Radial weights in the Mizohata–Takeuchi conjecture

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The Mizohata–Takeuchi conjecture – I

Let $\mathbb{S}$ be a (compact) piece of nice smooth hypersurface in $\mathbb{R}^n$ with a nice measure $\mathrm{d}\sigma$. Let $g \in L^2(\mathbb{S}, \mathrm{d}\sigma)$. We consider weighted inequalities of the form

$$\int_{\mathbb{R}^n} |\widehat{g\mathrm{d}\sigma}(x)|^2 w(x)\mathrm{d}x \lesssim \sup_T w(T) \int_{\mathbb{S}} |g|^2 \mathrm{d}\sigma$$

with the sup taken over some family of eccentric tubes.

Such an inequality acts as a rectilinear structure detector in $|\widehat{g\mathrm{d}\sigma}|^2$: it tells us something about the shape and location of the set where this function is large, not just its size (as do classical Fourier restriction estimates).

Usually we take the family of tubes to be all doubly-infinite tubes of cross-sectional diameter 1, (1-tubes). The corresponding inequality is often referred to as the Mizohata–Takeuchi conjecture (following Vega). But we will also be open to other functionals of w involving eccentric tubes of perhaps different sizes.

If we let $w = \chi_{B_R}$ and we consider 1-tubes then $\sup_T w(T) \sim R$ and the MT conjecture in this case is just the Agmon–Hörmander trace inequality (Plancherel's theorem).

The Mizohata–Takeuchi Conjecture – II

Let $\mathbb{S}$ denote a nice smooth hypersurface in $\mathbb{R}^n$ with a suitable surface measure σ.

Conjecture (MTV)

For all $w \geq 0$ we have

$$\int_{\mathbb{R}^n} |\widehat{g\mathrm{d}\sigma}(x)|^2 w(x)\mathrm{d}x \lesssim \sup_T w(T) \int_{\mathbb{S}} |g|^2 \mathrm{d}\sigma$$

where the sup is taken over all 1-tubes T in $\mathbb{R}^n$ with $T \perp$ supp g.

All presently understood examples are consistent with this.

A purely $L^2 - L^2$ inequality: Fairly accessible with lots of standard tools available ...?

Curvature is not mentioned here, so this conjecture should be more straightforward than those for Fourier extension operator which are curvature-dependent. Indeed, if $\mathbb{S}$ is a piece of hyperplane for example, it is easily seen to be true, and that $\sup_T w(T) < \infty$ is necessary. Equally plausible variants suggest themselves for $\mathbb{S}$ of general dimension. A flexible and basic conjecture in the spirit of the Agmon–Hörmander trace inequality?

A Stein-like conjecture

Conjecture (MTV)

For all $w \geq 0$ we have

$$\int_{\mathbb{R}^n} |\widehat{g d\sigma}(x)|^2 w(x) dx \lesssim \sup_T w(T) \int_{\mathbb{S}} |g|^2 d\sigma$$

where the sup is taken over all 1-tubes T in $\mathbb{R}^n$ with $T \perp \operatorname{supp} g$.

One could go a step further and speculate that we have the even stronger inequality

$$\int_{\mathbb{R}^n} |\widehat{g d\sigma}(x)|^2 w(x) dx \lesssim \int_{\mathbb{S}} |g(\xi)|^2 \sup_{T \parallel N(\xi)} w(T) d\sigma(\xi)$$

where the sup is taken over all 1-tubes T in $\mathbb{R}^n$ whose direction is parallel to the normal direction $N(\xi)$ to $\mathbb{S}$ at ξ. (The map $w \mapsto \sup_{T \parallel N(\xi)} w(T)$ is the Kakeya maximal function.)

This stronger, Stein-like conjecture has its roots in the parallel universe of inequalities for the disc multipliers, for which Stein conjectured L^2-weighted control by the Nikodym maximal function.

MTV and multilinear restriction

Stein's conjecture + Kakeya conjecture $\implies$ restriction conjecture.

Similarly, the Mizohata–Takeuchi conjecture + (true) endpoint Multilinear Kakeya Theorem $\implies$ endpoint Multilinear Restriction.

...One more reason for wanting the MTV conjecture to be tue.

Context for MT – the Stein–Tomas estimate

The Stein–Tomas theorem provides a point of reference for MT. Suppose now that $\mathbb{S}$ is compact and has nonvanishing curvature. Then we have

$$\|\widehat{g d\sigma}\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \lesssim \|g\|_2.$$

This is equivalent, by Hölder's inequality and its reverse, to

$$\int |\widehat{g d\sigma}|^2 w \lesssim \left(\int_{\mathbb{R}^n} w^{\frac{n+1}{2}}\right)^{\frac{2}{n+1}} \int |g|^2$$

for all g and all nonnegative w.

But this is a far cry from the MT conjecture because the functionals

$$\left(\int_{\mathbb{R}^n} w^{\frac{n+1}{2}}\right)^{\frac{2}{n+1}} \text{ and } \sup_{1-\text{tubes}\, T} w(T)$$

are very non-comparable.

Nevertheless this is the starting point for the first theme...

Comparing Stein–Tomas with MT

$$\int |\widehat{g d\sigma}|^2 w \lesssim \Big(\int_{\mathbb{R}^n} w^{\frac{n+1}{2}}\Big)^{\frac{2}{n+1}} \int |g|^2 \qquad \text{(S–T)}$$

$$\int |\widehat{g d\sigma}|^2 w \lesssim \sup_{1-\text{tubes}\, T} w(T) \int |g|^2 \qquad \text{(M–T)}$$

A question: can we replace $\Big(\int_{\mathbb{R}^n} w^{\frac{n+1}{2}}\Big)^{\frac{2}{n+1}}$ in ST by something smaller, featuring eccentric rectangles, so that the new inequality has more MT-like features?

An observation: interesting weights for MT will be those which satisfy

$$\sup_{1-\text{tubes}\, T} w(T) \ll \Big(\int_{\mathbb{R}^n} w^{\frac{n+1}{2}}\Big)^{\frac{2}{n+1}},$$

or, more informally, such that the mass in any 1-tube is small relative to the total mass.

These turn out to be related considerations.

Improved localised Stein–Tomas

We have the following MT-esque improvement over Stein–Tomas:

Theorem (AC, Marina Iliopoulou and Hong Wang, 2023/24)

Let $n \geq 2$. Suppose $\mathbb{S}$ is srtictly convex with nonvanishing curvature. Then

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^\epsilon \sup_{T \in \mathbb{T}_{R^{1/2}}:\ T \perp \mathrm{supp}\, g} \left(\int_T w^{\frac{n+1}{2}} \right)^{\frac{2}{n+1}} \int_{\mathbb{S}} |g|^2.$$

- Local at scale R, and incurs an R^ϵ-loss
- An improvement since the *worst* tube T is much smaller than B_R: much more information than Stein–Tomas if w is **not** concentrated on some $T \in \mathbb{T}_{R^{1/2}}$
- Sharp: we cannot raise the exponent $q = (n+1)/2$; for this $q = (n+1)/2$ we cannot 'narrow' $\mathbb{T}_{R^{1/2}}$ to consist of tubes of width $\ll R^{1/2}$ – just test on $g = \chi_S$ for an $R^{-1/2}$-cap $S \subseteq \mathbb{S}$.

A consequence: MT with power loss

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^{\epsilon} \left(\sup_{T\in\mathbb{T}_{R^{1/2}}} \int_T w^{(n+1)/2} \right)^{2/(n+1)} \int_{\mathbb{S}} |g|^2.$$

This has a simple consequence. Fix $T \in \mathbb{T}_{R^{1/2}}$. Then

$$\int_T w^{\frac{n+1}{2}} \le \|w\|_\infty^{\frac{n-1}{2}} w(T) \lesssim \|w\|_\infty^{\frac{n-1}{2}} R^{\frac{n-1}{2}} \sup_{S\ (1\times R)} w(S) \le R^{(n-1)/2} \left(\sup_{S\ (1\times R)} w(S) \right)^{\frac{n-1}{2}+1}$$

since wlog w is constant at scale 1, and thus $\|w\|_\infty \le \sup_{S\ (1\times R)} w(S)$. Hence we have

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^{\frac{n-1}{n+1}+\epsilon} \sup_{T\ (1\times R)} w(T) \int_{\mathbb{S}} |g|^2.$$

This is an improvement over Agmon–Hörmander which gives exponent 1 on R. Any exponent < 1 should probably be regarded as nontrivial. (There had been previous explicit and implicit intermediate improvements (Bourgain, Erdoğan, C–Seeger, Shayya, Du–Guth–Ou–Wang–Wilson–Zhang...) in dimensions $2, 3$).

Sets with limited tube-occupancy

Some years ago I constructed examples of sets with close to optimal limited tube occupancy properties as potential sources of interesting examples or indeed counterexamples to MT. These are sets of $\sim N \log N$ unit squares in an $N \times N$ grid for which no 1-tube meets more than $\log N$ of them.

However, the construction was probabilistic, and I wasn't able to use them as effective concrete test cases for MT.

But a couple of years ago in 2022 Larry Guth very cleverly used similar, slightly modified examples to (rather surprisingly) show that if one is allowed to use only the standard properties (calculus?) of wave packets, then the inequality

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^{(n-1)/(n+1)-\epsilon} \sup_{1-\text{tubes}\, T} w(T) \int |g|^2$$

fails for every $\epsilon > 0$.

So what about power exactly $(n-1)/(n+1)$? Can we take $\epsilon = 0$ in the CIW theorem?

Improved improved Stein–Tomas

Theorem (AC, Marina Iliopoulou and Hong Wang, 2023/24)

Let $n \geq 2$. Suppose $\mathbb{S}$ is strictly convex with nonvanishing curvature. Then

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^\epsilon \sup_{T\in\mathbb{T}_{R^{1/2}}:\, T\perp\mathrm{supp}\, g} \left(\int_T w^{\frac{n+1}{2}} \right)^{\frac{2}{n+1}} \int_{\mathbb{S}} |g|^2.$$

In fact,

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^\epsilon \left(\sum_{T\in\mathbb{T}_{R^{1/2}}:\, T\perp\mathrm{supp}\, g} w^{\frac{n+1}{2}}(T) \|g_T\|_2^2 \right)^{\frac{2}{n+1}} \|g\|_2^{\frac{2(n-1)}{n+1}}.$$

where $g = \sum_T g_T$ is the wave packet decomposition of g at scale $R^{-1/2}$.

The R^ϵ of the second inequality is **not** entirely removable: very simple number theory examples for the $l^2(\mathbb{Z}) - L^6(\mathbb{T}^2)$ extension inequality going back (at least) to Bob Vaughan's 1981 book on the Hardy–Littlewood method.

Vaughan's example and improved Stein–Tomas?

Vaughan's example – g with all wave packet coefficients 1 and then $w = |\widehat{g\mathrm{d}\sigma}|^{4/(n-1)}$ – shows we cannot entirely remove the ϵ in the red inequality

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^{\epsilon} \left(\sum_{T \in \mathbb{T}_{R^{1/2}}:\ T \perp \mathrm{supp}\, g} w^{\frac{n+1}{2}}(T) \|g_T\|_2^2 \right)^{\frac{2}{n+1}} \|g\|_2^{\frac{2(n-1)}{n+1}}.$$

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^{\epsilon} \sup_{T \in \mathbb{T}_{R^{1/2}}:\ T \perp \mathrm{supp}\, g} \left(\int_T w^{\frac{n+1}{2}} \right)^{\frac{2}{n+1}} \int_{\mathbb{S}} |g|^2$$

Passage from the red to the blue inequality is sharp if $w^{(n+1)/2}(T)$ is roughly constant over T: and indeed, by direct calculation,

$$\left\{ \int_T |\widehat{g\mathrm{d}\sigma}|^{2(n+1)/(n-1)} \right\}_T$$

is highly regularly distributed over tubes T, and thus Vaughan's example also contradicts $\epsilon = 0$ in the blue inequality. (Emerged in discussions with Po Lam Yung and Zane Li...)

Context for the CIW result

Theorem (AC, Marina Iliopoulou and Hong Wang, 2023/24)

Let $n \geq 2$. Suppose $\mathbb{S}$ is strictly convex with nonvanishing curvature. Then

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}|^2 w \lesssim R^{\epsilon} \left(\sum_{T \in \mathbb{T}_{R^{1/2}} :\, T \perp \mathrm{supp}\, g} w^{\frac{n+1}{2}}(T) \|g_T\|_2^2 \right)^{\frac{2}{n+1}} \|g\|_2^{\frac{2(n-1)}{n+1}}.$$

where $g = \sum_T g_T$ is the wave packet decomposition of g at scale $R^{-1/2}$.

This is **very** closely related to the *refined* decoupling theorem of Guth, Iosevich, Ou & Wang and of Du & Zhang. In fact, we use the refined decoupling theorem to prove it. Moreover the refined decoupling theorem can be deduced from it. So it is essentially a **reformulation** of refined decoupling, presented in an arguably less technical way.

If we could replace R^{ϵ} by $(\log R)^K$, by taking $w = |\widehat{g\mathrm{d}\sigma}|^{4/(n-1)}$, it would also imply the (stronger statement lying behind) the *improved* decoupling theorem of Guth, Maldague & Wang, and would give a further qualitative marginal improvement to it.

MT for radial weights – spectral theory in action

Theorem (J.A. Barceló, A. Ruiz & L. Vega, 1997)

If $\mathbb{S} = \mathbb{S}^{n-1}$ and $w(x) = \tilde{w}(|x|)$ is radial, then we have

$$\int_{\mathbb{R}^n} |\widehat{g\mathrm{d}\sigma}(x)|^2 w(x)\mathrm{d}x \lesssim \sup_T w(T) \int_{\mathbb{S}} |g|^2 \mathrm{d}\sigma.$$

where the sup is taken over all 1-tubes T in $\mathbb{R}^n$.

We lose the feature that the sup should be taken over all 1-tubes T in $\mathbb{R}^n$ with $T \perp \operatorname{supp} g$. For this, and various other reasons, the proof is not entirely satisfying.

MTV for radial weights – the set-up for the BRV Theorem

- We have a 1-parameter family of hypersurfaces Σ_r, on each of which the weight w is constant
- The Σ_r are isotropic dilates by $r > 0$ of a single **fixed** hypersurface $\Sigma = \mathbb{S}^{n-1}$
- The hypersurface $\mathbb{S} = \mathbb{S}^{n-1}$ is **spherically symmetric** and so $L^2(\mathbb{S})$ has the natural orthonormal basis of spherical harmonics $\{Y_k\}$
- The hypersurfaces $\mathbb{S}$ and Σ **coincide**
- For each $r > 0$, the spherical harmonics are **eigenfunctions** of the operator

$$g \mapsto \widehat{g\mathrm{d}\sigma}(r\cdot)$$

  mapping $L^2(\mathbb{S}^{n-1})$ to itself
- The eigenvalue corresponding to Y_k is $r^{-(n-2)/2}J_{k+(n-2)/2}(r) = \lambda_{k,n}(r)$.

With these observations in place, the argument essentially writes itself...

Proof of MTV in the radial case via spectral theory

Expand $g \in L^2(\mathbb{S}^{n-1})$ into spherical harmonics

$$g = \sum_k a_k Y_k \text{ with } \|g\|_2^2 = \sum_k |a_k|^2.$$

So

$$\widehat{g\mathrm{d}\sigma}(r\omega) = \sum_k a_k \widehat{Y_k \mathrm{d}\sigma}(r\omega) = \sum_k a_k \lambda_{k,n}(r) Y_k(\omega).$$

Therefore

$$\int_{\mathbb{S}^{n-1}} \int_0^\infty |\widehat{g\mathrm{d}\sigma}(r\omega)|^2 w(r) r^{n-1} \mathrm{d}r \mathrm{d}\sigma(\omega)$$

$$= \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |\sum_k a_k \lambda_{k,n}(r) Y_k(\omega)|^2 \mathrm{d}\sigma(\omega) \right) w(r) r^{n-1} \mathrm{d}r$$

$$= \int_0^\infty \sum_k |a_k|^2 \lambda_{k,n}(r)^2 w(r) r^{n-1} \mathrm{d}r \leq \sum_k |a_k|^2 \left(\sup_k \int_0^\infty \lambda_{k,n}(r)^2 w(r) r^{n-1} \mathrm{d}r \right).$$

And...

Bessel functions

We need to see that for radial weights w

$$\sup_k \int_0^\infty r^{-(n-2)} J_{k+(n-2)/2}(r)^2 w(r) r^{n-1} \mathrm{d}r \lesssim \sup_T w(T) \sim \sup_I \int_I w \mathrm{d}\lambda.$$

It is easy to show that for *radial* weights,

$$\sup_I \int_I w \mathrm{d}\lambda \sim \sup_{s>0} \int_s^\infty \frac{w(r) r^{1/2} \mathrm{d}r}{(r-s)^{1/2}}$$

And we know plenty enough about Bessel functions to enable us to conclude that we have

$$\sup_k \int_0^\infty J_{k+(n-2)/2}(r)^2 w(r) r \mathrm{d}r \lesssim \sup_{s>0} \int_s^\infty \frac{w(r) r^{1/2} \mathrm{d}r}{(r-s)^{1/2}}.$$

Altogether the argument is quite remarkable in how little geometric understanding it gives us – essentially none. It is entirely unstable under perturbations. Do we only need lines normal to supp g? ...

Weights at the other extreme...

Radial weights are "trivial" on concentric spheres but arbitrary with respect to the radii. An opposite scenario is where the weight is trivial (in some sense) with respect to radii but arbitrary with respect to the concentric spheres.

Theorem (J. Bennett, AC, F. Soria, A. Vargas, 2006)

Let $n = 2$. The MTV conjecture holds for arbitrary measures supported on circles $R\,\mathbb{S}^1 \subseteq \mathbb{R}^2$ for $R \gg 1$. In fact, the stronger Stein-like conjecture holds in this setting.

The proof relies strongly on Fourier Analysis on $\mathbb{S}^1 = \mathbb{T}$:

- $g \mapsto \widehat{g\mathrm{d}\sigma}(R\cdot)$ is given by convolution with $e^{iR\cos\theta}$
- Littlewood–Paley decomposition of the Fourier coefficients of the convolution kernel
- Mollification of the weight on $R\,\mathbb{S}^1$ at certain scales between 1 and $R^{1/2}$ corresponding to these dyadic pieces.

For R fixed, the most prominent scales occuring in the argument are $R^{1/2}$, and even more importantly, $R^{1/3}$. Tubes of size $R^{1/3} \times R^{2/3}$ feature strongly in the argument.

A more robust approach to the radial case?

Radial weights, by definition, are constant on origin-centred spheres. Working with a large spatial parameter R and restricting to $|x| \sim R$, we may assume that w is also constant on annuli of width ~ 1. Thus, w is essentially constant on slabs of size $1 \times R^{1/2} \times \cdots \times R^{1/2}$ which tesselate the annulus and are tangential to origin-centred spheres of radius R.

How far can we get under a (weaker-than-radial) hypothesis such as this? – perhaps with a family of such slabs tesselating an annular region $|x| \sim R$ but not necessarily being tangent to spheres? Can we recapture some of the geometry – tubes normal to supp g, tubes of various eccentricities? What about the Stein variant of the conjecture? Can we make a connection with the proof of the BCSV result?

Improved radial MT in dimension two

Theorem (AC, M. Iliopoulou, WIP)

Suppose w is constant on 1-neighbourhoods of spherical caps in $|x| \sim R$ of radius $R^{1/2}$ in $\mathbb{R}^2$. Then we have

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}(x)|^2 w(x)\mathrm{d}x \lesssim R^{\epsilon} \sup_{R^{1/3} \leq \alpha \leq R^{1/2}} \sup_{T \in \mathbb{T}_\alpha, T \perp \mathbb{S}} \frac{w(T)}{\alpha} \int_{\mathbb{S}^1} |g|^2 \mathrm{d}\sigma.$$

Moreover, we also have

$$\int_{B_R} |\widehat{g\mathrm{d}\sigma}(x)|^2 w(x)\mathrm{d}x \lesssim R^{\epsilon} \int_{\mathbb{S}^1} |g(\xi)|^2 \left[\sup_{1-\text{tubes } \mathrm{T}, \mathrm{T} \| \mathrm{N}(\xi)} w(T) \right] \mathrm{d}\sigma(\xi).$$

Recall that $\mathbb{T}_\alpha$ consists of tubes of dimensions $\alpha \times \alpha^2$. Note the sizes of the tubes featuring (cf. [BCSV]).

The arguments are robust.

Wave packets and orthogonality properties

We use a standard wave packet decomposition into tubes of size $R^{1/2} \times R$.

We have by Plancherel that distinct wave packets $\widehat{g_T \mathrm{d}\sigma}$ and $\widehat{g_{T'} \mathrm{d}\sigma}$ are (globally) orthogonal and that

$$\|\widehat{g_T \mathrm{d}\sigma}\|_2^2 \sim R\|g\|_2^2$$

(consistent with Agmon–Hörmander). But what about a more local orthogonality?

If distinct T and T' are both roughly normal to a $1 \times R^{1/2} \times \cdots \times R^{1/2}$ slab S, then the corresponding wave packets are orthogonal over S. This can be seen as a consequence of the interference of the oscillations on the respective wave packets ($\sin A + \sin B =$....)

This motivates the approach we take.

Strategy – I

Break up the annulus $|x| \sim R$ as a sum of disjoint union of $1 \times R^{1/2} \times \cdots \times R^{1/2}$ slabs S, on each of which we presume w is roughly constant. So

$$\int_{|x|\sim R} |\widehat{g d\sigma}|^2 w \sim \sum_S w_S \int_S |\widehat{g d\sigma}|^2 = \sum_S w_S \int_S |\sum_{T: T\cap S \neq \emptyset} \widehat{g_T d\sigma}|^2.$$

If distinct wave packets were orthogonal over each such S, we could continue this as

$$\sim \sum_S w_S \int_S \sum_{T: T\cap S \neq \emptyset} |\widehat{g_T d\sigma}|^2$$

and from here routine calculations would lead to

$$\sum_T \|g_T\|_2^2 \frac{w(T)}{R^{(n-1)/2}} \leq \int_{\mathbb{S}} |g(\xi)|^2 \sup_{T \perp N(\xi)} \frac{w(T)}{R^{(n-1)/2}} d\sigma(\xi)$$

– which is far too good to be true anyway. But we want to exploit the orthogonality that we do have...

Strategy – II

Consider

$$\sum_S w_S \int_S \Big| \sum_{T: T\cap S \neq \emptyset} \widehat{g_T \mathrm{d}\sigma} \Big|^2 = \sum_S w_S \int_S \Big| \sum_{k \geq 0} \sum_{T: T\cap S\neq\emptyset, \langle T, S \sim 2^{-k}} \widehat{g_T \mathrm{d}\sigma}\Big|^2$$

where $\langle T, S$ denotes the angle between the tube T and the hyperplane in which S lives.) The smallest angle occurring is effectively $R^{-1/2}$ so there are only $\sim \log R$ distinct k's, which we may therefore treat separately.

Consider the term corresponding to $k = 0$,

$$\sum_S w_S \int_S \Big| \sum_{T: T\cap S\neq\emptyset, \langle T, S \sim 1} \widehat{g_T \mathrm{d}\sigma}\Big|^2.$$

Here we have orthogonality of the wave packets involved over each S. and so we may proceed as before to get a favourable result.

And when $2^{-k} = R^{-1/2}$ and $n = 2$, for each S there is only a single T, so we don't need orthogonality at all and can proceed directly to a favourable result.

Intermediate values of 2^{-k} require more thought...

Intermediate values of the angles

We have a set of slabs S, and a collection of wave packet tubes T such that for each S and each T we have $\langle T, S \sim \lambda = 2^{-k}$ for some $R^{-1/2} \leq \lambda \leq 1$. Because we are in dimension two we can do an easy preliminary reduction to the case of g supported in a λ-cap on $\mathbb{S}^1$. And then we redecompose each such g into wave packets at a suitable scale.

$R^{-1/2} \leq \lambda \leq R^{-1/3}$: tangential case – we work on sub-balls of B_R of radius λ^{-2} and re-develop the corresponding parts of g in wave packet decompositions involving tubes of size $\lambda^{-1} \times \lambda^{-2}$. Such tubes are parallel so orthogonality is automatic.

$R^{-1/3} \leq \lambda \leq 1$: transversal case – we work on sub-balls of radius $R\lambda$ and and re-develop the corresponding parts of g in wave packet decompositions involving tubes of size $(R\lambda)^{1/2} \times R\lambda$. Such tubes are *not* parallel so orthogonality is **not** automatic. But we do have sufficient local orthogonality of the corresponding wave packets.

The fully transversal case $\lambda \sim 1$ works in all dimensions, but it is there that the local constancy hypothesis on the weight is used maximally.

So we still have plenty of work to do...

Thanks for coming!