# ENDPOINT SPARSE BOUNDS

MADISON LECTURES IN HARMONIC ANALYSIS

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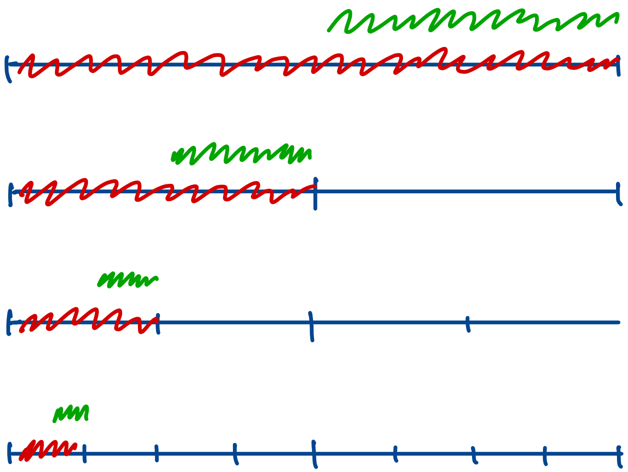
# Sparse domination

$$|\langle T f_1, f_2 \rangle| \lesssim \sum_{Q \in \mathcal{S}} |Q| \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q'}$$

$(p, q')$-sparse

$$\langle f \rangle_{Q,p} := \left( \frac{1}{|Q|} \int_Q |f|^p \right)^{1/p}$$



$\mathcal{S}$ is a $\gamma$- sparse family of dyadic cubes

$\exists\, E_Q \subseteq Q$ with

- $|E_Q| \geq \gamma |Q|$
- $\{E_Q\}_{Q \in \mathcal{S}}$ pairwise disjoint

- This concept gained relevance after Lerner's proof of the $A_2$- theorem for CZO.

- Lerner, Conde-Alonso – Rey, Lacey, Bérnicot – Frey - Petermichl, Culiuc – Di Plinio-Ou and many more.

# Modern measure of size

Assume **bilinear** :

$$|\langle T f_1, f_2 \rangle| \lesssim \sup_{\mathcal{S} : \gamma\text{-sparse}} \sum_{Q \in \mathcal{S}} \frac{|E_Q|}{\gamma} \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q'}$$

$$\lesssim \frac{1}{\gamma} \sup_{\mathcal{S} : \gamma\text{-sparse}} \sum_{Q \in \mathcal{S}} \int_{E_Q} M_p f_1(x) \cdot M_{q'} f_2(x) \, dx$$

$$\lesssim \frac{1}{\gamma} \sup_{\mathcal{S} : \gamma\text{-sparse}} \| M_p f_1 \|_r \, \| M_{q'} f_2 \|_{r'}$$

$$\lesssim \frac{1}{\gamma} \| f_1 \|_r \, \| f_2 \|_{r'} \qquad \text{if} \quad p < r < q$$

# Modern measure of size

Assume **bilinear** :

$$|\langle Tf_1, f_2\rangle| \lesssim \sup_{\mathcal{S}:\ \gamma\text{-sparse}} \sum_{Q\in\mathcal{S}} |Q| \langle f_1\rangle_{Q,p} \langle f_2\rangle_{Q,q'}$$

Then $\|Tf\|_r \lesssim \|f\|_r$ if $p<r<q$

$$\|Tf\|_{L^{p,\infty}} \lesssim \|f\|_{L^p}$$

$$\|Tf\|_{L^q} \lesssim \|f\|_{L^{q,1}}$$

# Off-diagonal necessary condition

$$Tf(x) = \int K(x,y)\, f(y)\, dy$$

localise $\quad K^{loc}(x,y) := K(x,y)\, \chi(x-y)\,, \qquad \operatorname{supp} \chi \subseteq \{ |z| \approx 1 \}$

$$T^{loc} f(x) := \int K^{loc}(x,y)\, f(y)\, dy$$

If $\quad |\langle T f_1, f_2 \rangle| \lesssim \sum_{Q \in S} |Q| \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q'}$

$$\Rightarrow \quad |\langle T^{loc} f_1, f_2 \rangle| \lesssim \sum_{Q \in S} |Q| \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q'}$$

$$\Rightarrow \quad \| T^{loc} \|_{L^p \to L^q} < \infty . \qquad (*)$$

$\operatorname{dist}(T^{loc} f_1, f_2) \approx 1 \;\rightarrow$ only one $|Q| \sim 1$ matters

# Modern measure of size : weighted estimates

**Proposition** ( Bérnicot - Frey - Petermichl )

Let $p, q \in [1, \infty]$, $p \leq q$. Assume $T$ is a sublinear operator satisfying a $(p, q')$-sparse bound. Then for any $p < r < q$ and $w \in A_{r/p} \cap RH_{(q/r)'}$

$$\| Tf \|_{L^r(w)} \lesssim C(w) \, \| f \|_{L^r(w)}$$

(quantitative $C(w)$ )

**Proposition** ( Frey - Nieraeth )

Let $p, q \in (1, \infty)$ with $p < q$. Assume $T$ is a sublinear operator satisfying a $(p, q')$-sparse bound. Then for $w \in A_1 \cap RH_{(q/p)'}$

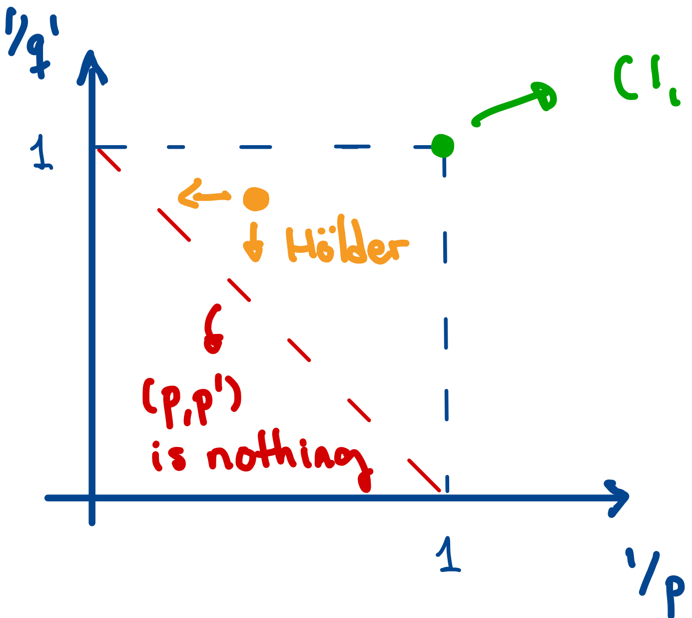
$$\| Tf \|_{L^{p,\infty}(w)} \lesssim C(w) \, \| f \|_{L^p}$$

# Modern measure of size : summary

Assume **bilinear** :

$$|\langle T f_1, f_2 \rangle| \lesssim \sup_{\mathcal{S} : \gamma\text{-sparse}} \sum_{Q \in \mathcal{S}} |Q| \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q'}$$

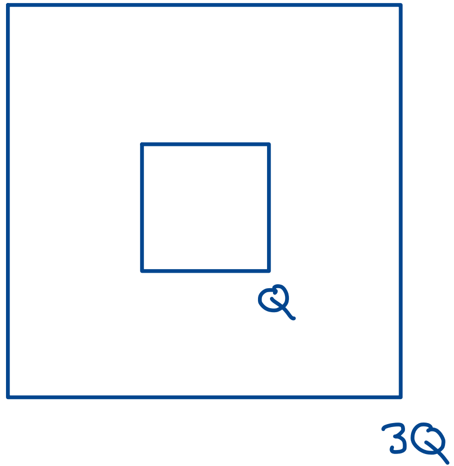
Lebesgue space bounds $\Longrightarrow$ Sparse bounds $\Longrightarrow$ Weighted $A_p$ inequalities




(1,1) is the best

region determined by $L^p \to L^q$ for local single scale pieces

# Single spatial scale sparse domination




① If $\left.\begin{array}{l} \operatorname{supp} f \subseteq Q \\ \ell(Q) \leq 2^j \end{array}\right\} \Longrightarrow \operatorname{supp} T_j f \subseteq 3Q.$

② $\| T_j f \|_q \lesssim 2^{-jd\left(\frac{1}{q}-\frac{1}{p}\right)} \| f \|_p$

$$\langle T_j(f_1 \mathbb{1}_Q), f_2 \rangle \underset{①}{=} \langle T_j(f_1 \mathbb{1}_Q), f_2 \mathbb{1}_{3Q} \rangle$$

$$\underset{\text{Hölder}}{\leq} \| T_j(f_1 \mathbb{1}_Q) \|_q \, \| f_2 \mathbb{1}_{3Q} \|_{q'}$$

$$\underset{②}{\lesssim} 2^{-jd\left(\frac{1}{q}-\frac{1}{p}\right)} \| f_1 \mathbb{1}_Q \|_p \, \| f_2 \mathbb{1}_{3Q} \|_{q'}$$

$$\approx |Q| \, \langle f_1 \rangle_{Q,p} \, \langle f_2 \rangle_{3Q,q'}$$

Breaking $f_1 = \sum_{Q \in \mathcal{Q}_j} f_1 \mathbb{1}_Q$ we get $\langle T_j f_1, f_2 \rangle \lesssim \sum_{Q \in \mathcal{Q}_j} |Q| \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{3Q,q'}$

↳ disjoint cubes of $\ell(Q) = 2^j$

# The easiest spatial multi-scale case

**Example** $\qquad M_{\mathcal{D}} f(x) = \sup_{\substack{Q \in \mathcal{D} \\ Q \ni x}} \langle f \rangle_{Q,1}$

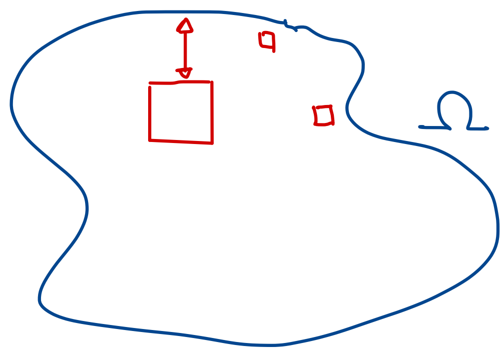
Assume $\quad \operatorname{supp}(f) \subseteq Q_0 \in \mathcal{D}$

Let $\quad \Omega = \left\{ x \in \mathbb{R}^d : M_{\mathcal{D}} f(x) > \dfrac{\| M_{\mathcal{D}} \|_{L^1 \to L^{1,\infty}}}{1-\gamma} \langle f \rangle_{Q_0,1} \right\}$

## Whitney decomposition

$\Omega = \bigcup_{W \in \mathcal{W}} W$ where

- $W$ dyadic
- $W$ disjoint
- $W$ maximal wrt inclusion




**Break** $\qquad M_{\mathcal{D}} f(x) \leq \underbrace{M_{\mathcal{D}} f(x) \mathbb{1}_{\mathbb{R}^d \setminus \Omega}}_{\lesssim \langle f \rangle_{Q_0,1}} + \sum_{W \in \mathcal{W}} \underbrace{M_{\mathcal{D}} (f \mathbb{1}_W)(x) \mathbb{1}_W(x)}_{\text{iterate}}$

(averages involving parents are always smaller).

Collection is sparse since $|\Omega| \leq (1-\gamma)|Q_0|$

# The sparse algorithm

Combining these two elementary approaches is the key idea behind most sparse results.

- Calderón – Zygmund : Lerner, Lerner - Nazarov, Conde - Alonso — Rey, Lacey ...
- Beyond Calderón - Zygmund : Lacey, ...

Example : $A_{2^j}$ your favourite average at scale $2^j$

$$M_{lac} f(x) := \sup_{j \in \mathbb{Z}} |A_{2^j} f(x)|$$

sparse bounds ?

# The sparse algorithm

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- Calderón – Zygmund : Lerner, Lerner - Nazarov, Conde - Alonso — Rey, Lacey ...
- Beyond Calderón - Zygmund : Lacey, ...

Key observation : if $f_k$ has $\operatorname{supp}(\widehat{f_k}) \subseteq \{ |\xi| \sim 2^k \}$

$$\boxed{\| A_{2^j} f_k \|_q \lesssim 2^{-jd\left(\frac{1}{q}-\frac{1}{p}\right)} \, 2^{-k\varepsilon} \, \| f \|_p}$$

$$\Rightarrow \quad \boxed{\| A_{2^j} f - A_{2^j}[f(\cdot - h)] \|_q \lesssim 2^{-jd\left(\frac{1}{q}-\frac{1}{p}\right)} \, |h|^{\varepsilon} \, \| f \|_p .}$$

**Thm** (B. - Roos - Seeger ; abstract spatial multi-scale sparse domination)

$B_1, B_2$ Banach spaces, $\{T_j\}_{j\in\mathbb{Z}}$, $T_j : S_{B_1} \to S_{B_2}$, $1 < p \le q < \infty$

- support : $T_j$ local at scale $2^j$
- weak-type $(p,p)$ : $\sup_{N_1 \le N_2} \| \sum_{j=N_1}^{N_2} T_j \|_{L^p_{B_1} \to L^{p,\infty}_{B_2}} \le A(p)$
- restricted strong-type $(q,q)$ : $\sup_{N_1 \le N_2} \| \sum_{j=N_1}^{N_2} T_j \|_{L^{q,1}_{B_1} \to L^q_{B_2}} \le A(q)$
- single-scale $(p,q)$ : $\sup_{j\in\mathbb{Z}} \| \mathrm{Dil}_{2^j} T_j \|_{L^p_{B_1} \to L^q_{B_2}} \le A_0(p,q)$
- single-scale $\varepsilon$-regularity : $\boxed{\sup_{j\in\mathbb{Z}} \| (\mathrm{Dil}_{2^j} T_j^{*}) \circ \Delta_h \|_{\substack{L^p_{B_1} \to L^q_{B_2} \\ L^{q'}_{B_2^*} \to L^{p'}_{B_1^*}}} \le B |h|^{\varepsilon} \text{ for all } |h| \le 1}$

where $\Delta_h f(x) = f(x+h) - f(x)$

Then, for any $N_1 \le N_2$,

$$| \langle \sum_{j=N_1}^{N_2} T_j f_1 , f_2 \rangle | \lesssim_{p,q,\varepsilon,d,\gamma} C \sup_{\mathcal{S} : \gamma\text{-sparse}} \sum_{Q\in\mathcal{S}} |Q| \langle f_1 \rangle_{Q,p,B_1} \langle f_2 \rangle_{Q,q',B_2^*}$$

# $\varepsilon$ - reg conditions

- Calderón - Zygmund / Whitney decomposition
- Multi-scale boundedness hypotheses for $g_1, g_2$

$$\boxed{\left| \sum_{j=N_1}^{N_2} \sum_{\substack{Q, Q' \\ L(Q) < j \\ L(Q') < j}} \langle T_j b_{1,Q}, b_{2,Q'} \rangle \right| \lesssim |Q_0| \langle f_1 \rangle_{p, Q_0} \langle f_2 \rangle_{q', Q_0}}$$

- $\varepsilon$ - reg allows sumability on the scale of $Q, Q'$.
- but beyond C-Z operators, to use these conditions typically means missing the endpoint.

$$\boxed{\| A_{2^j} f_k \|_q \lesssim 2^{-jd\left(\frac{1}{q} - \frac{1}{p}\right)} 2^{-k\varepsilon} \| f \|_p}$$

# Endpoint sparse domination

Very little is known beyond standard C-Z theory

- Rough-singular integrals, Bochner-Riesz at critical index $\frac{(n-1)}{2}$

  Conde-Alonso, Culiuc, Di-Plinio, Ou (refining Christ, Seeger)

- A portion of a boundary segment for Bochner-Riesz in $\mathbb{R}^2$ for $0 < \delta < 1/2$.

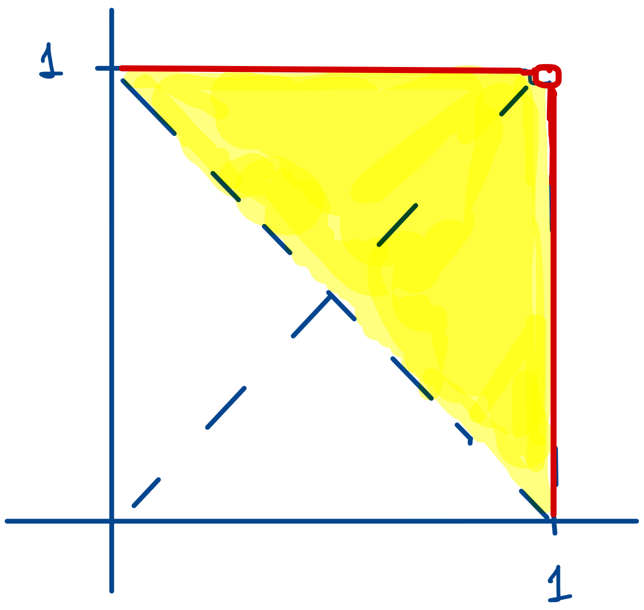
  Kesler-Lacey (refining Seeger)

# Sparse bounds for Bochner-Riesz

## Boundary (endpoint) bounds ?

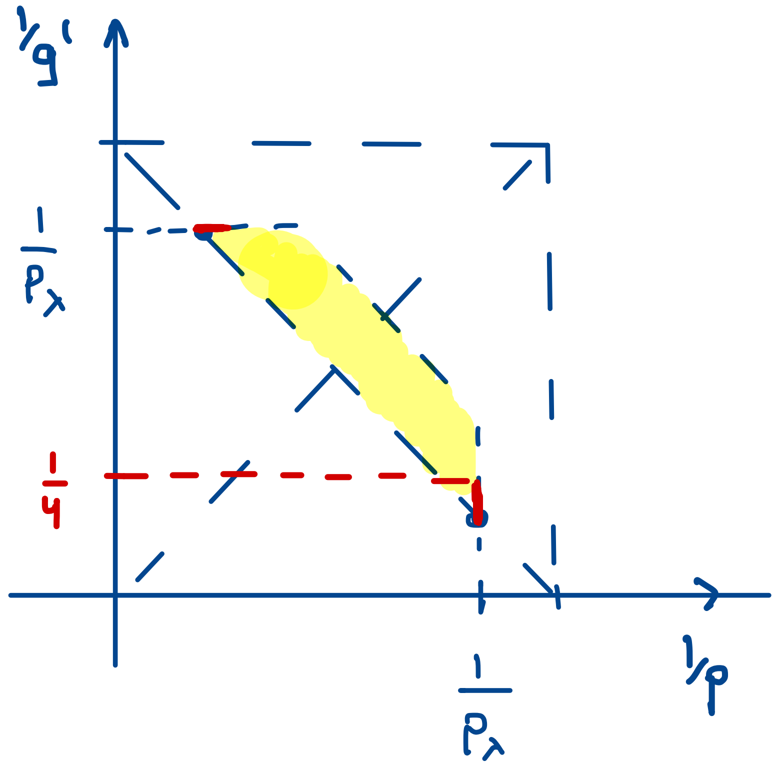
$$\lambda = \frac{d-1}{2}$$

Conde-Alonso, Culiuc, Di Plinio, Ou




$0 < \lambda < \frac{1}{2}$, $d = 2$, $q' > 4$

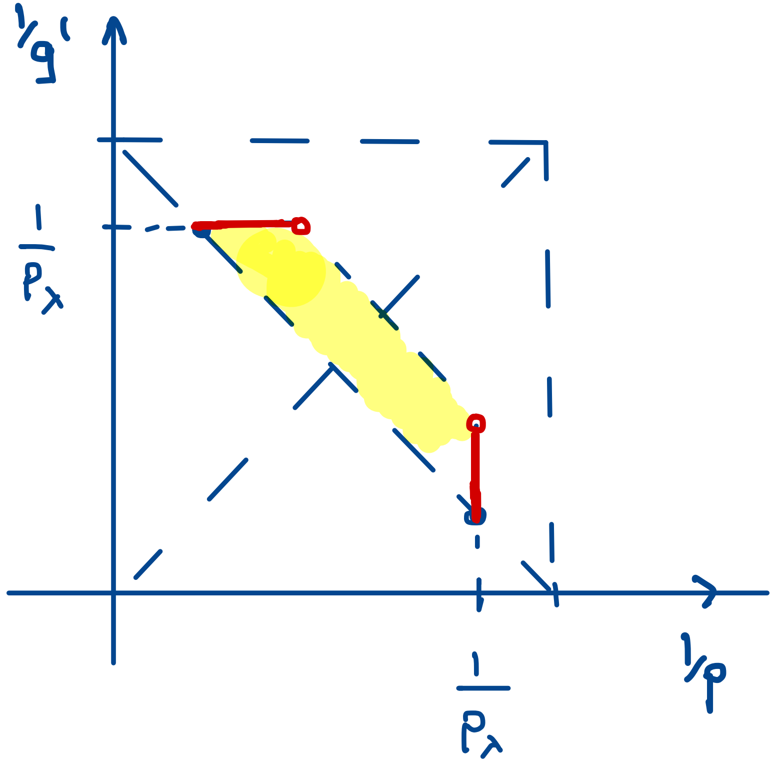
Kesler-Lacey



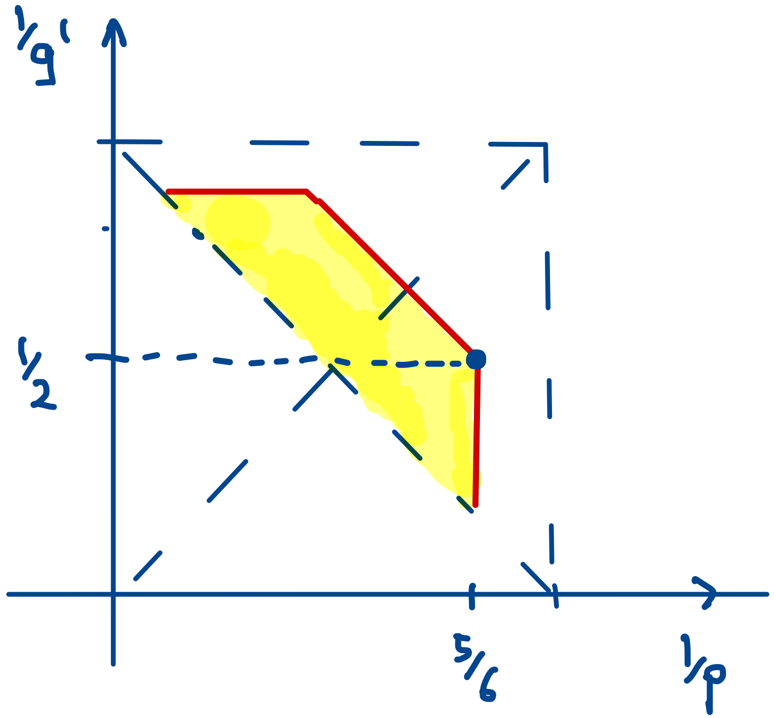
# New endpoint results ( $d = 2$ )

**Theorem** ( B. – Roos – Seeger )

$0 < \lambda < 1/2$




$\lambda = 1/6$




Similar results in higher dimensions conditional to knowledge on BR

## A very model case: oscillatory Fourier multipliers

For $a>0$, $a\neq 1$, $b\geqslant 0$,

$$\boxed{m_{a,b}(\xi) = \frac{e^{i|\xi|^a}}{|\xi|^b}\chi_\infty(\xi)}$$

$$\boxed{\text{These are } L^p\text{- bounded} \iff b \geqslant ad\left|\frac{1}{p}-\frac{1}{2}\right|, \quad 1<p<\infty}$$

Hirschman, Wainger, Fefferman, Fefferman-Stein, Miyachi ...

$\boxed{b > ad\left|\frac{1}{p}-\frac{1}{2}\right|}$ enough to study one single freq.

$$\| m_{a,b}(D) f_k \|_p \lesssim 2^{-k\varepsilon} \|f\|_p, \quad |\xi| \approx 2^k$$

$$K^k_{a,b}(x) = \int e^{i|\xi|^a - i\langle x,\xi\rangle} |\xi|^{-b} \varphi(2^{-k}\xi)\, d\xi$$

satisfies $$|K^k_{a,b}(x)| \lesssim \begin{cases} 2^{-kb}\, 2^{-kad/2}\, 2^{kd} & \text{if } \boxed{|x| \approx 2^{k(a-1)}} \\ \text{rapid decay} & \text{otherwise} \end{cases}$$

Interpolating with $L^2$

$$\boxed{\| m_{a,b}(D) f_k \|_{L^{p'}} \lesssim 2^{-kb + kd(2-a)(\frac{1}{p}-\frac{1}{2})} \|f\|_p}$$

$\boxed{b > ad\left(\frac{1}{p}-\frac{1}{2}\right)}$ becomes $2^{-k(a-1)d\left(\frac{1}{p'}-\frac{1}{p}\right) - k\varepsilon}$

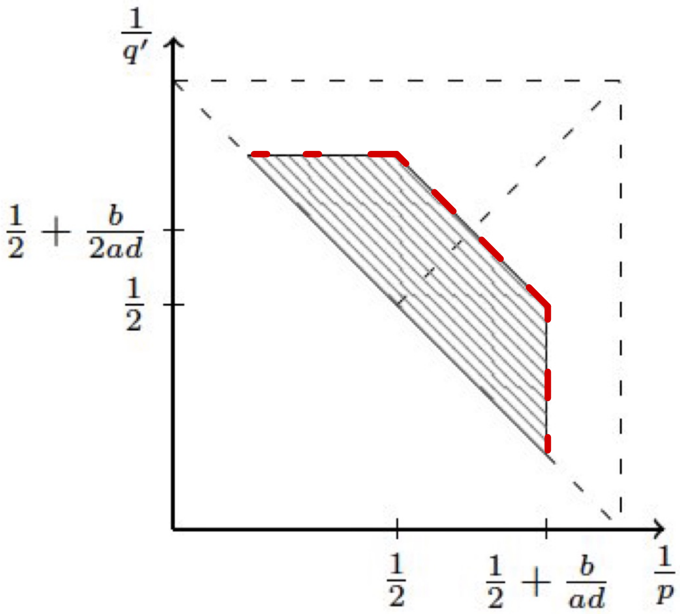
$\boxed{(p, q') - \text{sparse bounds}}$ via single-scale analysis, no need of C-Z algorithm

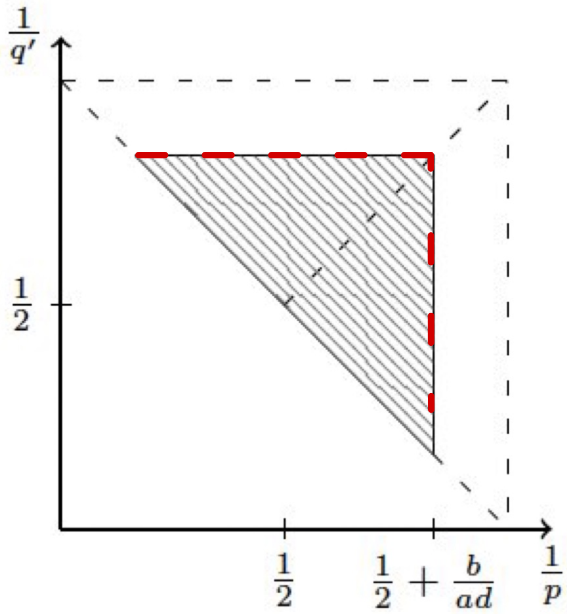
# A model case : oscillatory Fourier multipliers

**Thm** (B. - Cladek)

For $0 < b < \frac{da}{2}$, sparse bounds hold in the **interior** of the regions

$a \neq 1$




$m \in \mathrm{Miy}(a,b)$ $\qquad\qquad$ $m_{a,b}$

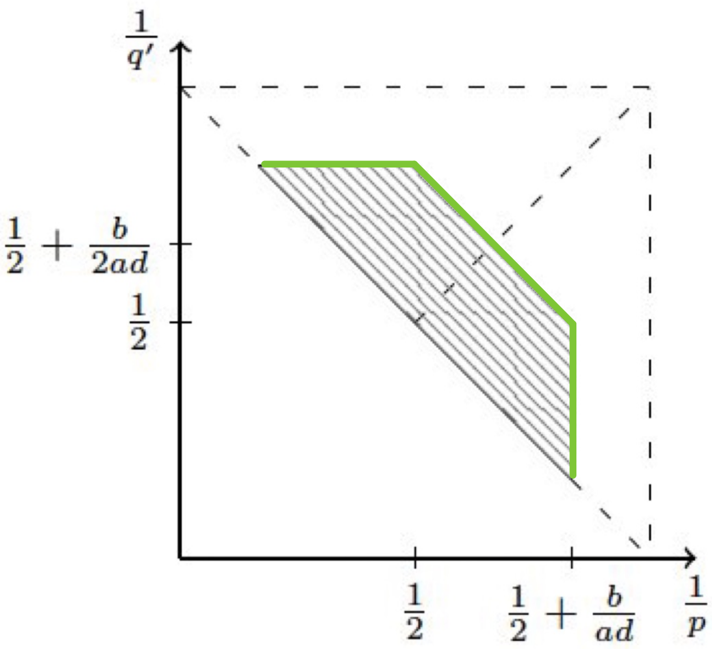
$$|D^\gamma m(\xi)| \lesssim |\xi|^{-b-|\gamma|(1-a)}, \qquad |\xi| \geq 1$$

# An endpoint result

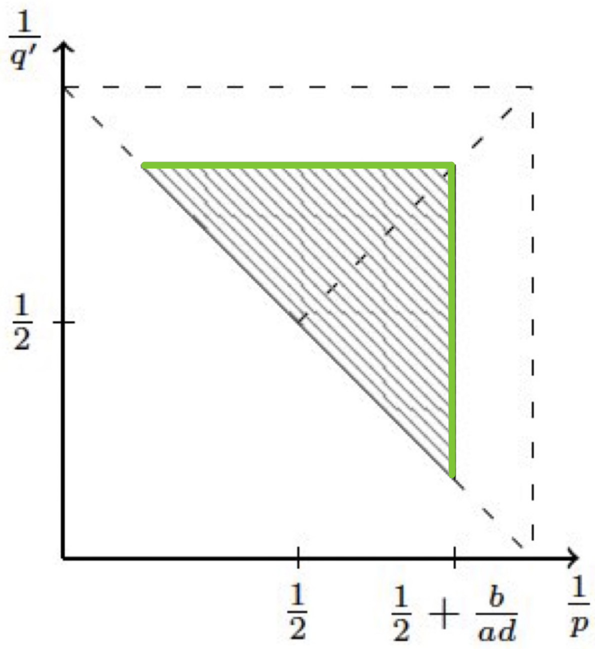
**Thm** (B. - Roos - Seeger)

For $0 < b < \frac{da}{2}$, sparse bounds hold in the **closure** of the regions

$a \neq 1$




$m \in \mathrm{Mi}_{\Upsilon}(a, b)$




$m_{a,b}$

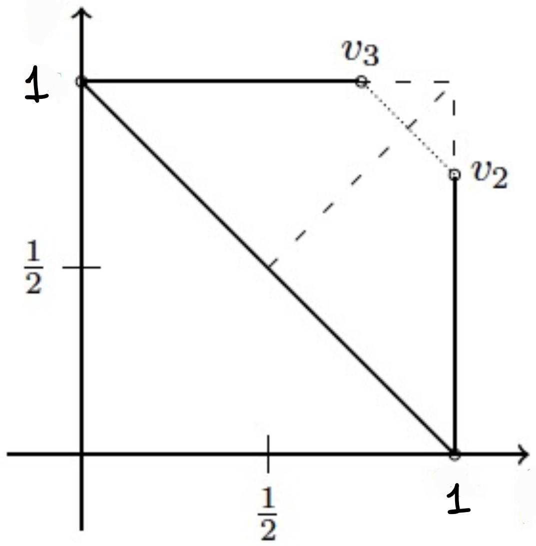
# It's a $p>1$ result

$m(D)$ bounded on $L^p$ for one Miy $(a,b)$ $\iff$ $b \geq ad \left|\frac{1}{p}-\frac{1}{2}\right|$

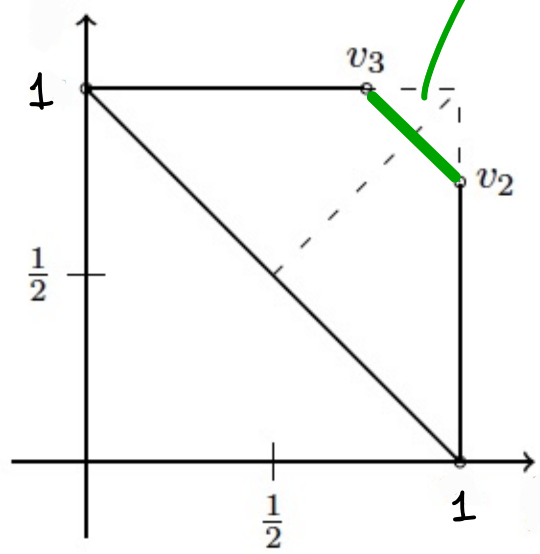
If $0 < b < ad/2$ we are in the regime $1 < p < \infty$.

For $\frac{ad}{2} \leq b \leq ad$

[BRS]




single-scale analysis




multi-frequency analysis

# Relationship between the "endpoint" and "multi-scale" result

- Non-endpoint bounds for $m_{a,b}(\xi) = \frac{e^{i|\xi|^a}}{|\xi|^b}\chi_\infty(\xi)$ are

  endpoint bounds frequency multi-scale [BRS]

frequency-single scale

$\not\Leftrightarrow$

spatial single scale

- General non-endpoint spatial multi-scale result can be applied to frequency single scale

$$\mathcal{M}_{a,b} f(x) := \sup_{j \in \mathbb{Z}} | m_{a,b}(2^j D) f(x) |$$

$$\boxed{\text{freq. multi-scale ?}}$$

$$\mathcal{S}_{a,b} f(x) := \left( \sum_{j \in \mathbb{Z}} | m_{a,b}(2^j D) f(x) |^2 \right)^{1/2}$$

$$\mathcal{H}_{a,b} f(x) := \sum_{j \in \mathbb{Z}} \varepsilon_j m_{a,b}(2^j D) f(x) \quad ; \quad |\varepsilon_j| \le 1$$

to obtain non-endpoint sparse results ; i.e., $\boxed{b > ad\left(\frac{1}{p} - \frac{1}{2}\right), \quad 1 < p \le 2}$

# Endpoint Lebesgue space bounds for multi-scale sums

Whilst it is classical that $m_{a,b}$ , $b = ad\left(\frac{1}{p}-\frac{1}{2}\right)$, $1<p\le 2$ is $L^p \to L^p$

and $\mathcal{M}_{a,b}$, $\mathcal{S}_{a,b}$, $\mathcal{H}_{a,b}$ $\quad b > ad\left|\frac{1}{p}-\frac{1}{2}\right|$, $1<p\le 2$ is $L^p \to L^p$

it was open the endpoint case $b = ad\left(\frac{1}{p}-\frac{1}{2}\right)$, $1<p<2$

**Thm** ( in progress )

If $b = ad\left(\frac{1}{p}-\frac{1}{2}\right)$, $1<p<2$, then $\mathcal{M}_{a,b}, \mathcal{S}_{a,b}, \mathcal{H}_{a,b} : L^p \to L^{p,\infty}$

Techniques for $\mathcal{H}_{a,b}$ different to those for $\mathcal{M}_{a,b}$, $\mathcal{S}_{a,b}$ .

# Endpoint sparse bound for multi-scale max. fn.

**Thm** ( in progress ) , $1<p<2$, $\varepsilon>0$.

If $b = ad\left(\frac{1}{p}-\frac{1}{2}\right)$, then $\mathcal{M}_{a,b}$ is $(p, p-\varepsilon)$ - sparse.

- First example of frequency and spatial multi-scale operator for which we can avoid using $\varepsilon$ - reg hypotheses.

- "Sharper" endpoint sparse result than that for $m_{a,b}(D)$

  $(p, q')$ sparse bound for $T \Rightarrow T : L^p \to L^{p,\infty}$.

# Proof idea for $(p,p)$-sparse bound for $m_{a,b}(D)$

The $L^p$-bounds for $b = ad\left(\frac{1}{p}-\frac{1}{2}\right)$ are obtained via BMO/Hardy space

**Idea**: since $1<p<2$, incorporate Hardy-space techniques into the sparse algorithm.

This means that instead of the usual

$$\Omega_1 := \{ x \in S_0 : M_{HL}|f_1|^p(x) \gtrsim \langle f_1\rangle^p_{S_0,p} \}$$

we should consider

$$\tilde{\Omega}_1 := \{ x \in S_0 : M_{HL}(F^p_{1,p})(x) \gtrsim \langle f_1\rangle^p_{S_0,p} \}$$

where $F^p_{1,p}$ is build up of $p$-atoms related to $f_1$.

# Atomic decomposition on $L^p$, $1<p\le 2$ (Chang-Fefferman)

Fix $S_0$ dyadic cube, $\operatorname{supp}(f) \subseteq S_0$

$$f(x) = \mathbb{E}_{1-L(S_0)} f(x) + \sum_{k \geq -L(S_0)} \mathbb{D}_k f(x) \quad ; \quad \mathbb{D}_k := \mathbb{E}_{k+1} - \mathbb{E}_k$$

$$\mathbb{G}_{S_0} f(x) = \left| \mathbb{E}_{1-L(S_0)} f(x) \right| + \left( \sum_{k \geq -L(S_0)} \sup_{y \in Q_k(x)} \left| \mathbb{D}_k f(y) \right|^2 \right)^{\frac{1}{2}}$$

$$\boxed{\Omega_\mu \equiv \Omega_\mu[f] := \{ x \in S_0 : \mathbb{G}_{S_0} f(x) > 2^\mu \}, \qquad \mu \in \mathbb{Z}}$$

$$|\Omega_\mu| \le 2^{-\mu p} \| \mathbb{G}_{S_0} f \|_p^p \lesssim 2^{-\mu p} \| f \|_{L^p(S_0)}^p$$

$$\sum_{\mu \in \mathbb{Z}} 2^{\mu p} |\Omega_\mu| \lesssim \| \mathbb{G}_{S_0} f \|_p^p \lesssim \| f \|_{L^p(S_0)}^p$$

- $\mathcal{W}_\mu$ : Whitney cubes of $\Omega_\mu$

- $\mathcal{R}_{W,\mu} := \{ R \in \mathcal{R}_\mu : R \subseteq W \}$

$\mathcal{R}_\mu \equiv \mathcal{R}_\mu[f]$ dyadic cubes $R \subsetneq S_0$ s.t. $\begin{cases} |R \cap \Omega_\mu| > |R|/2 \\ |R \cap \Omega_{\mu+1}| \leq |R|/2 \end{cases}$

- $\mathcal{W}_\mu$ : Whitney cubes of $\Omega_\mu$

- $\mathcal{R}_{W,\mu} := \{ R \in \mathcal{R}_\mu : R \subseteq W \}$

- $e_R \equiv e_R[f] := (\mathbb{D}_k f)\, \mathbb{1}_R = \mathbb{D}_k (f \mathbb{1}_R)$ if $L(R) = -k$

$\int e_R = 0$ subatoms / building blocks

$$\sum_{R \in \mathcal{R}_\mu} \| e_R \|_2^2 \lesssim 2^{2\mu} |\Omega_\mu|$$

and summing over $\mu \in \mathbb{Z}$ one gets $\| f \|_{L^2(S_0)}$

- $\mathcal{W}_\mu$ : Whitney cubes of $\Omega_\mu$

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- $e_R \equiv e_R[f] := (\mathbb{D}_k f)\, \mathbb{1}_R = \mathbb{D}_k (f\, \mathbb{1}_R)$ if $L(R) = -k$

- $a_{W,\mu} \equiv a_{W,\mu}[f] := \sum_{R \in \mathcal{R}_{W,\mu}} e_R[f]$

atoms ; $\quad f = \mathbb{E}_{1-L(S_0)} f + \sum_{\mu \in \mathbb{Z}} \sum_{W \in \mathcal{W}_\mu} a_{W,\mu}$

- $\mathcal{W}_\mu$ : Whitney cubes of $\Omega_\mu$

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- $a_{W,\mu} \equiv a_{W,\mu}[f] := \sum_{R \in \mathcal{R}_{W,\mu}} e_R[f]$

- $\gamma_{W,\mu} \equiv \gamma_{W,\mu}[f] := \left( \frac{1}{|W|} \sum_{R \in \mathcal{R}_{W,\mu}} \| e_R[f] \|_2^2 \right)^{1/2}$

$$= \frac{\| a_{W,\mu} \|_2}{|W|^{1/2}}$$

- $\mathcal{W}_\mu$ : Whitney cubes of $\Omega_\mu$

- $\mathcal{R}_{W,\mu} := \{ R \in \mathcal{R}_\mu : R \subseteq W \}$

- $e_R \equiv e_R[f] := (\mathbb{D}_k f)\, \mathbb{1}_R = \mathbb{D}_k (f \mathbb{1}_R)$ if $L(R) = -k$

- $a_{W,\mu} \equiv a_{W,\mu}[f] := \sum_{R \in \mathcal{R}_{W,\mu}} e_R[f]$

- $\gamma_{W,\mu} \equiv \gamma_{W,\mu}[f] := \left( \frac{1}{|W|} \sum_{R \in \mathcal{R}_{W,\mu}} \| e_R[f] \|_2^2 \right)^{1/2}$

$$\boxed{\| a_{W,\mu} \|_p \lesssim |W|^{1/p} \gamma_{W,\mu}}$$

$\ell^p$ in $W \in \mathcal{W}_\mu$ gives $2^\mu |\Omega_\mu|^{1/p}$

- $\mathcal{W}_\mu$ : Whitney cubes of $\Omega_\mu$

- $\mathcal{R}^K_{W,\mu} := \{ R \in \mathcal{R}_\mu : R \subseteq W \, , \, L(R) = -k \}$

- $e_R \equiv e_R[f] := (\mathbb{D}_k f)\, \mathbb{1}_R = \mathbb{D}_k (f \mathbb{1}_R)$ if $L(R) = -k$

- $a^K_{W,\mu} \equiv a^K_{W,\mu}[f] := \sum_{R \in \mathcal{R}^K_{W,\mu}} e_R[f]$

- $\gamma^K_{W,\mu} \equiv \gamma^K_{W,\mu}[f] := \left( \frac{1}{|W|} \sum_{R \in \mathcal{R}^K_{W,\mu}} \| e_R[f] \|_2^2 \right)^{1/2}$

$$\boxed{\| a^K_{W,\mu} \|_p \leq |W|^{1/p} \gamma^K_{W,\mu}}$$

For fixed $K$, there is spatial orthogonality if $1 < p \leq 2$

$$\boxed{\Big\| \sum_{\mu} \sum_{\substack{W \in \mathcal{W}_\mu \\ L(W) = -K+n}} a^K_{W,\mu} \Big\|_p \lesssim \Big( \sum_{\mu \in \mathbb{Z}} \sum_{\substack{W \in \mathcal{W}_\mu \\ L(W) = -K+n}} |W| \, (\gamma^K_{W,\mu}[f])^p \Big)^{1/p}}$$

can be summed in $\ell^2$ in $K$ to recover $\|f\|_{L^p(S_0)}$

(Littlewood-Paley theory)

Fix $L(W) = -K+n$ : can be summed in $\ell^p$ in $K$ to get $\|f\|_{L^p(S_0)}$

Need to sum in $n$ : $L^p \to L^r$, $r > p$, L.H.S. for $m_{a,b}$

$2^{-nd(\frac{1}{p}-\frac{1}{r})} \, 2^{Kd(\frac{1}{p}-\frac{1}{r})}$, R.H.S. $r = p' > 2$

**However**: we aim for a sparse $(p,p)$ bound

$$\sum_k \| f_1^k \|_p \, \| f_2^k \|_p \qquad \text{with} \quad 1 < p < 2$$

$$\lesssim \Big( \sum_k \| f_1^k \|_p^2 \Big)^{\frac{1}{2}} \Big( \sum_k \| f_2^k \|_p^2 \Big)^{\frac{1}{2}}$$

$$\lesssim \Big\| \Big( \sum | f_1^k |^2 \Big)^{\frac{1}{2}} \Big\|_p \, \Big\| \Big( \sum_k | f_2^k |^2 \Big)^{\frac{1}{2}} \Big\|_p$$

$1 \leq p < 2$

$$\lesssim \| f_1 \|_p \, \| f_2 \|_p$$

For those multipliers, no need to go down to atoms

To incorporate this technique into sparse, consider

$$\boxed{F_{1,p} := \Big( \sum_{\mu \in \mathbb{Z}} \sum_{w \in \mathcal{W}_\mu} (\gamma_{w,\mu}[f])^p \, \mathbb{1}_w(x) \Big)^{1/p}}$$

whose $L^p$ norm coincides with R.H.S. previous slides

Form C-Z cubes from

$$\Omega_1 = \{ x : M(F_{1,p}^p)(x) \gtrsim \langle f_1 \rangle_{S_0,p}^p \}$$

$$\Omega_2 = \{ x : M(f_2^p)(x) \gtrsim \langle f_2 \rangle_{3S_0,p}^p \}$$

## For $\mathcal{M}_{a,b}$

$$\mathcal{M}_{a,b} f(x) := \sup_{j \in \mathbb{Z}} | m_{a,b}(2^j D) f(x) |$$

If $\operatorname{supp} \widehat{f_k} \subseteq \{ |\xi| \sim 2^k \}$, we don't have $|x|$ localisation.

$$\sup_{j \in \mathbb{Z}} \leq \Big( \sum_{j \in \mathbb{Z}} | \quad |^{p'} \Big)^{1/p'}$$

Need to treat sum in $k$

sum in $j$

Atomic decomposition helps for the $j$-summability,

if we aim for a $(p, p-\varepsilon)$-sparse bound.


THANKS