Isoperimetric and Poincaré inequalities on the Hamming cube

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Joint work with Polona Durcik and Paata Ivanisvili

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Hamming cube $\{0,1\}^n$

- Space of length-n bitstrings $x = (x_1, \ldots, x_n) \in \{0,1\}^n$
- Equivalent multiplicative notation: $z \in \{-1,1\}^n$, $z_i = (-1)^{x_i}$

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- We only care about inequalities independent of dimension n

Boolean functions

- For $f : \{\pm 1\}^n \to \mathbb{R}$ let

$$\mathbf{E} f = 2^{-n} \sum_{z \in \{\pm 1\}^n} f(z)$$

- $|A| = \mathbf{E}\mathbf{1}_A$ normalized counting measure

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- Every $f : \{\pm 1\}^n \to \mathbb{R}$ is a multilinear polynomial:

$$f(z) = \sum_{S \subset \{\pm 1\}^n} \widehat{f}(S) z^S \quad \text{(Fourier expansion)}$$

$$z^S = \prod_{i \in S} z_i, \quad \widehat{f}(S) = \mathbf{E}_z(f(z) z^S)$$

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- "Majority" $f(z) = \mathrm{sgn}(z_1 + \cdots + z_n)$ ($\to$ Hamming ball)
- "Dictator" $f(z) = z_i$ ($\to$ half-cube)

Isoperimetric problem

Question

With $|A|$ fixed, how "small" can the "boundary" of A possibly be?

[2] e_i is ith unit vector

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Given $A \subset \{0,1\}^n$ need a notion of boundary size.

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Given $A \subset \{0,1\}^n$ need a notion of boundary size.
"Interior" vs "exterior" boundary[2]

$$\partial_{\mathrm{int}} A = \{x \in A : \exists\, i \text{ s.t. } x \oplus e_i \notin A\}$$

$$\partial_{\mathrm{ext}} A = \{x \notin A : \exists\, i \text{ s.t. } x \oplus e_i \in A\} = \partial_{\mathrm{int}}(A^c)$$

With $|A|$ fixed what is minimal size of $|\partial_{\mathrm{int}} A|$?

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Consider Hamming ball of radius $n/2$:

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$\implies$ No non-trivial isoperimetric inequality

Just counting boundary vertices is not the ‘right’ boundary measure

Taking into account number of boundary edges

Definition

$$h_A(x) = \mathbf{1}_A(x) \cdot \#\{i \,\text{s.t.}\, x \oplus e_i \notin A\}$$

$$= \textit{number of edges from } x \in A \textit{ connecting to outside of } A$$

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Edge boundary measure:

$$\mathbf{E} h_A = \mathbf{E} h_{A^c} = \text{normalized count of total boundary edges}$$

No longer minimized by Hamming balls.

Codimension k cube example

$$A = \{x \in \{0,1\}^n : x_1 = \cdots = x_k = 0\}$$

$|A| = 2^{-k}$

[3] $x^* = \min(x, 1-x)$.

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Classical isoperimetric inequality for the Hamming cube[3]:

$$\mathbf{E} h_A \geq |A|^* \log_2(1/|A|^*) \quad \text{for all } A \subset \{0,1\}^n$$

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Edge isoperimetric profile:

$$\mathcal{B}_1(x) = \inf_{n \geq 1} \inf_{\substack{A \subset \{0,1\}^n, \\ |A| = x}} \mathbf{E} h_A$$

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Edge isoperimetric profile

Hart '76:[4]

$$\mathcal{B}_1\left(\frac{k}{2^n}\right) = nk2^{-n} - 2^{-n+1}\sum_{j=1}^{k-1}(\text{binary digit sum of } j)$$

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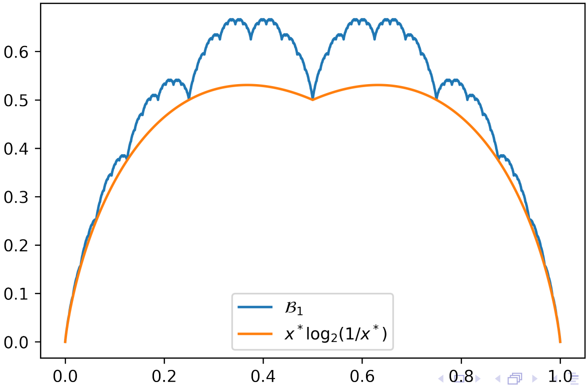

Interlude: Gaussian isoperimetric inequality

Gaussian space: $\mathbb{R}^n$ with

$$d\mu(t) = (2\pi)^{-n/2} e^{-|t|^2/2} dt = \varphi(t)\, dt.$$

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$$\mu^+(A) = \liminf_{h \to 0} h^{-1}(\mu(A_h) - \mu(A))$$

(A_h is the h-neighborhood of A)

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Sudakov-Tsirelson '75, Borell '78, Bobkov '97:

$$\mu^+(A) \geq \mu^+(H)$$

where H a half-space with $\mu(A) = \mu(H)$.

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I is also defined by $I'' \cdot I = -1, I(0) = I(1) = 1$.

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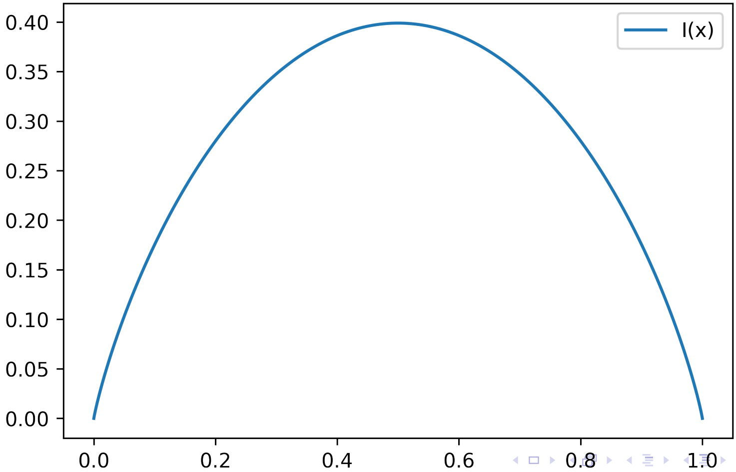

Bobkov '97: Proof via Boolean functions

Theorem (Bobkov's inequality)

For all $f : \{0,1\}^n \to [0,1]$*:*

$$I(\mathbf{E}f) \leq \mathbf{E}\sqrt{I(f)^2 + |\nabla f|^2}$$

where $|\nabla f|^2 = \sum_{i=1}^n |\frac{1}{2}(f(x \oplus e_i) - f(x))|^2$*.*

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- Proof by induction on n, case $n = 1$ only uses $I'' \cdot I = -1$
- Approx. argument gives Gaussian isoperimetric inequality
- When $f = \mathbf{1}_A$:

$$\mathbf{E}|\nabla \mathbf{1}_A| \geq I(|A|)$$

$$\frac{1}{2}(\mathbf{E}h_A^{1/2} + \mathbf{E}h_{A^c}^{1/2}) \geq I(|A|)$$

Not sharp!

Isoperimetric problem on the Hamming cube

For $\beta \geq 0$ consider

$$\mathbf{E} h_A^{\beta}$$

- $\beta = 0$: vertex boundary measure ("boring")
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Question

Given $\beta \in [1/2, 1]$, what is the best (largest) value $\mathcal{B}_\beta(|A|)$ such that

$$\mathbf{E} h_A^\beta \geq \mathcal{B}_\beta(|A|) \quad ?$$

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Question

Given $\beta \geq \frac{1}{2}$ and $x \in [0,1]$ what is the value of

$$\mathcal{B}_\beta(x) = \inf_{n \geq 1} \inf_{A \subset \{0,1\}^n, |A| = x} \mathbf{E} h_A^\beta$$

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Conjecture

For all $\beta \geq 0.5$ and $k \geq 1$:

$$\mathcal{B}_\beta(2^{-k}) = 2^{-k} k^\beta$$

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(What about other x?)

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Theorem (DIR '24)

For all $\beta \geq 0.50057$ and all x:

$$\mathcal{B}_\beta(x) \geq x^* (\log_2 \tfrac{1}{x}^*)^\beta$$

with equality for $x = 2^{-k}$.

State of the art for $\beta > 1/2$ (sharp bounds)

	β	$\mathcal{B}_\beta(x) \geq$	Sharp for $x =$
Classical '60s	1	$x^* \log_2(1/x^*)$	2^{-k}
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BIM '23	0.53	$8x(1-x)((1-\frac{2\sqrt{2}}{3})x+\frac{\sqrt{2}}{3}-\frac{1}{4})$	$\frac{1}{2}$

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DIR '24	0.50057	$b_\beta(x) \geq x^*(\log_2(1/x^*))^\beta$	2^{-k}

New bound for β near $1/2$

For $\beta = 0.50057$:

$$\mathcal{B}_\beta(x) \geq b_\beta(x) = \begin{cases} x(\log_2(1/x))^\beta & \text{for } x \in [0, \frac{1}{4}] \\ \frac{2}{3}x(1-x)(2^{2+\beta} - 3 + 4x(3 - 2^{1+\beta})) & \text{for } x \in [\frac{1}{4}, \frac{1}{2}] \\ \sqrt{2} \cdot w \cdot I(\frac{1-x}{w}) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

with $w > 0$ s.t. continuous at $x = 1/2$.

State of the art for $\beta = 1/2$ (no sharp bounds known)

	$\mathcal{B}_{\frac{1}{2}}(x) \geq$	$\mathcal{B}_{\frac{1}{2}}(\frac{1}{2}) \geq$
Talagrand '93	$\sqrt{2}\, x(1-x)$	0.35...
Conjectured	?	0.5

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BIM '23	$2\sqrt{2^{3/2}-2}\, x(1-x)$	$0.45\ldots$
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hypothetical best bound of form $Cx(1-x)$	$\frac{4\sqrt{2}}{3} x(1-x)$	$0.47\ldots$
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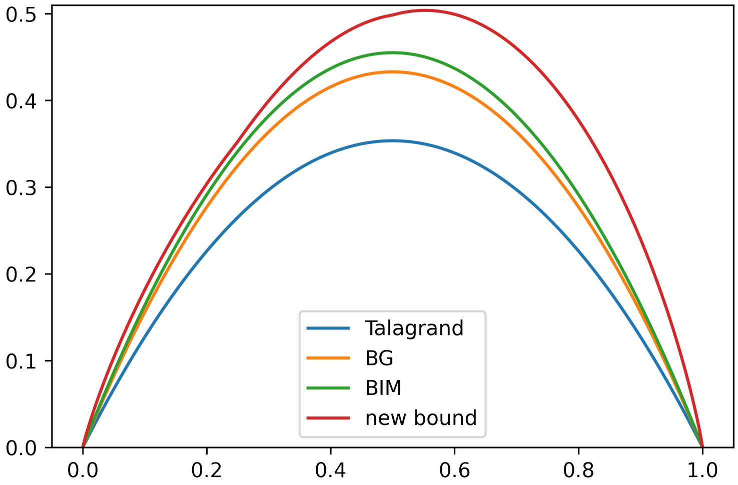
Bobkov–Götze '99	$\sqrt{3}\, x(1-x)$	$0.43\ldots$
BIM '23	$2\sqrt{2^{3/2}-2}\, x(1-x)$	$0.45\ldots$
hypothetical best bound of form $Cx(1-x)$	$\frac{4\sqrt{2}}{3} x(1-x)$	$0.47\ldots$
DIR '24	$0.997 \cdot b_{1/2}(x)$	0.4985
Conjectured	?	0.5

Known bounds for $\beta = 1/2$



Poincaré inequalities

Recall $|\nabla f| = \left(\sum_{j=1}^{n} |\frac{1}{2}(f(x \oplus e_j) - f(x))|^2 \right)^{1/2}$

Question

Let $p \geq 1$. What is the best C_p so that

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- BIM '23: Improvement when restricted to Boolean-valued f

$$C_{1,\text{B.v.}} > \sqrt{2/\pi}$$

$\mathbf{E}|\nabla f|^p \geq C_p \mathbf{E}|f - \mathbf{E}f|^p$

Say $f : \{0,1\}^n \to \{0,1\}$. Then $f = \mathbf{1}_A$.

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Theorem (DIR '24)

Conjecture holds for $p \geq 1.00114$.

1. Introduction & Results
   - Analysis on the Hamming cube
   - Isoperimetric problems
   - Poincaré inequalities

2. Two-point inequalities
   - Bobkov's inequality
   - Kahn–Park inequality
   - Computed envelopes

3. Computer-assisted verification
   - Automating lower bounds: Toy example
   - Rigorous numerics

Proposition (Bobkov)

$B : [0,1] \to [0,\infty)$ a given function. If

$$B(\mathbf{E}f) \le \mathbf{E}\sqrt{B(f)^2 + |\nabla f|^2}$$

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Generalization of Bobkov's inequality

$\mathcal{I}$ interval, $\|\cdot\|$ 'suitable' norm, D 'nice' sublinear operator, $B : \mathcal{I} \to [0, \infty)$ a function. If

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Variant of Bobkov's two-point inequality

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Idea: Increase β slightly ($\beta = 0.5 + 19 \cdot 2^{-15}$ is just enough) so that it holds on $[0.5, 1]$

Asymptotic behavior near 1

Recall $J(x) = \sqrt{2} \cdot w \cdot I((1-x)/w)$, $x_0 = 1 - w/2$.

Corollary

For all $A \subset \{0,1\}^n$ we have

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Using $B(x) = x\sqrt{\log_2(1/x)}$ (and $\mathcal{I} = [0, 1/2]$) gives:

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- Sharp inequality for $|A| = 1/2$ when β at or near $1/2$ requires a new ingredient

Improved two-point inequality (Kahn–Park '20)

Suppose $B : [0,1] \to [0,\infty)$ satisfies $B(0) = B(1) = 0$. Set $c_\beta = 2^\beta - 1$ and assume

$$(\mathrm{KP}) \quad \max\left(\left((y-x)^{\frac{1}{\beta}} + B(y)^{\frac{1}{\beta}}\right)^\beta, y - x + c_\beta B(y)\right) + B(x) \geq 2B\left(\tfrac{x+y}{2}\right)$$

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for all $0 \leq x \leq y \leq 1$.
Then $\mathcal{B}_\beta \geq B$, i.e. for all $A \subset \{0,1\}^n$:

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- Proof idea: Induction on n. For $A \subset \{0,1\}^{n+1}$ set

$$A_i = \{x : (x_1, \ldots, x_n, i) \in A\} \subset \{0,1\}^n.$$

Optimize carefully and use Jensen correctly.

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- So there is a largest function. Can we compute it?

Envelope function

Definition

Let $\mathfrak{B}_\beta : \mathcal{Q} \to [0, \infty)$ *be the largest function satisfying* (KP) *for all* $0 \leq x \leq y \leq 1$ *in* $\mathcal{Q} = \{k2^{-n} : n \geq 1, 0 \leq k \leq 2^n\}$.

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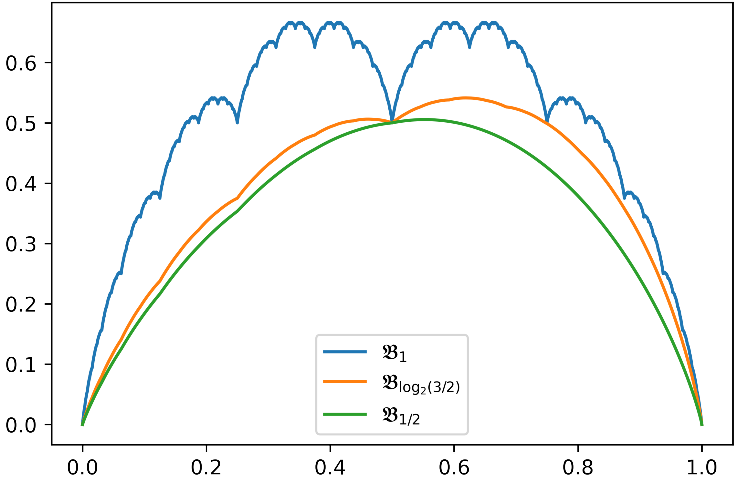
- Easy to see: $\mathfrak{B}_\beta(x) = \limsup_{n\to\infty} \mathfrak{B}_{\beta,n}(x)$
- $\mathfrak{B}_{\beta,n}$ can be computed by a 'greedy' approach improving guesses iteratively, for n small, say $n < 20$

Computed envelopes for different β



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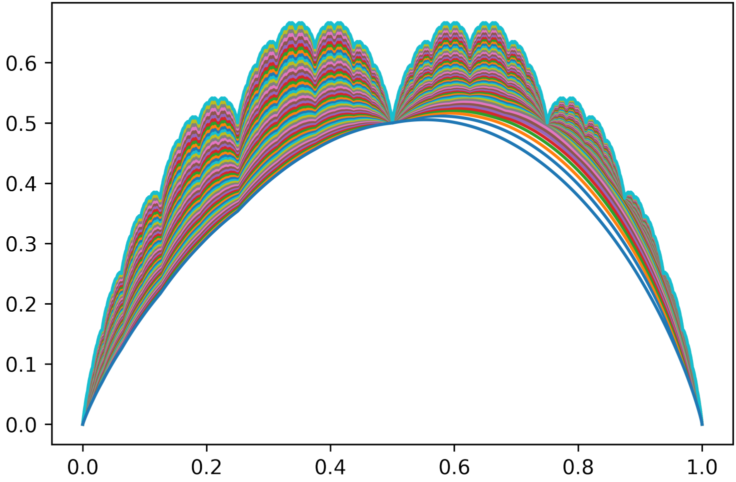
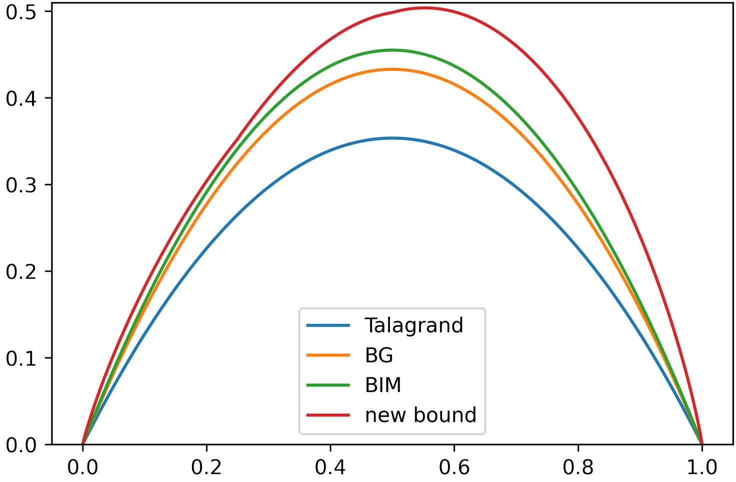


Known bounds vs. computed envelope for $\beta = 1/2$



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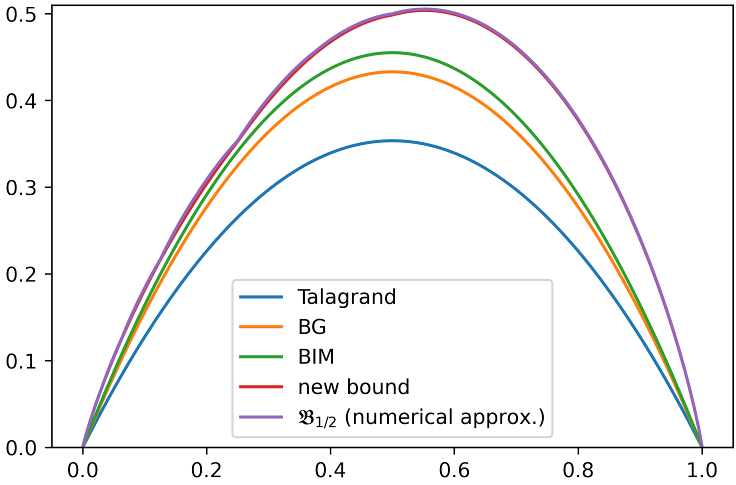

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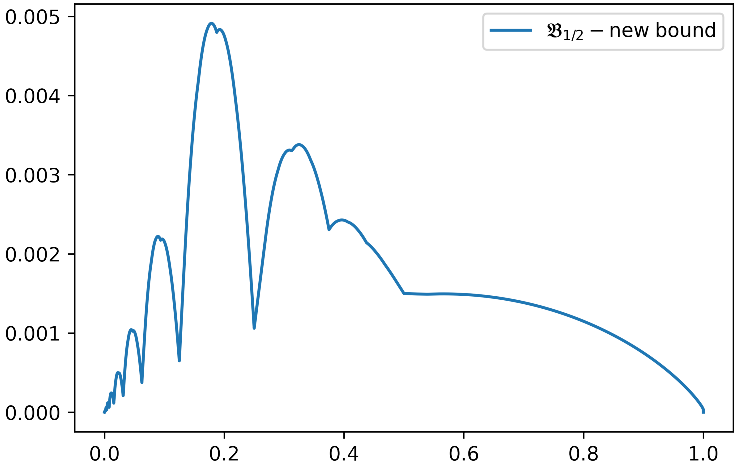

How to prove the two-point inequality?

Define

$$G_{B,\beta}(x,y) = \max(((y-x)^{\frac{1}{\beta}}+B(y)^{\frac{1}{\beta}})^{\beta}, y-x+c_\beta B(y))+B(x)-2B(\tfrac{x+y}{2})$$

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where $c_\beta = 2^\beta - 1$ and

$$b_\beta(x) = \begin{cases} L_\beta(x) = x(\log_2(1/x))^\beta & \text{for } x \in [0, \frac{1}{4}] \\ Q_\beta(x) = \frac{2}{3}x(1-x)(2^{2+\beta} - 3 + 4x(3 - 2^{1+\beta})) & \text{for } x \in [\frac{1}{4}, \frac{1}{2}] \\ J(x) = \sqrt{2} \cdot w_0 \cdot I(\frac{1-x}{w_0}) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

and $0 \le x \le y \le 1$ where $I = \varphi \circ \Phi^{-1}$ and w_0 is such that b_β is continuous at $x = 1/2$.

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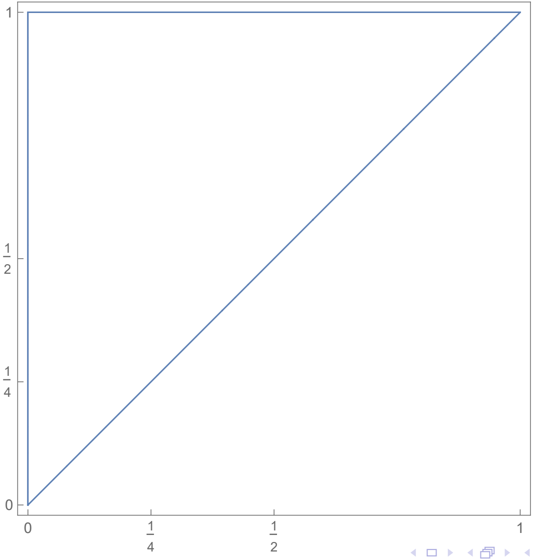
"Exercise"

For $\beta = 0.50057$ and all $0 \le x \le y \le 1$ show that

$$G_{b_\beta,\beta}(x,y) \ge 0.$$

Case distinction



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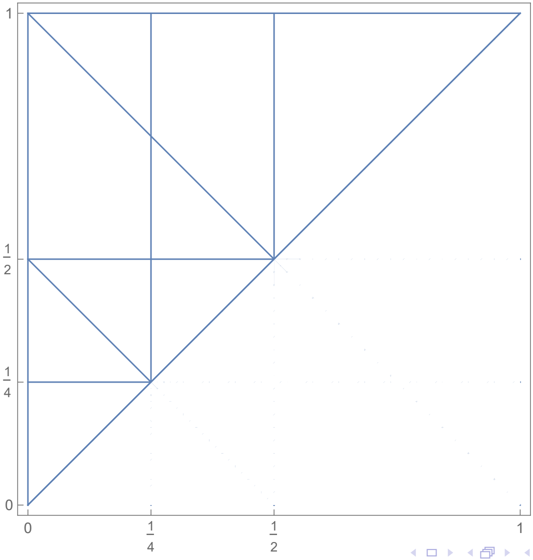
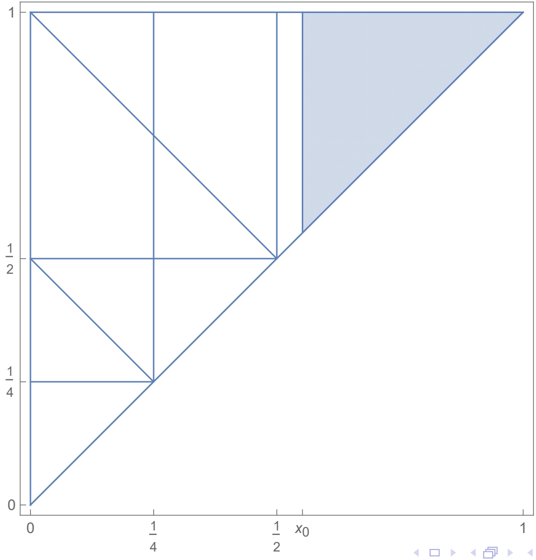
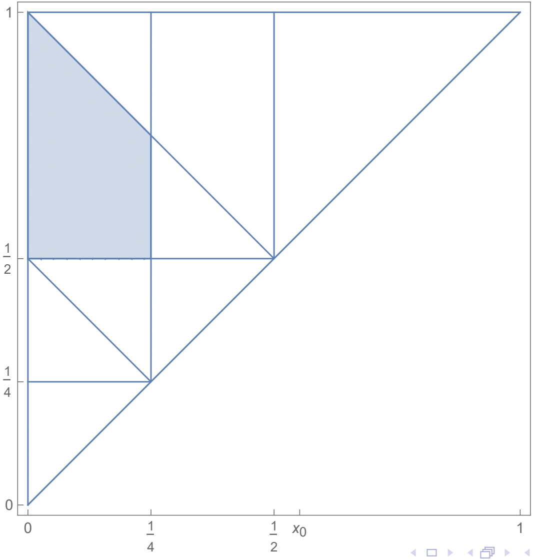


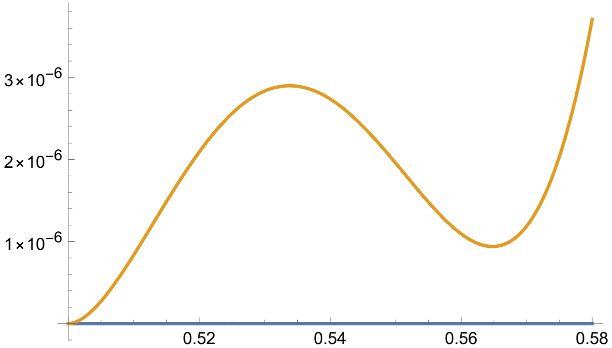
$((y-x)^2+J(y)^2)^{\frac{1}{2}}+J(x)-2J(\frac{x+y}{2})\geq 0$



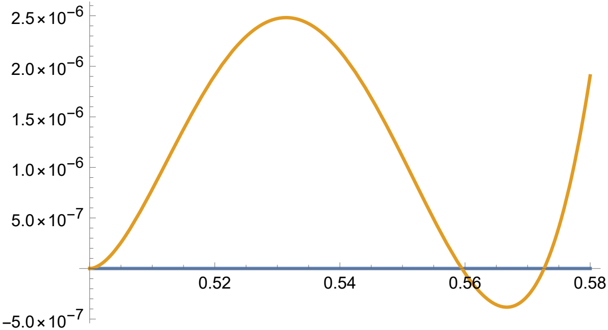
$\max(((y-x)^{\frac{1}{\beta}} + J(y)^{\frac{1}{\beta}})^{\beta}, y-x+c_\beta J(y)) + L_\beta(x) - 2Q_\beta(\frac{x+y}{2}) \geq 0$



$y \mapsto G_{b_\beta,\beta}(x,y)$ for $x = \frac{1}{2}$, $\beta = 0.5 + 19 \cdot 2^{-15}$



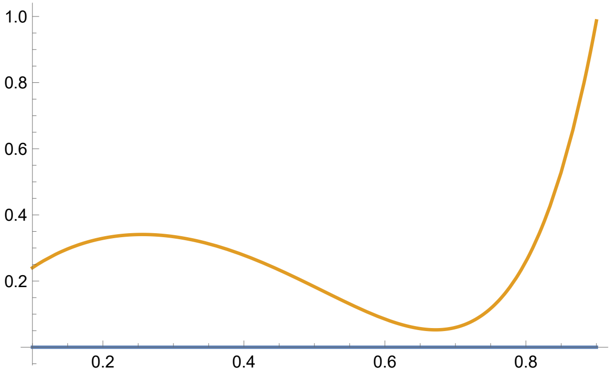
$y \mapsto G_{b_\beta,\beta}(x,y)$ for $x = \frac{1}{2}$, $\beta = 0.5 + 18 \cdot 2^{-15}$



Toy example

Exercise

Show that $f(x) = 2x\sqrt{\log\frac{1}{x}} + 3x^8 - 2\sqrt{x^3 - x^6} > 0$ *for* $x \in [0.1, 0.9]$.



$\mathfrak{I}$ set of closed subintervals of $[a, b]$.

Definition

$v : \mathfrak{I} \to \mathbb{R}$ is called a tight lower bound of f if

$$f(x) \geq v([\underline{x}, \overline{x}]) \quad \textit{for all } x \in [\underline{x}, \overline{x}]$$

and $v([\underline{x}, \overline{x}]) \to f(x)$ as $\overline{x} - \underline{x} \to 0$.

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$$\{0.1, 0.2, 0.3, 0.4, 0.45, 0.5, 0.55, 0.575, 0.6, 0.6125, 0.625, \ 0.6375,$$

$$0.65, 0.6625, 0.66875, 0.675, 0.68125, 0.6875, 0.69375, 0.7, \ 0.7125,$$

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- Can we trust computer evaluations using, say, Mathematica? **No!**
- Rounding (floating point) errors and numerical approximation errors propagate

Rump's example (80s)

$$\mathrm{Rump}(a, b) = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b}$$

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Correct value: $-0.8274(\pm 10^{-4})$

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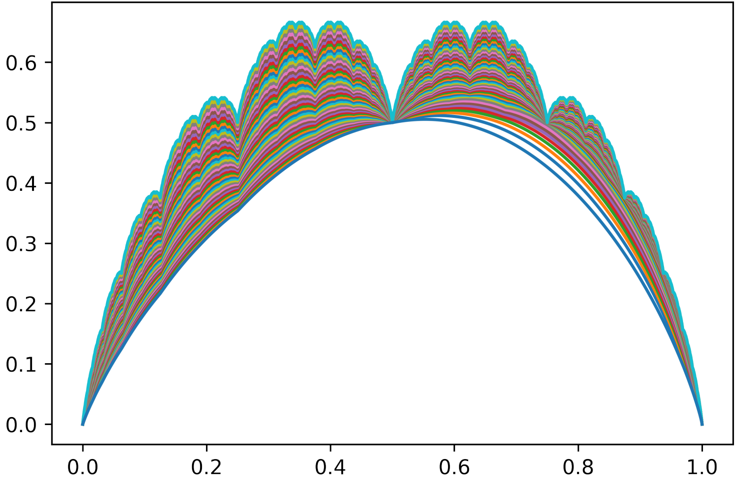
- flint/arb: open source library for rigorous numerics, relevant parts easy to verify



Thank you!