An FIO approach to L^p bounds for wave equations associated to sub-(Riemannian) Laplacians

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joint work with Alessio Martini

Madison Lectures in Fourier Analysis, May 2024

Sub-Laplacians

M a C^∞ - manifold of topological dimension $d = \dim M$

1.1. Hörmander's sum of squares operators ($\sim' 77$).

$X_1, \dots, X_m$ smooth real vector fields on M satisfying the bracket generating condition (BG):

these vector fields, along with their iterated commutators

$$X_1, \dots, X_m, [X_i, X_j], \dots, [X_{i_1}[X_{i_2}[\dots[X_{i_r}, X_{i_{r+1}}], \dots],],]$$

up to some step r, generate $T_x M$ at every point $x \in M$. Then

$$\mathcal{L} := -\sum_{j=1}^{m} X_j^2 \quad \text{is hypoelliptic} \quad \text{(Hörmander)}$$

[Hörmander, Rothschild/Stein, Nagel/Stein/Wainger, ...]

- Horizontal distribution $\mathcal{H}_x := \text{span}\{X_j|_x\}_{j=1,\dots,m}$
- A vector field X on M is called horizontal, if $X(x) \in \mathcal{H}_x$ for every $x \in M$.
- Horizontal gradient $\nabla_H f := \sum_{j=1}^m (X_j f)\, X_j$
- μ a smooth positive measure on M
- μ-divergence $\text{div}_\mu X$ of a smooth vector field $X \in \Gamma(TM)$:

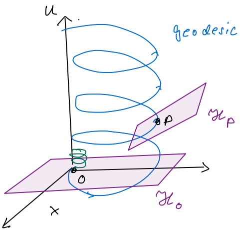
$$\int_M d\varphi(X)\, d\mu = -\int_M \varphi\, \text{div}_\mu X\, d\mu, \qquad \forall \varphi \in C_0^\infty(M)$$

- The associated Sub-Laplacian

$$\mathcal{L}f := -\text{div}_\mu(\nabla_H f) = \sum_{j=1}^m X_j^* X_j f = -(\sum_{j=1}^m X_j^2 + X_0) f$$

is essentially self-adjoint and non-negative on $L^2(M, d\mu)$

Sub-Riemannian geometry



- In the underlying sub-Riemannian geometry, only horizontal curves $\gamma : [0,1] \to M$ are admissible: $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$
- Associated Carnot-Carathéodory distance d_{CC} is defined by restricting to horizontal curves.
  [Chow, Rashevskii]: By (BG), $d_{CC}(x,y) < \infty$ for all $x, y \in M$
- CC-balls $B_r(x) := \{y \in M : d_{CC}(x,y) < r\}$

Associated functional calculus and wave propagators

L^2**-functional calculus:** m Borel measurable spectral multiplier:

$$\mathcal{L} = \int_0^\infty s\, dE_s, \qquad m(\mathcal{L}) := \int_0^\infty m(s)\, dE_s.$$

Wave equation on M: $\qquad (\partial_t^2 + \mathcal{L})u = 0$

Wave propagators:

- $u(x,t) := [\cos(t\sqrt{\mathcal{L}})f](x)$ solves wave equation on M
- $u_\pm(x,t) := exp(\mp it\sqrt{\mathcal{L}})f(x)$ solve the half-wave equations $(\partial_t \pm i\sqrt{\mathcal{L}})u_\pm = 0$

Main Question: Do Miyachi-Peral (M-P)-type estimates

$$(1) \qquad \|(1+t^2\mathcal{L})^{-\alpha/2} exp(\mp it\sqrt{\mathcal{L}})\|_{p\to p} \lesssim_{p,\alpha} 1, \quad \text{if } \alpha > (d-1)|\tfrac{1}{2} - \tfrac{1}{p}|,$$

hold true for time t sufficiently small $(1 < p < \infty)$?

Note: For $\mathcal{L} = -\Delta$ on $\mathbb{R}^d$ (after scaling to $t = 1$, etc.), (1) just means

$$\|e^{i\sqrt{-\Delta}} f\|_{L^p} \lesssim_{p,\alpha} \|(1-\Delta)^{\alpha/2} f\|_{L^p} \quad \text{if } \alpha > (d-1)|\tfrac{1}{2} - \tfrac{1}{p}|$$

Spectrally localized version of (M-P):

$$(2) \qquad \|\chi_1(t\sqrt{\mathcal{L}}/\lambda)\, exp(\mp it\sqrt{\mathcal{L}})\|_{p\to p} \lesssim_{p,\alpha} (1+\lambda)^\alpha, \quad \text{if } \alpha > (d-1)|\tfrac{1}{2}-\tfrac{1}{p}|,$$

for $|t| \le \varepsilon$, uniformly in $\lambda > 0$?

Note:

- [Martini /M./ Nicolussi Golo '2021] $\alpha \ge (d-1)|\frac{1}{2}-\frac{1}{p}|$ is necessary!
- The estimates (2) imply sharp Mihlin-Hörmander type multiplier theorems for functions $m(\mathcal{L})$

Results for elliptic $\mathcal{L}$:

- For the Laplacian $\mathcal{L} = -\Delta$ on $\mathbb{R}^d$, these estimates hold true even for the endpoint regularity $\alpha \ge (d-1)|\frac{1}{2}-\frac{1}{p}|$, with the Hardy space H^1 in place of L^1 for $p = 1$ [Miyachi, Peral '1980]
- Extensions to wide classes of FIOs by [Seeger/Sogge/Stein '91]; these apply, e.g., to Laplace-Beltrami operators on Riemannian manifolds.

Results for non-elliptic sub-Laplacians $\mathcal{L}$:

- (M-P)-type estimates for $\mathcal{L}$ satisfying Gaussian-type heat kernel bounds on a doubling space of upper regularity (often homogenous) dimension Q, with $\alpha > Q|\frac{1}{2} - \frac{1}{p}|$,[..., Hebisch '95,...]

  Note: $Q > d$ for non-elliptic $\mathcal{L}$!

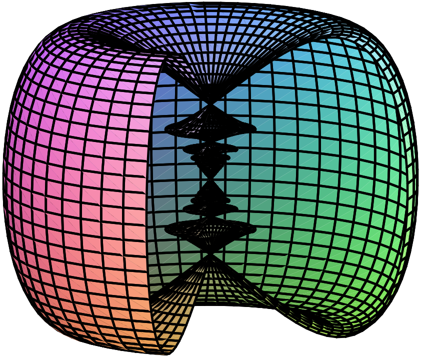
- (M-P)-type estimates on Heisenberg (-type) groups with $\alpha \geq (d-1)|\frac{1}{2} - \frac{1}{p}|$ [M./Stein '1999, M./Seeger '2015]

  Note: Proofs make use of either unitary representation theory, or of Mehler type formulas, so are "very non-classical" !

Example: Heisenberg group $\mathbb{H}_1$

- $$\mathbb{H}_1 = \mathbb{R}^2_x \times \mathbb{R}_u \text{ as manifold}$$
  $$(x,u)\cdot(x',u') = (x+x', u+u'+\langle Jx, x'\rangle/2), \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
- $$X_1 := \partial_{x_1} - \tfrac{x_2}{2}\partial_u,\ X_2 := \partial_{x_2} + \tfrac{x_1}{2}\partial_u,\ U := \partial_u$$
  form a basis of the Lie algebra of $\mathbb{H}_1$, which is identified with the space of left-invariant vector fields on $\mathbb{H}_1$. Then X_1, X_2 span the space of horizontal vector fields, and (BG) holds, since $[X_1, X_2] = U$
- $$\mathcal{L}_{\mathbb{H}_1} := -(X_1^2 + X_2^2)$$
  is left-invariant on the 2-step stratified (or Carnot) Lie group $\mathbb{H}_1$ and homogeneous under the dilating automorphisms
- $$\delta_r(x,u) := (rx, r^2u),\ r > 0.$$
- Note: $|B_r| = |\delta_r B_1| = r^Q |B_1|$ for CC-balls B, with $Q := 2 + 2\cdot 1 = 4 (> d = 3)$ the homogeneous dimension.

Singular support of a wave emenating from a single point in the Heisenberg group



2-step Carnot groups

Rothschild/Stein: Sub-Laplacians on Carnot groups, i.e., stratified Lie groups, are good local models for sub-Riemannian Laplacians!

- 2-step nilpotent group: $G = \mathbb{R}^{d_1}_x \times \mathbb{R}^{d_2}_u$ as mf., with (BCH)-product

$$(x,u)\cdot(x',u') := (x+x', u+u'+[x,x']/2),$$

  where $[x,x']_k = \langle J_k x, x'\rangle$, with $J_1, \dots J_{d_2}$ any skew-symmetric real $d_1 \times d_1$ matrices
- Let $J_\mu := \sum_{k=1}^{d_2} \mu_k J_k, \quad$ for $\mu \in \mathbb{R}^{d_2}$
- G is a 2-step Carnot group, if $J_\mu \neq 0$ for $\mu \neq 0$
- G is a Métivier group, if $\det J_\mu \neq 0$ for $\mu \neq 0$
- G is of Heisenberg type, if $J_\mu^2 = -|\mu|^2 \mathrm{Id}$
- $X_1, \dots, X_{d_1}$ left-invariant horizontal vector fields such that $X_j|_0 = \partial_{x_j}$.
- Sub-Laplacian on G: $\qquad \mathcal{L} := -\sum_{j=1}^{d_1} X_j^2$
- topological dimension $d = d_1 + d_2$; homogeneous dimension $Q = d_1 + 2d_2$

Theorem (Martini, M.)

Assume that G is a Métivier group. Then the spectrally localized (MP) estimates (2) *hold true. In particular,*

$$(3) \qquad \|\chi_1(t\sqrt{\mathcal{L}}/\lambda)\cos(t\sqrt{\mathcal{L}})\|_{1\to 1} \lesssim_\alpha (1+\lambda)^\alpha, \quad \textit{if } \alpha > (d-1)/2.$$

- Even more important than the given result: our new approach via FIO's with complex phase! Some key players for this approach:
- the symbol of $\mathcal{L}$: $|\xi + J_\mu x/2|^2$, and the associated
- Hamiltonian: $\mathcal{H}(\boldsymbol{x}, \boldsymbol{\xi}) := |\xi + J_\mu x/2|, \quad \boldsymbol{x} := (x, u), \ \boldsymbol{\xi} := (\xi, \mu)$
- I shall outline some key ideas of this approach mainly for the case of the Heisenberg group $\mathbb{H}_1$
- Note: by homogeneity, group invariance and finite propagation speed, we may reduce to a local estimate at time $t = 1$ for the convolution kernel:

$$(4) \qquad \|\mathbf{1}_{\overline{B_2(0)}}\chi_1(\sqrt{\mathcal{L}}/\lambda)\cos(\sqrt{\mathcal{L}})\delta_0\|_1 \lesssim_\varepsilon \lambda^{(d-1)/2}, \qquad (\lambda \gg 1)$$

1. Hamiltonian flow of $\mathcal{H}(\boldsymbol{x}, \boldsymbol{\xi})$: Integral curve of the Hamilton equations

$$\dot{x} = \nabla_\xi \mathcal{H}, \quad \dot{u} = \nabla_\mu \mathcal{H}, \quad \dot{\xi} = -\nabla_x \mathcal{H}, \quad \dot{\mu} = -\nabla_u \mathcal{H}$$

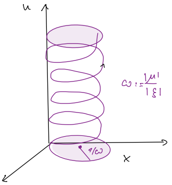
with initial datum $x(0) = 0, u(0) = 0, \xi(0) = \xi, \mu(0) = \mu$ and $\xi \neq 0$:

$$x^t = \frac{\exp(tJ_{\mu/|\xi|}) - I}{J_\mu}\xi, \qquad \xi^t = \frac{1}{2}(I + \exp(tJ_{\mu/|\xi|}))\xi,$$

$$u^t = \frac{1}{2}\int_0^t \left[\frac{\exp(\tau J_{\mu/|\xi|}) - I}{J_\mu}\xi, \exp(\tau J_{\mu/|\xi|})\frac{\xi}{|\xi|}\right] d\tau, \quad \mu^t = \mu.$$

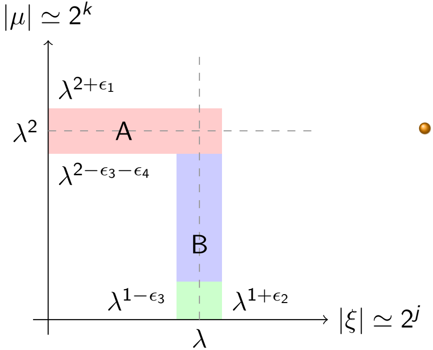
Note:

- the space projections $\boldsymbol{x}^t = (x^t, u^t)$ are sub-Riemannian geodesics
- for $G = \mathbb{H}_1$, these are spirals, whose rotational frequencies are of order $\omega := \frac{|\mu|}{|\xi|}$, with x-radius $1/\omega$.



2. Spectral versus frequency localisations

- localised wave propagator: $\mathbf{1}_{\overline{B}_2(0)}\chi_1(\sqrt{\mathcal{L}}/\lambda)\cos(\sqrt{\mathcal{L}})\delta_0$
- frequency decomposition $\delta_0 = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\chi_1(2^{-j}|D_x|)\chi_1(2^{-k}|D_u|)\delta_0$



- via cancellations (vanishing moments), reduce to regions
  - A where $|\xi| \lessapprox \lambda,\ |\mu| \approx \lambda^2$
  - B where $|\xi| \approx \lambda,\ |\mu| \lll \lambda^2$

3. In the "Anti-FIO" region A: one cannot apply FIO techniques!
But, since here $1/\omega = |\xi|/|\mu| \lesssim \lambda^{-1}$, a look at the spirals show that the convolution kernel should be supported in a thin cylinder, so that, by Cauchy-Schwarz, one can reduce to trivial L^2-estimates (in fact, we can use first-layer weighted L^2-estimates!).

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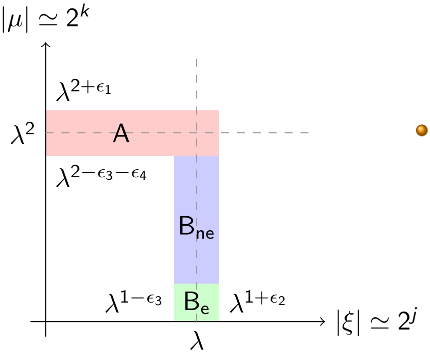

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4. **In the "FIO" region B:** further split into
   - the elliptic region B_e where $|\mu| \ll |\xi|$,
   - the non-elliptic region B_{ne} where $|\mu| \gtrsim |\xi|$
5. **In the elliptic region B_e,** the symbol of $\mathcal{L}$ satisfies $|\xi + J_\mu x/2|^2 \sim |\xi|^2 \sim |\boldsymbol{\xi}|^2$.
   - Thus, $\mathcal{L}$ is micro-locally elliptic and can be handled by classical FIO approach (cf. [M. & Müller & Nicolussi Golo '21] + [Seeger & Sogge & Stein '91])
6. **In the non-elliptic region B_{ne}:** reduced to proving (for $2^j \approx \lambda, \lambda \lesssim 2^k \lll \lambda^2$)

$$\|\mathbf{1}_{\overline{B}_4(0)} \cos(\sqrt{\mathcal{L}}) \chi_1(2^{-j}|D_x|) \chi_1(2^{-k}|D_u|) \delta_0\|_1 \lesssim_\varepsilon \lambda^{(d-1)/2}$$

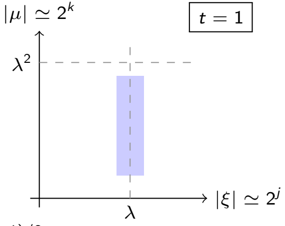
Nonelliptic region: parabolic scaling

- must prove

$$\|\mathbf{1}_{\overline{B}_4(0)} \cos(\sqrt{\mathcal{L}})\chi_1(2^{-j}|D_x|)\chi_1(2^{-k}|D_u|)\delta_0\|_1 \lessapprox \lambda^{(d-1)/2}$$

for

$$2^j \approx \lambda, \quad \lambda \lesssim 2^k \lll \lambda^2$$



- via parabolic scaling

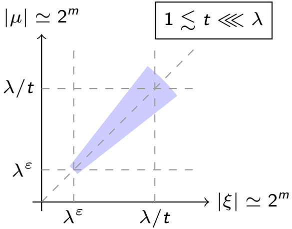
$$(\xi, \mu) \mapsto (2^\ell \xi, 2^{2\ell}\mu), \quad (x, u) \mapsto (2^{-\ell}x, 2^{-2\ell}u)$$

(with $\ell = k - j$, $m = 2j - k$) reduce to

$$\|\mathbf{1}_{\overline{B}(0,4\cdot 2^\ell)} \cos(2^\ell\sqrt{\mathcal{L}})\chi_1(2^{-m}|D_x|)\chi_1(2^{-m}|D_u|)\delta_0\|_1 \lessapprox \lambda^{(d-1)/2}$$

for

$$1 \lesssim 2^\ell \lll \lambda, \quad 1 \lll 2^m, \quad 2^{\ell+m} \lessapprox \lambda$$



- effectively we are reduced to the frequency region

$$|\xi| \simeq |\mu| \simeq 2^m,$$

but we need estimates for large time $t := 2^\ell$ (!)

Large-time FIO parametrix via complex phase
(à la [Laptev–Safarov–Vassiliev '94])

- represent

$$\cos(t\sqrt{\mathcal{L}})\chi_1(2^{-m}|D_x|)\chi_1(2^{-m}|D_u|)\delta_0 = \frac{Q_t^m + Q_{-t}^m}{2} + \text{l.o.terms},$$

$$Q_t^m(\boldsymbol{x}) = \int e^{i\phi(t,\boldsymbol{x},\boldsymbol{\xi})}\, q^{[m]}(t,\boldsymbol{\xi})\, \mathfrak{d}_\phi(t,\boldsymbol{x},\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

where the phase is defined in terms of the Hamiltonian flow for $\mathcal{H}$ as follows:

$$\phi(t,\boldsymbol{x},\boldsymbol{\xi}) := \langle \boldsymbol{x} - \boldsymbol{x}^t(\boldsymbol{\xi}), \boldsymbol{\xi}^t(\boldsymbol{\xi})\rangle + \frac{i}{4}\langle |J_\mu|(x - x^t(\boldsymbol{\xi})), x - x^t(\boldsymbol{\xi})\rangle$$

Remarks:

- the imaginary part of the phase is needed to ensure that the critical points of the phase are located exactly where $\boldsymbol{x} = \boldsymbol{x}^t(\boldsymbol{\xi})$ for some $\boldsymbol{\xi}$ ("no caustics")
- ϕ satisfies the eikonal equation $(\partial_t\phi)^2 = \sum_j (X_j\phi)^2$ only at the singular set $\boldsymbol{x} = \boldsymbol{x}^t$
- the associated density is $\mathfrak{d}_\phi(t,\boldsymbol{x},\boldsymbol{\xi}) = \sqrt{\partial_{\boldsymbol{x}}\partial_{\boldsymbol{\xi}}\phi(t,\boldsymbol{x},\boldsymbol{\xi})} \simeq \sqrt{1+|t|}$
- the amplitude is $q^{[m]}(t,\boldsymbol{\xi}) = \chi_1(2^{-m}|\xi|)\chi_1(2^{-m}|\mu|)+$ l.o.terms (problem: a priori, these seem to be larger in some frequency range than the leading term!)

How to control the large time behavior of $q^{[m]}$?

- the lower order terms of $q^{[m]}$ are obtained iteratively via transport equations
- **key problem:** show that no blow-up in time of phase/amplitude (and their derivatives) occurs !
- **key idea:** exploit a (weak) periodicity property!

Example $G = \mathbb{H}_1$: here $J_\mu = \mu J$, hence

$$\phi(t, \boldsymbol{x}, \boldsymbol{\xi}) := (\boldsymbol{x} - \boldsymbol{x}^t) \cdot \boldsymbol{\xi}^t + i\frac{|\mu|}{4}|x - x^t|^2$$

Hamiltonian flow: set $\theta := \frac{t|\mu|}{2|\xi|}$, $\bar{\mu} := \frac{\mu}{|\mu|}$, $\bar{\xi} := \frac{\xi}{|\xi|}$. Then

$$x^t = t\frac{\sin\theta}{\theta}(\cos\theta I + \sin\theta J_{\bar{\mu}})\bar{\xi},$$

$$\xi^t = \cos\theta(\cos\theta I + \sin\theta J_{\bar{\mu}})\xi,$$

$$u^t = \frac{t^2}{4\theta}\left[1 - \frac{\sin\theta}{\theta}\cos\theta\right]\bar{\mu}.$$

- potential blow-up in time:
  - $|\theta| \simeq |t| \simeq 2^\ell$, $\quad \theta$ is 0-homogeneous in $\boldsymbol{\xi}$
  - $\boldsymbol{\xi}$-differentiation of $\sin\theta$ and $\cos\theta$ pulls out factors t
  - u^t contains a term growing linearly in t
- further splitting into $\simeq 2^\ell$ pieces indexed by $k \in \mathbb{Z}, |k| \simeq |t| \simeq 2^\ell$:

$$\theta = k\pi + \tilde{\theta}, \qquad |\tilde{\theta}| \leq \pi$$

- change of variables: $\tilde{\theta} := \frac{\tilde{\mu}}{2|\xi|}$, i.e., from (ξ, μ) to $(\xi, \tilde{\mu})$:

$$\mu = \frac{2|\xi|k + \tilde{\mu}}{t} \qquad \text{(scaling and non-linear shearing!)}$$

- now $(\xi, \tilde{\mu})$-derivatives of $\sin(k\pi + \tilde{\theta})$ and $\cos(k\pi + \tilde{\theta})$ are no longer problematic!
- for each k, we then have $|\xi| \simeq 2^m$, $|\tilde{\mu}| \lesssim 2^m$, and now we can apply a modified Seeger-Sogge-Stein approach at scale $\tilde{\lambda} = 2^m$ to each of these pieces!
- the "Gaussian decay" from $\mathrm{Im}\,\phi$ gives a strong spatial localization in x !

Final remarks

- For Métivier groups, the weak periodicity property is encoded in the fact that the exponential mapping $\exp : \mathfrak{so}(\mathbb{R}, d_1) \to SO(\mathbb{R}, d_1)$ maps into a compact Lie group! In particular, $e^{sJ_\mu} \in SO(\mathbb{R}, d_1)$
- We can also already treat (work in progress) the perhaps simplest model of a non-Métivier group, where $J_\mu = \begin{pmatrix} 0 & -\mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a kernel, namely $G = \mathbb{H}_1 \times \mathbb{R}$.

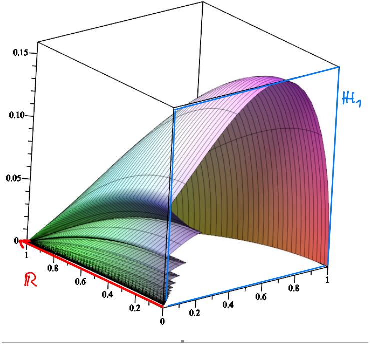


Figure: 3d-section of the singular support on $\mathbb{H}_1 \times \mathbb{R}$

References

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