column Reduction

- Think of $u^1(t), \ldots, u^k(t)$ as rows of a $k \times n$ matrix. Left multiplication by any $A(t) \in \mathbb{R}^{k\times k}$ of determinant 1 preserves wedge product.
- Right multiplication by constant matrix $B \in \mathbb{R}^{n\times n}$ of determinant 1 has effect of changing the basis e.
- It turns out to be useful to consider right multiplication which depends on some other parameter s.
- **Objective:** Given $M(t)$, find $A(t)$ and $B(s)$ such that $A(t)M(t)B(s)$ has canonical structure. **Reduce to force higher order terms in $(t-s)$ to appear.**

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & t_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & t_1 \\ 0 & 0 & 1 & 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & t_2 \\ 0 & 0 & 0 & 0 & 1 & t_1^2 & t_2^2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & t_1 - s_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & t_1 - s_1 \\ 0 & 0 & 1 & 0 & 0 & t_2 - s_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & t_2 - s_2 \\ 0 & 0 & 0 & 0 & 1 & t_1^2 - s_1^2 & t_2^2 - s_2^2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & t_1 - s_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & t_1 - s_1 \\ 0 & 0 & 1 & 0 & 0 & t_2 - s_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & t_2 - s_2 \\ -2t_1 & 0 & 0 & -2t_2 & 1 & -(t_1 - s_1)^2 & -(t_2 - s_2)^2 \end{bmatrix}$$

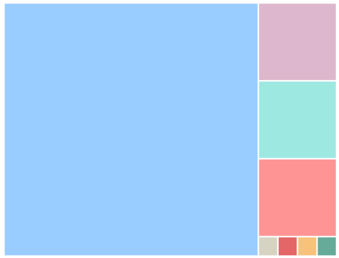
$$p(t,s)(t-s)^\alpha - p(s,s)(t-s)^\alpha = \sum_{|\beta|>0} \frac{(\partial_t^\beta p)(s,s)}{\beta!}(t-s)^{\alpha+\beta}$$

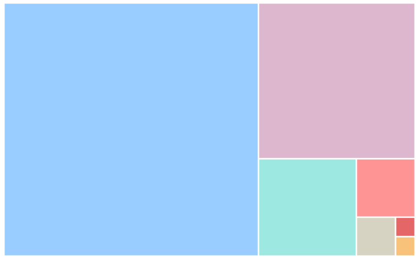
$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & t_1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & t_1 \\
0 & 0 & 1 & 0 & 0 & t_2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & t_2 \\
0 & 0 & 0 & 0 & 1 & t_1^2 & t_2^2
\end{bmatrix}
\rightsquigarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & t_1 - s_1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & t_1 - s_1 \\
0 & 0 & 1 & 0 & 0 & t_2 - s_2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & t_2 - s_2 \\
0 & 0 & 0 & 0 & 1 & t_1^2 - s_1^2 & t_2^2 - s_2^2
\end{bmatrix}
$$

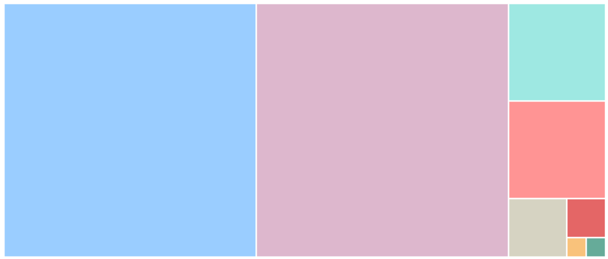
$\rightsquigarrow$ 

$$
p(t,s)(t-s)^\alpha - p(s,s)(t-s)^\alpha = \sum_{|\beta|>0} \frac{(\partial_t^\beta p)(s,s)}{\beta!}(t-s)^{\alpha+\beta}
$$

Greedy Decompositions are Generic







Hypothesis Setup

- Row/col reduction and block structure yields a family of $k \times n$ matrices with homogeneous poly entries in the variable $z := t - s$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & z_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & z_1 \\ 0 & 0 & 1 & 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & z_2 \\ -2t_1 & 0 & 0 & -2t_2 & 1 & -z_1^2 & -z_2^2 \end{bmatrix} \qquad \text{(THEMATRIX)}$$

- Act on these matrices by multiplication on left and right by $SL_k(\mathbb{R})$ and $SL_n(\mathbb{R})$, respectively.
- Act on z by $z \mapsto Pz$ for arbitrary $P \in GL_d(\mathbb{R})$.

Hypothesis Setup

- There is an action of $(A, B, P) \in \mathrm{SL}_k(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R})$
- Take the norm of $|\det P|^{-\sigma}(A, B, P) \circ \mathrm{THEMATRIX}$. Compute the infimum over A, B, P and call it THEINF.
- This family of matrices is “good” when $\mathrm{THEINF} > 0$.
- There are various criteria (e.g., Newton-diagram type) which characterize when such a lower bound exists.

THEINF **Generalizes Affine Curvature**

$$\begin{bmatrix} 1 & 0 & 0 & f_1(t) \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & f_k(t) \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & f_1(t) - f_1(s) \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & f_k(t) - f_k(s) \end{bmatrix}$$

$$f(t) - f(s) = -\sum_{|\alpha|>0} \frac{(s-t)^\alpha}{\alpha!} f^{(\alpha)}(t)$$

$$\rightsquigarrow \begin{bmatrix} m_{11}(t) & \cdots & m_{1k}(t) & -(s-t) \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \vdots & -\frac{(s-t)^{N-1}}{(N-1)!} \\ m_{k1}(t) & \cdots & m_{kk}(t) & -\sum_{|\alpha|=N} \frac{(s-t)^\alpha}{\alpha!} f^{(\alpha)}(t) \end{bmatrix}$$

Theorem (Setup)

Let $\Omega \subset \mathbb{R}^d$ be open and $u^1(t), \ldots, u^k(t)$ on Ω be smooth $\mathbb{R}^n$-valued functions such that $u(t) := u^1(t) \wedge \cdots \wedge u^k(t)$ is nonvanishing and is Nash of complexity at most K. Let $U(t)$ be the matrix whose rows are $u^1(t), \ldots, u^k(t)$. Suppose that $M(t,s) := A(t)U(t)B(s)$ admits block decomposition with formal degree D_{ij} for block ij.

Theorem (Conclusion)

Suppose there exists nonnegative $w(t)$ on Ω and some $\sigma > 0$ such that at every $t \in \Omega$,

$$(w(t))^{\sigma} \leq \mathrm{THEINF}(t).$$

Then there exists C depending only on n, k, d, and the D_{ij} such that

$$\int_{\Omega} \frac{w(t)dt}{[||u(t)||_e]^{1/(k\sigma)}} \leq CK^C$$

uniformly for all volume 1 bases e of $\mathbb{R}^n$.

Key Lemma: Differential Inequalities

For any polynomial $g(x_1, \ldots, x_k)$ of fixed degree, any set $E \subset \mathbb{R}^n$ and any weighted measure $W dx$, there are vector fields X_i^j on an open set Ω

$$\sup_{x_1 \in \Omega} \cdots \sup_{x_k \in \Omega} \left| X_1^{\alpha_1} \cdots X_k^{\alpha_k} g(x_1, \ldots, x_k) \right| \lesssim \sup_{x_1 \in E} \cdots \sup_{x_k \in E} \left| g(x_1, \ldots, x_k) \right|$$

such that $W(E \cap \Omega) \gtrsim W(E)$ and

$$W(x) \left| \det(X_1^j, \ldots, X_n^j) \right| \gtrsim W(E)$$

on a subset of E with W-measure $\gtrsim W(E)$.

Proof Sketch of the Estimation Theorem

- Fix basis $e^1, \ldots, e^n$. Key quantities are pairings $\left\langle u^1(t) \wedge \cdots \wedge u^k(t), e^{j_1} \wedge \cdots \wedge e^{j_k} \right\rangle$. **At all points t,**

$$\frac{1}{||u(t)||_e} \left| \left\langle u^1(t) \wedge \cdots \wedge u^k(t), e^{j_1} \wedge \cdots \wedge e^{j_k} \right\rangle \right| \leq 1.$$

- **We can pretend that $e^1, \ldots, e^n$ are not constant:**

$$\frac{1}{||u(t)||_e} \left| \left\langle u^1(t) \wedge \cdots \wedge u^k(t), e^{j_1}(s_1) \wedge \cdots \wedge e^{j_k}(s_k) \right\rangle \right| \leq 1.$$

- **Replace e^j by $\overline{v}^j$ aligned with block decomposition:**

$$\frac{1}{||u(t)||_e} \left| \left\langle u^1(t) \wedge \cdots \wedge u^k(t), \overline{v}^{j_1}(s_1) \wedge \cdots \wedge \overline{v}^{j_k}(s_k) \right\rangle \right| \lesssim 1.$$

- **Apply the differential inequalities in s variables:**

$$\frac{\left|\left\langle u^1(t) \wedge \cdots \wedge u^k(t), X^{\alpha_1}\overline{v}^{j_1}(s_1) \wedge \cdots \wedge X^{\alpha_k}\overline{v}^{j_k}(s_k)\right\rangle\right|}{||u(t)||_e} \lesssim 1.$$

- **Restrict to diagonal $s_1 = \cdots = s_k = t$ and produce $\overline{u}^1, \ldots, \overline{u}^k$ adapted to block decomposition so**

$$\frac{1}{||u(t)||_e}|\overline{u}^i(t) \cdot X^\alpha \overline{v}^j(t)|^k \lesssim 1.$$

- **Theorem's hypotheses imply** $|\overline{u}^i(t) \cdot X^\alpha \overline{v}^j(t)|^k \gtrsim |\det(X_1, \ldots, X_n)|^{\sigma k}(w(t))^{\sigma k}.$
- **Differential Inequality Lemma with $W := w/||u||_e^{1/(\sigma k)}$**

$$\left(\int \frac{w(t)}{||u(t)||_e^{1/(\sigma k)}} dt\right)^{\sigma k} \lesssim 1.$$

Corollary

Suppose T is an algebraic (or Nash) Radon-like transform. Let $X_1, \ldots, X_{d_X}$ and $Y_1, \ldots, Y_{d_Y}$ be usual annihilated vector fields associated to the double fibration formulation. Consider the bilinear map

$$\left(\sum_i u_i X_i, \sum_j v_j Y_j\right) \mapsto \sum_{ij} u_i v_j [X_i, Y_j] / (span\{X_1, \ldots, Y_1, \ldots\}).$$

T is a model operator if and only if this bilinear map is semistable.

Proposition (SVD-Like Condition)

Suppose $P(z)$ is a $k \times n$ matrix whose entries are polys of degree at most D in $z \in \mathbb{R}^d$. If P satisfies

$$\sum_{j=1}^{n} \sum_{|\alpha| \le D} \frac{1}{\alpha!} \left[\partial^\alpha P_{ij}(z) \partial^\alpha P_{i'j}(z) |_{z=0} \right] = \frac{C}{k} \delta_{ii'},$$

$$\sum_{i=1}^{k} \sum_{|\alpha| \le D} \frac{1}{\alpha!} \left[\partial^\alpha P_{ij}(z) \partial^\alpha P_{ij'}(z) |_{z=0} \right] = \frac{C}{n} \delta_{jj'},$$

$$\sum_{i=1}^{k} \sum_{j=1}^{q} \sum_{\substack{|\alpha|,|\alpha'| \le D \\ \alpha + e^{\ell'} = \alpha' + e^{\ell}}} \sqrt{\frac{\alpha_\ell \alpha'_{\ell'}}{\alpha! \alpha'!}} \left[\partial^\alpha P_{ij}(z) \partial^{\alpha'} P_{ij}(z) |_{z=0} \right] = \sigma C \delta_{\ell\ell'},$$

then THEINF *is attained at the identity.*

Corollary (Resolved: Nothing Sinister Happens)

- *Model (quadratic) operators exist for dimension $d \geq 1$ averages iff codimension $1 \leq k \leq d^2$.*
- *Translation-invariant model operators exist iff codimension $1 \leq k \leq \frac{d(d+1)}{2}$.*

Proof: Any nontrivial trilinear form T_{ijk} satisfying

$$\sum_{ij} T_{ijk} T_{ijk'} = \lambda_1 \delta_{kk'}, \sum_{ik} T_{ijk} T_{ij'k} = \lambda_2 \delta_{jj'}, \sum_{jk} T_{ijk} T_{i'jk} = \lambda_3 \delta_{ii'}$$

is semistable. If semistable T exists, then eqns have nontrivial solns. Construct explicit solns for all $\mathbb{R}^{d \times d \times k}$.

BUT...Sinister things do happen in overdetermined cases: no good 8d families of 2d averages in $\mathbb{R}^7$.

Questions

- When THEINF $= 0$, can higher-order behavior have an impact?
- Can weighted inequalities resolve (more) general behavior in low codimensions?
- Moving beyond from the outermost edges of the Riesz diagram

Thank You