Adjoint Brascamp–Lieb inequalities

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Brascamp–Lieb inequalities in (very) short...

Brascamp–Lieb inequalities are *Lebesgue space bounds on multilinear forms* – inequalities of the form

$$\Lambda(f_1, \dots, f_k) \leq C\|f_1\|_{L^{p_1}(X_1)} \cdots \|f_k\|_{L^{p_k}(X_k)}.$$

Here Λ is a multilinear form acting on suitable functions $f_1, \dots, f_k$.

Rather "generally" Λ has an integral representation of the form

$$\Lambda(f_1, \dots, f_k) = \int_{X_1 \times \cdots \times X_k} f_1(x_1) \cdots f_k(x_k) d\mu(x)$$

for some measure μ.

Brascamp–Lieb inequalities concern singular μ, usually supported on some subvariety of $X_1 \times \cdots \times X_k$.

The plan

1. A brief introduction to the classical Brascamp–Lieb inequalities
2. Adjoint Brascamp–Lieb inequalities
3. Applications to tomographic transforms

The Brascamp–Lieb inequalities are best introduced through examples...

Hölder's inequality

Theorem (Hölder's inequality on $\mathbb{R}^d$)

$$\int_{\mathbb{R}^d} f_1(x)\cdots f_k(x)dx \leq \|f_1\|_{p_1}\cdots\|f_k\|_{p_k},$$

whenever $\frac{1}{p_1}+\cdots+\frac{1}{p_k}=1$.

Equivalently,

$$\int f_1(x)^{c_1}\cdots f_k(x)^{c_k}dx \leq \left(\int f_1\right)^{c_1}\cdots\left(\int f_k\right)^{c_k}$$

where $c_1+\cdots+c_k=1$; here and throughout functions are assumed to be nonnegative.

The Loomis–Whitney inequality

Theorem (Loomis–Whitney 1948)

For $d \geq 2$,

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} f_i(x_1, \ldots, \widehat{x_i}, \ldots, x_d) dx \leq \prod_{i=1}^{d} \|f_i\|_{L^{d-1}(\mathbb{R}^{d-1})}$$

for all $f_1, \ldots, f_d : \mathbb{R}^{d-1} \to \mathbb{R}_+$. Here $\widehat{\ }$ denotes omission.

Equivalently,

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} f_i(x_1, \ldots, \widehat{x_i}, \ldots, x_d)^{\frac{1}{d-1}} dx \leq \prod_{i=1}^{d} \left(\int_{\mathbb{R}^{d-1}} f_i \right)^{\frac{1}{d-1}}.$$

Note the presence of the linear maps

$$\mathbb{R}^d \ni x \mapsto (x_1, \ldots, \widehat{x_i}, \ldots, x_d) \in \mathbb{R}^{d-1}$$

that *pull back* the f_i to functions on $\mathbb{R}^d$.

The affine-invariant Loomis–Whitney inequality

The Loomis–Whitney inequality

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} f_i(x_1, \ldots, \widehat{x_i}, \ldots, x_d)^{\frac{1}{d-1}}\, dx \leq \prod_{i=1}^{d} \left(\int_{\mathbb{R}^{d-1}} f_i \right)^{\frac{1}{d-1}},$$

reformulated as

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} f_i(P_{\langle e_i \rangle^\perp} x)^{\frac{1}{d-1}}\, dx \leq \prod_{i=1}^{d} \left(\int_{\langle e_i \rangle^\perp} f_i \right)^{\frac{1}{d-1}},$$

implies the *affine-invariant Loomis–Whitney inequality*,

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} f_i(P_{\langle \omega_i \rangle^\perp} x)^{\frac{1}{d-1}}\, dx \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d-1}} \prod_{i=1}^{d} \left(\int_{\langle \omega_i \rangle^\perp} f_i \right)^{\frac{1}{d-1}},$$

which holds for *any basis* of unit vectors $\omega_1, \ldots, \omega_d \in \mathbb{S}^{d-1}$.

Brascamp–Lieb inequalities

A natural common generalisation of the Hölder and Loomis–Whitney inequalities:

$$\int_{\mathbb{R}^d} \prod_{i=1}^k f_i(B_i x)^{c_i}\, dx \leq \mathrm{BL}(\mathbf{B}, \mathbf{c}) \prod_{i=1}^k \left(\int_{\mathbb{R}^{d_i}} f_i \right)^{c_i}$$

Here:

- $B_i : \mathbb{R}^d \to \mathbb{R}^{d_i}$ are linear maps, and the exponents $0 \leq c_i \leq 1$;
- $0 < \mathrm{BL}(\mathbf{B}, \mathbf{c}) \leq \infty$ is called the *Brascamp–Lieb constant*, and $(\mathbf{B}, \mathbf{c})$ the *Brascamp–Lieb datum*.

Geometrically, $\mathrm{BL}(\mathbf{B}, \mathbf{c})$ captures "information" about how the **kernels** (or **fibres**) of the B_i interact with each other – something manifest in the affine invariant Loomis–Whitney inequality

$$\int_{\mathbb{R}^d} \prod_{i=1}^d f_i(P_{\langle \omega_i \rangle^\perp} x)^{\frac{1}{d-1}}\, dx \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d-1}} \prod_{i=1}^d \left(\int_{\langle \omega_i \rangle^\perp} f_i \right)^{\frac{1}{d-1}}$$

for example.

Behaviour of the Brascamp–Lieb constant

Theorem (Lieb 1990)

$\mathrm{BL}(\mathbf{B},\mathbf{c}) = \mathrm{BL}_{\mathrm{g}}(\mathbf{B},\mathbf{c})$, *the gaussian Brascamp–Lieb constant (obtained by testing on centred gaussian inputs* f_i*).*

Theorem (B–Carbery–Christ–Tao 2007)

$\mathrm{BL}(\mathbf{B},\mathbf{c}) < \infty$ *if and only if* $\sum_{i=1}^k c_i d_i = d$ *and* $\sum_{i=1}^k c_i \dim(B_i V) \geq \dim(V)$ *for all subspaces* V *of* $\mathbb{R}^d$*.*

Recent applications in harmonic analysis begin with "perturbations" of the Brascamp–Lieb inequalities, notably *Brascamp–Lieb inequalities of Kakeya type*:

$$\int_{\mathbb{R}^d} \prod_{i=1}^k \left(\sum_{T_i \in \mathbb{T}_i} \chi_{T_i} \right)^{c_i} \lesssim \prod_{i=1}^k (\#\mathbb{T}_i)^{c_i}$$

where the $\mathbb{T}_i$ are families of (neighbourhoods) of subspaces/varieties of $\mathbb{R}^d$; B–Carbery–Tao 2006, Guth 2010/15, Bourgain–Guth 2011, Carbery–Valdimarsson 2013, B–Bez–Flock–Lee 2018, Zhang 2018, Zorin–Kranich 2020, Tao 2020, Maldague 2022,...

Uses further properties of $\mathrm{BL}(\mathbf{B},\mathbf{c})$, e.g. *continuity* (B–Bez–Cowling–Flock 2017).

Adjoint Brascamp–Lieb inequalities

The Brascamp–Lieb inequality

$$\int_{\mathbb{R}^d} \prod_{i=1}^k f_i(B_i x)^{c_i} dx \leq \mathrm{BL}(\mathbf{B}, \mathbf{c}) \prod_{i=1}^k \left(\int_{\mathbb{R}^{d_i}} f_i \right)^{c_i}$$

involves the **pullback** operation $f_i \mapsto f_i \circ B_i$, taking a function on $\mathbb{R}^{d_i}$ to a function on $\mathbb{R}^d$.

The adjoint of this is the **pushforward** operation $(B_i)_*$, taking a function on $\mathbb{R}^d$ to a function on $\mathbb{R}^{d_i}$:

$$(B_i)_* f(y) = \frac{1}{\sqrt{\det(B_i B_i^*)}} \int_{B_i^{-1}(\{y\})} f(x) dx; \quad y \in \mathbb{R}^{d_i}.$$

E.g. In the case of the first Loomis–Whitney map $\pi_1(x) = (x_2, \ldots, x_d)$,

$$(\pi_1)_* f(x_2, \ldots, x_d) = \int_{\mathbb{R}} f(x_1, x_2, \ldots, x_d) dx_1,$$

the first **marginal** of f.

Loose idea: A Brascamp–Lieb inequality involves pullbacks of low dimensional functions, so it ought to have an "adjoint" that involves pushforwards of high-dimensional functions.

An adjoint Loomis–Whitney inequality

Theorem (B–Tao 2023)

For all $0 < p \leq 1$ *and* $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\|f\|_p \leq \prod_{i=1}^d \|f_i\|_{\frac{p(d-1)}{d-p}}^{\frac{1}{d}}$$

where

$$f_i(x_1, \ldots, \widehat{x_i}, \ldots, x_d) := \int_{\mathbb{R}} f(x_1, \ldots, x_i, \ldots, x_d) dx_i.$$

Proof. If $q = \frac{p(d-1)}{d-p}$ then

$$\|f\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \left(\prod_{i=1}^d \left(f(x) f_i(\pi_i x)^{q-1} \right)^{\frac{1}{d}} \right)^p \left(\prod_{i=1}^d f_i^q(\pi_i x)^{\frac{1}{d-1}} \right)^{1-p} dx,$$

and the theorem follows by Hölder, Loomis–Whitney and the duality identity

$$\int_{\mathbb{R}^d} f(x) f_i(\pi_i x)^{q-1} dx = \int_{\mathbb{R}^{d-1}} f_i(y) f_i(y)^{q-1} dy = \|f_i\|_{L^q(\mathbb{R}^{d-1})}^q.$$

Adjoint Brascamp–Lieb inequalities

Definition (Adjoint Brascamp–Lieb inequality/constant)

Let $(\mathbf{B}, \mathbf{c})$ be a Brascamp–Lieb datum, $0 < p \leq 1$ and suppose $\theta_1, \ldots, \theta_k$ are positive real numbers that sum to 1. For $1 \leq i \leq k$ define $0 < p_i \leq 1$ by the formula

$$c_i \left(1 - \frac{1}{p}\right) = \theta_i \left(1 - \frac{1}{p_i}\right).$$

Let $\mathrm{ABL}(\mathbf{B}, \mathbf{c}, \theta, p)$ denote the best constant such that

$$\|f\|_{L^p(\mathbb{R}^d)} \leq \mathrm{ABL}(\mathbf{B}, \mathbf{c}, \theta, p) \prod_{i=1}^{k} \|(B_i)_* f\|_{L^{p_i}(\mathbb{R}^{d_i})}^{\theta_i}$$

holds for any $f : \mathbb{R}^d \to \mathbb{R}_+$. Define $\mathrm{ABL}_{\mathrm{g}}(\mathbf{B}, \mathbf{c}, \theta, p)$ similarly, but with f restricted to be gaussians.

- Inequality an identity when $p = 1$ with $\mathrm{ABL}(\mathbf{B}, \mathbf{c}, \theta, p) = 1$ (Fubini).
- $p \leq 1$ is necessary for an inequality of this type to hold (control of a function by its marginals).

The main theorem

Recall the adjoint Brascamp–Lieb inequality

$$\|f\|_{L^p(\mathbb{R}^d)} \leq \mathrm{ABL}(\mathbf{B}, \mathbf{c}, \theta, p) \prod_{i=1}^{k} \|(B_i)_* f\|_{L^{p_i}(\mathbb{R}^{d_i})}^{\theta_i}$$

Theorem (B–Tao 2023)

$$\left(p^{-\frac{d}{2p}} \prod_{i=1}^{k} p_i^{\frac{\theta_i d_i}{2p_i}} \right) \mathrm{BL}(\mathbf{B}, \mathbf{c})^{\frac{1}{p}-1} = \mathrm{ABL}_{\mathrm{g}}(\mathbf{B}, \mathbf{c}, \theta, p) \leq \mathrm{ABL}(\mathbf{B}, \mathbf{c}, \theta, p) \leq \mathrm{BL}(\mathbf{B}, \mathbf{c})^{\frac{1}{p}-1}$$

To prove the upper bound write $f_i = (B_i)_* f$ and

$$\|f\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \left(\prod_{i=1}^{k} \left(f(x) f_i(B_i x)^{p_i - 1} \right)^{\theta_i} \right)^p \left(\prod_{i=1}^{k} f_i^{p_i}(B_i x)^{c_i} \right)^{1-p} dx.$$

Then apply Hölder's inequality followed by the Brascamp–Lieb inequality and

$$\int_{\mathbb{R}^d} f(x) f_i(B_i x)^{p_i - 1} dx = \int_{\mathbb{R}^{d_i}} f_i(y) f_i(y)^{p_i - 1} dy = \|f_i\|_{L^{p_i}(\mathbb{R}^{d_i})}^{p_i}.$$

Relation to entropy subadditivity

Theorem (Carlen–Cordero–Erausquin 2009)

If $f : \mathbb{R}^d \to \mathbb{R}_+$ is a probability density and $h(f) := -\int f \log f$ then

$$h(f) \leq \sum_{i=1}^{k} c_i h((B_i)_* f) + \log \mathrm{BL}(\mathbf{B}, \mathbf{c}).$$

In the Loomis–Whitney case this is Shearer's inequality

$$h(f) \leq \frac{1}{d-1} \sum_{i=1}^{d} h(f_i),$$

where f_i is the ith marginal of the probability density $f : \mathbb{R}^d \to \mathbb{R}_+$.

These entropy inequalities follow by differentiating

$$p \longmapsto \frac{\mathrm{BL}(\mathbf{B}, \mathbf{c})^{\frac{1}{p}-1} \prod_{i=1}^{k} \|(B_i)_* f\|_{L^{p_i}(\mathbb{R}^{d_i})}^{\theta_i}}{\|f\|_{L^p(\mathbb{R}^d)}}$$

and using $\frac{d}{dp}\|f\|_p\Big|_{p=1} = -h(f)$. [Remark: this implication is effectively reversible.]

Applications

Two types at least:

1. To *structured functions* whose marginals are interesting for some reason (an application to Gowers uniformity norms - not today).
2. To *general functions*, interpreting marginals in terms of tomographic transforms...

Lower bounds on the X-ray transform

For $f : \mathbb{R}^d \to \mathbb{R}^+$ its *X-ray transform* $Xf : \mathcal{M}_{1,d} \to \mathbb{R}^+$ is given by the formula

$$Xf(\omega, v) := \int_{\mathbb{R}} f(v + t\omega)dt,$$

where $\omega \in \mathbb{S}^{d-1}$ and $v \in \langle \omega \rangle^\perp$ form the natural parametrisation of the Grassmannian manifold $\mathcal{M}_{1,d}$ of lines in $\mathbb{R}^d$, endowed with the obvious measure

$$\int_{\mathcal{M}_{1,d}} F(\omega, v) := \int_{\mathbb{S}^{d-1}} \left(\int_{\langle \omega \rangle^\perp} F(\omega, v)\ dv \right) d\sigma(\omega)$$

Theorem (Christ 1984)

Suppose that $p, q \geq 1$ and $d \geq 2$. Then there exists a positive constant $C = C_{p,q,d}$ such that

$$\|Xf\|_{L^q(\mathcal{M}_{1,d})} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

for all functions $f : \mathbb{R}^d \to \mathbb{R}$ if and only if

$$\frac{1}{d}\left(1 - \frac{1}{q}\right) = \frac{1}{d-1}\left(1 - \frac{1}{p}\right), \quad q \leq d + 1.$$

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Suppose that $p, q \leq 1$ and $d \geq 2$. Then there exists a positive constant $C = C_{p,q,d}$ such that

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for all functions $f : \mathbb{R}^d \to \mathbb{R}_+$ if and only if

$$\frac{1}{d}\left(1 - \frac{1}{q}\right) = \frac{1}{d-1}\left(1 - \frac{1}{p}\right).$$

Proof. If $\{\omega_1, \ldots, \omega_d\}$ is an orthonormal basis of $\mathbb{R}^d$ then by Loomis–Whitney,

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} f_i(P_{\langle\omega_i\rangle^\perp} x)^{\frac{1}{d-1}} \, dx \leq \prod_{i=1}^{d} \left(\int_{\langle\omega_i\rangle^\perp} f_i \right)^{\frac{1}{d-1}}$$

for any $f_i : \langle\omega_i\rangle^\perp \to \mathbb{R}_+$. This has as an adjoint

$$\|f\|_{L^p(\mathbb{R}^d)} \leq \prod_{i=1}^{d} \|(P_{\langle\omega_i\rangle^\perp})_* f\|_{L^q(\langle\omega_i\rangle^\perp)}^{\frac{1}{d}}.$$

However, $\|(P_{\langle\omega_i\rangle^\perp})_* f\|_{L^q(\langle\omega_i\rangle^\perp)} = \|Xf(\omega_i, \cdot)\|_{L^q(\langle\omega_i\rangle^\perp)}$,
and thus

$$\|f\|_{L^p(\mathbb{R}^d)}^q \leq \prod_{i=1}^{d} \|Xf(\omega_i, \cdot)\|_{L^q(\langle\omega_i\rangle^\perp)}^{\frac{q}{d}}.$$

which by the AM-GM inequality implies

$$\|f\|_{L^p(\mathbb{R}^d)}^q \leq \frac{1}{d} \sum_{i=1}^{d} \|Xf(\omega_i, \cdot)\|_{L^q(\langle\omega_i\rangle^\perp)}^q.$$

The theorem now follows on averaging over all such bases $\{\omega_1, \ldots, \omega_d\}$. □

Restricted X-ray transforms

Let's replace the uniform measure on $\mathbb{S}^{d-1}$ with a more general positive finite measure μ, and pull this back to a measure ν on $\mathcal{M}_{1,d}$ by

$$\int_{\mathcal{M}_{1,d}} F d\nu := \int_{\mathbb{S}^{d-1}} \left(\int_{\langle \omega \rangle^\perp} F(\omega, v)\, dv \right) d\mu(\omega).$$

Theorem (Lower bounds for restricted X-ray transforms)

Suppose $d \geq 2$ and that $0 < p, q \leq 1$ satisfy

$$\frac{1}{d}\left(1 - \frac{1}{q}\right) = \frac{1}{d-1}\left(1 - \frac{1}{p}\right).$$

Then

$$\|Xf\|_{L^q(d\nu)} \geq C(\mu) \|f\|_{L^p(\mathbb{R}^d)}$$

for all nonnegative $f : \mathbb{R}^d \to \mathbb{R}^+$, where

$$C(\mu) := \left(\int_{(\mathbb{S}^{d-1})^d} |\omega_1 \wedge \cdots \wedge \omega_d|^{\frac{dq}{d-1}\left(\frac{1}{p}-1\right)} d\mu(\omega_1) \cdots d\mu(\omega_d) \right)^{\frac{1}{dq}}.$$

Proof. Recall the affine-invariant Loomis–Whitney inequality

$$\int_{\mathbb{R}^{d-1}} \prod_{j=1}^{d} f_i(P_{\omega_i}x)^{\frac{1}{d-1}}\, dx \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d-1}} \prod_{i=1}^{d} \left(\int_{\langle \omega_i \rangle^\perp} f_i \right)^{\frac{1}{d-1}},$$

where $\omega_1, \ldots, \omega_d \in \mathbb{S}^{d-1}$. Taking an adjoint we have

$$\|f\|_{L^p(\mathbb{R}^d)} \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d-1}\left(\frac{1}{p}-1\right)} \prod_{i=1}^{d} \|(P_{\langle \omega_i \rangle^\perp})_* f\|_{L^q(\langle \omega_i \rangle^\perp)}^{\frac{1}{d}}.$$

Since

$$\|(P_{\langle \omega_i \rangle^\perp})_* f\|_{L^q(\langle \omega_i \rangle^\perp)} = \|Xf(\omega_i, \cdot)\|_{L^q(\langle \omega_i \rangle^\perp)},$$

this becomes

$$|\omega_1 \wedge \cdots \wedge \omega_d|^{\frac{dq}{d-1}\left(\frac{1}{p}-1\right)} \|f\|_{L^p(\mathbb{R}^d)}^{dq} \leq \prod_{i=1}^{d} \|Xf(\omega_i, \cdot)\|_{L^q(\langle \omega_i \rangle^\perp)}^{q}.$$

It just remains to integrate with respect to μ in all variables. □

Note the role played by the Brascamp–Lieb constant.

k-plane transform results

The *k-plane transform* is given by

$$T_{k,d}f(\pi, y) := \int_\pi f(x+y)\,d\lambda_\pi(x),$$

where the Grassmannian manifold $\mathcal{M}_{k,d}$ of affine k-planes is parametrised by a k-dimensional subspace π and an element $y \in \pi^\perp$.

Theorem (B–Tao 2023)

Suppose $d \geq 2$, $0 < p \leq 1$ and that the exponent p_k is given by

$$\frac{1}{d}\left(1 - \frac{1}{p_k}\right) = \frac{1}{d-k}\left(1 - \frac{1}{p}\right)$$

for each $1 \leq k \leq d-1$. Then $\|T_{k,d}f\|_{L^{p_k}(\mathcal{M}_{k,d})}$ is nondecreasing in k. In particular,

$$\|T_{k,d}f\|_{L^{p_k}(\mathcal{M}_{k,d})} \geq \|f\|_{L^p(\mathbb{R}^d)}.$$

The upper bounds on $T_{k,n}$ (where $p, q \geq 1$) are also due to Christ 1984.

Theorem (B–Tao 2023)

Suppose $d \geq 2$, $0 < p \leq 1$ and that the exponent p_k is given by

$$\frac{1}{d}\left(1 - \frac{1}{p_k}\right) = \frac{1}{d-k}\left(1 - \frac{1}{p}\right)$$

for each $1 \leq k \leq d-1$. Then $\|T_{k,d}f\|_{L^{p_k}(\mathcal{M}_{k,d})}$ is nondecreasing in k. In particular,

$$\|T_{k,d}f\|_{L^{p_k}(\mathcal{M}_{k,d})} \geq \|f\|_{L^p(\mathbb{R}^d)}.$$

Differentiating with respect to p at $p = 1$ we obtain:

Corollary

For a probability density $f : \mathbb{R}^d \to \mathbb{R}_+$, the sequence of normalised entropies

$$\frac{1}{d-k} h(T_{k,d}f)$$

is nondecreasing, and in particular,

$$h(T_{k,d}f) \geq \left(\frac{d-k}{d}\right) h(f)$$

Reverse inequalities and $p \geq 1$

When $p \geq 1$ our adjoint Brascamp–Lieb inequalities take the *reverse form*

$$\|f\|_{L^p(\mathbb{R}^d)} \geq \mathrm{BL}(\mathbf{B}, \mathbf{c})^{\frac{1}{p}-1} \prod_{i=1}^{k} \|(B_i)_* f\|_{L^{p_i}(\mathbb{R}^{d_i})}^{\theta_i}$$

(provided all but one of the θ_j are negative).

These also have applications to tomographic transforms...

Theorem

Suppose $d \geq 2$, $1 < p < \infty$ and $0 < q < 1$. If $1 < r < \infty$ satisfies

$$\left(\frac{1}{q} - \frac{1}{p}\right)\left(1 - \frac{1}{r}\right) = \frac{1}{d-1}\left(1 - \frac{1}{p}\right)\left(\frac{1}{q} - 1\right), \tag{1}$$

then there exists a constant $C > 0$ such that

$$C\|Xf\|_{L^\infty_\omega L^r_v}^{\frac{1}{q}-\frac{1}{p}} \leq \|f\|_p^{\frac{1}{q}-1} \|Xf\|_{L^q_{\omega,v}}^{1-\frac{1}{p}} \tag{2}$$

for all nonnegative functions $f : \mathbb{R}^d \to \mathbb{R}^+$. Morever, if the condition (1) is not satisfied then (2) fails for all positive C.

Further results

- *Adjoint discrete Brascamp–Lieb inequalities.* There are discrete Brascamp–Lieb inequalities

$$\sum_G \prod_{i=1}^k f_i^{c_i} \circ B_i \leq \mathrm{BL}(\mathbf{B}, \mathbf{c}) \prod_{i=1}^k \Big(\sum_{G_i} f_i\Big)^{c_i}$$

that also admit adjoint versions. Here $\mathrm{ABL}(\mathbf{B}, \mathbf{c}, p, \theta) \leq \mathrm{BL}(\mathbf{B}, \mathbf{c})^{\frac{1}{p}-1}$ holds with equality.

- *Nonlinear adjoint Brascamp–Lieb inequalities.* The process of taking adjoints doesn't care about linear/algebraic structure. For example, one can take an adjoint of the "spherical Brascamp–Lieb inequality"

$$\int_{\mathbb{S}^2} f_1(x_1)^{\frac{1}{2}} f_2(x_2)^{\frac{1}{2}} f_3(x_3)^{\frac{1}{2}} \, d\sigma(x) \leq \left(\int_{-1}^1 f_1\right)^{\frac{1}{2}} \left(\int_{-1}^1 f_2\right)^{\frac{1}{2}} \left(\int_{-1}^1 f_3\right)^{\frac{1}{2}}$$

of Carlen–Lieb–Loss (2004). These have applications to *spherical Radon transforms*...

Thank you for listening!