

Problems on spherical means and associated maximal functions

ICM, Analysis Section
July 24, 2026

- Discuss selected open problems on spherical maximal operators.

I. Behavior on and near L^1

II. Sharp L^p bounds for intermediate dilation sets

III. Weighted estimates and L^p improving

IV. Further problems (fractal local smoothing).

Problems on spherical maximal functions.

Survey article with Joris Roos, to appear in Bull. Amer. Math. Soc. arXiv 2511.11283.

Spherical means

σ : normalized surface measure on $S^{d-1} \subset \mathbb{R}^d$ ($d \geq 2$)

$$A_t f(x) = \int f(x - ty) d\sigma(y)$$

- Spherical means are translation invariant model cases for particularly nice classes of generalized Radon transforms.
- PDE: Spherical means (or certain derivatives) show up in solutions to IVP's for the **wave equation** $u_{tt} = \Delta u$.
- **Differentiation**. Let $f \in L^p(\mathbb{R}^d)$.
Does $A_t f(x) \rightarrow f(x)$ a.e. as $t \rightarrow 0$ within a set $\mathcal{E}$ of radii?

We need *quantitative bounds* for a maximal function

$$M_{\mathcal{E}} f(x) = \sup_{t \in \mathcal{E}} |A_t f(x)|.$$

Emphasis will be put on the impact of the dilation set $\mathcal{E}$.

Two classical results

Let $d \geq 2$. Almost every convergence results for all $f \in L^p$ are consequences of L^p estimates

$$\|M_{\mathcal{E}}f\|_p \lesssim \|f\|_p \quad (*)$$

or the weak type (p, p) version $\sup_{\alpha>0} \alpha |\{M_{\mathcal{E}}f(x) > \alpha\}|^{1/p} \lesssim \|f\|_p$.

- E. M. Stein '76 ($d \geq 3$), Bourgain '86 ($d = 2$): $\mathcal{E} = (0, \infty)$: $(*)$ holds iff $p > \frac{d}{d-1}$.

Weak type (p, p) fails for $p = \frac{d}{d-1}$. Bourgain: RWT holds when $d \geq 3$. Not for $d = 2$ (STW)

- C.P. Calderón, Coifman & Weiss ('77, '78)

$\mathcal{E}^{\text{lac}} = \{2^{-k} : k \in \mathbb{Z}\}$: $(*)$ holds for all $p > 1$.

I. $M_{\mathcal{E}^{\text{lac}}}$ on spaces near L^1 .

Open problem: Is $M_{\mathcal{E}^{\text{lac}}}$ of weak type $(1, 1)$?

- This is a model problem for a large class of singular or maximal Radon transforms (with known L^p bounds for $p > 1$).

Spaces near L^1 . Want to beat a weak type $L \log L$ inequality.

- Christ '88: $H^1 \rightarrow L^{1,\infty}$ estimate (with the Hardy space H^1).
- S., Tao, Wright '03: Weak type $L \log \log L$ inequality.
- Cladek, Krause '24:

Weak type $L(\log \log \log L)(\log \log \log \log L)^{1+\varepsilon}$ inequality.

- Based on sophisticated versions of Calderón-Zygmund theory

II. Restricted approach sets – Intermediate $\mathcal{E}$

Question: What can be said about $A_t f \rightarrow f$ a.e. if $t \rightarrow 0$ within $\mathcal{E}$, and control of the associated maximal function $M_{\mathcal{E}} f(x) = \sup_{t \in \mathcal{E}} |A_t f(x)|$?

- For $E \subset [1, 2]$ let $N(E, \delta)$ be the minimal number of intervals of length δ needed to cover E .
- Let $\mathcal{E}_R = \frac{1}{R}(\mathcal{E} \cap [R, 2R])$.

Definition (Dilation invariant version of Minkowski dimension)

$\text{Dim}_{\text{M}} \mathcal{E} \equiv \beta(\mathcal{E})$ is the smallest number such that for all $\varepsilon > 0$:

$$\sup_R N(\mathcal{E}_R, \delta) \leq C_{\varepsilon} \delta^{-\beta-\varepsilon} \quad \forall \delta \in (0, 1).$$

Boundedness ranges of $M_{\mathcal{E}}$

Theorem (S., Wainger, Wright '95:)

Let $p(\mathcal{E}) = 1 + \frac{\beta(\mathcal{E})}{d-1}$. Then $M_{\mathcal{E}}$ is bounded on L^p for $p > p(\mathcal{E})$ and unbounded for $p < p(\mathcal{E})$.

- The definition of dimension alone cannot settle the natural question of what happens at $p(\mathcal{E})$. Given any $\mathcal{E}$ can we **characterize** the p -range for weak type (p, p) (or L^p -boundedness) of $M_{\mathcal{E}}$?

Conjecture (For $1 < p < \frac{d}{d-1}$)

$$\|M_{\mathcal{E}}\|_{L^p \rightarrow L^{p,\infty}} \approx \sup_R \sup_{\delta < 1} \delta^{\frac{d-1}{p'}} N(\mathcal{E}_R, \delta)^{\frac{1}{p}}$$

Upgrade to L^p -boundedness for the local variant with $\mathcal{E} \subset [1, 2]$

Note: Equivalence for **restricted** weak type of $M_{\mathcal{E}}$ is easy.

Examples

For simplicity, just consider instead of $\mathcal{E}$ a dilation set $E \subset [1, 2]$ and think of $\mathcal{E} = \cup_k 2^k E$.

Examples:

- E Cantor middle third set in $[1, 2]$. Then $\beta(E) = \log_3 2$.
- $E = \{1 + n^{-a}\}$: Then $\beta(E) = (a + 1)^{-1}$

In both cases, for small δ ,

$$\delta^{\frac{d-1}{p'}} N(E, \delta)^{\frac{1}{p}} \approx 1 \quad \text{for} \quad p = p(E) \equiv 1 + \frac{\beta}{d-1}.$$

Similarly, $\delta^{\frac{d-1}{p'}} N(\mathcal{E}_R, \delta)^{\frac{1}{p}} \approx 1$ for the global analogues $\mathcal{E} = \cup_k 2^k E$.

Elementary bound

Assume j large and that $\text{supp}(\widehat{f}) \subset \{\xi : |\xi| \approx 2^j\}$.

Let $\{t_\nu\}$ a set of 2^{-j} -separated times in $E \subset [1, 2]$.

Then for $1 \leq p \leq 2$

$$\left\| \sup_{t_\nu \in E} |A_{t_\nu} f| \right\|_p \lesssim 2^{-j \frac{d-1}{p}} N(E, 2^{-j})^{\frac{1}{p}} \|f\|_p.$$

Suffices to check the cases $p = 1$ and $p = 2$.

For $p = 2$ use $|\widehat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$.

Towards the endpoint

The relevant quantity before was $\kappa_{j,p}(E) = 2^{-j\frac{d-1}{p'}} N(E, 2^{-j})^{\frac{1}{p}}$ and we have, for $1 \leq p \leq \frac{d}{d-1}$,

$$\sup_j |\kappa_{j,p}(E)| \lesssim \|M_E\|_{L^p \rightarrow L^p} \lesssim \sum_j \kappa_{j,p}(E)$$

Intermediate conditions involve the ℓ^q norm of $\{\kappa_{j,p}(E)\}_{j=1}^\infty$, the upper bounds works with $q = p'$ (use atomic decompositions etc. [SWW95]).

- Conjecture states that it should work with $q = \infty$ (and $1 < p < \frac{d}{d-1}$).
- Conjecture holds for radial functions. [S-Wainger-Wright '97]
- Conjecture holds for E a convex sequence. [S-Tao-Wright '03].
- Conjecture is [open](#) for self-similar sets.

III. Weighted L^p

$$\|f\|_{L^p(w)} = \left(\int |f(x)|^p w(x) dx \right)^{1/p}.$$

Problem: (Almost) characterize $L^p(w)$ boundedness of $M_{\mathcal{E}}$ for given p, w . [Wide open in general.](#)

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- **Power weights** $w(x) = |x|^a$ (Duoandikoetxea–Vega '96, Juyoung Lee '25), Duoandikoetxea–Seijo '02.

Almost characterization, i.e. for all $\mathcal{E}$ an almost sharp (p, a) range by Fraccaroli-Roos-S. '26.

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Problem: (Almost) characterize $L^p(w)$ boundedness of $M_{\mathcal{E}}$ for given p, w .

- Sparse domination inequalities for $M_{\mathcal{E}}$ (Lerner, Lacey ...) give weighted inequalities for large classes.
- Almost sharp sparse domination inequalities for $M_{\mathcal{E}}$ are almost equivalent with almost sharp $L^p \rightarrow L^q$ improving bounds, $p \leq q$, for local maximal operators M_E .

For $E = \{1\}$ (Strichartz '70), and $E = [1, 2]$, Schlag 97, Schlag–Sogge '97, Lee '03,
More general E : Anderson–Hughes–Roos–S '21, Roos–S. '23.,...

L^p improving questions for M_E

The **type set** is given by



$$\mathcal{T}_E = \{(\frac{1}{p}, \frac{1}{q}) \in [0, 1]^2 : M_E : L^p \rightarrow L^q\}$$

$\mathcal{T}_E$ is always convex (by interpolation). Here we'll ignore endpoints, so we'll only care about $\overline{\mathcal{T}_E}$.

Question (1)

For given $E \subset [1, 2]$ what is $\overline{\mathcal{T}_E}$?

Question (2)

Which closed convex sets can arise as $\overline{\mathcal{T}_E}$ for some $E \subset [1, 2]$?

- [Roos-S. '23] Complete and somewhat surprising answer.

Upper Assouad spectrum of dimensions

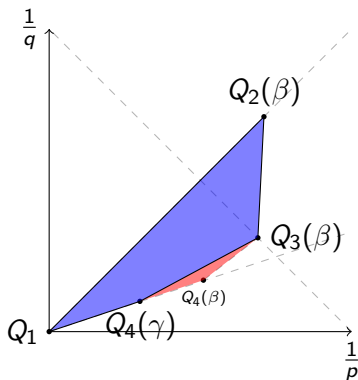
Definition (Fraser–Yu '18, Fraser–Hare–Hare–Troscheit–Yu '19)

For $\theta \in [0, 1)$ let $\overline{\dim}_{A,\theta} E$ be smallest $\alpha > 0$ such that $\forall \varepsilon > 0$

$$N(E \cap I, \delta) \leq C_\varepsilon (\delta/|I|)^{-\alpha-\varepsilon} \quad \text{for all } \delta \in (0, 1), \text{ all } I \text{ with } |I| \geq \delta^\theta$$

- $\theta \mapsto \overline{\dim}_{A,\theta} E$ is continuous, increasing on $[0, 1)$.
- $\overline{\dim}_{A,\theta} E$ is intermediate between the Minkowski dimension β and the (quasi)-Assouad dimension γ .
- $\beta = \gamma = \overline{\dim}_{A,\theta} E$ for self similar Cantor sets.
- E convex sequence: $\gamma = 1$, $\overline{\dim}_{A,\theta} E = \begin{cases} (1-\theta)^{-1}\beta & \text{if } 0 \leq \theta \leq 1-\beta \\ 1 & \text{if } 1-\beta \leq \theta < 1 \end{cases}$
- Both examples are (β, γ) -quasi-Assouad regular.

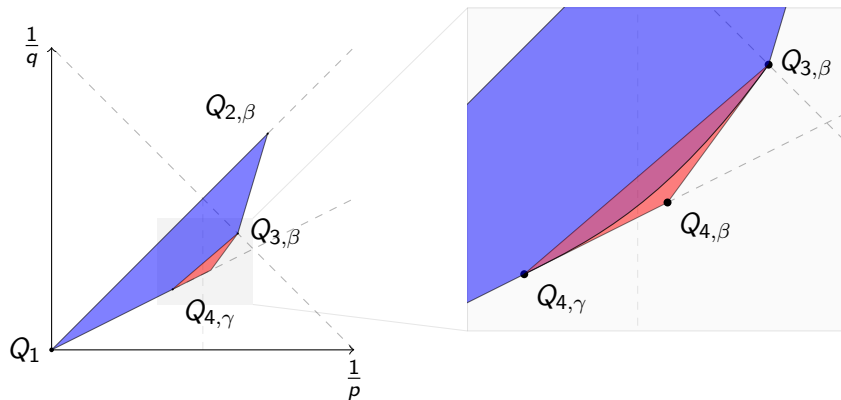
$L^p \rightarrow L^q$ (AHRS21, RS23), sharp for qA -regular sets



$$Q_1 = (0, 0), \quad Q_2(\beta) = \left(\frac{d-1}{d-1+\beta}, \frac{d-1}{d-1+\beta} \right),$$

$$Q_3(\beta) = \left(\frac{d-\beta}{d-\beta+1}, \frac{1}{d-\beta+1} \right), \quad Q_4(\gamma) = \left(\frac{d(d-1)}{d^2-2\gamma+1}, \frac{d-1}{d^2-2\gamma+1} \right).$$

Which type sets are possible? (Roos-S '23)



A convex, closed $Q(\beta, \gamma) \subset A \subset Q(\beta, \beta) \Rightarrow \exists E \subset [1, 2] : A = \overline{\mathcal{T}_E}$

Main open problem: Given general $E \subset [1, 2]$, determine $\overline{\mathcal{T}_E}$.

Another dimensional quantity (Legendre–Assouad function)

Definition (Beltran–Roos–S.'24+)

For $\alpha \in \mathbb{R}$ let $\nu_E^\sharp(\alpha)$ be the infimum over $\xi \geq 0$ such that $\forall \varepsilon > 0$

$$\sup_{\delta < |I| < 1} N(E \cap I, \delta) |I|^{-\alpha} \leq C_\varepsilon \delta^{-\xi - \varepsilon} \quad \forall \delta > 0.$$

- $\nu_E^\sharp(\alpha) = \beta$ for $\alpha \leq 0$
- $\nu_E^\sharp(\alpha) = \alpha$ for $\alpha \geq \gamma$.
- $\nu_E^\sharp$ convex, increasing, continuous.

Based on long Knapp examples: A necessary condition ([RS23]) for $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{Q}(\beta, \beta) \setminus \mathcal{Q}(\beta, \gamma)$ to be in $\overline{\mathcal{T}}_E$ can be rewritten as

$$\frac{1}{q} \nu_E^\sharp\left(\frac{q(d-1)}{2}\left(1 - \frac{1}{p} - \frac{1}{q}\right)\right) \leq \frac{d-1}{2} - \frac{d+1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)$$

Conjecture: This is also sufficient.

Legendre-Assouad function $\nu_E^\sharp$

Rewriting the definition

$$\nu_E^\sharp(\alpha) = \limsup_{\delta \rightarrow 0} \frac{\log \left(\sup_{\delta \leq |I| \leq 1} |I|^{-\alpha} N(E \cap I, \delta) \right)}{\log \frac{1}{\delta}}$$

is Legendre-Assouad function in [Beltran–Roos–Rutar–S.'25].

Let

$$\nu_E(\theta) = -(1 - \theta) \dim_{A, \theta} E$$

which is increasing in θ (but not necessarily convex). Then

$$\nu_E^\sharp(\alpha) = \sup_{0 \leq \theta \leq 1} (\theta \alpha - \nu_E(\theta))$$

i.e., $\nu_E^\sharp$ is the Legendre transform on ν_E .

IV. Other problems

The Legendre-Assouad function keeps coming up in other problems.

- Power weight inequalities for spherical maximal functions M_E , $M_{\mathcal{E}}$ [FRS] where the (closure of the) set

$$\mathcal{W}_p = \{(\frac{1}{p}, \frac{\alpha}{p}) : M_E \text{ bounded on } L^p(|x|^\alpha)\}$$

is described in terms of the $\nu_E^\sharp$ or its pseudo-inverse.

- [Fractal local smoothing conjecture](#) [BRRS '25] which addresses optimal space-time estimates for the wave equation for $2 < p < \frac{2d}{d-1}$.

Local smoothing conjecture for the wave equation on $\mathbb{R}^{d+1}$

Let $\{t_\nu\}$ a set of 2^{-j} -separated times in $[1, 2]$. Assume $\text{supp}(\widehat{f}) \subset \{|\xi| \approx 2^j\}$,

Conjecture (Sogge 91', Heo-Nazarov-S. '11)

For $q > \frac{2d}{d-1}$, $s = (d-1)(\frac{1}{2} - \frac{1}{q})$,

$$\left(\sum_{\nu} \|e^{it_{\nu}\sqrt{-\Delta}} f\|_q^q \right)^{1/q} \lesssim 2^{js} \|f\|_q$$

Non-endpoint version solved for $d = 2$ (Guth–Wang–Zhang 20)

For a fractal version want to replace $\{t_\nu\}$ with a 2^{-j} -discretization E_j of $E \subset [1, 2]$, i.e. any set of 2^{-j} -separated points in E .

Now $2 < q \leq \frac{2d}{d-1}$ is the interesting range. For fractal E can we do better than $s > 1/q$ in this q -range?.

Fractal local smoothing conjecture

Recall $\nu_E^\#(\alpha) = \limsup_{\delta \rightarrow 0} \frac{\log \left(\sup_{\delta \leq |I| \leq 1} |I|^{-\alpha} N(E \cap I, \delta) \right)}{\log \frac{1}{\delta}}$

Conjecture (Beltran, Roos, Rutar, S. '25)

For $2 \leq q \leq \frac{2d}{d-1}$, $s_q = (d-1)(1/2 - 1/q)$, $\text{supp}(\hat{f}) \subset \{|\xi| \approx 2^j\}$,

$$\left(\sum_{t \in E_j} \|e^{it\sqrt{-\Delta}} f\|_q^q \right)^{1/q} \lesssim 2^{js} \|f\|_q, \quad s > \frac{\nu_E^\#(qs_q)}{q}$$

where E_j is any 2^{-j} -discretization of E .

It is optimal (up to endpoints) and it holds for radial functions.

Thanks for your attention!