When the Schrödinger equation meets number theory

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Madison, May 2024

The (free) Schrödinger equation on the manifold $\mathcal{M}$.
Here x =space, t =time, $u : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$

$$\begin{cases} 2\pi i u_t(x,t) = \Delta_x u(x,t), & (x,t) \in \mathcal{M} \times \mathbb{R} \\ u(x,0) = \phi(x), & x \in \mathcal{M} \end{cases} .$$

Most common examples are $\mathcal{M} = \mathbb{R}^{n-1}$ (Euclidean) and $\mathcal{M} = \mathbb{T}^{n-1}$ (periodic).

Questions/problems that are typically asked/considered about this equation include: is there a local/global in time solution, prove Strichartz estimates, does the solution u converge to the initial data ϕ as t approaches 0?

A Strichartz estimate seeks to dominate the norm $\|u\|_{L^p_x L^q_t}$ using some norm (typically L^2) of the initial data ϕ.

The resolution of the case $p = q$ when $X = \mathbb{R}^{n-1}$ (Strichartz 1977) has been a defining moment in *Fourier Restriction* and *PDEs*. This is equivalent to the (so-called *restriction*) estimate

$$\|\widehat{d\sigma}\|_{L^p(\mathbb{R}^n)} \lesssim \|\sigma\|_{L^2(\mathbb{P}^{n-1})}, \quad p \geq \frac{2(n+1)}{n-1}$$

where σ is the pull-back of ϕ to the paraboloid (singular measure)

$$\mathbb{P}^{n-1} = \{(\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \ldots + \xi_{n-1}^2)\}$$

and

$$\widehat{d\sigma}(x_1, \ldots, x_n) = \int_{\mathbb{P}^{n-1}} e(x \cdot \xi) d\sigma(\xi)$$

Notation: $e(z) = e^{iz}, \; z \in \mathbb{R}$

The proof of

$$\|\widehat{d\sigma}\|_{L^p(\mathbb{R}^n)} \lesssim \|\sigma\|_{L^2(\mathbb{P}^{n-1})}, \quad p \geq \frac{2(n+1)}{n-1}$$

is very "elementary" by today's standards. It uses TT^* and the decay of the Fourier transform of the surface measure on the paraboloid. The range $p \geq \frac{2(n+1)}{n-1}$ is sharp for this estimate.

For the periodic Schrödinger equation

$$\begin{cases} 2\pi i u_t(x,t) = \Delta_x u(x,t), \ \ (x,t) \in \mathbb{T}^{n-1} \times \mathbb{R} \\ u(x,0) = \phi(x), \ \ x \in \mathbb{T}^{n-1} \end{cases}.$$

The solution is easily seen to be an exponential sum

$$u(x_1, \ldots, x_{n-1}, t)$$

$$= \sum_{\xi_1,\ldots,\xi_{n-1}\in\mathbb{Z}} \hat{\phi}(\xi_1, \ldots, \xi_{n-1}) e(x_1\xi_1 + \ldots + x_{n-1}\xi_{n-1} + t(\xi_1^2 + \ldots + \xi_{n-1}^2))$$

$$= \sum_{\xi \in \mathbb{P}^{n-1} \cap \mathbb{Z}^n} a_\xi e((x,t)\cdot\xi)$$

where a_ξ is the Fourier coefficient of the initial data ϕ.
E.g. when $n = 2$ this is a weighted Gauss sum $\sum_n a_n e(nx + n^2 t)$.

Notation: $P_N \lesssim_\epsilon N^\epsilon Q_N$, or $P_N \lessapprox Q_N$, means that for each $\epsilon > 0$ there is some constant C_ϵ independent of N such that $P_N \le C_\epsilon N^\epsilon Q_N$. For example

$$\log N \lesssim_\epsilon N^\epsilon.$$

Theorem (Bourgain, D. 2015, Strichartz estimates for tori)

Let $\phi \in L^2(\mathbb{T}^{n-1})$ *with* $\text{supp}(\hat{\phi}) \subset [-N,N]^{n-1}$. *Then for each* $\epsilon > 0$ *we have*

$$\|u\|_{L^p(\mathbb{T}^{n-1}\times[0,1])} \lesssim_\epsilon \begin{cases} N^\epsilon \|\phi\|_2, & 2 \le p \le \frac{2(n+1)}{n-1} \\ N^{\frac{n-1}{2}-\frac{n+1}{p}} \|\phi\|_2, & p > \frac{2(n+1)}{n-1} \text{ no } \epsilon \text{ loss} \end{cases}.$$

Prior to this, in 1993 Bourgain proved this theorem when $n = 2$ and $n = 3$ using elementary methods (the key is that the critical exponent $\frac{2(n+1)}{n-1}$ is an even integer in these cases). This approach fails in higher dimensions.

In early 90's Bourgain attacked this problem by combining Strichartz's approach from the Euclidean case (estimates for the kernel and its Fourier transform) with number theory.

The resolution of the Strichartz estimates on tori came via a purely Fourier analytic method, called *decoupling*. This was introduced by Tom Wolff in 2000, for the cone. Bourgain and I were able to prove (essentially) sharp estimates in the sharp range.

Theorem (Bourgain, D. 2015, Decoupling for the paraboloid)

Let σ be a measure supported on $\mathbb{P}^{d-1} \cap B(0,1)$. Given $R \gg 1$, we cover the paraboloid with $1/\sqrt{R}$-caps θ. Then for each ball $B_R \subset \mathbb{R}^d$ of radius R

$$\|\widehat{\sigma}\|_{L^p(B_R)} \lesssim_\epsilon R^\epsilon \Big(\sum_\theta \|\widehat{\sigma 1_\theta}\|^2_{L^p(B_R)}\Big)^{1/2}$$

Question (The "Carleson problem")

For what range of s is it true that $\phi \in H^s(\mathbb{R}^{n-1})$ *implies that*

$$\lim_{t \to 0} u(x,t) = \phi(x)$$

for almost every $x \in \mathbb{R}^{n-1}$*?*

When $n = 2$, $s \geq 1/4$ was known for a long time to be sharp (Carleson 1979, Dahlberg-Kenig 1981) (convergence for $s \geq 1/4$, divergence for $s < 1/4$).

In 2016 Bourgain proved divergence for $s < \frac{n-1}{2n}$.

The recent papers (Du, Guth, Li, Ann. 2017 of Math.) and (Du, Zhang, Ann. of Math. 2019) proved convergence for $s > \frac{n-1}{2n}$ when $n = 3$, then $n \geq 3$. The endpoint is open, likely very difficult.

These papers rely on the fact that it suffices to prove sharp L^p bounds for the associated Schrödinger maximal function (this is a Strichartz estimate!)

$$x \mapsto \sup_{t>0} |u(x,t)|$$

$$\|u\|_{L^p_x L^\infty_t} \lesssim \|\phi\|_{L^2}.$$

These bounds are proved using wave-packet decompositions, refined decoupling and incidence estimates from the multi-linear Kakeya problem.

Like the non-mixed Strichartz estimates, Carleson's problem has generated a lot of interest, and a lot of powerful mathematics.

Carleson's problem has an analogue on the torus. Strikingly, it is open even in the case $n = 2$.

Conjecture (Schrödinger maximal function on $\mathbb{T}$)

Assume $\|a_n\|_{l^2} = 1$. *Then for each* $p \leq 4$

$$\|\sup_t |\sum_{n=1}^{N} a_n e(nx + n^2 t)|\|_{L^p([0,1],dx)} \lessapprox N^{1/4}. \qquad (1)$$

The best known upper bound in (1) for any $p \leq 4$ is $\lessapprox N^{\frac{1}{3}}$ (Moyua and Vega, 2008), valid in the range $1 \leq p \leq 6$. It is essentially a consequence of Bourgain's Strichartz estimate on $\mathbb{T}^2$

$$\|\sum_{n=1}^{N} a_n e(nx + n^2 t)\|\|_{L^6([0,1]^2,dxdt)} \lessapprox \|a_n\|_{l^2}.$$

The conjecture was recently solved in the case $a_n \equiv 1$, first by Barron then by Baker. The latter identified the precise magnitude

$$\| \sup_t | \sum_{n=1}^{N} e(nx + n^2 t)| \|_{L^p([0,1],dx)} \sim N^{a(p)} (\log N)^{b(p)}$$

for each $p \geq 1$.

The wave packet analysis that was successful in the Euclidean case is ill-fitted for the periodic case. Wave packets are not sensitive enough to distinguish the parabola (on which (n, n^2) lies) from other strictly convex curves (e.g. (t, t^3)), Jarnick's curve, whose appropriate dilates contain many lattice points). Fu, Ren and Wang recently proved that $N^{1/3}$ is sharp within the class of convex sequences, a category covered by decoupling methods.

The conjecture can be reformulated using level set estimates.

Conjecture

Assume $\|a_n\|_{l^2} = 1$ *and* $N^{\frac{1}{4}} \leq \lambda \leq N^{\frac{1}{2}}$. *(outside this range the estimate is trivial) Then*

$$|E_\lambda := \{x \in [0,1] : \sup_t |\sum_{n=1}^{N} a_n e(nx + n^2 t)| \geq \lambda\}| \lessapprox \frac{N}{\lambda^4}.$$

This offers a parallel, independent track of conjectures. No such estimate appeared in the literature. In particular, sharp level-set estimates do not follow from a sub-optimal estimate

$$\|\sup_t |\sum_{n=1}^{N} a_n e(nx + n^2 t)|\|_{L^4([0,1],dx)} \lessapprox N^{\frac{1}{4}+\beta}, \quad \beta > 0. \tag{2}$$

On the other hand, for example, the sub-optimal bound

$$|E_\lambda| \lessapprox \frac{N^{4/3}}{\lambda^4}, \text{ for all } \lambda \gtrsim N^{1/3}$$

would imply the best known exponent $\frac{1}{4} + \beta = \frac{1}{3}$ in (2).

Conjecture

Assume $\|a_n\|_{l^2} = 1$ *and* $N^{\frac{1}{4}} \le \lambda \le N^{\frac{1}{2}}$. *Then*

$$|\{x \in [0,1] : \sup_t |\sum_{n=1}^{N} a_n e(nx + n^2 t)| \ge \lambda\}| \lessapprox \frac{N}{\lambda^4}.$$

An argument of Bourgain (On Λ_p subsets for squares, Israel J Math) can be used to prove the conjecture for $\lambda \ggg N^{\frac{1}{4}+\frac{1}{8}}$ (Alex Barron, private communication). This argument breaks down even for $\lambda \sim N^{\frac{1}{4}+\frac{1}{8}}$.

I proved

Theorem

Assume $\|a_n\|_{l^2} = 1$ *and* $N^{\frac14+\frac1{10}} \lessapprox \lambda \le N^{\frac12}$. *Then*

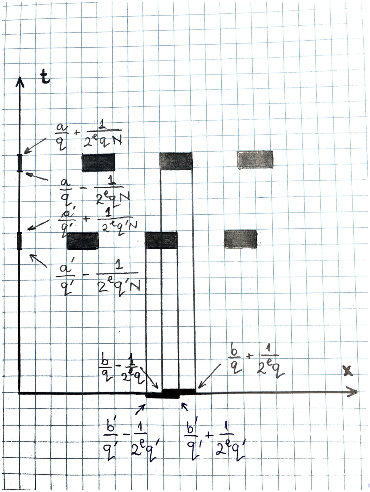
$$|\{x \in [0,1] : \sup_t |\sum_{n=1}^N a_n e(nx + n^2 t)| \ge \lambda\}| \lessapprox \frac{N}{\lambda^4}.$$

Via discretization, a variant of TT^* and standard Gauss sum estimates this would follow from another conjecture with arithmetic/additive combinatorial structure

Let $\mathcal{A}(q) = \{1 \le a \le q-1 : \gcd(a,q) = 1\}$. For dyadic $1 \le Q \le N$ and for $1 \le 2^l \lesssim \frac{N}{Q}$ we write

$$S_{Q,l} = \bigcup_{q \sim Q} \bigcup_{a \in \mathcal{A}(q)} \bigcup_{0 \le b \le q-1} [\frac{b}{q} - \frac{1}{2^l q}, \frac{b}{q} + \frac{1}{2^l q}] \times [\frac{a}{q} - \frac{1}{2^l qN}, \frac{a}{q} + \frac{1}{2^l qN}].$$

The t-intervals are pairwise disjoint, while the x-intervals overlap.



$$S_{Q,l} = \bigcup_{q\sim Q} \bigcup_{a\in\mathcal{A}(q)} \bigcup_{0\le b\le q-1} [\frac{b}{q} - \frac{1}{2^l q}, \frac{b}{q} + \frac{1}{2^l q}] \times [\frac{a}{q} - \frac{1}{2^l qN}, \frac{a}{q} + \frac{1}{2^l qN}].$$

Conjecture

Assume $1 \le 2^l \lesssim \frac{N}{Q}$ *and* $1 \le K \le 2^{l/2}$.
Let $z_r = (x_r, t_r) \in [0,1]^2$, $1 \le r \le R$, *with* x_r $1/N$*-separated, satisfy*

$$R^2 \lesssim K \sum_{r,r'=1}^{R} 1_{S_{Q,l}}(z_r - z_r').$$

Then we have

$$R \lesssim K^2 \frac{N}{2^l}.$$

The point is that $S_{Q,l}$ does not have enough additive structure. E.g. it does not contain a large AP× AP

$$S_{Q,l} = \bigcup_{q\sim Q}\bigcup_{a\in\mathcal{A}(q)}\bigcup_{0\le b\le q-1}[\frac{b}{q}-\frac{1}{2^l q},\frac{b}{q}+\frac{1}{2^l q}]\times[\frac{a}{q}-\frac{1}{2^l qN},\frac{a}{q}+\frac{1}{2^l qN}].$$

Conjecture

Assume $1 \le 2^l \lesssim \frac{N}{Q}$ *and* $1 \le K \le 2^{l/2}$. *Let* $z_r = (x_r, t_r) \in [0,1]^2$, $1 \le r \le R$, *with* x_r $1/N$*-separated, satisfy*

$$R^2 \lesssim K \sum_{r,r'=1}^{R} 1_{S_{Q,l}}(z_r - z_r').$$

Then we have

$$R \lesssim K^2 \frac{N}{2^l}.$$

A tight example: Take z_r of the form $(\frac{a}{q},\frac{a}{q})$ with $q \sim \sqrt{Q}$ prime and $1 \le a \le q-1$. Then $R \sim Q$ and $K \sim 1$, since $q \ne q' \implies \frac{a'}{q'} - \frac{a}{q} = \frac{A'}{Q'}$ with $Q' \sim Q$ and $(A', Q') = 1$.

Note that $R \sim Q \lesssim \frac{N}{2^l} = K^2 \frac{N}{2^l}$

$$S_{Q,l} = \bigcup_{q\sim Q} \bigcup_{a\in\mathcal{A}(q)} \bigcup_{0\le b\le q-1} [\frac{b}{q} - \frac{1}{2^l q}, \frac{b}{q} + \frac{1}{2^l q}] \times [\frac{a}{q} - \frac{1}{2^l qN}, \frac{a}{q} + \frac{1}{2^l qN}].$$

$$X_{Q,l} = \bigcup_{q\sim Q} \bigcup_{0\le b\le q-1} [\frac{b}{q} - \frac{1}{2^l q}, \frac{b}{q} + \frac{1}{2^l q}]$$

Fourier analysis on [0,1] and Gauss sum estimates give the following.

Theorem (A. Barron, private communication)

Assume $1 \le 2^l \lesssim \frac{N}{Q}$, $1 \le K \le 2^{l/2}$ *and* $2^l \gtrsim N^{1/2}$
Let $x_r \in [0,1]$, $1 \le r \le R$, *with* x_r $1/N$*-separated, satisfy*

$$R^2 \lesssim K \sum_{r,r'=1}^{R} 1_{X_{Q,l}}(x_r - x_{r'}). \quad (3)$$

Then we have

$$R \lesssim K^2 \frac{N}{2^l}. \quad (4)$$

This theorem (i.e using information only about the x-component) is false for $2^l \lesssim N^{1/2}$. For example, $X_{\sqrt{N},\sqrt{N}}$ has measure ~ 1, and is a union of intervals of length $\sim 1/N$. Thus, $x_r = r/N$, $r \le R = N$ satisfies (3) with $K = 1$. However (4) is false, since $2^l \sim \sqrt{N}$.

Here is my result.

Theorem

Assume $1 \le 2^l \lesssim \frac{N}{Q}$, $1 \le K \le 2^{l/2}$ *and* $2^l \gtrsim N^{2/5}$.
Let $z_r = (x_r, t_r) \in [0,1]^2$, $1 \le r \le R$, *with* x_r $1/N$*-separated, satisfy*

$$R^2 \lesssim K \sum_{r,r'=1}^{R} 1_{S_{Q,l}}(z_r - z_r'). \tag{5}$$

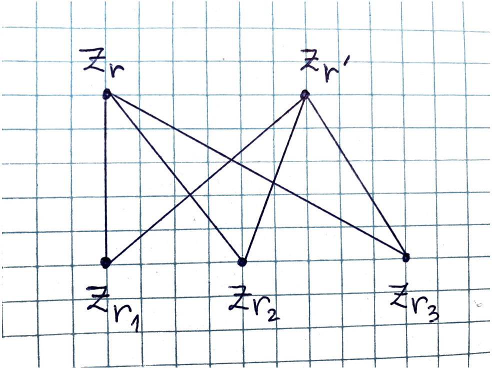
Then we have

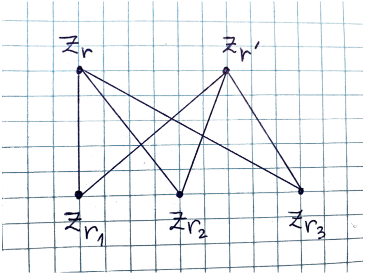
$$R \lesssim \frac{K^2}{2^l} N. \tag{6}$$

My proof uses no Fourier analysis. One obstacle for using Fourier analysis on $[0,1]^2$ is that it does not detect the false counter-example: choose z_r to be the $R \sim Q^2$ points $(\frac{b}{q}, \frac{a}{q})$ with q fixed and $1 \le a, b \le q-1$. Note that $z_r - z_{r'} \in S_{Q,l}$ for each pair, so (5) is satisfied with $K \sim 1$, but (6) is false. But x_r are not $1/N$-separated!

Instead, I use number theory and a little bit of graph theory.

Let G be the graph with vertices z_r, $1 \leq r \leq R$ and an edge between z_r and z_{r_1} if $z_r - z_{r_1} \in S_{Q,l}$. There are $\gtrsim R^2K^{-1}$ edges. This easily implies that there is a pair $(z_r, z_{r'})$ that shares $\gtrsim R/K^2$ neighbors $z_{r_1}, z_{r_2}, \ldots$.





Since there is an edge between z_{r_1} and z_r, ther must exist a, b, q such that $z_{r_1} - z_r \in [\frac{b}{q} - \frac{1}{2^l q}, \frac{b}{q} + \frac{1}{2^l q}] \times [\frac{a}{q} - \frac{1}{2^l qN}, \frac{a}{q} + \frac{1}{2^l qN}]$. I write this as $z_{r_1} - z_r \approx (\frac{b}{q}, \frac{a}{q})$. Similarly, $z_{r_1} - z_{r'} \approx (\frac{b'}{q'}, \frac{a'}{q'})$.Thus

$$z_r - z_{r'} \approx (\frac{b'}{q'} - \frac{b}{q}, \frac{a'}{q'} - \frac{a}{q})$$

I prove that for "many" values (x, t), the number of solutions (a, b, q, a', b', q') with $q, q' \sim Q$ to

$$(x, t) \approx (\frac{b'}{q'} - \frac{b}{q}, \frac{a'}{q'} - \frac{a}{q})$$

is $\lesssim Q$. Thus, a typical pair of vertices $(z_r, z_{r'})$ has at most Q joint neighbors. Since we can pick a pair with R/K^2 joint neighbors, we find

$$R/K^2 \lesssim Q \lesssim N/2^l \text{ or } R \lesssim \frac{K^2}{2^l} N.$$

Complications:

The number of solutions to

$$(x, t) \approx (\frac{b'}{q'} - \frac{b}{q}, \frac{a'}{q'} - \frac{a}{q})$$

is only $\lesssim Q$ if $\gcd(q, q') = 1$, and it gets larger if this condition fails. Pigeonholing is used to find a dyadic value D such that the graph G is dominated by pairs of neighbors with $\gcd(q, q') \sim D$. Two other (more subtle) parameters are also needed besides $\gcd(q, q')$ and they also need to be pigeonholed.

In a certain range of these parameters, I need to identify subgraphs of G with high additive structure, and localize analysis to them (a-la induction on scales). I get better estimates in these cases.

My argument using joint neighbors of two vertices can only prove the conjecture for $2^l \gtrsim N^{2/5}$. The enemy scenario is when the graph is dominated by $D \sim 1$. I recover the suboptimal estimate

$$|E_\lambda| \lessapprox \frac{N^{4/3}}{\lambda^4}, \text{ for all } \lambda \gtrsim N^{1/3}$$

that is equivalent to

$$\|\sum_{n=1}^{N} a_n e(nx + n^2 t)\|\|_{L^4([0,1]^2, dxdt)} \lessapprox N^{1/3}\|a_n\|_{l^2}.$$

The novelty is that it produces sharp estimates for $\lambda \gtrsim N^{\frac{1}{4}+\frac{1}{10}}$.

Working with three or more neighbors has the potential to fully solve the conjecture. For example, heuristics show that the three-neighbor argument would produce sharp estimates for $2^l \gtrsim N^{1/3}$. This means $\lambda = (N2^l)^{1/3} \gtrsim N^{1/3}$.
Executing the 3-neighbor argument boils down to controlling the number of solutions $(a, b, q, a', b', q', a'', b'', q'')$ for

$$\begin{cases}(x', t') \approx (\frac{b'}{q'} - \frac{b}{q}, \frac{a'}{q'} - \frac{a}{q}) \\ (x'', t'') \approx (\frac{b''}{q''} - \frac{b}{q}, \frac{a''}{q''} - \frac{a}{q})\end{cases}$$

Even getting a Q^ϵ gain over the trivial count seems difficult. Any such gain would very likely lead to an exponent β smaller than $1/3$ in

$$\|\sum_{n=1}^{N} a_n e(nx + n^2 t)\|\|_{L^4([0,1]^2, dxdt)} \lessapprox N^\beta \|a_n\|_{l^2}.$$